In this lecture we pause our discussion of the proof of the Weil conjectures to sketch the definition of étale cohomology. More details can be found in [1, Chap. 1].

36.1 Étale maps

Étale maps play the role in algebraic geometry that local diffeomorphisms do in differential geometry, or unbranched covers of Riemann surfaces in complex analysis, or unramified extensions of number fields in algebraic number theory. The full definition is a little involved, but we can give a version for nonsingular algebraic varieties without too much difficulty.

Definition 36.1. Let X and Y be nonsingular algebraic varieties over an algebraically closed field k, and let $f : X \to Y$ be a regular map. We say that f is *étale* at a point $x \in X$ if the map $df : T_x(X) \to T_{f(x)}(Y)$ on tangent spaces is an isomorphism, and we say that f is *étale* if it is étale at every point of X.

For example, if $f : \mathbb{A}^n \to \mathbb{A}^n$ if given by polynomials $(f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n))$ then we can determine whether f is étale by testing if the Jacobian matrix is nonsingular; this is the same condition as the inverse function theorem for testing that a map of manifolds is a local diffeomorphism.

More generally, the definition of étale is made up of two ingredients.

Flat maps

Definition 36.2. A ring homomorphism $f : A \to B$ is *flat* if the functor $M \mapsto M \otimes_A B$ from A-modules to B-modules is exact. Namely, if whenever

$$0 \to M \to N \to P \to 0$$

is an exact sequence of A-modules, then

$$0 \to M \otimes_A B \to N \otimes_A B \to P \otimes_A B \to 0$$

is exact.

A map $\phi : X \to Y$ of schemes (or varieties) is *flat* if the local homomorphisms $\mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ are flat for all $x \in X$.

It turns out that the only part of exactness that can fail is the injectivity of $M \otimes_A B \to N \otimes_A B$, so flatness of $A \to B$ is equivalent to injectivity of $M \otimes_A B \to N \otimes_A B$ for all injections $M \to N$. For example, the map $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is not flat since the multiplication map $[2] : \mathbb{Z} \to \mathbb{Z}$ becomes 0 after tensoring with $\mathbb{Z}/2\mathbb{Z}$.

Flat morphisms behave nicely with respect to dimensions of fibers: if $f : X \to Y$ is flat then the fiber $X_y = f^{-1}(y)$ has dimension $\dim(X) - \dim(Y)$ whenever it is nonempty. The converse is true when X and Y are nonsingular. If we further assume that $X \to Y$ is finite, then $f : X \to Y$ is flat if and only if the inverse image $f^{-1}(y)$ always has the same number of points (counting multiplicities). Open immersions $X \hookrightarrow Y$ are flat, but closed immersions $X \hookrightarrow Y$ are only flat when they are also open (so a connected component).

Unramified maps

Definition 36.3. Let A and B be local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$. A ring homomorphism $f : A \to B$ is *local* if $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$. A local homomorphism is *unramified* if $B/f(\mathfrak{m}_A)B$ is a finite separable field extension of $A/\mathfrak{m}_A$ or, equivalently, if

1. $f(\mathfrak{m}_A)B = \mathfrak{m}_B$ and
2. the field $B/\mathfrak{m}_B$ is finite and separable over $A/\mathfrak{m}_A$.

A map $\phi : X \to Y$ of schemes (or varieties) is *unramified* if the local homomorphisms $\mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ are unramified for all $x \in X$.

For example, we can localize the inclusion $\mathbb{Z} \to \mathbb{Z}[i]$ at different primes to see both ramified and unramified behavior.

