PLAYING BALL WITH CANTOR: A BEGINNER'S EXPLORATION OF CANTOR WINNING SETS

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Abstract. Diophantine approximation is the study of approximating the irrational numbers with rationals. Surprisingly, many games played in metric spaces, such as Schmidt's game, generate winning sets that are relevant in Diophantine approximation and have many properties in common. This paper aims to provide a friendly introduction to Cantor winning sets and the Cantor game, which are recent inventions in the style of Schmidt's game. The paper has a special focus on making the topic more accessible for unfamiliar readers, providing definitions for many concepts and rephrasing ideas intuitively.

Contents

1. Introduction

There exist certain games, played in metric spaces between two players, whose winning sets reveal patterns from Diophantine approximation and number theory. Stemming from such games, like the (α, β)-game and Schmidt's game, the Cantor game and Cantor winning sets are a recent addition to this framework. The Cantor game, which is based on the splitting, fractal qualities of the Cantor set, allows us to understand and make even more connections between number theory and these games.

Section 2, Background, introduces basic concepts necessary for understanding this paper, such as Diophantine Approximation, badly approximable numbers, and games. Section 3 discusses several examples of these games in detail, describing their common structure and the connections between their winning sets. Subsection 3.3 introduces the Cantor game and related concepts, like the Cantor set, splitting

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structures, and generalized Cantor sets. Finally, Section 4 discusses Cantor winning sets, how they relate to the Cantor game, and how they relate to other winning sets. (Note: there is a distinction between *Cantor winning sets*, which are discussed in Section 4, and *winning sets for the Cantor game*, which are first discussed in Section 3. This distinction is explained in further detail at the beginning of Section 4.)

At the end of the paper is a collection of number theory and analysis definitions that the unfamiliar reader may want to reference. References to these definitions will be provided throughout the paper where the relevant concepts are discussed.

2. BACKGROUND

2.1. **Diophantine Approximation.** This paper deals with a subject in *Diophantine approximation*, which is the study in number theory of approximating irrational numbers using rational numbers. It is similar to when we define the real numbers by extending the rational numbers. Diophantine approximation is named after Diophantus of Alexandria, whose 13-book series *Arithmetica* granted him the title "founder of algebra," and whose work formed the foundation of many results in geometry, abstract algebra, and number theory.

One of the most fundamental theorems in Diophantine approximation is not due to Diophantus, but Dirichlet. Unsurprisingly, it is called

THEOREM 2.1. *Dirichlet's theorem. If $\alpha, Q \in \mathbb{R}$ and $Q > 0$, there exist $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $q \leq Q$ and*

$$|q\alpha - p| < \frac{1}{Q}.$$

In other words, Dirichlet's theorem states that it is possible to approximate the real number α arbitrarily well using a rational number $\frac{p}{q}$. Q serves as a measure of how close $\frac{p}{q}$ can get to α, and since it can be any real number, the approximation can be arbitrarily good.

By choosing $q = Q$, and dividing both sides by q, we get the following corollary:

COROLLARY 2.2. *For all irrational numbers α, there exist $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $|\alpha - \frac{p}{q}| < \frac{1}{q^2}$.*

2.2. **Badly Approximable Numbers.** While Dirichlet's theorem tells us that rational numbers approximate irrational numbers arbitrarily well, there may still be limitations for how closely a particular rational number, e.g. one with a certain denominator, can approximate α. In particular, there is a set of numbers that are *badly approximable.*

DEFINITION 2.3. $\alpha \in \mathbb{R}$ is *badly approximable* if there is a real constant $c > 0$ such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$,

$$\left|\alpha - \frac{p}{q}\right| > \frac{c}{q^2}.$$

The inequality in Definition 2.3 closely resembles the one in Corollary 2.2. The constant c, which must be between 0 and 1, is a bound on how closely you can approximate α with a given denominator q.

An example of a badly approximable number is the golden ratio $\varphi = \frac{1+\sqrt{5}}{2}$, for which $c \leq \frac{1}{\sqrt{5}}$. Other examples include $\sqrt{2}, \sqrt{5}$, and $\sqrt{8}$. We will call the set of badly approximable real numbers **Bad**.

