18.782 Introduction to Arithmetic Geometry **Fall 2020**
Problem Set #6 **Due: 10/16/2020**

These problems are related to the material covered in Lectures 14-16. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due by 5pm on 10/16/2020 and should be submitted electronically as a pdf-file e-mailed to zzyzhang@mit.edu and roed@mit.edu (please include "18.782" in the subject of the email). You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions and to identify collaborators and any sources not listed in the syllabus.

Problem 1. Homogeneous ideals and projective algebraic sets (50 points)

Let k be a perfect field and fix an algebraic closure $\overline{k}$. Fix a positive integer n, let R be the polynomial ring $\overline{k}[x_0, \ldots, x_n]$, and let R^h be the set of all homogeneous polynomials in R. Recall from lecture that (1) a homogenous ideal in R is an ideal generated by elements of R^h, (2) for any subset $S \subset R$ the algebraic set Z_S is the zero locus in $\mathbb{P}^n$ of $S \cap R^h$, and (3) for any subset $Z \subset \mathbb{P}^n$ the ideal $I(Z)$ is the homogeneous ideal generated by the polynomials in R^h that vanish at every point in Z.

(a) Prove the "homogeneous *Nullstellensatz*," which says that if $I \subseteq R$ is a homogeneous ideal and $f \in R^h$ is a nonconstant polynomial that vanishes at every point in Z_I, then $f^r \in I$ for some $r > 0$ (hint: re-interpret the problem in $\mathbb{A}^{n+1}$ and use the usual *Nullstellensatz*).

(b) Let R_+ denote the set of all polynomials in R with no constant term. Prove that for any homogeneous ideal $I \subseteq R$ the following are equivalent:

(i) Z_I is the empty set;

(ii) Either $\sqrt{I} = R$ or $\sqrt{I} = R_+$.

(iii) I contains every polynomial in R^h of degree d, for some $d > 0$.

(c) Prove the following:

(i) If $S \subseteq T$ are subsets of R^h then $Z_T \subseteq Z_S$.

(ii) If $Y \subseteq Z$ are subsets of $\mathbb{P}^n$ then $I(Z) \subseteq I(Y)$.

(iii) For any two subsets Y, Z in $\mathbb{P}^n$ we have $I(Y \cup Z) = I(Y) \cap I(Z)$.

(iv) For any algebraic set $Z \subseteq \mathbb{P}^n$ we have $Z_{I(Z)} = Z$.

(v) For any homogeneous radical ideal I for which $Z_I \neq \emptyset$ we have $I(Z_I) = I$.

(d) Conclude from (a), (b), and (c) that there is a one-to-one inclusion-reversing correspondence between algebraic sets in $\mathbb{P}^n$ and homogeneous radical ideals in R that are not equal to R_+, given by the maps $Z \to I(Z)$ and $I \to Z_I$.[1]

(e) Prove that an algebraic set $Z \in \mathbb{P}^n$ is irreducible if and only if $I(Z)$ is a prime ideal.

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[1]The ideal R_+ that does not appear in this correspondence is called the *irrelevant* ideal.

Problem 2. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.