- The prime above (2) in $\mathbb{Z}[i]$ is the ideal $(1+i)$, and the induced map $\mathbb{Z}_{(2)} \to \mathbb{Z}[i]_{(1+i)}$ is ramified since $2\mathbb{Z}[i] \neq (1+i)\mathbb{Z}[i]$. Indeed, $\mathbb{Z}[i]/2\mathbb{Z}[i]$ is not a field: the image of $(1+i)$ is a zero divisor.
- If $p \equiv 1 \pmod 4$ then $p = (a+bi)(a-bi)$ for $a, b \in \mathbb{Z}_{>0}$. Then $(a+bi)$ and $(a-bi)$ are distinct prime ideals in $\mathbb{Z}[i]$ and the induced maps $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}[i]/(a \pm bi)\mathbb{Z}[i]$ are isomorphisms. Thus $\mathbb{Z}_{(p)} \to \mathbb{Z}[i]_{(a \pm b)}$ is unramified.
- Finally, if $p \equiv 3 \pmod 4$ then $p\mathbb{Z}[i]$ is prime and $\mathbb{Z}[i]/p\mathbb{Z}[i]$ is a quadratic extension of $\mathbb{Z}/p\mathbb{Z}$. Again, $\mathbb{Z}_{(p)} \to \mathbb{Z}[i]_{(p)}$ is unramified.

Étale maps

Definition 36.4. A morphism $X \to Y$ of schemes is *étale* if it is flat and unramified, and a homomorphism $f : A \to B$ is *étale* if the induced map $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is étale. Equivalently, it is étale if

1. B is a finitely generated A-algebra,
2. The map $A \to B$ is flat.
3. For all maximal ideals $\mathfrak{m}$ of B, $B_\mathfrak{m}/f(\mathfrak{p})B_\mathfrak{m}$ is a finite separable field extension of $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, where $\mathfrak{p} = f^{-1}(\mathfrak{m})$.

We give some examples of étale maps, sorted based on increasing algebraic complexity of A.

If A is a field and B/A is a finite separable extension of fields then $A \to B$ is étale. More generally, if $B_1, \dots, B_k$ are all finite separable extensions of A then the diagonal embedding $A \to B_1 \times \cdots \times B_k$ is étale. In fact, every étale k-algebra is of this form. Étale k-algebras are a nice category to work with since they behave nicely under base change: if E is an étale k-algebra and k'/k is any field extension then $E \otimes_k k'$ is an étale k'-algebra of the same degree. This property fails for field extensions: $\mathbb{C}/\mathbb{R}$ is a field extension but $\mathbb{C} \times_\mathbb{R} \mathbb{C}$ is not a field extension.

If A is a Dedekind domain, let K be its field of fractions, L any finite separable extension of K, B the integral closure of A in L, $\mathfrak{P}$ a prime ideal of B and $\mathfrak{p} = A \cap \mathfrak{P}$. Then $\mathfrak{p}$ is a prime ideal of A and the map $A_\mathfrak{p} \to B_\mathfrak{P}$ is unramified if and only if

1. In the factorization of $\mathfrak{p}B$ into prime ideals the ideal $\mathfrak{P}$ occurs exactly once, and

2. the residue field extension $B/\mathfrak{P} \supset A/\mathfrak{p}$ is separable.

If $b \in B$ is contained in all ramified prime ideals of B then $A \to B[b^{-1}]$ is étale, and every étale A-algebra is a finite product of algebras of this type.

More generally, suppose A is any ring, $f(T) \in A[T]$ is monic and $b \in A[T]/(f(T))$ has the property that $f'(T)$ is invertible in $(A[T]/(f(T)))[b^{-1}]$. Then $A \to (A[T]/(f(T)))[b^{-1}]$ is étale; étale maps of this form are called *standard*. Every étale morphism of schemes is locally standard: if $f : X \to Y$ is étale then for each $x \in X$ there are open affine neighborhoods $U \ni x$ and $V \ni f(x)$ so that $f(U) \subseteq (V)$ and the restriction of f to U is given by a standard étale map on the corresponding rings.

36.2 Grothendieck topologies

The notion of open covering is a fundamental one in topology, central to the definition of a compact space for example. A Grothendieck topology is a generalization of this, where the role of open subsets of a topological space X is replaced by an arbitrary category $\mathcal{C}$.

