18.782 Introduction to Arithmetic Geometry **Fall 2020**
Problem Set #7 **Due: 10/23/2020**

These problems are related to the material covered in Lectures 13-18. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due by 5pm on 10/23/2020 and should be submitted electronically as a pdf-file e-mailed to zzyzhang@mit.edu and roed@mit.edu (please include "18.782" in the subject of the email). You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions and to identify collaborators and any sources not listed in the syllabus.

Problem 1. Short answer questions (25 points)

Please justify each of the following assertions in 25 words or less. They all have short easy proofs and are meant to check your understanding, not to challenge you.

1. For any ideals $I, J \subseteq \overline{k}[x_1, \ldots, x_n]$ the affine algebraic sets $Z_{I\cup J}$ and Z_{I+J} are equal.
2. For any ideals $I, J \subseteq \overline{k}[x_1, \ldots, x_n]$ the affine algebraic sets $Z_{I\cap J}$ and Z_{IJ} are equal.
3. Every nonempty open set in $\mathbb{A}^n$ is dense.
4. Every affine variety is connected.
5. The closure of the image of a morphism of affine varieties is a variety.
6. A morphism of affine varieties is dominant if and only if the corresponding morphism of coordinate rings is injective.
7. $\mathbb{A}^n$ and $\mathbb{P}^n$ are birationally equivalent (prove this explicitly by writing down the rational maps; indicate in each case whether your rational map is a morphism or not).

Problem 2. Sheaves (25 points)

In this problem and the next all rings are commutative with identity. This includes the zero ring, which is the unique ring with exactly one element $0 = 1$.

Definition 1. A *presheaf* (of rings) $\mathcal{F}$ on a topological space X assigns to each open set U a ring $\mathcal{F}(U)$, and to each nested pair of open sets $U \subseteq V$ a ring homomorphism

$$\rho_{VU} \colon \mathcal{F}(V) \to \mathcal{F}(U),$$

(note the direction) such that ρ_{UU} is the identity map and for all open sets $U \subseteq V \subseteq W$,

$$\rho_{VU} \circ \rho_{WV} = \rho_{WU}.$$

In the motivating examples for presheaves, each $\mathcal{F}(U)$ is a ring of functions defined on U, and the maps ρ_{VU} simply restrict the domains of these functions from V to U. For this reason the maps ρ_{VU} are called *restriction maps*, and for each $f \in \mathcal{F}(U)$ the element $\rho_{VU}(f) \in \mathcal{F}(V)$ is called the *restriction* of f to V.[1]

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[1] One can define presheaves for any category: just replace "ring" with your favorite type of object and "ring homomorphism" with the appropriate morphism; in general a presheaf is just a contravariant functor from the category of open sets of X (whose morphisms are inclusion maps) to any category you like.

Now let X be an affine variety over $k = \overline{k}$ with function field $k(X)$.

(a) Show that for any open set $U \subseteq X$ the set

$$\mathcal{O}(U) = \{f \in k(X) : U \subseteq \operatorname{dom}(f)\}$$

is a subring of $k(X)$ that contains k (hence is a k-algebra), and has fraction field $k(X)$. Prove that $\mathcal{O}(X)$ is equal to the coordinate ring $k[X]$.

(b) For open sets U in X, let $\mathcal{F}(U) = \mathcal{O}(U)$, and for nested open sets $U \subseteq V$ in X, let ρ_{VU} be the inclusion map $\mathcal{O}(V) \subseteq \mathcal{O}(U)$. Show that $\mathcal{F}$ is a presheaf.

Definition 2. An *open cover* of an open set U is a collection of open sets $\{U_\alpha\}_{\alpha\in A}$ whose union is U. A presheaf $\mathcal{F}$ is a *sheaf* if it satisfies the *sheaf axiom*.

Sheaf axiom: For every open cover $\{U_a\}_{a\in A}$ of $U = \bigcup_a U_a$ and every collection $\{f_a\}$ with $f_a \in \mathcal{F}(U_\alpha)$ such that for all $a, b \in A$ the restrictions of f_a and f_b to $U_a \cap U_b$ are equal, there is a *unique* $f \in \mathcal{F}(U)$ whose restriction to U_a is f_a for all $a \in A$.

(c) Prove that in any sheaf of rings $\mathcal{F}$, the ring $\mathcal{F}(\emptyset)$ must be the zero ring.[2]

(d) Modify the presheaf $\mathcal{F}$ defined in part (b) so that $\mathcal{F}(\emptyset)$ is the zero ring and $\rho_{U\emptyset}$ is the unique homomorphism to the zero ring (for each open set U). The resulting presheaf $\mathcal{O}_X$ is called the *structure sheaf* of X. Prove that $\mathcal{O}_X$ is indeed a sheaf.[3]

Definition 3. A pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X is called a *ringed space*.

