

1 **THE DISCRETE BEGINNING OF THE LAGRANGE SPECTRUM**  
2 **AND ITS CONNECTION TO MARKOV THEORY**

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ABSTRACT. This paper provides an expository survey of the classical structure of the Lagrange spectrum  $\mathcal{L}$ , with a specific focus on its discrete characteristics in the beginning of the spectrum. The Lagrange spectrum characterizes the constants of Diophantine approximation for irrational numbers, which represent how badly a real number can be approximated by rationals. While the Lagrange spectrum is mostly known as containing a continuous subset named Hall's ray, the beginning of the spectrum has, in fact, a surprising characteristic of discreteness. This paper mainly discusses the structure of the Lagrange spectrum for values below 3. A combinatorial analysis of the symbolic sequences of partial quotients is provided to demonstrate that such values correspond exclusively to purely periodic continued fractions. Furthermore, we explore the underlying connection between the discrete values in the Lagrange spectrum and the positive integer solutions to the Markov equation  $x^2 + y^2 + z^2 = 3xyz$ . By tracing the repeated generation of Markov triples through the Markov tree structure, we show how the convergence of the algebraic values towards the upper threshold point of 3 links to the increasing complexity of the associated symbolic number blocks.

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11 1. INTRODUCTION

12 Diophantine approximation is the study that deals with the approximation of ir-  
13 rational numbers through rational numbers. The theory is driven by a fundamental  
14 trade-off dilemma, which is how closely we are able to approximate an irrational  
15 number  $\alpha$  by a rational  $p/q$  while keeping a reasonable cost of using a large denom-  
16 inator.

17 As introduced in Dirichlet's Approximation Theorem (1842), for any irrational  $\alpha$   
18 and any real  $Q > 1$ , there exist integers  $p, q$  such that  $1 \leq q < Q$  and  $|\alpha q - p| \leq \frac{1}{Q}$ .

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19 An essential [Sch80, Corollary 1B] of the theorem is that there exist infinitely many  
 20 pairs of relatively prime integers  $p, q$  satisfying

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

21 This means that Dirichlet’s Theorem guarantees an error bound of  $q^{-2}$ , serving as  
 22 a baseline for the Lagrange spectrum  $\mathcal{L}$ . The Lagrange spectrum aims to improve  
 23 such a constant by replacing the 1 in the numerator through finding the largest  
 24  $L(\alpha)$ .

25 An interesting observation in the study of the Lagrange spectrum is its transition  
 26 from ordered elements to continuous chaos, which in fact has been a long-standing  
 27 mystery for decades. It is now known that, for values larger than approximately  
 28 4.5224, the spectrum acts as a solid line which can also be seen as a continuous ray,  
 29 on which any real number can be served as an approximation constant. However,  
 30 at the beginning of the Lagrange spectrum, the values are, in contrast, remarkably  
 31 sparse, which is made up of a sequence of isolated, discrete points starting at  $\sqrt{5}$ .

32 This leads us to the central guiding questions of this paper: Why does the  
 33 Lagrange spectrum exhibit such discrete beginning before it eventually transitions  
 34 into a continuous ray? More specifically, what underlying arithmetic structure  
 35 causes these specific isolated values to appear?

36 The exploration of the Lagrange spectrum below 3 reveals a tight interdepen-  
 37 dence between Diophantine approximation, symbolic dynamics, and number theory.  
 38 To investigate this relationship, we have organized the paper as follows:

39 In **Section 2**, we demonstrate the connection between the symbolic dynamics of  
 40 partial quotients and the quality of rational approximations.

41 In **Section 3**, we formally introduce the definition of the Lagrange spectrum  
 42 alongside its key properties, including the mysterious transition from discrete values  
 43 to the continuous Hall’s Ray.

