APPROACHES TO THE LONELY RUNNER CONJECTURE: REDUCTIONS, COMPUTATIONS, AND HEURISTICS

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Abstract. The Lonely Runner Conjecture is a long-standing open problem in number theory. It states that for $n+1$ runners moving at distinct constant speeds on a unit circle, each runner is at some time at distance at least $\frac{1}{n+1}$ from all others. The conjecture is known only for small values of n despite its simple formulation.

This paper presents an expository overview of approaches to the conjecture, focusing on reductions and computational techniques. We describe the reduction to integer velocities and its reformulation in a number-theoretic setting, and briefly discuss bounds and probabilistic intuition related to the conjectured distance. We also survey computational methods that help restrict the search space for finite cases. We conclude by sharing some insights on how Diophantine approximation connects to this problem.

Contents

1. Introduction

The Lonely Runner Conjecture of Wills [Wil67] and Cusick [CP84] is a deceptively simple problem concerning the motion of points on a circle. Informally, it asserts that if several runners move at distinct constant speeds around a circular track, then each runner will eventually be "lonely," in the sense of being sufficiently far away from all others. Despite its elementary formulation, the conjecture has proven to be remarkably deep and has connections to several areas of mathematics, including Diophantine approximation and additive combinatorics.

We now state the conjecture formally.

Conjecture 1.1. [Lonely Runner Conjecture] Suppose $n+1$ runners move around a circular track at distinct constant speeds. Then for each runner there

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exists a time at which that runner is at least $\frac{1}{n+1}$ of the track away from every other runner.

An illustration of the problem for three runners is shown in Figure 1. At any given time, the runners occupy positions on the unit circle, and the conjecture asserts that each runner eventually finds itself in a sufficiently large gap between the others.

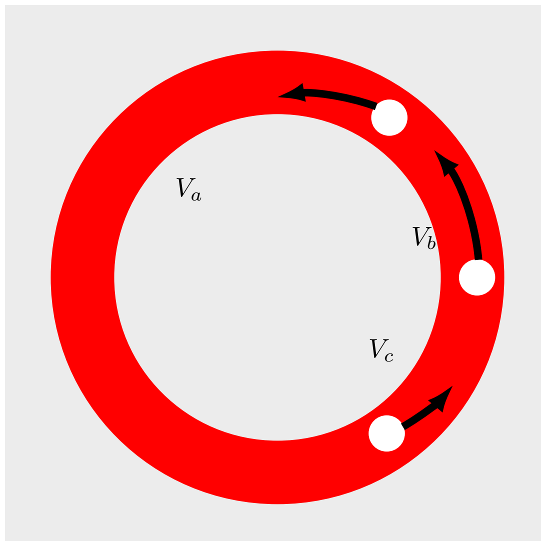

FIGURE 1. Lonely runner problem for three runners : Generated using TikZ.

A key feature of the conjecture is that only relative positions matter. By shifting to a reference frame moving with one of the runners, one may assume that one runner is stationary and the remaining n runners move with distinct nonzero velocities. Thus, the problem reduces to understanding how the positions of these n runners evolve modulo 1 over time.

Another useful way to interpret the conjecture is in terms of spacing on the circle. At any fixed time, the runners determine a configuration of $n + 1$ points on the unit circle, which partitions the circle into arcs. Since the total length of the circle is 1, the average arc length is $\frac{1}{n+1}$. The conjecture asserts that for each runner, there exists a time at which the arc containing that runner has length at least $\frac{1}{n+1}$. In other words, each runner eventually lies in a sufficiently large gap between the others.

Although the conjecture is easy to verify for small values of n, the general case remains open. It is known to hold for $n \leq 12$, with proofs combining combinatorial arguments and, in larger cases, computer-assisted methods. For larger values of n, only partial results are known, and the gap between lower and upper bounds remains significant.

A central challenge in the problem is to understand how the arithmetic structure of the velocities influences the distribution of the points $\{tv_i\}$ on the circle. On the one hand, probabilistic intuition suggests that these points should behave somewhat like independent random variables, leading to a natural scale of order $\frac{1}{2n}$ for the minimal distance. On the other hand, the conjecture predicts a stronger bound of $\frac{1}{n+1}$, indicating that there are nontrivial correlations that can be exploited.