3. Games

Some patterns that appear in Diophantine approximation, like the set **Bad**, can be generated using *games* that are played between two players in metric spaces. Games provide a useful framework in which we can study these patterns in an interactive, intuitive way.

The games discussed in this section involve each player increasingly limiting their opponent into smaller and smaller pockets of the metric space, so that eventually gameplay might converge at a single point. In these games, there exist *winning sets* such that if the endpoint lands in the set, player A wins. These winning sets have surprisingly relevant properties.

3.1. Schmidt's Game.

This is the most fundamental of these games, and all others can be viewed as variations on this theme.

We first define the foundation of Schmidt's game, which is called the (α, β)-game. (Note: for the sake of this paper, whenever a ball is mentioned, that ball is closed; see Definition A.1.)

Definition 3.1. The (α, β)-*game* is played between two players, A and B, in a complete metric space (X, d). There are two parameters: α and β, which each lie between 0 and 1, exclusive. There is also a set $S \subseteq X$ that is A's "target." The game begins like so:

1. B goes first. On their first turn, they choose a closed ball $B_0 \subseteq X$, with radius denoted $r(B_0)$.
2. A goes next. On their turn, they choose a closed ball $A_0 \subseteq B_0$, which has radius $\alpha \cdot r(B_0)$.
3. B goes next. On their turn, they choose a closed ball $B_1 \subseteq A_0$, which has radius $\beta \cdot r(A_0)$, or $\beta\alpha \cdot r(B_0)$.

The game continues infinitely, with A and B taking turns choosing closed balls nested in one another's previous plays, with radii determined from B's very first play. A and B's plays produce a sequence of nested balls $B_0 \supseteq A_0 \supseteq B_1 \supseteq A_1 \ldots$, which eventually converge to a single point in space. If this point is in S, A wins.

Some notes about the game:

- $r(B_i) = (\beta\alpha)^i \cdot r(B_0)$.
- $r(A_i) = (\beta\alpha)^{i-1}\alpha \cdot r(B_0)$.
- The actual value of the radius $r(B_0)$ can be ignored without loss of generality due to scaling.

Now we can work through the incremental differences between the (α, β)-game and Schmidt's game to finally arrive at the definition of a *winning set*.

Definition 3.2. $S \subseteq X$ is a (α, β)-*winning set* of an (α, β)-game if, no matter how B plays, A has a strategy to win the game.

Definition 3.3. The α-*game* is an (α, β)-game where B gets to choose the value of β.

Definition 3.4. $S \subseteq X$ is a α-*winning set* of an α-game if, no matter how B plays, A has a strategy to win the game.

Definition 3.5. *Schmidt's game* is an α-game where A gets to choose the value of α.

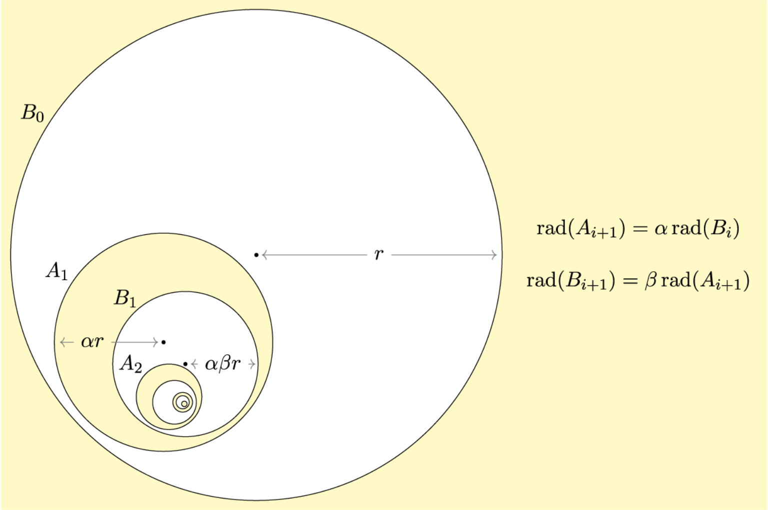


FIGURE 1. An illustration of Schmidt's game. Courtesy of Badziahan, Harrap, Nesharim, and Simmons [BHNS24].