44 In **Section 4**, we provide a combinatorial proof of the spectrum’s discreteness  
 45 below the value 3. We illustrate that these values must be generated from purely  
 46 periodic sequences, proving that as the symbolic complexity of the continued frac-  
 47 tions grows, the Lagrange values monotonically approach the upper limit of 3. We  
 48 show that the discreteness of the initial Lagrange spectrum is a direct consequence  
 49 of the arithmetic isolation of certain quadratic irrationals. Specifically, when the  
 50 partial quotients of an irrational number are restricted to compositions of small  
 51 integers—such as pairs of 2s separated by varying blocks of 1s—the resulting ap-  
 52 proximation constants are forced into a countable sequence of values. This marks  
 53 the boundary where the spectrum begins its transition toward its dense, chaotic  
 54 section.

55 Finally, in **Section 5**, we map these symbolic sequences to branches of the Markov  
 56 Tree. We show that this structural rigidity is perfectly captured by the Markov  
 57 equation:

$$x^2 + y^2 + z^2 = 3xyz.$$

58 The mapping from a Markov number  $m$  to the value  $\sqrt{9m^2 - 4}/m$  provides  
 59 a bridge between the binary mutation of integer triples and the discrete values  
 60 of the Lagrange spectrum. This proves that values such as  $\sqrt{5}$ ,  $\sqrt{8}$ , and  $\sqrt{221}/5$   
 61 are not arbitrary coincidences, but are reflections of an underlying algebraic tree.  
 62 Ultimately, this study emphasizes how the most badly approximatable numbers are

63 captured by the beauty of elegant algebraic laws, revealing that the gap between  
 64  $\sqrt{5}$  and the accumulation threshold at 3 is a direct manifestation of the Markov  
 65 tree's architecture.

## 66 2. BACKGROUND

67 The field of Diophantine approximation originates from the motivating question  
 68 of how effectively irrational numbers can be represented by rational fractions. Be-  
 69 cause the set of rational numbers  $\mathbb{Q}$  is dense in the realm of  $\mathbb{R}$ , we would be able to  
 70 find a rational number  $p/q$  that is arbitrarily close to any irrational  $\alpha$ . However, a  
 71 fundamental balance needs to be sought between the accuracy of the approximation  
 72 and the complexity of the denominator  $q$ . To make this precise, we first formalize  
 73 what it means for an approximation to be good in [Section 2.1](#), then introduce the  
 74 theory of continued fractions as the natural tool for systematically generating best  
 75 approximations [Section 2.2](#).

76 **2.1. The Quality of Approximation.** A central question in the theory is that,  
 77 for a given  $\alpha$ , what is the smallest error we can make  $|\alpha - p/q|$  relative to the  
 78 size of  $1/q^2$ ? Dirichlet's Approximation Theorem in [[Sch80](#), Corollary 1B] provides  
 79 one with a baseline, stating that, for any irrational number  $\alpha$ , there exist infinitely  
 80 many rational numbers  $p/q$  such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{q^2}.$$

81 This result suggests that the order of approximation for irrational numbers is at  
 82 least  $1/q^2$ . The Lagrange spectrum arises when we attempt to refine this result by  
 83 seeking for the largest constant  $L(\alpha)$  that can replace the 1 in the numerator while  
 84 still maintaining infinitely many solutions. This constant

$$L(\alpha) = \limsup_{p,q \rightarrow \infty} \frac{1}{q|q\alpha - p|}$$

85 measures how badly an irrational number is approximated, demonstrating how  
 86 numbers with small  $L(\alpha)$  are considered the badly approximable numbers.

87 **2.2. Continued Fractions.** To analyze the approximation constants more sys-  
 88 tematically, we need a method for generating the best rational approximations.  
 89 The theory of continued fractions in [[Sch80](#), pp. 12-13] provides a standard way to  
 90 represent real numbers through a sequence of integers. For any irrational number  
 91  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , we can write it in the form of

$$\alpha = [a_0; a_1, a_2, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \ddots}},$$

92 where  $a_0 \in \mathbb{Z}$  and  $a_i \in \mathbb{N}$  for  $i \geq 1$  is the  $i$ th partial quotient of  $\alpha$ .