We first need the notions of slice categories and fiber product.[1] Suppose $\mathcal{C}$ is a category.

Definition 36.5. Given two morphisms $f : X \to Z$ and $g : Y \to Z$ in a category $\mathcal{C}$, the fiber product $X \times_Z Y$ of X and Y over Z is an object equipped with morphisms $p : X \times_Z Y \to X$ and $q : X \times_Z Y \to Y$ with the following properties:

1. $g \circ q = f \circ p$ and

2. For any other object T equipped with morphisms $x : T \to X$ and $y : T \to Y$, there is a **unique** map $a : T \to X \times_Z Y$ so that $x = q \circ a$ and $y = p \circ a$.

It's much easier to see what's going on using a commutative diagram. Implicit in such a diagram is that any sequence of arrows that start and end at the same point have the same composition.

$$\begin{tikzcd}
T \arrow[drr, bend left, "x"] \arrow[ddr, bend right, "y"'] \arrow[dr, dotted, "\exists! a"] & & \\
& X \times_Z Y \arrow[r, "p"] \arrow[d, "q"] & X \arrow[d, "f"] \\
& Y \arrow[r, "g"] & Z
\end{tikzcd}$$

Depending on the category and/or on the objects, fiber products may or may not exist. But the uniqueness of the map a guarantees that, if the fiber product of X and Y over Z does exist then it is unique up to unique isomorphism (at least, up to isomorphism that commutes with the maps to X and Y). In many categories that have a faithful forgetful functor to sets (such as abelian groups, commutative rings, R-modules), any two objects have a fiber product: $\{(x, y) \in X \times Y : f(x) = g(y)\}$.

We can also interpret the fiber product as just a normal product in a different category.

[1] A fiber products is also called a pullback or a Cartesian square; the dual notion is referred to as a pushout

Definition 36.6. If X is an object of $\mathcal{C}$ then the slice category $\mathcal{C}/X$ is the category whose objects are morphisms $Y \to X$ in $\mathcal{C}$. Given two objects $y : Y \to X$ and $z : Z \to X$, the morphisms between them are morphisms $f : Y \to Z$ in $\mathcal{C}$ such that $y = z \circ f$. We write $\mathrm{Hom}_X(Y, Z)$ for the set of such morphisms.

For example, the category of schemes over a field k is just the slice category of the category of all schemes over the object $\mathrm{Spec}(k)$. We can now define a Grothendieck topology.

Definition 36.7. Let $\mathcal{C}$ be a category. A *Grothendieck topology* on $\mathcal{C}$ is a collection of distinguished families of maps that we call *coverings*: for each object U of $\mathcal{C}$ we give a set of families of maps $(U_i \to U)_{i\in I}$ satisfying the following properties.

1. For any covering $(U_i \to U)_{i\in I}$ and any morphism $V \to U$, the fiber products $U_i \times_U V$ exist, and $(U_i \times_U V \to V)_{i\in I}$ is a covering of V.
2. If $(U_i \to U)_{i\in I}$ is a covering of U, and if for each $i \in I$ $(V_{ij} \to U_i)_{j\in J_i}$ is a covering of U_i, then the family $(V_{i,j} \to U)_{i,j}$ is a covering of U.
3. For any U in $\mathcal{C}$, the family $(U \xrightarrow{\mathrm{id}} U)$ consisting of a single map is a covering of U.

A category equipped with a Grothendieck topology is known as a *site*.

For example, if X is a topological space then we can consider the category of all open subsets of X, with morphisms given by inclusions. In this case, we say that a family $(U_i \subseteq U)_{i\in I}$ cover U if their union is all of U inside X. The other example that will play a central role for us will be the site $X_{\acute{e}t}$ whose category whose objects consist of étale morphisms $U \to X$ for some fixed scheme X (with morphisms as in the slice category), and where the coverings are families of étale morphisms $(\varphi_i : U_i \to U)_{i\in I}$ that are jointly surjective in the sense that $U = \bigcup_{i\in I} \varphi_i(U_i)$.