The goal of this paper is to present an expository account of several approaches to the Lonely Runner Conjecture, with a focus on reductions and computational techniques. We begin by reducing the problem to the case of integer velocities and reformulating it in a number-theoretic framework in Section 2. We then examine the upper bound (in Section 3) and lower bounds (in Section 4) for the conjectured distance, highlighting both combinatorial constructions and measure-theoretic arguments. In Section 5, we discuss computational approaches that aim to verify the conjecture for finite ranges of parameters by reducing the search space. Additionally, in Section 6 we discuss how the problem deviates significantly from the independence assumption when viewed from a probabilistic lens. Finally, we conclude with insights into how this problem can be viewed as a Diophantine approximation-avoidance problem in Section 7.

2. Reduction to a number-theoretic statement

Our first goal would be to formalize it in a number theoretic language via reducing the problem into the following scenario.

We have n non-stationary runners with integer speeds (rather than reals) and one runner who is not moving. Reducing to this instance will help us understand how Diophantine approximation come up in this problem. Formally we want to reduce to the following claim.

Claim 2.1. *Let $v_1, \ldots, v_n$ be distinct positive integers representing runner speeds. There exists a time t such that*

$$\|tv_i\| \geq \frac{1}{n+1}$$

for all i, where $\|x\|$ denotes the distance from x to the nearest integer.

First, assume that every runner's velocity is rational. We can reduce from this case to Claim 2.1 in the following manner.

Let $v_i = p_i/q_i$ for $i \in \{0, 1, 2, \ldots, n\}$. Now rescale the time by a factor of $q_0 q_1 q_2 \ldots q_n$. Then the velocities become all integers. Note that only relative velocities matter in this problem. Hence WLOG we can reduce to $v_0 = 0$. Now note that Claim 2.1 has $||tv_i||$, which takes the same value if we replace v_i by $-v_i$, hence we can assume $v_1, v_2, \ldots, v_n$ are positive integers. This completes our reduction from rational velocities to Claim 2.1.

The reduction from irrational velocities to rational velocities was given in Lemma 8 in Section 4 of [BHK01] by Bohman, Holzman and Kleitman. The above mentioned reduction involves some linear-algebraic manipulations, which we omit.

A natural next question would be to ask whether the distance used in Conjecture 1.1 is the correct distance. We will answer this question by proving upper and lower bounds in the next two sections.

3. Optimal Upper Bound

In this section, we prove the following theorem, which will give us the optimal upper bound (i.e. matches Conjecture 1.1).

THEOREM 3.1. *$1/(n+1)$ is an upper bound for the distance that we use in Conjecture 1.1.*

Proof. We will do this by building a counterexample for general n.

Let $v_i = i$ for $i \in \{0, 1, 2, \ldots, n\}$. Let $\{x\}$ be the fractional part of x. For the sake of contradiction, assume that there exists a time t such that $\{v_i t\} \in (1/(n+1), n/(n+1))$, for all $i \in [n]$ (we define $[n] := \{1, 2, 3, \ldots, n\}$). If we look at where runners 1 to n are at time t, there should be $n-1$ gaps and note that the size of the interval $(1/(n+1), n/(n+1))$ is $\frac{n-1}{n+1}$. The pigeonhole principle implies that there are two runners with gap at most

$$\frac{\frac{n-1}{n+1}}{n-1} = \frac{1}{n+1}.$$

Therefore, there exist distinct $i, j \in [n]$, such that

$$\{v_i t\} \leq \{v_j t\} < \{v_i t\} + 1/(n+1).$$

Note that $|v_j - v_i| = |j - i| = k = v_k$ for some $k \in [n]$ because $i, j \in [n]$. Therefore, $||v_k t|| = ||(v_j - v_i)t|| < 1/(n+1)$, which is a contradiction. An example with 10 runners can be found in Figure 2. Here we see that the minimum distance over all non-stationary runners to the origin reaches $1/(n+1) = 1/10 = 0.10$.