DEFINITION 3.6. $S \subseteq X$ is a *winning set* of Schmidt's game if, no matter how B plays, A has a strategy to win the game. In other words, S is winning if, for (α, β)-games played in X, A can win for all β for at least one value of α.

Now, for a surprising result:

THEOREM 3.7. ***Bad** is a α-winning set of Schmidt's game in $\mathbb{R}$ when $\alpha \leq \frac{1}{2}$.*

3.7, discovered by Schmidt in 1966 [Sch66, Theorem 3], sheds light on some of the properties shared by winning sets in Euclidean space.

(1) All winning sets in Euclidean space are dense in their spaces and have full Hausdorff dimension (Def A.9).
(2) Countable intersections of α-winning sets are α-winning (see Def A.2 for countable).
(3) Images of winning sets under bi-Lipschitz maps (Def A.4) are winning.

Building off the above, we see the interesting fact that **Bad** is "small" in that it has Lebesgue measure 0. (The Lebesgue measure can be thought of as the infimum of the total length of countable intervals that cover the set; in other words, **Bad** is composed of singletons.) However, **Bad** is "big" in that it has full Hausdorff dimension 1.

As we will see, other games generate their own winning sets, with properties that mirror or extend those of Schmidt's game winning sets.

3.2. Other Games.

3.2.1. *The Absolute Game.*

This game is very similar to the (α, β)-game, except:

- Instead of choosing balls A_i that B must play within, A now chooses balls A_i (still contained in B's balls B_i) wherein B is *not* allowed to play.
- There is now one parameter β. $r(A_{i+1}) \leq \beta \cdot r(B_i)$ for all valid i, but $r(B_{i+1}) \geq \beta \cdot r(B_i)$. In other words, A's balls are bounded above in size, and B's are bounded below.

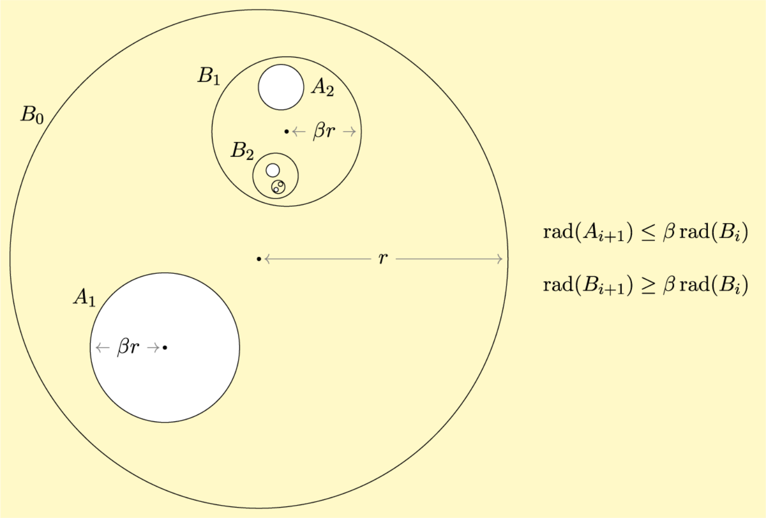


Figure 2. An illustration of the absolute game. Courtesy of Badziahan, Harrap, Nesharim, and Simmons [BHNS24].

A consequence of B's balls not being bounded above in size is that the radii of B's balls may not approach 0; if B's balls do not converge to a single point, A wins by default. An *absolute winning set* S is one where A always has a strategy such that they either win by default, or they force the single-point convergence of B's balls into S.

