

HOMOGENEOUS DYNAMICS AND OPPENHEIM'S CONJECTURE

ZACK WEST

ABSTRACT. This expository paper examines Oppenheim's 1929 conjecture, which asserts that an indefinite, nondegenerate, irrational quadratic form Q in $n \geq 3$ variables has a value set $Q(\mathbb{Z}^n)$ that is dense in \mathbb{R} . We present Margulis's 1987 proof and develop the necessary background in homogeneous dynamics from the ground up. The proof represents the problem as a statement about orbits of $H = SO(2, 1)$ on the space of unimodular lattices $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$; irrationality forces the orbit to be non-closed; Dani's nondivergence theorem prevents it from escaping to the boundary of the space; and Ratner's theorem then forces the orbit to be dense in the full space of lattices. We conclude with the quantitative refinement of Eskin, Margulis, and Mozes, which counts how many integer points map into a given interval.

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1. INTRODUCTION

A quadratic form in n variables is a homogeneous polynomial of degree two,

$$Q(x_1, \dots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j = x^T A x,$$

for a symmetric matrix $A \in \text{Sym}_n(\mathbb{R})$ with $\det A \neq 0$. A natural question is how the value set $Q(\mathbb{Z}^n) = \{Q(x) : x \in \mathbb{Z}^n\}$ is distributed in \mathbb{R} . We can see that if Q is definite, $Q(\mathbb{Z}^n)$ is bounded away from zero, and that if $Q = \lambda R$ for $\lambda \in \mathbb{R}$ and R with rational coefficients, then $Q(\mathbb{Z}^n) \subseteq \lambda\mathbb{Z}$. In either case the values are discrete, not dense. Oppenheim's 1929 conjecture asserts that these are the *only* obstructions, provided $n \geq 3$.

CONJECTURE 1.1 (Oppenheim [1], 1929). Let Q be an indefinite nondegenerate quadratic form in $n \geq 3$ variables. If Q is not proportional to a form with rational coefficients, then $Q(\mathbb{Z}^n)$ is dense in \mathbb{R} .

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We require that $n \geq 3$; Section 2 includes a proof that irrational binary forms can be bounded away from zero on \mathbb{Z}^2 , via a Diophantine approximation argument that fails in higher dimensions.

The conjecture resisted proof for over fifty years. The breakthrough came in 1987, when Margulis [2, pp. 377–398] proved it in full generality using methods from homogeneous dynamics. The central observation, due to Raghunathan, is that density of $Q(\mathbb{Z}^n)$ is equivalent to a statement about orbits of isometry-preserving transformations acting on the space of unimodular lattices $X = SL(3, \mathbb{R})/SL(3, \mathbb{Z})$. Writing $Q(v) = Q_0(gv)$ for $g \in SL(3, \mathbb{R})$ and standard form $Q_0 = x^2 + y^2 - z^2$, density of $Q(\mathbb{Z}^3)$ reduces to the orbit $H \cdot [g]$ being dense in X , where $H = SO(2, 1)$ is the symmetry group of Q_0 . The proof then proceeds in three steps:

$$Q \text{ irrational} \implies \text{orbit not closed} \xrightarrow{\text{nondivergence}} \text{orbit doesn't escape} \implies \text{orbit closure equals full space of lattices} \implies Q(\mathbb{Z}^n) \text{ dense.}$$

Each arrow corresponds to a nontrivial theorem. Dani's nondivergence theorem [3, §2] guarantees that unipotent orbits spend only a negligible fraction of their time near the degenerate regions of X . Ratner's theorem [4, Theorem 1] then forces the closure of any non-closed unipotent orbit to be an entire homogeneous subspace, i.e., all of X .

Beyond density, in [5, Theorem 1.3], Eskin, Margulis, and Mozes proved a quantitative refinement: the number of integer points x with $\|x\| < T$ and $a < Q(x) < b$ grows asymptotically like $c_{Q,a,b} \cdot T^{n-2}$. This supersedes the original result of density with a precise count that reflects the geometry of the problem.

The paper is organized as follows. Section 2 motivates the conjecture through examples showing how rationality, irrationality, and the number of variables interact. Section 3 develops the necessary background: quadratic forms, the space of lattices X , the stabilizer $H = SO(2, 1)$, and the nondivergence property of unipotent flows. Section 4 presents the proof of density, explaining why the dynamical reformulation is the right one and how Ratner's theorem closes the argument. Section 5 discusses the Eskin–Margulis–Mozes counting theorem.

2. MOTIVATING EXAMPLES

We introduce the distinction between rational and irrational quadratic forms through simple examples.

EXAMPLE 2.1 (Rational form). Let $Q(x, y, z) = x^2 + y^2 - 2z^2$. Every coefficient is an integer, so for any $(x, y, z) \in \mathbb{Z}^3$ the value $Q(x, y, z)$ is an integer. Thus $Q(\mathbb{Z}^3) \subseteq \mathbb{Z}$ is discrete, and in particular not dense in \mathbb{R} . This illustrates the general phenomenon for rational forms: if $Q = \lambda Q_{\text{rat}}$ with Q_{rat} having rational coefficients, then $Q(\mathbb{Z}^n) \subseteq \lambda\mathbb{Q}$, so the value set lies in a discrete subgroup of \mathbb{R} .