□

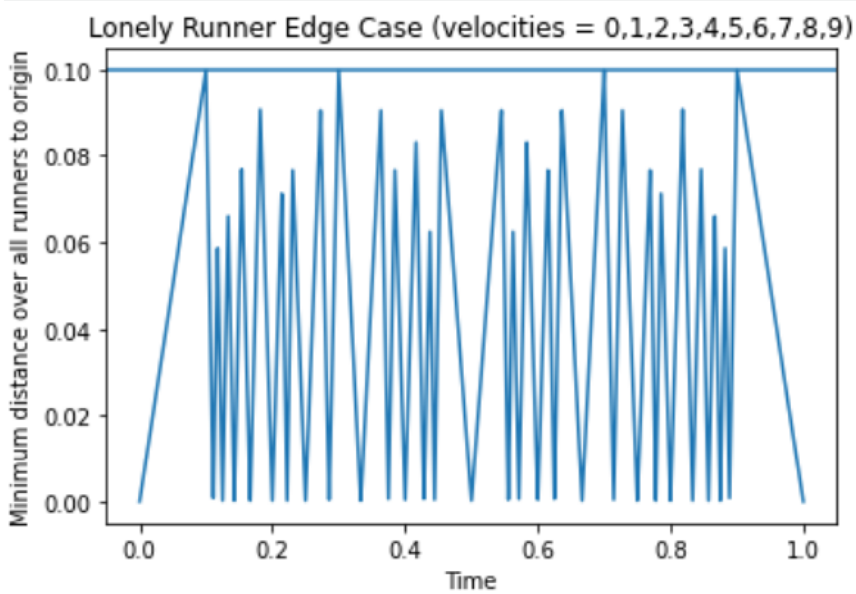

FIGURE 2. Example edge case for 10 runners : The figure was generated via Python. Updates position of each runner using small timesteps, and plots the minimum distance to origin.

Now that we have proved the optimal upper bound, let us focus our attention into lower bounds.

4. Lower Bounds

We now prove a lower bound of $\frac{1}{2n}$. To do so, we first recall several measure-theoretic definitions and results that will be used in the proof.

First, note that we can work in $\mathbb{R}/\mathbb{Z}$, because the lonely runner conjecture is based on a circular track. We will now have to introduce some notation for our convenience. Fortunately what we need is related to the idea of Bohr sets, and hence we can use them as notation. The rank-one Bohr set $B(v;\delta)$ is defined as follows:

$$B(v;\delta) := \{t \in \mathbb{R}/\mathbb{Z} : ||tv|| \leq \delta\}.$$

We can interpret this as the set of times t in $\mathbb{R}/\mathbb{Z}$, where the runner with velocity v is at most δ away from the stationary runner. But this notion is useful only when we have one non-stationary runner. To handle more runners, we will need the notion of higher-ranked Bohr sets. Higher-ranked Bohr sets are defined through intersections in the following way :

$$B(v_1, v_2, \ldots, v_r; \delta_1, \delta_2, \ldots, \delta_r) = B(v_1;\delta_1) \cap B(v_2;\delta_2) \cap \cdots \cap B(v_r;\delta_r).$$

We can interpret this as the set of times t in $\mathbb{R}/\mathbb{Z}$, where runner i is at most δ_i away from the stationary runner for all i. But this is not our set of interest. In fact, what we need is the intersection of the complements of these rank-one Bohr Sets. $B(v_1;\delta_1)^c \cap B(v_2;\delta_2)^c \cap \cdots \cap B(v_n;\delta_n)^c$ is the set of times t where for all $i \in [n]$, runner i will be at least δ_i far from the stationary runner. We need this set to be non-empty. Note that it is the same as saying

$$m((B(v_1;\delta_1)^c \cap B(v_2;\delta_2)^c \cap \cdots \cap B(v_n;\delta_n)^c)^c) \leq 1.$$

By De-Morgan's law :

$$(B(v_1;\delta_1)^c \cap B(v_2;\delta_2)^c \cap \cdots \cap B(v_n;\delta_n)^c)^c = B(v_1;\delta_1) \cup B(v_2;\delta_2) \cup \cdots \cup B(v_n;\delta_n).$$

For the sake of Conjecture 1.1, we are interested in the case of all δ_i's being the same. Therefore we restrict our attention to the uniform case where $\delta_i = \delta_n$ for all $i \in [n]$.