93 As shown in the algebraic theory of such expansions in [[Cas57](#), pp. 6-7], the  
 94 truncated fractions

$$p_n/q_n = [a_0; a_1, \dots, a_n],$$

95 named convergents of  $\alpha$ , provide the best rational approximations to  $\alpha$ . More  
 96 specifically, any rational number  $p/q$  satisfying condition  $|\alpha - p/q| < 1/(2q^2)$  must  
 97 be a convergent of  $\alpha$ . This fact allows one to move from a general search over  
 98 the realm of all rational numbers to a more discrete study of the partial quotients

99  $\{a_i\}$ . Therefore, the study of the Lagrange spectrum can be transformed into the  
 100 study of the symbolic properties of these integer sequences, providing the neces-  
 101 sary foundation for the Markov theory, which will be discussed in the subsequent  
 102 sections.

103

### 3. LAGRANGE SPECTRUM

104 The classical theory of Diophantine approximation is centered on the question of  
 105 how closely irrational numbers can be approximated by rational numbers. For an  
 106 irrational number  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ , we are interested in discussing the existence of infinitely  
 107 many rational fractions  $p/q$  that satisfy some specific proximity constraints. We  
 108 begin by formally defining the Lagrange constant and the Lagrange spectrum in  
 109 [Section 3.1](#) followed by examining two key structural properties that characterize  
 110 the spectrum: the lower bound established by Hurwitz's Theorem in [Section 3.2.1](#)  
 111 and the transition to a continuous ray proved by Hall in [Section 3.2.2](#).

112 **3.1. Definition.** The best constant of Diophantine approximation for a positive  
 113 real number  $\alpha$ , denoted as the **Lagrange constant**, is defined as:

$$L(\alpha) := \limsup_{p,q \rightarrow \infty} \frac{1}{|q(q\alpha - p)|}.$$

114 In other terms,  $L(\alpha)$  represents the largest constant such that the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{L(\alpha)q^2}.$$

115 has infinitely many solutions  $(p, q) \in \mathbb{N}^2$ . Using this value, we define the **Lagrange**  
 116 **spectrum** in [[JMM24](#), Section 1] as the set of all such best constants

$$\mathcal{L} := \{L(\alpha) \mid \alpha \in \mathbb{R} \setminus \mathbb{Q}\}.$$

117 **3.2. Key Properties.** The Lagrange spectrum exhibits a unique topological struc-  
 118 ture that transitions from a discrete set of isolated points to a continuous line seg-  
 119 ment. This behavior is governed by the size and distribution of partial quotients in  
 120 the continued fraction expansions of real numbers. We examine the classical lower  
 121 bound  $\sqrt{5}$  arising from Hurwitz's Theorem, which grounds the discrete beginning  
 122 of the spectrum before eventually turning to Hall's Ray.

123 **3.2.1. Hurwitz's Theorem and the Lower Bound.** The structure of the Lagrange  
 124 spectrum  $\mathcal{L}$  is bounded by a classic result in Diophantine approximation.

125 **THEOREM 3.1** ([\[MM21](#), Hurwitz's Theorem]). *For every irrational number  $\alpha$ ,*  
 126 *there exist infinitely many rational numbers  $p/q$  such that*

$$\left| \alpha - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.$$

127 This theorem establishes that  $\inf(\mathcal{L}) = \sqrt{5}$ . Because the error of approximation  
 128 is roughly inversely proportional to the size of the partial quotients, the numbers  
 129 that are badly approximable are the ones with the smallest possible quotients.

130 The golden ratio would be one of the most extreme cases, where its continued  
 131 fraction expansion  $\phi = [1; 1, 1, \dots]$  is entirely consisted of the smallest possible  
 132 number in  $\mathbb{N}$ . For this sequence  $\lambda_n(\phi) \rightarrow \sqrt{5}$  as  $n$  approaches  $\infty$ ,  $\sqrt{5}$  is thus the  
 133 absolute floor of the spectrum.

134 The gap between  $\sqrt{5}$  and the next value,  $\sqrt{8}$  (which corresponds to  $L(\alpha)$  of the  
 135 continued fraction  $\alpha = [2; 2, 2, \dots]$ ), demonstrates the sparse characteristic of the  
 136 initial section of the spectrum. As it will be introduced in the following sections,  
 137 this discreteness persists for all values below 3, along with some grammar of the  
 138 partial quotients in order to avoid exceeding the upper threshold.