3.2.2. *The Potential Game.* This game is played with two parameters $c, \beta > 0$. B again starts by picking a ball $B_0 \subseteq X$, but now on A's turn, they choose a *collection* of closed balls of positive radius contained in B_i such that, if there are k such balls forming the collection A_{i+1} on A's $i+1$th turn, the following condition is true:

$$\sum_k r(A_{i+1,k})^c \leq (\beta \cdot r(B_i))^c.$$

Then, during B's $i+1$th turn, they choose a ball $B_{i+1} \subseteq B_i$ such that $r(B_{i+1}) \geq \beta \cdot r(B_i)$. A wins by default if the radii of B's balls do not approach 0 or if the intersection of B's balls falls within one of A's balls. A *potential winning set* S is one where A always has a strategy such that they either win by default, or they force the single-point convergence of B's balls into S.

3.2.3. *Connections.* It turns out that α-winning sets, absolute winning sets, and potential winning sets are related in numerous ways. For example:

- If $S \in \mathbb{R}^N$ is absolute winning, then S is α-winning for all $\alpha \in (0, \frac{1}{2})$. [BHNS24, Prop. 1.1]
- If X is a complete doubling metric space (Def A.5), and $S \subseteq$ X is c'-potential winning, then S is absolute winning. [BHNS24, Prop. 1.3]

Here, c-potential winning simply means that A can win no matter what B chooses for $c > c'$ and $\beta > 0$.

As such, we begin to see the links between these games' winning sets, badly approximable numbers, and Diophantine approximation.

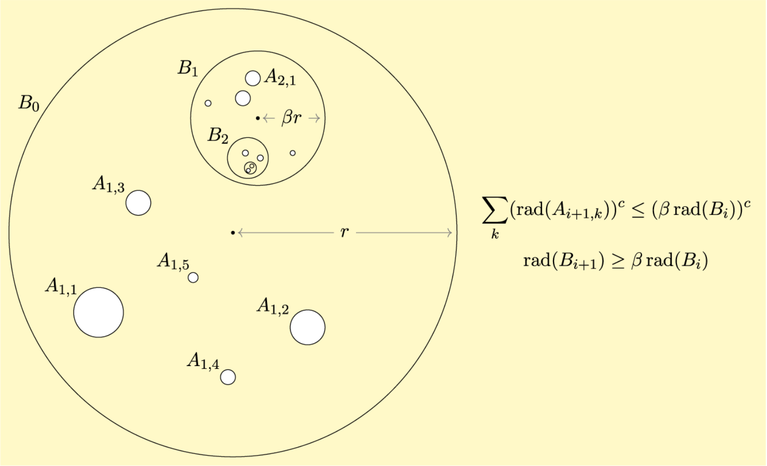


FIGURE 3. An illustration of the potential game. Courtesy of Badziahan, Harrap, Nesharim, and Simmons [BHNS24].

3.3. **Cantor Game.**

3.3.1. *The Cantor Set.* A different game variation, the Cantor game, is based loosely on another concept in number theory called the Cantor set. This set was popularized by George Cantor, a brilliant mathematician with an eccentric reputation, in 1883 [Can83]. The set's construction, with its iterative, fractal structure, serves as inspiration for the Cantor game, which too involves repeated splitting of space.

The canonical Cantor ternary set, $\mathcal{C}$, is constructed by the following steps:

1. Take the interval [0,1] in $\mathbb{R}$.
2. Remove its middle third, leaving the set $[0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.
3. Remove the middle third of the two remaining segments, leaving $[0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$.
4. Continue indefinitely, removing the middle third of every segment.

FIGURE 4. An illustration of $\mathcal{C}$, where each black line is a segment of $\mathbb{R}$. Courtesy of [The07].

There are several unintuitive properties that arise in relation to the Cantor set:

- The Cantor set is non-empty; in fact, it has uncountably many elements.
- The Cantor set has Lebesgue measure 0; there are no segments of any appreciable length.
- The Cantor set has fractal properties.
- The Cantor set has Hausdorff dimension $\frac{\log 2}{\log 3}$.

3.3.2. *Generalized Cantor Set.* We can generalize the repeated splitting-style construction of the canonical Cantor set from $\mathbb{R}$ to any metric space X. To do so, we replace the line segments with balls, where all the points within radius distance of the center are part of the ball. This generalization from $\mathbb{R}$ to all metric spaces will help us define the Cantor game.