Theorem 4.1. *For every choice of distinct positive integer velocities $v_1, \ldots, v_n$, there exists a time $t \in \mathbb{R}/\mathbb{Z}$ such that*

$$\|tv_i\| \geq \frac{1}{2n}$$

for all $i \in [n]$.

Proof. Note that since v_i's are integers, tv_i's are also distributed in the same way as t with respect to the Lebesgue measure (m). Hence, for all $i \in [n]$,

$$m(B(v_i;\delta_n)) = 2\delta_n.$$

By Union Bounding,

$$
\begin{aligned}
m(B(v_1;\delta_n)\cup\cdots\cup B(v_n;\delta_n)) &\le \sum_{i\in[n]} m(B(v_i;\delta_n)),\\
&\le \sum_{i\in[n]} 2\delta_n,\\
&\le 2n\delta_n.
\end{aligned}
$$

Now to get the best possible bound for δ_n in this proof, we need to take $m(B(v_1;\delta_n)\cup B(v_2;\delta_n)\cup\cdots\cup B(v_n;\delta_n)) = 1$.

Thus, the optimal loneliness distance δ_n satisfies

$$\delta_n \ge \frac{1}{2n}.$$

□

Now we have proven that $\frac{1}{2n} \le \delta_n \le \frac{1}{n+1}$.

The lower bound can be improved. Note that we used a union-bound, which is considered to be one of the most crude-bounding techniques. It is possible to better this bound for sufficiently large n by exploiting correlations between the events $\{\|tv_i\| \le \delta_n\}$, rather than treating them independently. For example, in 2016, Perarnau and Serra [PS16] proved the following bound for sufficiently large n:

$$\delta_n \ge \frac{1}{2n-2+o(1)}.$$

In 2018, Tao [Tao18] proved that there exists an absolute constant $c > 0$ such that

$$\delta_n \ge \frac{1}{2n} + \frac{c\log n}{n^2(\log\log n)^2}.$$

Unfortunately, regardless of the constant c the above bounds are much closer to our original lower bound $\frac{1}{2n}$ and far away from our upper bound $\frac{1}{n+1}$. An illustration is available in Figure 3.

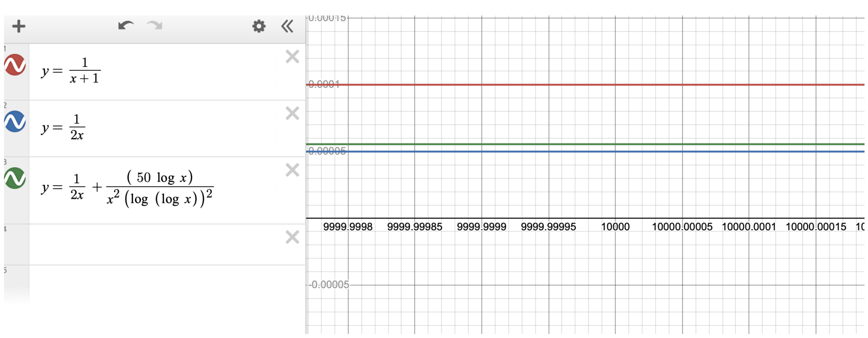

FIGURE 3. $c = 50$ was used here, zoomed in around $n = 10000$: The figure was generated using Desmos graphing calculator.

5. Computational Approach

Recall that we were able to reduce the problem down to positive integer velocities. Since integers (or even tuples of integers) are finite, when bounded from above, this suggests the possibility of a computational approach. Tao [Tao18] proved the following theorem which tells us that a computational approach is possible.

Theorem 5.1. *There exists an absolute and explicitly computable constant $C_0 > 0$ such that the following assertions are logically equivalent for every natural number $n_0 \geq 1$:*

- *One has $\delta_n = \frac{1}{n+1}$ for all $n \leq n_0$*
- *One has $\delta(v_1, v_2, \dots, v_n) \geq \frac{1}{n+1}$ for all $n \leq n_0$ and every tuple $(v_1, v_2, \dots, v_n)$ of nonzero distinct integers with $|v_i| \leq n^{C_0 n^2}$ for all $i \in [n]$.*

The main point of Theorem 5.1 is that, for each fixed value of n_0, the Lonely Runner Conjecture can in principle be reduced to a finite verification problem. Instead of checking all possible integer velocity tuples, which form an infinite set, it is enough to check only those tuples whose entries are bounded by $n^{C_0 n^2}$.