EXAMPLE 2.2 (Irrational form). The values of $Q(x, y, z) = x^2 + y^2 - \sqrt{2}z^2$ on \mathbb{Z}^3 lie in $\{a - \sqrt{2}b : a, b \in \mathbb{Z}\}$. Since $\sqrt{2}$ is irrational, the set $\{a - \sqrt{2}b : a, b \in \mathbb{Z}\}$ is dense in \mathbb{R} , e.g. by Dirichlet's approximation theorem. Thus, unlike the rational case, the values of Q are not confined to a discrete subset, and can approximate any real number arbitrarily well.

2.1. Analogy with linear forms. For irrational α , Dirichlet's approximation theorem [?, Ch. 1, Theorem 1] gives

$$\{m\alpha - n : m, n \in \mathbb{Z}\} \text{ is dense in } \mathbb{R}.$$

The irrationality of α gives the linear form $L(m, n) = m\alpha - n$ a continuous set of outputs. Inductively, results from classical number theory suggest that the irrationality of Q prevents $Q(\mathbb{Z}^n)$ from being discrete.

For $n = 2$ the conjecture fails. Let $Q(x, y) = x^2 - \phi y^2$ where $\phi = \frac{1+\sqrt{5}}{2}$. Since $\sqrt{\phi}$ is an algebraic irrational of degree 4, it is badly approximable by Roth's theorem [7, Theorem 2.1]: there exists $c > 0$ such that $|\sqrt{\phi} - p/q| \geq c/q^2$ for all $p, q \in \mathbb{Z}$ with $q \neq 0$. Factoring $Q = (x - \sqrt{\phi}y)(x + \sqrt{\phi}y)$ and applying this bound gives $|x - \sqrt{\phi}y| \geq c/|y|$ and $|x + \sqrt{\phi}y| \geq \frac{\sqrt{\phi}}{2}|y|$ for $|y|$ large, so

$$|Q(x, y)| \geq \frac{c}{|y|} \cdot \frac{\sqrt{\phi}}{2}|y| = \frac{c\sqrt{\phi}}{2} > 0.$$

For bounded $|y|$ the values are bounded away from zero since $\sqrt{\phi} \notin \mathbb{Q}$. Thus $\inf_{(x,y) \neq 0} |Q(x, y)| > 0$ and density fails.

The geometric reason is clear: for $n = 2$ the zero set $\{Q = 0\}$ is a pair of lines, and \mathbb{Z}^2 can avoid them when their slopes are badly approximable. For $n \geq 3$ the zero set is a cone of dimension $n - 1 \geq 2$, and the extra degrees of freedom force \mathbb{Z}^n to probe it closely enough to produce density.

3. DEFINITIONS AND BACKGROUND

3.1. Quadratic forms.

DEFINITION 3.1. A *quadratic form* in n variables is a homogeneous polynomial of degree 2:

$$Q(x_1, \dots, x_n) = \sum_{1 \leq i \leq j \leq n} a_{ij} x_i x_j = x^T A x,$$

where $A = (a_{ij})$ is the associated symmetric matrix. The form is *nondegenerate* if $\det(A) \neq 0$, *rational* if some nonzero scalar multiple has all rational coefficients (otherwise *irrational*), and its *signature* is the pair (p, q) where p and q count the positive and negative eigenvalues of A . We call Q *definite* if $p = 0$ or $q = 0$, and *indefinite* if $p \geq 1$ and $q \geq 1$.

Geometrically, the zero set of a definite form is just the origin, while that of an indefinite form is a cone; level sets $Q(x) = c \neq 0$ are compact ellipsoids in the definite case and unbounded hyperboloids in the indefinite case. For a quadratic form of signature (p, q) with $p + q = n$, the *special orthogonal group* $SO(p, q)$ is the group of determinant-one real matrices preserving it. The compact group $SO(n) = SO(n, 0)$ preserves the standard positive definite form, while $SO(2, 1)$ preserves an indefinite form and is non-compact.

The reason $SO(2, 1)$ is the right group to study is that any indefinite form Q of signature $(2, 1)$ can be written as $Q(v) = Q_0(gv)$ for some $g \in SL(3, \mathbb{R})$, where $Q_0 = x^2 + y^2 - z^2$ is the standard form (all forms of the same signature are linearly equivalent). The symmetries of Q_0 , or the matrices preserving Q_0 , define $H = SO(2, 1)$, and acting by these symmetries transforms the lattice $g\mathbb{Z}^3$ without changing which values Q take on integer vectors. Studying how H moves the lattice $g\mathbb{Z}^3$ around the space X is therefore equivalent to studying the values $Q(\mathbb{Z}^3)$.

EXAMPLE 3.2. The forms from Section 2 are classified as follows.

- (a) $Q = x^2 + y^2 + z^2$: definite, signature $(3, 0)$, rational. Values on \mathbb{Z}^3 are non-negative integers, hence not dense.