In order to perform this generalization, we need a new tool for metric spaces, which is called a splitting structure [BHNS24, Section 2].

Definition 3.8. A *splitting structure* is a tuple $(X, \mathcal{S}, U, f)$ where

- X is a metric space;
- $U \subseteq \mathbb{N}$ is a multiplicatively closed set with infinitely many elements, i.e. if $u, v \in U$ and u divides v, then $\frac{v}{u} \in U$;
- $f : U \to \mathbb{N}$ is a totally multiplicative function, i.e. $f(1) = 1$ and $f(ab) = f(a)f(b)$ for all $a, b \in U$.
- $\mathcal{S} : \mathcal{B}(X) \times U \to \mathcal{P}(\mathcal{B}(X))$, where $\mathcal{B}(X)$ is the set of all balls in X, is a map such that for all $B \in \mathcal{B}(X)$ and $u \in U$, $\mathcal{S}(B, u)$ consists solely of balls $S \subseteq B$ of radius $\frac{r(B)}{u}$.

The following properties must also apply to the splitting structure:

- The cardinality of $S(B, u)$ is equal to $f(u)$.
- If balls $S_1, S_2 \in \mathcal{S}(B, u)$ and $S_1 \neq S_2$, then S_1 and S_2 may only intersect on their boundaries.
- For all $u, v \in U$,

$$\mathcal{S}(B, uv) = \bigcup_{S \in \mathcal{S}(B,u)} \mathcal{S}(S, v).$$

  In other words, evaluating $\mathcal{S}$ with a ball B and a product uv of $u, v \in U$ yields the union of all the balls from evaluating $\mathcal{S}$ with balls in $\mathcal{S}(B, u)$ and with v.

Example 3.9. The standard splitting structure on $X = \mathbb{R}^N$ uses the supnorm, or Chebyshev, distance metric in $\mathbb{R}^N$, where $d(x, y)$ is the greatest difference between the corresponding coordinates of x and y. In this case, balls in $\mathbb{R}^N$ are cubes. To evaluate $\mathcal{S}(B, u)$, the cube B is split into $f(u) = u^N$ smaller cubes, each with side length u times smaller than that of B. In essence, imagine a large cube B split into several smaller cubes based on the value of u. The full splitting structure tuple is $(\mathbb{R}^N, \mathcal{S}, \mathbb{N}, f)$.

When constructing a *generalized Cantor set* with the aid of our metric space's splitting structure, we begin with a closed ball $B_0 \subseteq X$ and $R \in U$. Then

$$\mathbf{r} := (r_{m,n}), m, n \in \mathbb{Z}_{\geq 0}, m \leq n$$

is a sequence of non-negative real numbers with two parameters, m and n. We consider the set $1/R\{B_0\}$, which is the set of balls contained in B_0 that have roughly the radius $\frac{r(B_0)}{R}$. Similarly to in the construction of the ternary Cantor set, we begin to remove parts of the set. We remove the collection $\mathcal{A}_{0,0}$, which contains $r_{0,0}$ balls, calling the surviving collection $\mathcal{B}_1$. This process continues ad infinitum, with the following rules:

1. $\mathcal{B}_{n+1} := (1/R\{\mathcal{B}_n\}) \setminus \bigcup_{m=0}^{n} \mathcal{A}_{m,n}$
2. $\mathcal{A}_{m,n} \subseteq 1/R^{n-m+1}\mathcal{B}_m$ for $0 \leq m < n$