Although we have finite computation to do in terms of tuples, brute-force checking will take super-exponential time, to be more precise $\exp(C_0 n^2 \log n)$ choices for velocity for each runner, which means the exponent is faster than a quadratic polynomial. Therefore, it would not be useful even for moderate values of n.

In [Tao18], Tao proved that the lonely runner conjecture holds if you restrict to the case $|v_i| \leq 1.2n$ for all $i \in [n]$. But $1.2n$ and $n^{C_0 n^2}$ has a massive gap even at small values of n (Figure 4). Therefore, even for smaller n the search space is very large.

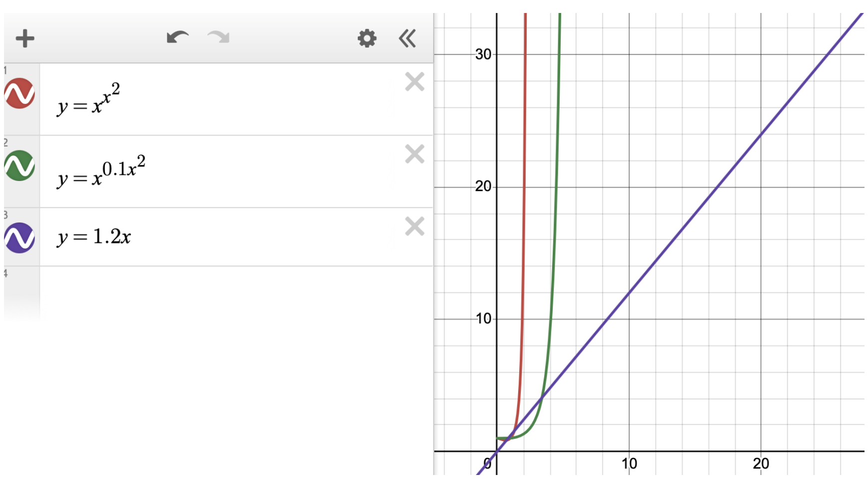

Figure 4. $C_0 = 1$ and $C_0 = 0.1$ was used here : The figure was generated using Desmos graphing calculator.

This was improved by Malikiosis, Santos and Schymura [MSS25] recently. They proved that it is sufficient to check the integer velocity tuples such that,

$$|v_1 v_2 \dots v_n| \leq \left[\frac{\binom{n+1}{2}^{n-1}}{n} \right]^n .$$

Variations of the above computational approach have been used to prove the lonely runner conjecture for small n. These variations deviate from a full brute-force search.

Considering the prime-factors of the product $v_1 v_2 \dots v_n$, Rosenfield proved the lonely runner conjecture for $n = 7$ by a reduced computer search. By introducing a sieve lemma that makes checking for a prime factor more efficient, Trakulthongchai [Tra25] proved the lonely runner conjecture for $n = 8, 9$. Recently, by refining this method by new sieving techniques and employing a polynomial method argument, Sungkawichai and Takulthongchai [ST26] proved the lonely runner conjecture for $n = 10, 11, 12$.

The computational approach promises to be successful for small n if such reductions of search space are further continued. Yet, we have to acknowledge that this approach is slow and we will need to see a different approach to prove the lonely runner conjecture for all n or even for moderately large values of n. In the rest of the paper we will give some other perspectives to look at the lonely runner conjecture.

6. A Probabilistic Heuristic

Recall that the $\frac{1}{2n}$ lower bound in Theorem 4.1 arose from a union bound, which treats the relevant events in a maximally pessimistic way. In this section, we discuss a complementary probabilistic heuristic that suggests why this bound is natural, and why improving it requires exploiting additional structure.

