18.782 Introduction to Arithmetic Geometry **Fall 2020**
Problem Set #10 **Due: 11/13/2020**

These problems are related to the material covered in Lectures 26-28. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due by the start of class on 11/13/2020 and should be submitted electronically as a pdf-file e-mailed to zzyzhang@mit.edu and roed@mit.edu (please include "18.782" in the subject of the email). You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions and to identify collaborators and any sources not listed in the syllabus.

Recall that we have defined a *curve* as a smooth projective variety of dimension one (and varieties are defined to be irreducible algebraic sets).

Problem 1. Bezout's theorem (50 points)

In this problem k is an algebraically closed field.
A curve in $\mathbb{P}^2$ is called a *plane curve*.[1]

(a) Prove that every plane curve X/k is a hypersurface, meaning that its ideal $I(X)$ is of the form (f), where f is a homogeneous polynomial in $k[x, y, z]$. Then show that every generator for $I(X)$ has the same degree.

The *degree* of X (denoted $\deg X$) is the degree of any generator for its homogeneous ideal.

(b) Let F/k be a function field, let P be a place of F, and let $f \in \mathcal{O}_P$. Prove that the ring $\mathcal{O}_P/(f)$ is a k-vector space of dimension $\operatorname{ord}_P(f)$.

Given a nonconstant homogeneous polynomial $g \in k[x, y, z]$ that is relatively prime to f, we can represent g as an element of the local ring $\mathcal{O}_{P,X}$ of functions in X that are regular at P by picking a homogeneous polynomial h that does not vanish at P and representing g as g/h reduced modulo $I(X)$, an element of $k(X)$. Note that in terms of computing $\operatorname{ord}_P(g)$ it makes no difference which h we pick, $\operatorname{ord}_P(g)$ will always be equal to the order of vanishing of g at P, a nonnegative integer. We then define the *divisor* of g in $\operatorname{Div}_k X$ to be

$$\operatorname{div}_X g = \sum \operatorname{ord}_P(g)P.$$

Note that $\operatorname{div}_X g$ is *not* a principal divisor.[2] Indeed, $\deg \operatorname{div}_X g$ is never zero.

(c) Prove that $\deg \operatorname{div}_X g$ depends only on $\deg g$ (i.e. $\deg \operatorname{div}_X g = \deg \operatorname{div}_X h$ whenever g and h have the same degree and are both relatively prime to f). Then prove that $\deg \operatorname{div}_X g$ is a linear function of $\deg g$.

Now suppose that g is irreducible and nonsingular, so it defines a plane curve Y/k.

(d) Prove that $\deg \operatorname{div}_Y f = \deg \operatorname{div}_X g$.

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[1] Plane curves are not usually required to be smooth or irreducible, but ours are.
[2] By varying h locally we eliminate the poles that would be present if we fixed a global choice for h.

Definition 1. Let f and g be two nonconstant homogeneous polynomials in $k[x, y, z]$ with no common factor, and let P be a point in $\mathbb{P}^2$. The *intersection number* of f and g at P is

$$I_P(f, g) := \dim_k \mathcal{O}_{P,\mathbb{P}^2}/(f, g)$$

Here $\mathcal{O}_{P,\mathbb{P}^2}$ denotes the ring of functions in $k(\mathbb{P}^2)$ that are regular at P, and f and g are represented as elements of this ring by choosing homogeneous denominators of appropriate degree that do not vanish at P, exactly as described above.

As above, let X/k and Y/k denote plane curves defined by relatively prime homogeneous polynomials f and g, and let $I(f, g) = \sum_P I_P(f, g)$.

(e) Prove that $I(f, g)$ is equal to $\deg \operatorname{div}_X g = \deg \operatorname{div}_Y f$.

(f) Prove **Bezout's Theorem** for plane curves:

$$I(f, g) = \deg f \deg g.$$

In fact Bezout's theorem holds even when f and g are not necessarily irreducible and nonsingular, but you need not prove this. It should be clear that f and g do not need to be irreducible; just factor them and apply the theorem to all pairs of factors. You proof should also handle cases where just one of f or g is singular; it takes a bit more work to handle the case where both f and g are singular and intersect at a common singularity. The assumption that $k = \overline{k}$ is necessary, in general, but the inequality $I(f, g) \leq \deg f \deg g$ always holds.

Problem 2. Derivations and differentials (50 points)

A *derivation* on a function field F/k is a k-linear map $\delta\colon F \to F$ such that

$$\delta(fg) = \delta(f)g + f\delta(g).$$

for all $f, g \in F$.

(a) Prove that the following hold for any derivation δ on F/k:

(i) $\delta(c) = 0$ for all $c \in k$.

(ii) $\delta(f^n) = nf^{n-1}\delta(f)$ for all $f \in F^\times$ and $n \in \mathbb{Z}$.

(iii) If k has positive characteristic p then $\delta(f^p) = 0$ for all $f \in F$.

(iv) $\delta(f/g) = (\delta(f)g - f\delta(g))/g^2$ for all $f, g \in F$ with $g \neq 0$.

To simplify matters, we henceforth assume that k has characteristic zero.[3]

The simplest example of a derivation is in the case where $F = k(x)$ is the rational function field and $\delta\colon F \to F$ is the map defined by $\delta(f) = \partial f/\partial x$. We want to generalize this example to arbitrary function fields.

[3]For those who are interested, the key thing that changes in characteristic $p > 0$ is that everywhere we require an element x to be transcendental we need to additionally require it to be a *separating element*, which means that $F/k(x)$ is a separable extension.