Problem 3. Affine schemes (50 points)

As in Problem 2, all rings in this problem are commutative rings with identity.

Let R be a ring whose elements f we shall call *regular functions*. Let

$$X = \operatorname{Spec} R$$

be the set of prime ideals of R. We shall call the elements P of X *points*. For any point $P \in X$, the *residue field* of P is the fraction field of the integral domain R/P. For any regular function $f \in R$ and any point $P \in X$ we define $f(P)$ to be the image of f in the residue field of P (this means we reduce f mod P to get an element $\bar{f}$ of R/P that we then regard as the element $\bar{f}/1$ of the fraction field). Notice that $f(P)$ is defined at every point $P \in X$, which is why we call f a regular function. The field in which $f(P)$ lies will vary with P, but in any case we can meaningfully ask whether $f(P) = 0$ or not. We now define an *algebraic set* Z as the zero locus of a set of regular functions $S \subseteq R$:

$$Z_S = \{P \in X : f(P) = 0 \text{ for all } f \in S\}.$$

We now endow X with the Zariski topology by letting the closed sets be the algebraic sets.

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[2] Some authors include this restriction in the definition of a presheaf, but this is unnecessary.

[3] While this modification may seem artificial, it can actually be justified by noting that if you restrict the domain of a set of functions with the same codomain to the empty set they become a single function (there is exactly one function from the empty set to a given set S, it is the function whose graph is the empty set).

(a) Prove that this actually defines a topology by showing that the empty set and X are both closed, and that arbitrary intersections and finite unions are closed.

(b) Show that for $S \subseteq R$ the algebraic set Z_S is precisely the set of prime ideals of R (points of X) that contain S, and that $Z_S = Z_I$, where I is the ideal generated by S.

(c) We call a point $P \in X$ a *closed point* if the set $\{P\}$ is closed in the Zariski topology. Prove that the closed points in X are precisely the maximal ideals of R.

(d) Let R be the polynomial ring $k[x_1, x_2]$ for some field $k = \overline{k}$. Give a bijection between closed points P in X and elements (a_1, a_2) of $\mathbb{A}^2(k)$, and show that under this bijection we have $f(P) = f(a_1, a_2)$ for any $f \in R$ and closed point P. Describe the points of X that are not closed and evaluate $3x_1^2 + 2x_1x_2 - 5x_2^2$ at two such points.

(e) Let $R = \mathbb{Z}$. Describe the open sets in X and the residue fields at points $P \in X$. Evaluate $15 \in R$ at the points $(0), (2), (3), (5) \in X$, and describe the zero locus of 15.

Now let $V \subseteq \mathbb{A}^n$ be an affine variety of dimension 1 over a field $k = \overline{k}$, let $R = k[V]$ be its coordinate ring, let $k(V)$ be its function field, and let $X = \operatorname{Spec} R$.

(f) Prove that every point in V corresponds to a maximal ideal of $k[x_1, \ldots, x_n]$ that contains $I(V)$, which in turn corresponds to a maximal ideal of R, hence a closed point in X. We may therefore identify V with the set of closed points in X.

(g) Prove that the only point in X that is not in V is the zero ideal and that this point is dense in X and contained in every nonempty open set in X.

(h) Let us say that an element $f = g/h$ of the function field $k(V)$ is *regular* at $P \in X$ if $h(P) \neq 0$ (where $h(P)$ is defined as above). Argue that for points $P \in V$ this definition agrees with the corresponding definition given in Lecture 15. Then show that every $f \in k(V)$ is regular at the point $P = (0)$ and that we may thus add (0) to $\operatorname{dom}(f)$ for every $f \in k(V)$.

Definition 4. An *affine scheme* is a ringed space $(X, \mathcal{O}_X)$ where $X = \operatorname{Spec} R$ for a ring R.

(i) Use (h) to define a structure sheaf $\mathcal{O}_X$ for $X = \operatorname{Spec} k[V]$, as in part (d) of Problem 2, thereby constructing an affine scheme corresponding to V.

(j) Use (e) to define a structure sheaf $\mathcal{O}_X$ for $X = \operatorname{Spec} \mathbb{Z}$.

Just in case you are interested, here is the general definition of a scheme.

Definition 5. A *scheme* is a ringed space $(X, \mathcal{O}_X)$ for which every point has a neighborhood U for which $(U, \mathcal{O}_U)$ is an affine scheme, where $\mathcal{O}_U$ is $\mathcal{O}_X$ restricted to U.

Problem 4. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.