139 **3.2.2. Hall's Ray and Its Continuity.** One of the most intriguing features of the  
 140 Lagrange spectrum is its transition from a sparse, discrete set to a continuous  
 141 set. In 1947, Marshall Hall Jr. proved that the spectrum contains an infinite line  
 142 segment, which is now known as **Hall's ray**.

143 **THEOREM 3.2** ([Hal47]). *Any real number  $\alpha$  can be written in the form*

$$\alpha = a + [0; b_1, b_2, \dots] + [0; c_1, c_2, \dots],$$

144 *where  $a$  is an integer and the partial quotients  $b_i$  and  $c_i$  for  $i = 1, 2, \dots$  are positive*  
 145 *integers not exceeding 4 ( $1 \leq b_i, c_i \leq 4$ ).*

146 Hall's theorem indicates that once the partial quotients are sufficiently varied,  
 147 the gaps in the spectrum will then disappear. While Hall proved the existence  
 148 of the infinite ray, the point where the spectrum becomes a solid line was more  
 149 precisely determined a few decades after by Freiman in 1962 in [Fre62] to be ap-  
 150 proximately 4.5278..., known as Freiman's constant. This eventually continuous  
 151 behavior in Lagrange spectrum gives a sharp contrast to the segment below 3, where  
 152 the spectrum is comprised solely of isolated points.

#### 153 4. THE DISCRETE BEGINNING OF THE SPECTRUM

154 While the Lagrange spectrum  $L$  exhibits a rich, complex structure, it ultimately  
 155 lies within a continuous interval. Surprisingly, the initial part of the spectrum  
 156 appears to behave differently. For values below 3, instead of containing dense  
 157 intervals, the spectrum is formed by some sparse and countable sequences of values,  
 158 which is thus referred to as the discrete part of the spectrum.

159 This behavior is formalized through the following theorem, stating that for any  
 160 threshold below 3, there are only countably many symbolic sequences that can  
 161 produce such values.

162 **THEOREM 4.1** ([CF89, Theorem 5]). *Given any  $\varepsilon > 0$ , the set of doubly infinite*  
 163 *sequences  $A$  such that  $\lambda_i(A) < 3 - \varepsilon$  for all  $i$  is finite. Every such sequence  $A$  is*  
 164 *purely periodic.*

165 *The set of numbers less than 3 in the Markov or Lagrange spectrum is countable*  
 166 *and discrete, with 3 as its only limit point.*

167 To sketch why the spectrum below 3 is discrete, we outline the combinatorial  
 168 argument from [CF89, pp. 8-9], built on the foundation of the following two theo-  
 169 rems.

170 **THEOREM 4.2** ([CF89, Theorem 3]). *In order to have  $\lambda_i(A) < 3$  for all  $i$  and*  
 171  *$A \neq \bar{1}$ , it is necessary and sufficient that  $A$  have the form*

$$\cdots 2 2 1_{2r(-1)} 2 2 1_{2r(0)} 2 2 1_{2r(1)} 2 2 \cdots,$$

172 *where the  $r(i)$  are nonnegative integers with the properties:*

173 (A)  $|r(i) - r(j)| \leq 1$  for all  $i$  and  $j$ ;

174 (B) if  $r(i+1) - r(i)$  is  $-1$  or  $+1$ , respectively, then the first of the integers

$$r(i+j+1) - r(i-j) \quad (j = 1, 2, \dots)$$

175 which is not zero is  $+1$  or  $-1$ , respectively.

176 THEOREM 4.3 ([CF89, Theorem 4]). If  $\lambda_i(A) < 3$  for all  $i$  and  $A \neq \bar{1}$ , then there  
177 is a nonnegative integer  $r$  such that the sequence

$$\dots, r(-1), r(0), r(1), \dots$$

178 is either

$$\dots, r, r, r, \dots$$

179

$$\dots, r+1, r_{s(-1)}, r+1, r_{s(0)}, r+1, r_{s(1)}, r+1, \dots$$

180 where the  $s(i)$  are nonnegative integers with the properties:

181 (A)  $|s(i) - s(j)| \leq 1$  for all  $i$  and  $j$ ;

182 (B) if  $s(i+1) - s(i)$  is  $-1$  or  $+1$ , respectively, then the first of the integers

$$s(i+j+1) - s(i-j) \quad (j = 1, 2, \dots)$$

183 which is not zero is  $+1$  or  $-1$ , respectively.