Definition 36.8. A *presheaf of sets* on a site is a contravariant functor $\mathcal{C} \to \mathsf{Sets}$. That is, to each object we associate a set $\mathcal{F}(U)$ and to each morphism $Y \to Z$ we associate a map of sets $\mathcal{F}(Z) \to \mathcal{F}(Y)$, which we will sometimes write $a \to a\big|_Y$ because of our motivating example of open subsets of a topological space. Note that the definition of a presheaf only depends on the category, not the coverings.

Similarly, a presheaf of groups, abelian groups or rings is a contravariant functor $\mathcal{C} \to \mathsf{Grps}$, $\mathcal{C} \to \mathsf{Ab}$ or $\mathcal{C} \to \mathsf{Rings}$.

A *sheaf* on a site is a presheaf $\mathcal{F}$ so that

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I\times I} \mathcal{F}(U_i \times_U U_j)$$

is exact for every covering $(U_i \to U)_{i\in I}$. That is, $\mathcal{F}$ is a sheaf if the map $\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i)$ identifies $\mathcal{F}(U)$ with the subset of the product consisting of families (α_i) so that

$$\alpha_i\big|_{U_i\times_U U_j} = \alpha_j\big|_{U_i\times_U U_j}$$

for all $i, j \in I$.

The notation $\mathcal{F}(X)$ suggests that $\mathcal{F}$ is fixed and X is varying. We will frequently instead be fixing X and varying $\mathcal{F}$, so we define another notation for the same thing:

$$\Gamma(X, \mathcal{F}) := \mathcal{F}(X).$$

Here are some examples of sheaves on $X_{\acute{e}t}$.

1. The structure sheaf $\mathcal{O}_U$, which associates to each $U \to X$ the ring of regular functions $\Gamma(U, \mathcal{O}_U) = \mathcal{O}_U(U)$.
2. For any fixed scheme Z, the presheaf defined by $\mathcal{F}_Z(U) = \operatorname{Hom}_X(U, Z)$ is a sheaf. Moreover, if Z has a group structure then $\mathcal{F}_Z$ is a sheaf of groups. For example, if $Z = \mu_n$ is the variety defined by the single equation $T^n - 1 = 0$ then $\mu_n(U)$ is the group of nth roots of unity in $\Gamma(U, \mathcal{O}_U)$. If $Z = \mathbb{G}_a$ is the affine line regarded as a group under addition then $\mathbb{G}_a(U) = \Gamma(U, \mathcal{O}_U)$ regarded as an abelian group. If $Z = \mathbb{G}_m$ is the affine line missing the origin, regarded as a group under multiplication, then $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^\times$.
3. If X is a variety and Λ is any set we can define $\mathcal{F}_\Lambda(U) = \Lambda^{\pi_0(U)}$, the product of copies of Λ indexed by the connected components of U. With the obvious restriction maps this is a sheaf, called the *constant sheaf* defined by Λ.

36.3 Derived functors

Étale cohomology is one instance of a more general framework known as *right and left derived functors*. This will allow us to associate, to any left-exact or right-exact functor $\mathcal{F} : \mathsf{A} \to \mathsf{B}$ between two abelian categories a sequence of functors, of which $\mathcal{F}$ is the 0th. We will now define abelian categories, exact functors and derived functors, and give multiple examples (including étale cohomology).

Abelian categories

The notion of abelian category is modeled on the category of abelian groups, and is essentially a category where you can add morphisms (the set of homomorphisms between any two objects is an abelian group) and compute kernels and cokernels of maps. Here is a definition (there are multiple equivalent ways to define it), but I encourage you to focus on the examples at first.