Let x be a transcendental element of F/k. Any $y \in F$ is then algebraic over $k(x)$ and has a minimal polynomial $\lambda \in k(x)[T]$. After clearing denominators we can assume that $\lambda \in k[x, T]$. We now formally define

$$\frac{\partial y}{\partial x} := -\frac{\partial\lambda/\partial x}{\partial\lambda/\partial T}(y) \in k(x, y) \subseteq F$$

and let the map $\delta_x\colon F \to F$ send y to $\partial y/\partial x$.

One can show (but you are not asked to do this) that δ_x is a derivation on F/k. Note that we get a derivation δ_x for each transcendental x in F. Now let D_F be the set of all deriviations on F/k.

(b) Let x be a transcendental element of F/k. Prove that for any $\delta_1, \delta_2 \in \mathrm{D}_F$ we have $\delta_1(x) = \delta_2(x) \Rightarrow \delta_1 = \delta_2$. Conclude that δ_x is the unique $\delta \in \mathrm{D}_F$ for which $\delta(x) = 1$.

(c) Prove the following:

- (i) For all $\delta_1, \delta_2 \in \mathrm{D}_F$ the map $(\delta_1 + \delta_2)\colon F \to F$ defined by $f \mapsto \delta_1(f) + \delta_2(f)$ is a derivation (hence an element of D_F).
- (ii) For all $f \in F$ and $\delta \in \mathrm{D}_F$ the map $(f\delta)\colon F \to F$ defined by $g \mapsto f\delta(g)$ is a derivation (hence an element of D_F).
- (iii) Every $\delta \in \mathrm{D}_F$ satisfies $\delta = \delta(x)\delta_x$ (in particular, the *chain rule* $\delta_y = \delta_y(x)\delta_x$ holds for any transcendental $x, y \in F/k$).

It follows that we may view D_F as one-dimensional F-vector space with any δ_x as a basis vector. But rather than fixing a particular basis vector; instead, let us define a relation on the set S of pairs (u, x) with $u, x \in F$ and x transcendental over k:

$$(u, x) \sim (v, y) \iff v = u\delta_y(x). \tag{1}$$

(d) Prove that $\sim$ is an equivalence relation on S.

For each transcendental element $x \in F/k$, let the symbol dx denote the equivalence class of $(1, x)$, and for $u \in F$ define udx to be the equivalence class of (u, x); we call dx a *differential.* It follows from part (iii) of (d) that every derivation δ can be uniquely represented as $\delta = udx$ for some $u \in F$, but now we have the freedom to change representations; we may also write $\delta = vdy$ for any transcendental element y, where $v = u\delta_y(x) = u\partial x/\partial y$.

(e) Prove that $d(x + y) = dx + dy$ and $d(xy) = xdy + ydx$ for all $x, y \in F/k$ so that $x, y, x + y$ (in the first case) and xy (in the second case) are transcendental.

Let us now extend our differential notation to elements of F that are not transcendental over k. Recall that k is algebraically closed in F, so we only need to consider elements of k.

(f) Prove that defining $da = 0$ for all $a \in k$ ensures that (e) holds for all $x, y \in F$, and that no other choice does.

Now momentarily forget everything above and just define Δ_F to the F-vector space generated by the set of formal symbols $\{dx : x \in F\}$, subject to the relations

$$(1)\ d(x + y) = dx + dy, \qquad (2)\ d(xy) = xdy + ydx, \qquad (3)\ da = 0 \text{ for } a \in k.$$

Note that x and y denote elements of F (functions), not free variables, so Δ_F reflects the structure of F and will be different for different function fields.

(g) Prove that $\dim_F \Delta_F = 1$, and that any dx with $x \notin k$ is a basis.

The set $\Delta = \Delta_F$ is often used as an alternative to the set of Weil differentials Ω. They are both one-dimensional F-vector spaces, hence isomorphic (as F-vector spaces). But in order to be useful, we need to associate divisors to differentials in Δ, as we did for Ω.

For any differential $\omega \in \Delta$ and any place P, we may pick a uniformizer t for P and write $\omega = w dt$ for some function $w \in F$ that depends on our choice of t; note that t is necessarily transcendental over k, since it is a uniformizer. We then define $\operatorname{ord}_P(\omega) := \operatorname{ord}_P(w)$, and the divisor of ω is then given by

$$\operatorname{div} \omega := \sum_P \operatorname{ord}_P(\omega) P.$$

As in Problem 1, the value $\operatorname{ord}_P(\omega)$ does not depend on the choice of the uniformizer t.

(h) Prove that $\operatorname{div} u dv = \operatorname{div} u + \operatorname{div} dv$ for any $u, v \in F$ with v transcendental over k. Conclude that the set of nonzero differentials in Δ constitutes a linear equivalence class of divisors.

(i) Let $F = k(t)$ be the rational function field. Compute $\operatorname{div} dt$ and prove that it is a canonical divisor. Conclude that a divisor $D \in \operatorname{Div}_k \mathbb{P}^1$ is canonical if and only if $D = \operatorname{div} df$ for some transcendental $f \in F$.

Part (i) holds for arbitrary curves, but you are not asked to prove this. It follows that the space of differentials Δ plays the same role as the space of Weil differentials Ω, and it has the virtue of making explicit computations much easier.

(j) Prove that the curve $x^2 + y^2 + z^2$ over $\mathbb{Q}$ has genus 0 (even though it is not isomorphic to $\mathbb{P}^1$ because it has no rational points) by explicitly computing a canonical divisor.

Problem 3. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.