- (b) $Q = x^2 + y^2 - 2z^2$: indefinite, signature $(2, 1)$, rational. Values on \mathbb{Z}^3 lie in \mathbb{Z} , hence not dense.
- (c) $Q = x^2 + y^2 - \sqrt{2}z^2$: indefinite, signature $(2, 1)$, irrational. By Margulis' proof of Oppenheim's conjecture (shown in Section 4), values on \mathbb{Z}^3 are dense in \mathbb{R} .
- (d) $Q = x^2 - y^2 - \sqrt{2}z^2$: indefinite, signature $(1, 2)$, irrational. Up to a linear change of variables, this is equivalent to a form of signature $(2, 1)$, and its values on \mathbb{Z}^3 are likewise dense in \mathbb{R} .

3.2. The space of lattices. The dynamical proof takes place on a space with points representing lattices. We construct this space from $G = SL(n, \mathbb{R})$.

DEFINITION 3.3. A *lattice* $L \subset \mathbb{R}^n$ is a discrete subgroup of \mathbb{R}^n of rank n .

$$L = \mathbb{Z}v_1 + \cdots + \mathbb{Z}v_n,$$

where v_1, \dots, v_n is a basis of \mathbb{R}^n .

The *covolume* of L is defined by

$$\text{covol}(L) = |\det[v_1 \ \cdots \ v_n]|.$$

This quantity is the Lebesgue measure of a fundamental domain of L and is independent of the choice of basis (any change of basis $A \in GL(n, \mathbb{Z})$ preserves the covolume). A lattice is called *unimodular* if $\text{covol}(L) = 1$.

Since $G = SL(n, \mathbb{R})$ acts on \mathbb{R}^n by matrix multiplication, each $g \in G$ carries the standard lattice \mathbb{Z}^n to a new lattice $g\mathbb{Z}^n$ with covolume $|\det g| = 1$, hence unimodular. Two matrices g and $g\gamma$ with $\gamma \in \Gamma = SL(n, \mathbb{Z})$ produce the same lattice since $g\gamma\mathbb{Z}^n = g\mathbb{Z}^n$, so the coset $g\Gamma$ is the correct equivalence class. This gives a bijection between the quotient

$$X = G/\Gamma, \quad G = SL(n, \mathbb{R}), \quad \Gamma = SL(n, \mathbb{Z}),$$

and the space of unimodular lattices, where the point $[g] = g\Gamma \in X$ represents the lattice $g\mathbb{Z}^n$. For a general reference on lattices and the geometry of G/Γ , see [7, Ch. 1].

The space X is not compact: a lattice degenerates when its shortest nonzero vector shrinks to zero, and such lattices escape every compact subset of X . A subset of X is *relatively compact* if its closure ($\bar{A} = A \cup \{\text{all limit points of } A\}$) is compact, and Mahler's criterion characterizes this by a uniform lower bound on the lengths of nonzero vectors.

THEOREM 3.4 (Mahler's Compactness Criterion [7, p. 121]). *A subset $S \subseteq G/\Gamma$ is relatively compact if and only if there exists $\delta > 0$ such that every lattice L represented by a point in S satisfies $\|v\| \geq \delta$ for all nonzero $v \in L$.*

This means the non-compact ends of G/Γ , i.e., cusps, correspond to lattices with vectors of arbitrarily small length. We'll use Mahler's Criterion to classify when a set of lattices stays bounded away from these cusps.

For instance, consider the set of unimodular lattices constructed by stretching two axes equally and oppositely:

$$L_t = \begin{pmatrix} t & 0 & 0 \\ 0 & 1/t & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{Z}^3, \quad t > 0.$$

Each matrix operating on \mathbb{Z}^n has determinant 1. As $t \rightarrow \infty$, the shortest nonzero vector in L_t has length $\min\{t^{-1}, 1\} \rightarrow 0$, so the lattices leave every compact subset of $SL(3, \mathbb{R})/SL(3, \mathbb{Z})$. In other words, $[L_t]$ approaches a cusp of X .

3.3. The stabilizer $SO(2, 1)$. We now specialize to $n = 3$ and fix the standard indefinite form

$$Q_0(x, y, z) = x^2 + y^2 - z^2.$$

The strategy for proving density of $Q(\mathbb{Z}^3)$ is to realize the equivalence of Q as a transformation of Q_0 and then study the orbit of a lattice under Q_0 's symmetry group. To do this, we'll introduce the group $H = SO(2, 1)$ that preserves Q_0 (which will act on the space of lattices X and whose orbits we will show are dense via Ratner's theorem), the structure of its Lie algebra $\mathfrak{so}(2, 1)$ (which tells us what kinds of one-parameter subgroups are available), and the unipotent subgroups of H (whose polynomial-growth will prevent orbits from escaping to the cusps of X and thus guarantee the recurrence needed for Ratner's theorem to apply).

DEFINITION 3.5. The *special orthogonal group of signature (2, 1)* is

$$H = SO(2, 1) = \{g \in SL(3, \mathbb{R}) : Q_0(gx) = Q_0(x) \text{ for all } x \in \mathbb{R}^3\}.$$

The group H is the set of volume-preserving linear transformations that map each hyperboloid $Q_0 = c$ to itself. In contrast to $SO(3)$, which preserves the definite form $x^2 + y^2 + z^2$ and is compact, $SO(2, 1)$ is non-compact and preserves the indefinite measure.