184 Detailed proofs of Theorem 4.2 and Theorem 4.3 can be found in [CF89, pp. 6-7].  
185 The central observation from the two theorems is that, in order to keep the value  
186 of  $\lambda_i(A)$  strictly below 3, the digits of the sequence  $A$  are physically restricted  
187 to be within the set  $\{1, 2\}$ . Any occurrence of a digit  $a_i \geq 3$  would immediately  
188 push  $\lambda_i(A)$  to be greater than 3 due to the additive nature of continued fraction  
189 expansion  $[a_i; \dots] + [0; \dots]$ .

190 The proof excludes the trivial cases, where  $A$  is the constant sequence  $\bar{1}$  or  $\bar{2}$ ,  
191 which gives the first two values of the spectrum:  $\sqrt{5}$  and  $\sqrt{8}$ , such that both 1s  
192 and 2s would appear. For any other nontrivial sequences below 3, a strict grammar  
193 appears to state as follows: 2s must come in pairs, which must also be separated  
194 by blocks of 1s. This allows one to view the sequence  $A$  as a sequence of blocks

$$A = \dots 2, 2, \underbrace{1, 1, \dots, 1}_{r(i)}, 2, 2, \dots,$$

195 where  $r(i)$  is a non-negative integer that denotes the number of 1s between the  $i$ -th  
196 and  $(i+1)$ -th pair of 2s.

197 A recursive pattern emerges as the core idea of the proof. Cusick shows in his  
198 work that if  $A$  were to remain below  $3 - \varepsilon$ , the sequence of block-lengths  $\{r(i)\}$   
199 must itself follow a rigid pattern similar to  $A$ . Taking one step further, by treating  
200 the sequence of  $\{r(i)\}$  as a new alphabet, a second-order sequence  $\{s(i)\}$  can be  
201 created which describes the frequency of these  $r$ -blocks by looking for gaps within  
202 these gaps.

203 This process thus generates a hierarchy of sequences

- 204 •  $A$ : The primary sequence of digits  $\{1, 2\}$ .
- 205 •  $\{r(i)\}$ : The sequence of counts of 1s.
- 206 •  $\{s(i)\}$ : The sequence of counts of  $r$ -blocks.
- 207 •  $\{t(i)\}, \{u(i)\}, \dots$ : Higher-order descriptions of such pattern.

208 For instance, let

$$A = (\dots, \underline{2}, \underline{2}, 1, 1, \underline{2}, \underline{2}, 1, 1, \underline{2}, \underline{2}, 1, 1, 1, 1, \underline{2}, \underline{2}, 1, 1, \underline{2}, \underline{2}, 1, 1, \underline{2}, \underline{2}, 1, 1, 1, \dots)$$

209 then the sequences

$$\begin{aligned} \{r(i)\} &= (\dots, 2, 2, 4, 2, 2, 4, 2, 2, 4, \dots) \\ \{s(i)\} &= (\dots, 1, 1, 1, 1, \dots) \end{aligned}$$

211 are generated following the rules defined above. At this third layer, the sequence  
212 has finally become constant.

213 It is demonstrated that, for a fixed  $\varepsilon > 0$ , the nesting cannot continue indefi-  
214 nitely with varying terms. By establishing that the number of possible symmetric  
215 arrangements is bounded by a function defined as  $f(\varepsilon)$ , the hierarchy shown above  
216 must eventually collapse into a sequence of identical and repeating terms. When  
217 a sequence in this hierarchy becomes constant, meaning that at one point all el-  
218 ements in the sequence are equal, the original sequence  $A$  is hence forced to be  
219 purely periodic. Therefore, the Lagrange spectrum below 3 is consisted solely of  
220 values derived from periodic continued fractions, resulting in such a discrete and  
221 countable set.