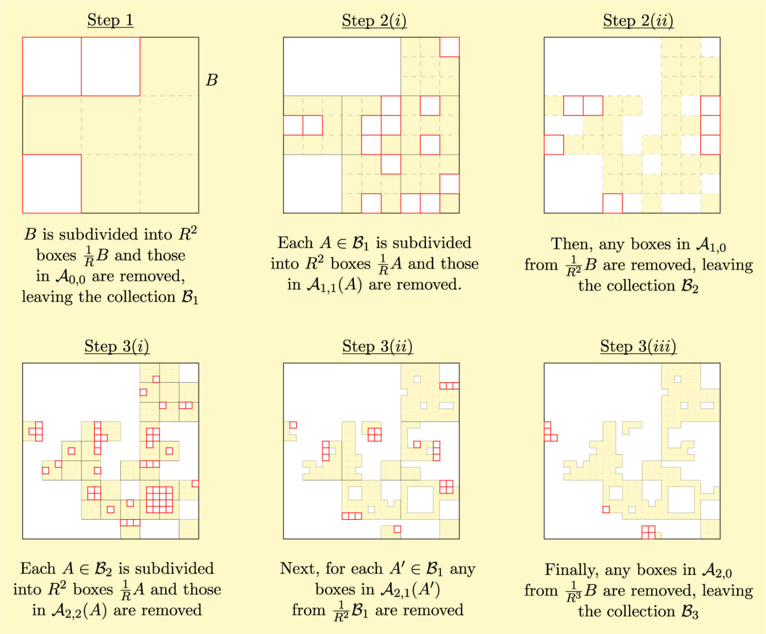
(3) $|\mathcal{A}_{m,n}| \leq r_{m,n}$.

You can think about $\mathcal{A}_{m,n}$ as the set of balls removed from a subdivided $\mathcal{B}_n$ to create $\mathcal{B}_{n+1}$, where $\mathcal{A}_{m,n}$ follows rules about cardinality and size dependent on $\mathbf{r}$, R, n, and m.

Definition 3.10. The *limit set* is

$$\bigcap_{n=0} \bigcup_{B \in \mathcal{B}_n} B,$$

denoted by $\mathcal{K}(B_0, R, \mathbf{r})$.



Step 1

B is subdivided into R^2 boxes $\frac{1}{R}B$ and those in $\mathcal{A}_{0,0}$ are removed, leaving the collection $\mathcal{B}_1$

Step 2(i)

Each $A \in \mathcal{B}_1$ is subdivided into R^2 boxes $\frac{1}{R}A$ and those in $\mathcal{A}_{1,1}(A)$ are removed.

Step 2(ii)

Then, any boxes in $\mathcal{A}_{1,0}$ from $\frac{1}{R^2}B$ are removed, leaving the collection $\mathcal{B}_2$

Step 3(i)

Each $A \in \mathcal{B}_2$ is subdivided into R^2 boxes $\frac{1}{R}A$ and those in $\mathcal{A}_{2,2}(A)$ are removed

Step 3(ii)

Next, for each $A' \in \mathcal{B}_1$ any boxes in $\mathcal{A}_{2,1}(A')$ from $\frac{1}{R^2}\mathcal{B}_1$ are removed

Step 3(iii)

Finally, any boxes in $\mathcal{A}_{2,0}$ from $\frac{1}{R^3}B$ are removed, leaving the collection $\mathcal{B}_3$

Figure 5. An illustration of such a construction in $\mathbb{R}^2$. Courtesy of Badziahan, Harrap, Nesharim, and Simmons [BHNS24].

We call $\mathcal{K} = \mathcal{K}(B_0, R, \mathbf{r})$ *a generalized Cantor set*. For simpler notation, we can simply call $\mathcal{K}$ $(B_0, R, \mathbf{r})$ Cantor.

3.3.3. *The Cantor Game.* Finally, we can define the Cantor game. It is notably similar to the construction of a generalized Cantor set, and it touches on a new type of set: the Cantor winning set.

Definition 3.11. The ε-*Cantor game* is played as follows:

Let X be a complete metric space, and let $(X, \mathcal{S}, U, f)$ be a splitting structure. ε is a real constant in $(0, 1]$.

(1) B begins by choosing $R \in U$ such that $R \geq 2$. B also chooses a closed ball $B_0 \subseteq X$.
(2) Following B's ith ball B_i, A will choose a collection of balls $\mathcal{A}_{i+1} \subseteq 1/R\{B_i\}$ such that the cardinality of $\mathcal{A}_{i+1}$ is less than or equal to $f(R)^{1-\varepsilon}$.
(3) After A's play $\mathcal{A}_{i+1}$, B chooses a ball $B_{i+1} \in \frac{1}{R}\{B_i\} \setminus \mathcal{A}_{i+1}$.