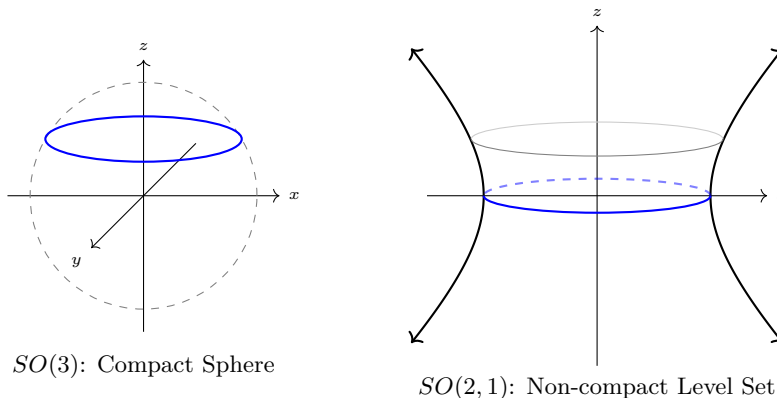


FIGURE 1. Geometric comparison of stabilizers. In $SO(3)$, the orbit is a compact 1-sphere. In the indefinite $SO(2, 1)$ case, the level set is a non-compact hyperboloid that extends infinitely in the z -direction. The blue circle represents a compact $SO(2)$ subgroup orbit.

The Lie algebra $\mathfrak{so}(2, 1)$ consists of matrices $X \in \mathfrak{sl}(3, \mathbb{R})$ satisfying $X^T J + JX = 0$ where $J = \text{diag}(1, 1, -1)$ is the matrix operator of the indefinite form; it is three-dimensional, and any nilpotent element $N \in \mathfrak{so}(2, 1)$ satisfies $N^3 = 0$, which we'll use to define a tractable polynomial action on our transformed lattice gL .

3.4. Unipotent Elements. The proof of density of $Q(\mathbb{Z}^3)$ uses the fact that unipotent elements of H act on lattices with polynomially bounded displacement, preventing orbits from escaping to the cusps of X . A *unipotent* element $u \in H$ is one whose eigenvalues are all 1, and it arises as the exponential of a *nilpotent* element $N \in \mathfrak{so}(2, 1)$, i.e., $u = \exp(N)$. For a nilpotent $N \in \mathfrak{so}(2, 1)$, all eigenvalues are zero; in dimension three this implies that N is nilpotent of index at most three, so $N^3 = 0$. Therefore, the exponential map $u(t) = \exp(tN)$ terminates as a quadratic polynomial:

$$u(t) = \exp(tN) = \sum_{k=0}^{\infty} \frac{(tN)^k}{k!} = I + tN + \frac{t^2}{2!}N^2 + \frac{t^3}{3!}N^3 + \dots$$

Substituting $N^3 = 0$ (and consequently $N^k = 0$ for all $k \geq 3$), the series terminates exactly at the quadratic term:

$$I + tN + \frac{t^2}{2}N^2 = \begin{pmatrix} 1 + \frac{t^2}{2} & -\frac{t^2}{2} & t \\ \frac{t^2}{2} & 1 - \frac{t^2}{2} & t \\ t & -t & 1 \end{pmatrix}$$

This gives a $u(t)$ strictly polynomial in t , and is specific to nilpotent elements. For a semisimple (diagonalizable over \mathbb{C} , with no nilpotent part) element $S \in \mathfrak{so}(2, 1)$, no power of S vanishes, so $\exp(tS)$ remains a convergent infinite series whose entries grow or decay exponentially in t , as illustrated in Figure 2.

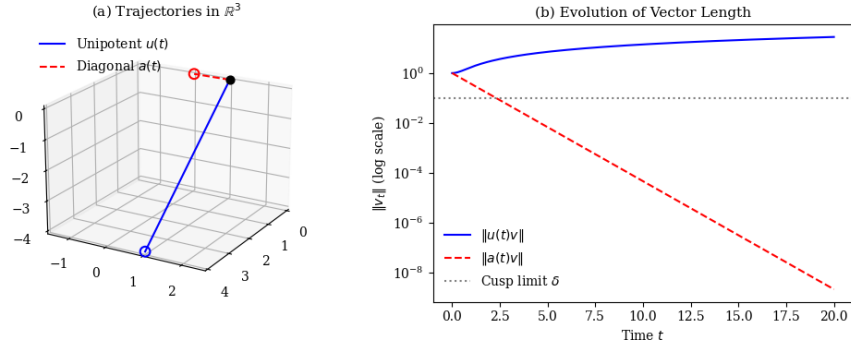


FIGURE 2. Comparison of $SO(2,1)$ flows, generated using `matplotlib` in Python. Unipotent polynomial growth maintains recurrence; diagonal orbits contract exponentially while polynomial orbits remain in the compact bulk of X .

To see why polynomial growth is useful, we introduce the notion of (C, α) -goodness, which measures the fraction of time a trajectory spends near zero.

Let $\lambda(A)$ denote the standard *Lebesgue measure* of a set $A \subseteq \mathbb{R}$, representing its total length.