Fix $\delta > 0$, and consider a uniformly random choice of $t \in \mathbb{R}/\mathbb{Z}$. For each $i \in [n]$, the event that the i-th runner lies within distance δ of the origin has probability 2δ. We denote the indicator random variable corresponding to this event at time t as $\mathbf{1}_{\{\|tv_i\| \leq \delta\}}$. Thus, the expected number of runners that are within distance δ of the origin at time t is

$$\mathbb{E}\left[\sum_{i=1}^{n} \mathbf{1}_{\{\|tv_i\| \leq \delta\}} \right] = 2n\delta.$$

From this perspective, the condition $2n\delta < 1$ ensures that the expected number of runners in the interval $[-\delta, \delta]$ is less than one. This suggests that there should exist times at which no runner lies in this interval, which is consistent with the bound $\delta \geq \frac{1}{2n}$ obtained earlier.

A more refined heuristic is obtained by informally modeling the events $\{\|tv_i\| \leq \delta\}$ as independent. Under this assumption, the probability that no runner lies within distance δ of the origin is approximately

$$(1 - 2\delta)^n.$$

This quantity remains positive for all $\delta < \frac{1}{2}$, but decays rapidly once $2n\delta$ becomes large. In particular, the regime $2n\delta \approx 1$ again emerges as the threshold (Figure 5) beyond which one should not expect to find a time t avoiding all such events. Thus, both the first-moment calculation and the independence heuristic point to a natural scale of order $\frac{1}{2n}$.

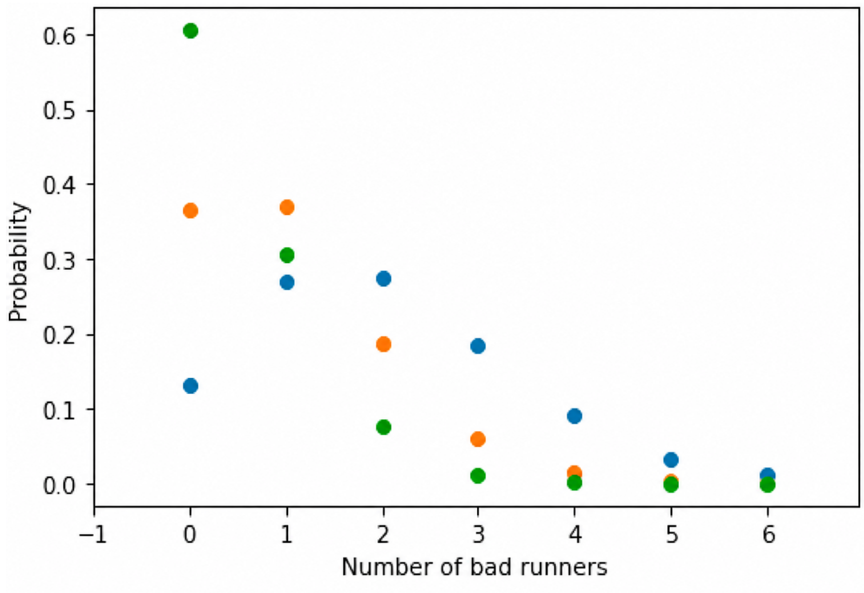

FIGURE 5. A bad runner is defined as a runner who is atmost δ_n distance away from origin. $\delta_n = 1/n$ in blue, $\delta_n = 1/2n$ in orange, $\delta_n = 1/4n$ in green : The figure was generated as a plot using Python. (Corresponds to a binomial random variable)

However, the Lonely Runner Conjecture asserts that one can take $\delta = \frac{1}{n+1}$, which is significantly larger than this heuristic threshold. The reason for this discrepancy is that the events $\{\|tv_i\| \leq \delta\}$ are not independent, and in fact exhibit strong correlations arising from the arithmetic structure of the velocities. These correlations can lead to substantial overlap among the corresponding sets of times, thereby reducing the size of their union compared to what would be predicted under independence.

In particular, if there are additive relations among the velocities, the associated events can align in such a way that large portions of $\mathbb{R}/\mathbb{Z}$ are simultaneously avoided. This phenomenon is not captured by either the union bound or the independence heuristic, both of which ignore such structure. As a result, any improvement over the $\frac{1}{2n}$ barrier must rely on a more detailed analysis of these correlations.