Again, just as we did with the winning sets of Schmidt's game, we build up one step at a time from the ε-Cantor game to the definition of the winning sets of the Cantor game.

Definition 3.12. A set $S \subseteq X$ is *winning for the ε-Cantor game* if A has a strategy ensuring that the final intersection of B's balls lands in S in a round of the ε-Cantor game.

Definition 3.13. The *Cantor game* consists of an ε-Cantor game where A is allowed to choose the value of ε.

Definition 3.14. A set $S \subseteq X$ is *winning for the Cantor game* if A has a strategy ensuring that the final intersection of B's balls lands in S in a round of the Cantor game.

Remark 3.15. To understand the Cantor game more intuitively, it is possible to play a version of it, resembling the absolute game, without a splitting structure. With parameters $c \geq 0$ and $0 < \beta < 1$, B begins by choosing a ball B_0. On their ith turn, A removes at most β^{-c} closed balls of radius $\beta \cdot r(B_i)$. B then chooses for their next play a ball $B_{i+1} \subset B_i$, also with radius $\beta \cdot r(B_i)$.

Just as with Schmidt's game, the potential game, and the absolute game, this game generates its own winning sets with fascinating properties.

4. Cantor Winning Sets

Here is where confusion often arises: there is a distinction between *winning sets for the Cantor game*, which we have just seen, and *Cantor winning* sets. In fact, Badziahin and Harrap originated the idea of Cantor winning sets [BH17, Section 4] in relation to sets with winning properties in 2015, and Badziahin, Harrap, Nesharim, and Simmons later developed the Cantor game [BHNS24, Subsection 3.1] to provide a more familiar framework in which to situate the sets. Cantor winning sets can then be thought of as an offshoot of sets that win the Cantor game, or as their own notion, which is defined in relation to generalized Cantor sets.

As such, while Cantor winning sets are not *identical* in definition to winning sets for the Cantor game, they are strongly connected in meaning. The following result relates them:

Theorem 4.1. *Let $B_0 \subseteq X$ be a closed ball, and $\varepsilon_0 \in (0,1]$. Then if $S \subseteq X$ is ε-Cantor winning on B_0, then S is winning for the ε_0-Cantor game on B_0.*

We formally define *Cantor winning sets* by the following definitions, that once again progress from specific to general:

Definition 4.2. Given a ball $B_0 \subseteq X$ and a parameter $\varepsilon_0 \in (0,1]$, $S \subseteq X$ is ε_0-*Cantor winning on B_0* (with respect to the splitting structure) if for all $\varepsilon \in (0, \varepsilon_0)$, there exists $R_\varepsilon \in U$ such that for all $R \in U$ where $R \geq R_\varepsilon$, S contains a generalized $(B_0, R, \mathbf{r})$ Cantor set that satisfies

$$r_{m,n} \leq f(R)^{(n-m+1)(1-\varepsilon)} \text{ for all } m, n \in \mathbb{N}, m \leq n.$$

Definition 4.3. S is ε_0 *Cantor winning* if it is ε_0-Cantor winning on B_0 for all balls $B_0 \subseteq X$.

Definition 4.4. S is *Cantor winning* if it is ε_0-Cantor winning for some $\varepsilon_0 \in (0,1]$.

Just like between the winning sets of Schmidt's game, the absolute game, and the potential game, there are a number of connections between those winning sets and Cantor winning sets:

- [BHNS24, Prop. 1.4] Absolute winning sets in $\mathbb{R}^N$ are 1-Cantor winning.
- [BHNS24, Thms. 1.7, 1.8] Cantor winning sets and (Schmidt) winning sets are not the same: there exists a Cantor winning set in $\mathbb{R}$ that is not winning, and there exists a winning set in $\mathbb{R}$ that is not Cantor winning.
- [BHNS24, Thm. 1.11] The intersection in $\mathbb{R}^N$ of a (Schmidt) winning set and a Cantor winning set has full Hausdorff dimension N.