DEFINITION 3.6. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \rightarrow \mathbb{R}$ is called (C, α) -good if for all $\varepsilon > 0$,

$$\lambda(\{t \in I : |f(t)| < \varepsilon\}) \leq C \left(\frac{\varepsilon}{\sup_{t \in I} |f(t)|} \right)^\alpha \lambda(I),$$

where $C, \alpha > 0$ depend only on the class of functions (e.g. degree bounds), not on ε or I [9, p. 344].

A trajectory v_t in the space of lattices X stays in the cusp if its length $\|v_t\|$ remains below some threshold ε for a disproportionate amount of time.

In contrast, a non-zero polynomial $P(t)$'s time near zero is controlled by the (C, α) -good property:

$$\lambda(\{t \in [0, T] : |P(t)| < \varepsilon\}) \leq C \left(\frac{\varepsilon}{\sup_{t \in [0, T]} |P(t)|} \right)^\alpha T.$$

$\sup |P(t)|$ grows as T^d (with $d = \deg P$), so the time during which $|P(t)| < \varepsilon$ occupies only a negligible fraction of the total flow time T .

EXAMPLE 3.7. Let $P(t) = t^2$ on the interval $I = [0, T]$. The measure of time where the trajectory is within ε of the origin is:

$$\lambda(\{t \in [0, T] : t^2 < \varepsilon\}) = \lambda([0, \sqrt{\varepsilon}]) = \sqrt{\varepsilon}.$$

Applying the (C, α) -good formula with $d = 2$:

$$\sqrt{\varepsilon} \leq C \left(\frac{\varepsilon}{T^2} \right)^{1/2} T = C\sqrt{\varepsilon}.$$

Unlike the exponential case, the measure of time spent near zero is independent of the total flow time T . Even if ε is extremely small, the polynomial quickly leaves the neighborhood of zero, and the associated lattice $u(t)L$ returns to the compact region of $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$.

The (C, α) -good estimate above controls the proportion of time the orbit $u(t)L$ spends in the cusp, but it does so relative to $\sup_{t \in [0, T]} |P(t)|$. In particular, if the initial lattice L is already deep in the cusp (i.e., if some nonzero vector $v \in L$ already satisfies $\|v\| \ll 1$) then the estimate does not immediately guarantee that the orbit returns to the compact bulk quickly. The estimate does guarantee that the time in the cusp is a negligible fraction of T as $T \rightarrow \infty$, regardless of the starting point.

To obtain a uniform non-divergence statement that applies to any starting point $[g] \in X$, we use the following theorem.

THEOREM 3.8 (Dani's Non-Divergence Theorem, [3, §2]). *Let $G = SL(n, \mathbb{R})$, $\Gamma = SL(n, \mathbb{Z})$, and let $U = \{u(t)\}$ be a one-parameter unipotent subgroup of G . Then for every $x \in G/\Gamma$ and every $\varepsilon > 0$,*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \lambda(\{t \in [0, T] : u(t)x \notin K_\varepsilon\}) < \varepsilon,$$

where $K_\varepsilon \subset G/\Gamma$ is the compact set of lattices with shortest vector of length $\geq \varepsilon$.

Before Theorem 3.8, we had only a polynomial estimate valid for lattices whose shortest nonzero vector is bounded away from zero. Theorem 3.8 upgrades this to a uniform statement valid for any starting lattice in X , including lattices with degenerate vectors. For semisimple one-parameter evolutions like the diagonal subgroup $a(t) = \text{diag}(e^t, e^{-t}, 1)$, orbits can diverge because coordinates decay exponentially, so recurrence is not guaranteed. In contrast, unipotent flows $u(t)L$ remain recurrent in the compact bulk of X regardless of the initial lattice, including those starting deep in the cusp; this recurrence is exactly the input that Ratner's theorem requires.

4. ORBIT CLOSURES AND THE PROOF OF DENSITY

Throughout this section, $g \in SL(3, \mathbb{R})$ remains fixed by the condition $Q(v) = Q_0(gv)$; the group $H = SO(2, 1)^\circ$ acts by left multiplication on the coset $[g] \in X$, and $[g]$ itself does not move. We now use the nondivergence established in Section 3 to classify the closure of the full orbit $H \cdot [g]$ and show density of $Q(\mathbb{Z}^3) = \{Q_0(v) : v \in L \setminus \{0\}\}$. Since $Q(hx) = Q(x)$ for all $h \in H$, all lattices in the H -orbit give the same value set, so density of the orbit in X implies density of the values in \mathbb{R} .

4.1. Closed orbits correspond to rational forms. The following proposition connects the rationality of Q to the topology of the orbit $H \cdot [g]$.

PROPOSITION 4.1. *The orbit $H \cdot [g]$ is closed in X if and only if Q is a scalar multiple of a rational form.*

Sketch. If Q is a scalar multiple of a rational form, then after rescaling g we may assume Q has integer coefficients. The stabilizer $\Gamma_Q = \{\gamma \in SL(3, \mathbb{Z}) : Q \circ \gamma = Q\}$ is an arithmetic subgroup of $SO(Q)$, and the Borel–Harish-Chandra theorem [6, Proposition 4.3] implies that Γ_Q is a lattice in $SO(Q)$. Translating back via g , the stabilizer $H \cap g\Gamma_Q g^{-1}$ is a lattice in H , which forces $H \cdot [g]$ to be closed and to have a finite H -invariant measure.