222 The values below 3 represent the sparse beginning of the Lagrange spectrum.  
223 As illustrated in [Table 1](#) below, as the sequences grow in complexity, for example,  
224  $A_k = \overline{221}_{2k}$ , the values  $M(A_k)$  increase monotonically towards 3, which shows that  
225 3 is not only a upper bound, but also the first accumulation point of the Lagrange  
226 spectrum.

TABLE 1. The smallest values in the Lagrange spectrum indexed by Markov Triples

Sequence $A$	$M(A)$	Decimal (approx.)	Markov Triple $(m_1, m_2, m)$
$\overline{1}$	$\sqrt{5}$	2.236068...	(1, 1, 1)
$\overline{2}$	$\sqrt{8}$	2.828427...	(1, 1, 2)
$\overline{2211}$	$\sqrt{221}/5$	2.973214...	(1, 2, 5)
$\overline{221111}$	$\sqrt{1517}/13$	2.996053...	(1, 5, 13)
$\overline{222211}$	$\sqrt{7565}/29$	2.999207...	(2, 5, 29)

227 The data in [Table 1](#) reveals an intriguing pattern: each value  $M(A)$  in the La-  
228 grange spectrum appears to be closely tied to a triple of positive integers  $(m_1, m_2, m)$   
229 as listed in the final column. These triples are not arbitrary but instead satisfy a  
230 remarkable algebraic relationship that will be made precise shortly. We call such  
231 triples Markov triples, and their connection to the Lagrange spectrum turns out to  
232 be far more than a numerical coincidence, as we explore in Section 5.

## 233 5. MARKOV NUMBERS: THE UNDERLYING STRUCTURE

234 The pattern observed in [Table 1](#) reflects a deep structural fact about the Lagrange  
235 spectrum. As noted above, the discreteness of the spectrum's beginning is not a  
236 mere coincidence. Instead, it arises from a deep connection to Markov's Theory.  
237 The transition from sequences of  $\{1, 2\}$  discussed above to the numerical values of  
238 the spectrum is largely related to the Markov Equation

$$x^2 + y^2 + z^2 = 3xyz.$$

239 The positive integer solutions  $(m, m_1, m_2)$  are known as Markov triples, and such  
240 integers appearing in the triples are called Markov numbers. We show how all

241 Markov triples can be systematically generated through a recursive binary tree  
 242 structure in [Section 5.1](#) and establish the bijection between this algebraic framework  
 243 and the symbolic sequences of  $\{1, 2\}$  that index the spectrum [Section 5.2](#), where  
 244 the following theorem explicitly maps these algebraic solutions to the values in the  
 245 Lagrange spectrum.

246 **THEOREM 5.1** ([\[CF89, Theorem 6\]](#)). *The Markov (or Lagrange) spectrum below*  
 247 *3 consists of the numbers  $\sqrt{9m^2 - 4}/m$ , where  $m$  is a positive integer such that*

$$m^2 + m_1^2 + m_2^2 = 3mm_1m_2, \quad m_1 \leq m, \quad m_2 \leq m$$

248 *holds for some positive integers  $m_1$  and  $m_2$ .*

249 **5.1. The Markov Tree.** Every Markov triple can be derived from the first Markov  
 250 triple  $(1, 1, 1)$  by repeatedly applying mutations to each Markov number. Given a  
 251 Markov triple  $(m_1, m_2, m)$ , we can fix the first two elements and treat the third as  
 252 a variable in the quadratic equation. More specifically, let  $x_1 = m$  be one solution  
 253 of variable  $x$  in the quadratic equation with  $m_1$  and  $m_2$  as fixed constants. We  
 254 have

$$x^2 - (3m_1m_2)x + m_1^2 + m_2^2 = 0.$$

255 Through Vieta's formulas, it is implied that  $x_2 = 3m_1m_2 - m$ , and thus  $(m_1, m_2, 3m_1m_2 -$   
 256  $m)$  is also a solution to the Markov Equation. This process generates a binary tree  
 257 structure, which is known as the Markov tree as shown in [Figure 1](#) below.

