Transfer of transfert

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The conjectures (Langlands-Shelstad)

(All this talk: for Standard endoscopy).
G, H – endoscopic groups over a non-archimedean field F.
The ‘smooth transfer’ conjecture: for any $f \in C_c^\infty(G)$, there exists $f^H \in C_c^\infty(H)$ such that for all $\gamma_H \in H(F)^{G-rss}$ and $\gamma_G \in G(F)$ in a matching conjugacy class in G,

$$O^{st}_{\gamma_H}(f^H) = \sum_{\gamma' \sim \gamma_G} \kappa(\gamma', \gamma_H) O_{\gamma'}(f),$$

(This is for γ_H near 1; otherwise need a central extension $\tilde{H}$ of H and a character on the centre of $\tilde{H}$).

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the Fundamental Lemma

Assume here for simplicity G, H unramified.
K_G, K_H – hyperspecial maximal compacts.
Then:

- The 'unit element': for $f = \mathbf{1}_{K_G}$ – the characteristic function of K_G, $f^H = \mathbf{1}_{K_H}$.
- The version of this for Lie algebras.
- Explicit matching for the basis of $\mathcal{H}(G//K_G)$ with elements of $\mathcal{H}(G//K_H)$ using Satake.

The reductions in characteristic zero

- The FL for the group reduces to FL for the Lie algebra (Langlands-Shelstad)
- The FL for the full Hecke algebra reduces to the unit element (Hales, 1995), and
- If FL holds for $p >> 0$, then it holds for all p (global argument).
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The logical implications

- FL for Lie algebras, $\mathrm{char} F > 0$ (Ngô) $\Rightarrow$ FL for $\mathrm{char} F = 0$, $p >> 0$ (Waldspurger $p > n$), Cluckers-Hales-Loeser $p >> 0$,
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What's left

- FL for the full Hecke algebra for $\mathrm{char} F > 0$ (proved extending Ngô's techniques by A. Bouthier, 2014).
  Transfer from characterstic zero using model theory (for $p >> 0$), Jorge Cely's thesis (exp. 2016)
- Smooth transfer conjecture in positive characteristic. We prove it for $p >> 0$ (the bound is determined by root data of G, H, roughly speaking) by transfer based on model theory. (2015, this talk).
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Language of rings

The language of rings has:

- 0, 1 – symbols for constants;
- $+$, $\times$ – symbols for binary operations;
- countably many symbols for variables.

The formulas are built from these symbols, the standard logical operations, and quantifiers. Any ring is a structure for this language.

Example

A formula: ’$\exists y, f(y, x_1, \ldots, x_n) = 0$’, where $f \in \mathbb{Z}[x_0, \ldots, x_n]$.

Ax-Kochen transfer principle

A first-order statement in the language of rings is true for all $\mathbb{Q}_p$ *with* $p >> 0$ *off it is true in* $\mathbb{F}_p((t))$ *for* $p >> 0$. (Depends only on the residue field).

Example

For each positive integer d there is a finite set P_d of prime numbers, such that if $p \notin P_d$, every homogeneous polynomial of degree d over $\mathbb{Q}_p$ in at least $d^2 + 1$ variables has a nontrivial zero.

First-order means, all quantifiers run over definable sets in the structure (e.g. cannot quantify over statements). (In the Example, cannot quantify over d, it is a separate theorem for each d).

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Denef-Pas Language (for the valued field)

Formulas are allowed to have variables of three sorts:

- valued field sort, $(+, \times,$ '0','1', ac$(\cdot)$, ord$(\cdot)$)
- value sort ($\mathbb{Z}$), $(+,$ '0', '1', $\equiv_n$, $n \geq 1$)
- residue field sort, (language of rings: $+$, $\times$, '0', '1')

Formulas are built from arithmetic operations, quantifiers, and symbols ord$(\cdot)$ and ac$(\cdot)$. **Example:**
$\phi(y) =$ '$\exists x, y = x^2$', or, equivalently,

$$\phi(y) = \text{'ord}(y) \equiv 0 \mod 2 \wedge \exists x : \text{ac}(y) = x^2\text{'}.$$

Cluckers-Loeser transfer principle

Cluckers and Loeser defined a class of *motivic functions* which is stable under integration. Motivic functions are made from definable functions (but are not themselves definable). A motivic function f on a definable set X gives a $\mathbb{C}$-valued function f_F on $X(F)$ for all fields F of sufficiently large residue characteristic.

Theorem

(Cluckers-Loeser, 2005). Let f be a motivic function on a definable set X. Then there exists M_f such that when $p > M_f$, whether f_F is identically zero on $X(F)$ or not depends only on the residue field of F.

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The challenges

For the FL: express both sides as motivic functions, FL says that their difference vanishes identically. For smooth transfer, two problems:

- Do not know anything about f^H
- Groups, etc. depend on a lot of parameters, and we can only transfer statements with universal quantifiers.

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Reduction of Smooth transfer to FL is done in two steps:

- (Langlands-Shelstad): it suffices to prove that κ-Shalika germs (transferred from G) lie in the space spanned by the stable Shalika germs on H. Their proof works in positive characteristic.
- (Waldspurger) Proves the statement about Shalika germs, using TF on the Lie algebra. This is the statement we transfer.
- To transfer this statement we need to transfer a statement about linear dependence. Run into difficulties because cannot transfer statements about linear independence. A vey difficult argument circumvents this.
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