Conversely, if $H \cdot [g]$ is closed, then by a theorem of Dani–Margulis [8, Theorem 1.4] the stabilizer $H \cap g\Gamma_Q g^{-1}$ is a lattice in H . Pulling back to \mathbb{Z}^3 , this lattice consists of integer linear maps preserving Q , and the existence of this type of lattice in $SO(Q)$ forces Q to correspond to a rational form. \square

We use the contrapositive that if Q is irrational, then Proposition 4.1 implies that $H \cdot [g]$ is not closed. Dani’s theorem (Theorem 3.8) then prevents the orbit from escaping, and Ratner’s theorem (Theorem 4.2) implies that its closure is all of X .

THEOREM 4.2 (Ratner [4, Theorem 1]). *Let G be a connected Lie group, $\Gamma \subset G$ a lattice, and $F \subset G$ a subgroup generated by one-parameter unipotent subgroups. Then for every $x \in G/\Gamma$, there is a closed connected subgroup $S = S(x)$ with $F \subseteq S \subseteq G$ such that*

$$\overline{F \cdot x} = S \cdot x,$$

and the orbit $S \cdot x$ supports a finite S -invariant measure.

The claim from Ratner’s theorem is that orbit closures of unipotent flows are the full orbit of some intermediate closed subgroup S . The theorem does not tell us which S occurs for a given starting point; that depends on the geometry of the problem. Our task is to determine which intermediate subgroups S with $H \subseteq S \subseteq SL(3, \mathbb{R})$ are actually possible.

Since $H = SO(2, 1)^\circ$ is generated by its unipotent one-parameter subgroups, Theorem 4.2 applies with $F = H$, giving a closed connected subgroup S depending on $[g]$ such that

$$H \subseteq S \subseteq SL(3, \mathbb{R}), \quad \overline{H \cdot [g]} = S \cdot [g].$$

We now show the only possibilities are $S = H$ and $S = SL(3, \mathbb{R})$.

To classify the intermediate subgroups, we look at Lie algebras, which associate to each matrix Lie group G a vector space

$$\mathfrak{g} = \{X : e^{tX} \in G \text{ for all } t \in \mathbb{R}\},$$

encoding all directions one can move while staying in G . A *Lie subalgebra* is a linear subspace closed under the bracket $[X, Y] = XY - YX$, and every closed subgroup of G corresponds to exactly one such subspace, reducing the group problem to linear algebra. This means classifying intermediate groups S with $H \subseteq S \subseteq SL(3, \mathbb{R})$ is equivalent to classifying linear subspaces \mathfrak{s} with

$$\mathfrak{so}(2, 1) \subseteq \mathfrak{s} \subseteq \mathfrak{sl}(3, \mathbb{R}).$$

By Dynkin's classification [10, §2–§3] of maximal subalgebras of $\mathfrak{sl}(3, \mathbb{R})$, the only candidates are $\mathfrak{so}(3)$ (dimension 3, which equals $\dim \mathfrak{so}(2, 1)$ and so cannot strictly contain it) and $\mathfrak{gl}(2, \mathbb{R})$ (dimension 4).

Neither can contain $\mathfrak{so}(2, 1)$. The algebra $\mathfrak{so}(3)$ is the Lie algebra of the compact rotation group $SO(3)$, whose one-parameter subgroups all trace out bounded orbits; since $\mathfrak{so}(2, 1)$ contains the hyperbolic boost directions

$$b = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

whose one-parameter subgroups e^{tb} and e^{tc} move points along unbounded hyperboloids, a compact algebra cannot contain a non-compact one. For $\mathfrak{gl}(2, \mathbb{R})$, although $\mathfrak{so}(2, 1) \cong \mathfrak{sl}(2, \mathbb{R})$, the boost generators b and c have nonzero entries in the third row and column, placing them outside the upper-left 2×2 block entirely, so no rearrangement can fit $\mathfrak{so}(2, 1)$ inside $\mathfrak{gl}(2, \mathbb{R})$. Therefore no proper intermediate subgroup exists, so $S = H$ or $S = SL(3, \mathbb{R})$.

COROLLARY 4.3 (Ratner's dichotomy for H -orbits in X). *For every $[g] \in X$, exactly one of the following holds:*

- (i) $\overline{H \cdot [g]} = H \cdot [g]$, and the orbit is closed; or
- (ii) $\overline{H \cdot [g]} = X$, and the orbit is dense.

4.2. Proof of Oppenheim's Conjecture. Proposition 4.1 links irrationality of Q to the orbit $H \cdot [g]$ being non-closed; Corollary 4.3 then forces $\overline{H \cdot [g]} = X$; it remains to convert density of the orbit in X into density of $Q(\mathbb{Z}^3)$ in \mathbb{R} .

Proof of Conjecture 1.1. Choose $g \in SL(3, \mathbb{R})$ such that $Q(v) = Q_0(gv)$. Since Q is irrational, Proposition 4.1 implies $\overline{H \cdot [g]} = X$, so Corollary 4.3 gives $\overline{H \cdot [g]} = X$.