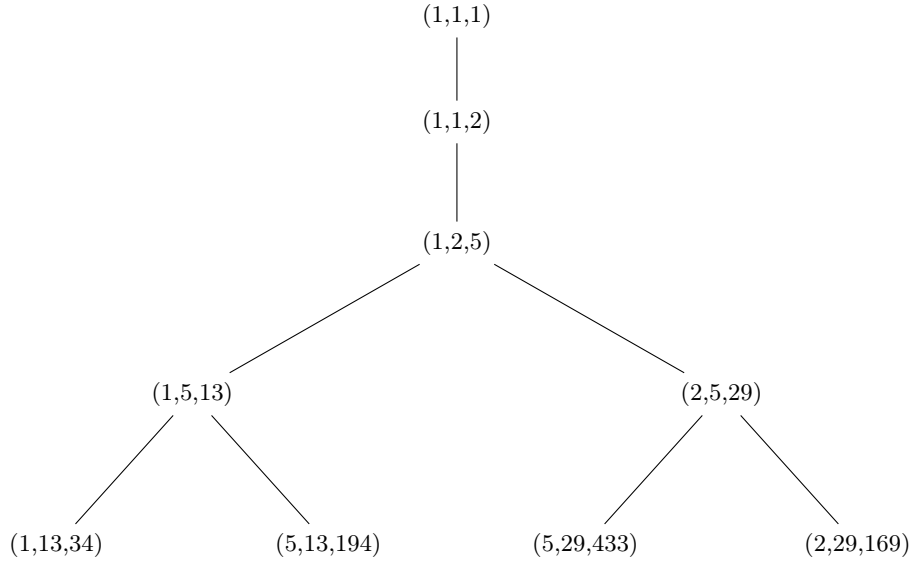


FIGURE 1. First layers of Markov tree structure generated recursively from Markov triples

- 258 • First Markov triple:  $(1, 1, 1)$ , corresponding to the sequence  $\bar{1}$ , applying the  
 259 equation provided in the theorem  $\sqrt{9m^2 - 4}/m$  gives

$$\frac{\sqrt{9 \cdot (1)^2 - 4}}{1} = \sqrt{5}.$$

- 260 • First branch:  $(1, 1, 2)$ , corresponding to the sequence  $\overline{2}$ , applying the same  
261 equation gives

$$\frac{\sqrt{9 \cdot (2)^2 - 4}}{2} = \sqrt{8}.$$

- 262 • Second branch:  $(1, 2, 5)$ , corresponding to the sequence  $\overline{2211}$ , applying the  
263 equation gives

$$\frac{\sqrt{9 \cdot 5^2 - 4}}{5} = \frac{\sqrt{221}}{5}.$$

- 264 • Subsequent branches:  $(13, 5, 1), (29, 5, 2), \dots$

265 As we move along the branches of the Markov tree, the Markov number  $m$   
266 increases, and the value of  $L(\alpha)$  matches the result of  $\sqrt{9m^2 - 4}/m$ , which also  
267 strictly increases toward the limit point 3. This explains the nesting behavior  
268 discussed in the [previous section](#), demonstrating that the tree structure provides  
269 the exact roadmap of how the sequences of digits  $\{1, 2\}$  need to be arranged.

270 **5.2. The Bijection between combinatorics and algebra.** Another essential  
271 insight of Markov's theory is the bijection between the periodic symbolic sequences  
272  $A$  and the Markov triples.

273 PROPOSITION 5.2 (Symbolic Characterization [CF89]). *For every Markov num-*  
274 *ber  $m$ , there exists a unique purely periodic sequence  $A_m$  consisting only of the*  
275 *digits  $\{1, 2\}$  (up to shift and reversal) such that the Lagrange value  $L(\alpha)$  for any*  
276  *$\alpha = [a_0; A_m]$  is exactly  $\sqrt{9m^2 - 4}/m$ .*

277 In other words, if  $m$  is a Markov number, the corresponding quadratic form has  
278 a root  $\alpha$  whose continued fraction is purely periodic, whereas the partial quotients  
279 are in fact determined by the path taken through the Markov tree to reach  $m$ .

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