Definition 36.9. A *preadditive category* is one where every homset is an abelian group, and composition is bilinear: $\gamma(\alpha + \beta) = \gamma\alpha + \gamma\beta$ and $(\alpha + \beta)\gamma = \alpha\gamma + \alpha\beta$. We write 0 for the zero morphism, with the domain and codomain to be understood from context.

An *additive category* is a preadditive category with finite products (as in our definition of fiber product, but without Z) and coproducts (the dual notion).

A *zero object* in a category is an object 0 that is both initial and terminal: for every object X in the category there is a unique morphism $0 \to X$ and a unique morphism $X \to 0$.

The *kernel* of a morphism $f : X \to Y$ is a morphism $k : K \to X$ so that $f \circ k = 0$ and with the following universal property:

$$\begin{tikzcd}
T \arrow[drr, bend left, "k'"] \arrow[dr, dotted, "\exists! u"] \arrow[ddr, bend right, "0"'] & & \\
& K \arrow[r, "k"] \arrow[d, "0"] & X \arrow[dl, "f"] \\
& Y &
\end{tikzcd}$$

A *monomorphism* is the categorical version of an injection: a morphism $f : X \to Y$ with the property that $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$ for any morphisms $g_1, g_2 : T \to X$.

The *cokernel* of a morphism $f : X \to Y$ is a morphism $q : Y \to Q$ so that $q \circ f = 0$ and with the following universal property:

$$\begin{tikzcd}
T & & \\
& Q \arrow[ul, dotted, "\exists! u"'] & Y \arrow[l, "q"'] \arrow[ull, bend right, "q'"'] \\
& X \arrow[u, "0"'] \arrow[ur, "f"'] \arrow[uul, bend left, "0"] &
\end{tikzcd}$$

An *epimorphism* is the categorical version of a surjection: a morphism $f : X \to Y$ with the property that $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$ for any morphisms $g_1, g_2 : Y \to T$.

An *epi-mono factorization* of a morphism $f : X \to Y$ is a decomposition of f as $X \xrightarrow{s} Z \xrightarrow{i} Y$, with s an epimorphism and i a monomorphism.

An *abelian category* is an additive category that has a zero object, where every morphism has a kernel and cokernel, where every monomorphism is the kernel of some morphism and every epimorphism is the cokernel of some morphism, and where ever morphism has an epi-mono factorization.

The examples to keep in mind: abelian groups, k-vector spaces, R-modules over a commutative ring R, finitely generated modules over a Noetherian ring R. Most relevantly for us, the category of sheaves of abelian groups on a site is an abelian category, as is the category of presheaves on a site with values in any abelian category.[2]

Injective and projective objects

The key feature of an abelian category that allows one to apply the machinery of derived functors is that is has enough injective or projective objects. Here is the definition and some examples.

Definition 36.10. An object Q is *injective* if, for every monomorphism $f : X \to Y$ and **any** morphism $g : X \to Q$, there is an extension h of g to Y:

$$\begin{tikzcd}
X \arrow[r, hook, "f"] \arrow[d, "g"] & Y \arrow[dl, dotted, "\exists h"] \\
Q &
\end{tikzcd}$$

An object P is *projective* if, for every epimorphism $f : X \to Y$ and **any** morphism $g : P \to Y$, there is a lift of g to X.

$$\begin{tikzcd}
Y & X \arrow[l, two heads, "f"] \\
P \arrow[u, "g"] \arrow[ur, dotted, "\exists h"] &
\end{tikzcd}$$

[2]Technically you need an additional assumption that the site is *small*

Note that, unlike most of the universal properties that we have seen so far, the morphism h is usually not unique.

The statement that all sets are projective is equivalent to the axiom of choice. In the category of abelian groups injective objects are *divisible groups*[3] (multiplication by any nonzero integer is surjective) and projective objects are free abelian groups ($\mathbb{Z}^r$ for example). For vector spaces, every object is both injective and projective. An R-module P is projective if there another R-module Q so that $P \oplus Q$ is free.