And, as we see below, there is a big result connecting winning sets of Schmidt's game with the winning sets of the Cantor game:

Theorem 4.5. [BHNS24, Thm. 1.11] *If $S \subseteq \mathbb{R}$ is $\frac{1}{2}$-winning, then it is 1-Cantor winning.*

Corollary 4.6. [BHNS24, Corr. 1.12] *The following are equivalent statements about $S \subseteq \mathbb{R}$:*

(1) S is $\frac{1}{2}$-winning;
(2) S is 1-Cantor winning;
(3) S is absolute winning.

Appendix A. Various Definitions

Definition A.1. A *closed ball* of radius r centered at point x in a metric space consists of all points of distance r or less from x. [Soh03, Definition 5.1.14]

Definition A.2. A set is *countable* if it has the same or smaller cardinality than $\mathbb{N}$. For example, $\mathbb{Z}$ and $\mathbb{Q}$ are countable because bijections can be made from them to $\mathbb{N}$, meaning they have the same cardinality. A set is *uncountable* if no such bijection can be made to $\mathbb{N}$. For example, $\mathbb{R}$ is uncountable.

Definition A.3. A map or function $f : X \to Y$, where (X, d_X) and (Y, d_Y) are metric spaces, is *Lipschitz-continuous* (or just *Lipschitz*) if there exists a nonnegative real constant K such that

$$d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2)$$

for all $x_1, x_2 \in X$.

Definition A.4. A map is *bi-Lipschitz* if it is Lipschitz, injective, and its inverse is also Lipschitz.

Definition A.5. A *doubling metric space* is one where there is a constant $N \in \mathbb{N}$ such that for all $r > 0$, every ball of radius $2r$ can be covered by a collection of at most N balls of radius r.

The Hausdorff dimension of a set, defined intuitively, is a measure of its roughness, or fractal dimension. Hausdorff dimension relates to the number of balls of radius r it takes to cover a set as r approaches 0. It is an extension of the idea of dimension in real vector spaces. For a formalization, see Definitions A.6 through A.9 below.

Definition A.6. A *separable metric space* X is one that has a countable dense subset, i.e. it has a subset D that is both countable and has elements in every ball of X, no matter how small said balls are.

Definition A.7. A map $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is an *outer measure* on set X if it satisfies the following conditions:

- $\mu^*(\emptyset) = 0$.
- $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$.
- For any sequence (A_n) of subsets of X,

$$\mu^*\big(\bigcup_{n=1}^{\infty} A_n\big) \leq \sum_{n=1}^{\infty} \mu^*(A_n).$$

[Soh03, Definition 12.1.6]

Definition A.8. To define the *Hausdorff outer measure* μ_p^* of a separable metric space (X, d) where $p \geq 0$, we introduce for each $\varepsilon > 0$ the collection

$$\mathcal{C}_\varepsilon := \{C \subset X : 0 < \delta(C) < \varepsilon\},$$

where $\delta(C)$ is the diameter of C, i.e. $\delta(C) := \sup\{d(x, y) : x, y \in C\}$. Define $\mu_{p\varepsilon}$ such that $\mu_{p\varepsilon}(\emptyset) = 0$ and $\mu_{p\varepsilon}(C) := (\delta(C))^p$ for each $C \in \mathcal{C}_\varepsilon$. Let $\mu_{p\varepsilon}^*$ be the corresponding outer measure.

Finally, define

$$\mu_p^*(S) := \sup\{\mu_{p\varepsilon}^*(S) : \varepsilon > 0\},$$

where $S \subset X$.

[Soh03, Subsection 12.1]

Definition A.9. The *Hausdorff dimension* of a set S in a separable metric space X is the unique number

$$\dim_H(S) := \inf\{p \geq 0 : \mu_p^*(S) = 0\} = \sup\{p \geq 0 : \mu_p^*(S) = \infty\}.$$

In particular, if $0 < \mu_p^* < \infty$, $\dim_H(S) = p$. [Definition 12.1.22][Soh03]

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