Fix a target value $c \in \mathbb{R}$ and tolerance $\varepsilon > 0$. Since Q_0 is indefinite, there exists $w \in \mathbb{R}^3$ with $Q_0(w) = c$; choose $h \in SL(3, \mathbb{R})$ so that w is a primitive vector of $h\mathbb{Z}^3$. Then $[h] \in X$ represents a lattice on which Q_0 takes the value c .

Since $\overline{H \cdot [g]} = X$, there exist $h_n \in H$ and $\gamma_n \in SL(3, \mathbb{Z})$ with $h_n g \gamma_n \rightarrow h$ in $SL(3, \mathbb{R})$. Define $v_n = \gamma_n^{-1} g^{-1} h_n^{-1} w \in \mathbb{Z}^3$; since $w \neq 0$ and all factors are invertible, $v_n \neq 0$. Since $h_n \in H = SO(Q_0)$ preserves Q_0 ,

$$Q(v_n) = Q_0(gv_n) = Q_0(h_n^{-1}w).$$

As $h_n g \gamma_n \rightarrow h$ we have $h_n^{-1}w \rightarrow h^{-1}w$, so by continuity of Q_0 ,

$$Q(v_n) \rightarrow Q_0(h^{-1}w) = Q_0(w) = c.$$

For n large enough $|Q(v_n) - c| < \varepsilon$, and since c was arbitrary, $Q(\mathbb{Z}^3)$ is dense in \mathbb{R} . \square

Summarizing the full argument:

$$\underbrace{Q \text{ irrational}}_{\text{Prop. 4.1}} \implies H \cdot [g] \text{ not closed} \xrightarrow{\text{Thm. 3.8} + \text{Thm. 4.2}} \underbrace{H \cdot [g] = X}_{\text{Cor. 4.3}} \implies \underbrace{Q(\mathbb{Z}^3) \text{ dense in } \mathbb{R}}_{\text{continuity of } Q_0}.$$

Remark 4.4 (Extension to higher dimensions). For $n \geq 4$, $\mathfrak{so}(p, q)$ is no longer maximal in $\mathfrak{sl}(n, \mathbb{R})$, so intermediate subgroups can appear and the dichotomy of Corollary 4.3 does not directly apply. The conjecture still holds in this case, but requires additional argument to rule out intermediate orbit closures.

5. QUANTITATIVE REFINEMENTS AND CONNECTIONS

5.1. The Eskin–Margulis–Mozes theorem. Margulis’s result establishes density, but we can do better. In [5, Thm. 1.3], Eskin, Margulis, and Mozes refined the qualitative density theorem to a precise asymptotic.

THEOREM 5.1 (Eskin–Margulis–Mozes [5, Thm. 1.3]). *Let Q be a nondegenerate, indefinite, irrational quadratic form in $n \geq 3$ variables. For any interval $(a, b) \in \mathbb{R}$ with $b > a$,*

$$\#\{x \in \mathbb{Z}^n : \|x\| < T, a < Q(x) < b\} \sim c_{Q,a,b} \cdot T^{n-2} \quad \text{as } T \rightarrow \infty,$$

where $c_{Q,a,b}$ is a positive constant depending on Q and the interval.

This set is just the number of nonzero integer vectors $x \in \mathbb{Z}^n$ with $\|x\| < T$ and $Q(x) \in (a, b)$. Now, not only do values of Q at integer points fall in (a, b) , but the number of such points grows at an explicit polynomial rate in T , which we’ll relate to the regions bounded by forms with values a and b .

The exponent $n-2$ comes from the hyperboloid $Q(x) = c$ having dimension $n-1$: the shell $\{a < Q(x) < b\} \cap \{\|x\| < T\}$ has volume proportional to $T^{n-2}(b-a)$, and the theorem says that integer points distribute in proportion to this volume. The constant

$$c_{Q,a,b} = \lambda(\{x \in \mathbb{R}^n : \|x\| < 1, a < Q(x) < b\}),$$

where λ denotes Lebesgue measure on \mathbb{R}^n [5, p. 94], is the fraction of the unit-ball slice contained in the shell. This is computed by writing $x = r\theta$ in spherical coordinates and using the homogeneity $Q(r\theta) = r^2Q(\theta)$ to separate the integral into a radial part and an integral over the surface area \mathbb{S}^{n-1} , which is evaluated numerically.

Inherently it may seem difficult to quantify the intersection of structured points from \mathbb{Z}^n and high-dimensional bounded hyperbolic shell regions defined by irrational coefficients that we already know bring uncertainty in asymptotics. In this case, it will help to first compute the average behavior over all unimodular lattices, and then show that \mathbb{Z}^n matches this average. The bridge is Siegel’s mean value theorem [12, Theorem 1], which says that for any integrable function f on \mathbb{R}^n , the sum of f over the nonzero vectors of a lattice Λ ,

$$\Phi_f(\Lambda) = \sum_{v \in \Lambda \setminus \{0\}} f(v),$$

then averaging over all unimodular lattices $\Lambda \in X_n$ recovers the Lebesgue integral:

$$\int_{X_n} \Phi_f(\Lambda) d\mu(\Lambda) = \int_{\mathbb{R}^n} f(x) dx.$$

Taking $f = \mathbf{1}_{\Omega_{T,a,b}}$ to be the indicator of the shell $\Omega_{T,a,b} = \{\|x\| < T, a < Q(x) < b\}$, the sum $\Phi_f(\Lambda)$ counts how many nonzero vectors of the lattice Λ are inside $\Omega_{T,a,b}$. Integrating this count over all unimodular lattices with respect to Haar measure μ on our space of lattices gives the average count across all lattices, while the right side $\lambda(\Omega_{T,a,b})$ is simply the Euclidean volume of the shell. Therefore, Siegel's theorem just says that on average, over our space of lattices, the number of lattice points in the shell equals its volume.