We will be particularly interested in finding a monomorphism from an arbitrary object X in our abelian category to an injective object (or, dually, finding an epimorphism from a projective object to X). If this is always possible then we say that the abelian category has *enough injectives* (or *enough projectives*). Most of our example abelian categories have enough injectives[4] and enough projectives,[5] but finitely generated R-modules do not have enough injectives (nonzero divisible groups are not finitely generated).

When you have enough injectives you can form an *injective resolution* of any object in your category. This is an exact sequence

$$0 \to X \to I_0 \to I_1 \to \cdots$$

where each I_i is injective. A projective resolution is an exact sequence

$$\cdots \to P_1 \to P_0 \to X \to 0$$

where each P_i is projective. Injective and projective resolutions are not unique, but the choice of resolution will not end up mattering for our purposes.

Exact functors

Abelian categories allow us to consider exact sequences, since we have kernels and cokernels. The extent to which functors between abelian categories preserve exactness is an important attribute when working with them.

Definition 36.11. A covariant additive[6] functor $F : \mathsf{A} \to \mathsf{B}$ between abelian categories is *left exact* if, for all exact sequences

$$0 \to A \to B \to C$$

in A, the sequence

$$0 \to F(A) \to F(B) \to F(C)$$

is exact. It is *right exact* if, for all exact sequences

$$A \to B \to C \to 0$$

in A, the sequence

$$F(A) \to F(B) \to F(C) \to 0$$

is exact. A contravariant additive functor G is left exact if $A \to B \to C \to 0$ exact implies $0 \to G(C) \to G(B) \to G(A)$ is exact; right exactness is defined similarly. We say a functor is *exact* if it is both left and right exact.

[3] one implication requires the axiom of choice

[4] For R-modules the injective object is called the injective hull, and for sheaves it is called the Godement resolution

[5] assuming the axiom of choice

[6] a functor is additive if it acts as a group homomorphism on homsets

Here are some important examples that will form the starting point for various derived functors.

1. If $\mathcal{C}$ is an abelian category and A an object of $\mathcal{C}$, the functor $\mathcal{C} \to \mathsf{Ab}$ defined by $X \mapsto \mathrm{Hom}(A, X)$ is left exact, and is right exact if and only if A is projective. The functor defined by $X \to \mathrm{Hom}(X, A)$ is a contravariant left exact functor, and is right exact if and only if A is injective.

2. If A is a commutative ring and B an A-algebra then the functor $A{-}\mathsf{mod} \to B{-}\mathsf{mod}$ defined by $X \mapsto X \otimes_A B$ is right exact, and is left exact if B is flat.

3. If G is a group and R a commutative ring, the category of G-modules[7] is an abelian category and the "invariants" functor $R[G]{-}\mathsf{mod} \to R{-}\mathsf{mod}$ defined by $M \mapsto M^G$ is left exact. Here $M^G = \{x \in M : gx = x \text{ for all } g \in G\}$.

4. The functor from sheaves of abelian groups on $X_{ét}$ to Ab defined by $\mathcal{F} \mapsto \mathcal{F}(X)$ is left exact. This is called the *global sections* functor.

36.3.1 Derived functors and cohomology

Suppose now that we have a covariant left exact functor $F : \mathsf{A} \to \mathsf{B}$ between two abelian categories, and that A has enough injectives. If A is an object of A then we can find an injective resolution

$$0 \to A \to I_0 \to I_1 \to I_2 \to \cdots$$

Applying F to this resolution yields a sequence

$$0 \to F(A) \to F(I_0) \xrightarrow{\varphi_0} F(I_1) \xrightarrow{\varphi_1} F(I_2) \xrightarrow{\varphi_2} \cdots$$

While this sequence will usually not be exact, it will have the property that the composition of any two maps is zero (since left exact functors are additive). We define the right derived functors R^iF by setting

$$R^iF(A) = \begin{cases} \ker(\varphi_0) & \text{if } i = 0 \\ \ker(\varphi_i)/\operatorname{im}(\varphi_{i-1}) & \text{otherwise} \end{cases}$$

