

These problems are related to the material covered in Lectures 19-22. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due by 11:59pm on 10/30/2020 and should be submitted electronically as a pdf-file e-mailed to zzyzhang@mit.edu and roed@mit.edu (please include "18.782" in the subject of the email). You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions and to identify collaborators and any sources not listed in the syllabus.

Problem 1. Products of varieties (20 points)

1. Define the map $\phi: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^3$ by

$$\phi((x_0 : x_1), (y_0 : y_1)) = [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1].$$

Prove that ϕ is a morphism whose image V is a variety (specify $I(V)$ explicitly). Then prove that $\mathbb{P}^1 \times \mathbb{P}^1 \simeq V$ by giving an inverse morphism.

2. Let x_0, \dots, x_m and y_0, \dots, y_n be homogeneous coordinates for \mathbb{P}^m and \mathbb{P}^n respectively, and for $0 \leq i \leq m$ and $0 \leq j \leq n$ let z_{ij} be homogeneous coordinates for \mathbb{P}^N , where $N = (m+1)(n+1) - 1$. Prove that the *Segre morphism* from $\varphi: \mathbb{P}^m \times \mathbb{P}^n \rightarrow \mathbb{P}^N$ defined by $z_{ij} = x_i y_j$ is an isomorphism to a projective variety $V \subseteq \mathbb{P}^N$.
3. The field \mathbb{C} is an affine \mathbb{R} -algebra (an integral domain finitely generated over \mathbb{R}). Show that the tensor product of \mathbb{R} -algebras $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is not an integral domain, hence not an affine \mathbb{R} -algebra. Thus Lemma 16.4, which states that a tensor product of affine algebras is again an affine algebra, depends on the assumption that k is algebraically closed. Explain exactly where in the proof this assumption is used.

Problem 2. Valuation rings (50 points)

An *ordered abelian group* is an abelian group Γ with a total order \leq that is compatible with the group operation. This means that for all $a, b, c \in \Gamma$ the following hold:

$$\begin{aligned} a \leq b \leq a &\implies a = b && \text{(antisymmetry)} \\ a \leq b \leq c &\implies a \leq c && \text{(transitivity)} \\ a \not\leq b &\implies b \leq a && \text{(totality)} \\ a \leq b &\implies a + c \leq b + c && \text{(compatibility)} \end{aligned}$$

Note that totality implies reflexivity ($a \leq a$). Given an ordered abelian group Γ , we define the relations $\geq, <, >$ and the sets $\Gamma_{\leq 0}, \Gamma_{\geq 0}, \Gamma_{< 0}$, and $\Gamma_{> 0}$ in the obvious way.

A *valuation* v on a field K is a surjective homomorphism $v: K^\times \rightarrow \Gamma$ to an ordered abelian group Γ that satisfies $v(x+y) \geq \min(v(x), v(y))$ for all $x, y \in K^\times$. The group Γ is called the *value group* of v , and when $\Gamma = \{0\}$ we say that v is the *trivial valuation*.

1. Let R be a valuation ring with fraction field F , and let $v: F^\times \rightarrow F^\times/R^\times = \Gamma$ be the quotient map. Show that the relation \leq on Γ defined by

$$v(x) \leq v(y) \iff y/x \in R,$$

makes Γ an ordered abelian group and that v is a valuation on F .

2. Let F be a field and let $v: F^\times \rightarrow \Gamma$ be a non-trivial valuation. Prove that the set

$$R_v := v^{-1}(\Gamma_{\geq 0}) \cup \{0\}$$

is a valuation ring of F and that $v(x) \leq v(y) \iff y/x \in R_v$ (note that this includes proving that R_v is actually a ring).

3. Let Γ an ordered abelian group, let k be a field, and let A be the k -algebra generated by the set of formal symbols $\{x^a : a \in \Gamma_{\geq 0}\}$, with multiplication defined by $x^a x^b = x^{a+b}$. This means that A consists of all sums of the form

$$\sum_{a \in \Gamma_{\geq 0}} c_a x^a$$

with $c_a \in k$ and only finitely many nonzero. Let F be the fraction field of A and define the function $v: F^\times \rightarrow \Gamma$ by

$$v\left(\frac{\sum c_a x^a}{\sum d_a x^a}\right) = \min\{a : c_a \neq 0\} - \min\{a : d_a \neq 0\}.$$

Prove that v is a valuation of F with value group Γ .

4. Let $v: F^\times \rightarrow \Gamma_v$ and $w: F^\times \rightarrow \Gamma_w$ be two valuations on a field F , and let R_v and R_w be the corresponding valuation rings. Prove that $R_v = R_w$ if and only if there is an order preserving isomorphism $\rho: \Gamma_v \rightarrow \Gamma_w$ for which $\rho \circ v = w$ (in which case we say that v and w are equivalent). Thus there is a one-to-one correspondence between valuation rings and equivalence classes of valuations.
5. Let D be an integral domain that is properly contained in its fraction field F , and let \mathcal{R} be the set of local rings that contain D and are properly contained in F . Partially order \mathcal{R} by writing $R_1 \leq R_2$ if $R_1 \subseteq R_2$ and the maximal ideal of R_1 is contained in the maximal ideal of R_2 .¹ Zorn's lemma states that any partially ordered set in which every chain has an upper bound contains at least one maximal element. Use this to prove that \mathcal{R} contains a maximal element R , and then show that any such R is a valuation ring (note: the hypothesis of Zorn's lemma is understood to include the empty chain, so you must prove that \mathcal{R} is nonempty).
6. Prove that every valuation ring R is *integrally closed*. This means that every element of the fraction field of R that is the root of a monic polynomial in $R[x]$ lies in R .

Problem 3. Singularities (30 points)

Let X be the projective closure of the affine curve $y^2 = x^5$ over an algebraically closed field of characteristic 0.

1. Find the singularities of X .
2. Find a smooth projective curve Y that is birational to X .

¹This is known as the *dominance* ordering.

Problem 4. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = “mind-numbing,” 10 = “mind-blowing”), and how difficult you found the problem (1 = “trivial,” 10 = “brutal”). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.