This heuristic perspective thus highlights a key feature of the problem: while probabilistic reasoning suggests a natural barrier at scale $\frac{1}{2n}$, the conjecture predicts that the underlying arithmetic structure allows one to surpass this barrier. Understanding and quantifying this structure is therefore central to all known approaches to the Lonely Runner Conjecture.

7. Connection to Diophantine Approximation

We conclude by returning to the number-theoretic formulation introduced in Section 2. After reducing to the case of integer velocities, the Lonely Runner Conjecture asks whether, for any distinct positive integers $v_1, \ldots, v_n$, there exists a time $t \in \mathbb{R}/\mathbb{Z}$ such that

$$\|tv_i\| \geq \frac{1}{n+1} \quad \text{for all } i \in [n].$$

Thus the problem is not only about runners on a circular track, but also about the simultaneous distribution of the multiples

$$tv_1, \ldots, tv_n \pmod 1.$$

This formulation places the conjecture near classical Diophantine approximation, but with an important reversal in perspective. Classical Diophantine approximation often studies when quantities such as $\|q\alpha\|$ can be made small. For example, Dirichlet's approximation theorem guarantees good rational approximations, or equivalently guarantees many times at which certain fractional parts come close to integers. The Lonely Runner Conjecture asks for the complementary phenomenon: instead of finding a time when one or more multiples are close to integers, we seek a time when all of the multiples tv_i simultaneously avoid being too close to any integer. In this sense, the conjecture is a problem of simultaneous approximation avoidance.

A simple example illustrates the difference between approximation and avoidance. Suppose the velocities are

$$v_1 = 1, \qquad v_2 = 2.$$

Taking t very close to 0 makes both $\|t\|$ and $\|2t\|$ small, so this is a good approximation time but a bad lonely-runner time. On the other hand, at $t = \frac{1}{3}$ we have

$$\left\|\frac{1}{3}\right\| = \frac{1}{3}, \qquad \left\|\frac{2}{3}\right\| = \frac{1}{3}.$$

Thus both runners are simultaneously at distance at least $\frac{1}{3}$ from the stationary runner, exactly matching the Lonely Runner bound for $n = 2$. This example shows the basic tension: Diophantine approximation naturally produces times of closeness, while the Lonely Runner Conjecture asks for times of uniform separation.

Thus, the Lonely Runner Conjecture can be seen as a meeting point of several themes discussed in this paper. The reduction to integer velocities turns the problem into a question about fractional parts; the lower-bound argument studies the measure of approximation sets; the probabilistic heuristic suggests what one would expect under independence; and the conjecture itself asserts that arithmetic structure forces more overlap than independence would predict. From this perspective, the conjecture is not merely a problem about runners, but a problem about how simultaneous approximation and simultaneous avoidance interact on the circle.

References

[BHK01] Tom Bohman, Ron Holzman, and Daniel Kleitman. Six lonely runners. *Electronic Journal of Combinatorics*, 8(2):R3, 2001.

[CP84] Thomas W. Cusick and Carl Pomerance. View-obstruction problems III. *Journal of Number Theory*, 19(2):131–139, 1984.

[MSS25] Romanos Diogenes Malikiosis, Francisco Santos, and Matthias Schymura. Linearly-exponential checking is enough for the lonely runner conjecture and some of its variants. *Forum of Mathematics, Sigma*, 13:e164, 2025.

[PS16] Guillem Perarnau and Oriol Serra. Correlation among runners and some results on the lonely runner conjecture. *Electronic Journal of Combinatorics*, 23(1):P1.50, 2016.

[ST26] Touch Sungkawichai and Tanupat Trakulthongchai. Eleven, twelve, and thirteen lonely runners. *arXiv preprint*, 2026. arXiv:2604.23906.

[Tao18] Terence Tao. Some remarks on the lonely runner conjecture. *Contributions to Discrete Mathematics*, 13(2), 2018.

[Tra25] Tanupat Trakulthongchai. Nine and ten lonely runners. *arXiv preprint*, 2025. arXiv:2511.22427.

[Wil67] Jörg M. Wills. Zwei sätze über inhomogene diophantische approximation von irrationalzahlen. *Monatshefte für Mathematik*, 71(3):263–269, 1967. doi:10.1007/BF01298332.

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