The EMM theorem focuses this average statement towards \mathbb{Z}^n specifically. Because Q is irrational, the H -orbit of $[g]$ equidistributes in X_n , so the lattice point counts for \mathbb{Z}^n agree with the space average over X_n . The time average (the point count for \mathbb{Z}^n) converges to the space average (the volume), giving

$$N_Q(T, a, b) \sim \lambda(\Omega_{T,a,b}) \sim c_{Q,a,b} \cdot T^{n-2}.$$

To see why the exponent is $n - 2$ and not $n - 1$, note that the gradient $\|\nabla Q\|$ grows as $O(T)$ on the ball of radius T , which thins the width of the shell as distance from the origin increases. This thinning reduces the volume growth by one power of T relative to the $(n - 1)$ -dimensional hyperboloid.

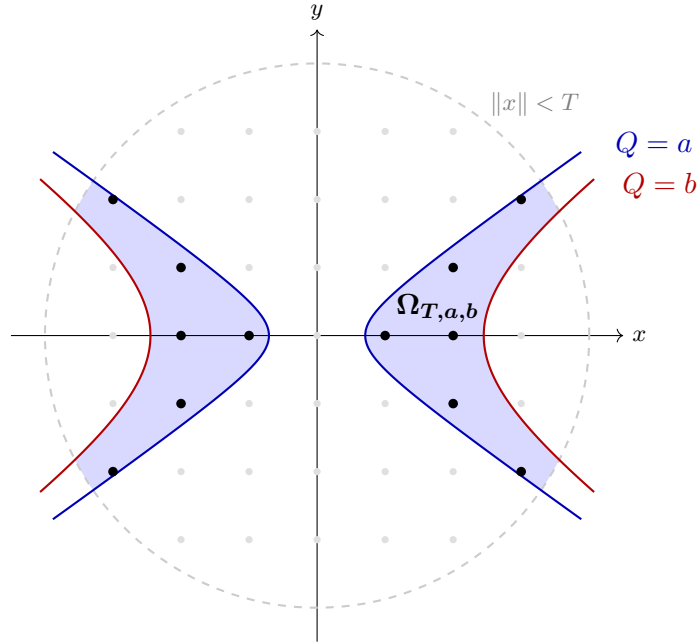


FIGURE 3. A visualization of the hyperboloid shell for a different example in $n = 2$ variables. The shell $\Omega_{T,a,b} = \{\|x\| < T, a < Q(x) < b\}$ (shaded blue) for $Q(x, y) = x^2 - 2y^2$, $T = 4$, $a = \frac{1}{2}$, $b = 6$. The shell is bounded by the level curves $Q = a$ (blue) and $Q = b$ (red) and clipped to the ball $\|x\| < T$ (dashed). Black dots are the integral points lying in $\Omega_{T,a,b}$ and are the points counted by $N_Q(T, a, b)$. Gray dots are integer points in the ball but outside the shell, and do not contribute to N_Q .

For $n \geq 3$, volume grows as T^{n-2} , and equidistribution of the H -orbit converts this into the integer point count. Lindenstrauss–Mohammadi–Wang–Yang [11] later established polynomial bounds on the rate of equidistribution, making the asymptotic effective.

5.2. Further directions. Progressing from Dirichlet’s study of linear forms to Oppenheims work on the distribution of quadratic forms at integral points leads naturally to the question: for which degrees $d \geq 3$ and homogeneous polynomials P in n variables, not proportional to a rational form, is $P(\mathbb{Z}^n)$ dense in \mathbb{R} ? The dynamical approach does not extend because the relevant group actions are no longer unipotent and neither Dani’s nondivergence theorem nor Ratner’s theorem applies. The best result in this direction is from Birch [13, Theorem 1], who showed that a nondegenerate homogeneous form of odd degree d with rational coefficients has a nontrivial zero over \mathbb{Z}^n , provided the number of variables is sufficiently large relative to d . In 2017, Browning and Prendiville showed that this holds for forms with $n \geq (d - \frac{\sqrt{d}}{2})2^d$ variables [14, Theorem 1.1]. However, guaranteeing a vector $x \in \mathbb{Z}^n$ exists such that $P(x) = 0$ is weaker than proving density in \mathbb{R} , and no proof of density is known for any specific irrational form of degree $d \geq 3$.

Whether analogous effective counting results to those of Eskin–Margulis–Mozes can be obtained for higher degree forms, even in special cases, remains an active area of research.

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