Note that, since F is left exact, $R^0F = F$. Of course, there's a lot to check: this doesn't depend on the choice of injective resolutions, that the results are functors.... The most important property of left derived functors is the following:

Theorem 36.12. *If $F : \mathsf{A} \to \mathsf{B}$ is a left exact functor[8] and*

$$0 \to A \to B \to C \to 0$$

is an exact sequence in A then

$$0 \to F(A) \to F(B) \to F(C) \to R^1F(A) \to R^1F(B) \to R^1F(C) \to R^2F(A) \to R^2F(B) \to \cdots$$

[7] left R-modules with an action of G by R-linear maps; if G is a topological group you can upgrade this by requiring that the action be continuous

[8] and A has enough injectives so that the R^iF are defined

There is a dual notion: if $F : \mathsf{A} \to \mathsf{B}$ is a right exact functor and A has enough projectives then you can take a projective resolution to defined left derived functors $L_i F$ with so that an analogue of theorem 36.12 holds: for any short exact sequence you get a long exact sequence

$$\cdots \to L_2F(B) \to L_2F(C) \to L_1F(A) \to L_1F(B) \to L_1F(C) \to F(A) \to F(B) \to F(C) \to 0.$$

One can also perform a similar process with contravariant functors.

Right derived functors measure the failure of the functor to also be left exact, and left derived functors measure the failure of the functor to also be right exact.

This notion appears in many guises.

1. The right derived functors[9] of $\mathrm{Hom}(A, —)$ are denoted $\mathrm{Ext}^i(A, —)$. The name comes from the fact that $\mathrm{Ext}^1(M, N)$ classifies extensions, that is objects E that fit into a sequence
$$0 \to N \to E \to M \to 0$$
up to equivalence.

2. The right derived functors of the fixed-point functor[10] for G-modules is group cohomology. In particular, if G is a Galois group then this specializes to Galois cohomology, which plays a central role in arithmetic geometry since it helps determine how isomorphism classes of objects over the algebraic closure of a field k break up into isomorphism classes over k itself.

3. If R is a commutative ring and A is an R-module then $B \mapsto A \otimes_R B$ is a right exact functor from $R-\mathsf{mod}$ to Ab. Its left derived functors are denoted $\mathrm{Tor}_i^R(A, B)$.

4. And finally, if $X_{ét}$ is the étale site and $\Gamma(X, —)$ is the left exact functor of global sections then the étale cohomology groups are defined as
$$H^i_{ét}(X, \mathcal{F}) = R^i\Gamma(X, \mathcal{F}).$$

36.4 A Weil cohomology theory

Of course, we're not quite there. To get a Weil cohomology theory we needed a vector space over a field of characteristic 0. And what is $\mathcal{F}$?

There's one more small step. In what is sometimes referred to ℓ-adic cohomology we define

$$H^i_{ét}(X, \mathbb{Z}_\ell) = \varprojlim_{m} H^i_{ét}(X, \underline{\mathbb{Z}/\ell^m\mathbb{Z}}),$$
$$H^i_{ét}(X, \mathbb{Q}_\ell) = H^i_{ét}(X, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Here $\underline{\mathbb{Z}/\ell^m\mathbb{Z}}$ is the constant sheaf associated to $\mathbb{Z}/\ell^m\mathbb{Z}$. Note that taking the projective limit does not commute with taking cohomology, so we cannot just take coefficients to be the constant sheaf $\underline{\mathbb{Z}_\ell}$ (in fact, cohomology with this coefficient sheaf is usually 0). And now the real work begins: proving that this definition satisfies the axioms of a Weil cohomology theory.

[9] You can also construct them as the left derived functors of the contravariant Hom functor on the other coordinate

[10] this is actually a special case of the previous example, since $M^G = \mathrm{Hom}_{R[G]}(R, M)$ where G acts trivially on R

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