THE LONELY RUNNER CONJECTURE: THE CASE FOR 6 RUNNERS

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Abstract. This is an expository paper in which we look at the lonely runner conjecture through the lens of diophantine approximations. Specifically, we will explore different reformulations and give some insight into how these can be used to prove the conjecture for 6 runners. The conjecture states that given n runners running at different constant speeds in a circle of circumference 1, having all started at the same time and place, each one will be lonely at some time. That is, they will be at a distance of at least $\frac{1}{n}$ from any other runner. We begin by rephrasing it in terms of integer speeds, as opposed to real-valued speeds. Then, we present a proof of the irrational case of the conjecture for 6 runners. Finally, we discuss the edge cases.

1. Introduction

The Lonely Runner Conjecture, first proposed by Wills in 1968 [Wil68], is best pictured by the formulation that gave it its name, as introduced in 1996 [Bie98].

Conjecture 1.1. Imagine there are n people running around a circle of circumference 1. They all start at the same spot at the same time, and each person runs at their own constant pace, different from that of others. For each runner, there will be a time where they are "lonely": all the other runners will be at a distance of at least $\frac{1}{n}$ from them.

Note that this is not a strict inequality: when the speeds are $1, 2, ..., n$, each runner is always at a distance of at least $\frac{1}{n}$ from at least one other runner (but they are sometimes exactly $\frac{1}{n}$ away, keeping them "lonely") [BHK01, Theorem 3]. So far, this conjecture has been proved for small values of n. In this paper, we will focus on the proof of the conjecture for the irrational case of $n = 6$, and how it brought a new perspective to the problem. In order to do so, we will first set up the conjecture and notation, looking at reformulations of it (section 2). From there, we will move to the distinction between the rational and irrational cases. Specifically, we will prove that we can reduce the problem for n runners to the problems for n and $n-1$ runners where we only consider integer speeds, as presented in [BHK01, Section 4] (section 3). Then, we will explain the proof of [Bie98], highlighting the main ideas of approaching casework by manipulating runners (sections 4). We will finally talk about the extremal cases (section 5).

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2. Reformulations

This section serves the double purpose of setting up notation, the definitions of *good* times and *distant* positions (those terms will be pretty self-descriptive), as well as introducing different perspectives on the conjecture. To begin with, because the runners loop around a distance of 1, we will use $\{x\}$ to denote $x \bmod 1$:

Definition 2.1. For any real number x, we use the notation

$$\{x\} := x - \lfloor x \rfloor.$$

If we have n runners, we denote the speed vector of the runners by $\mathbf{v} := (v_1, v_2, ..., v_n)$.

Our first way to simplify this problem (1.1) is to look at it from the perspective of any one of the runners, without loss of generality, say the one with speed v_n. In this way, we will only need to worry about this specific runner being lonely at some point. From their reference frame, the relative speed vector is

$$\mathbf{v}' = (v_1 - v_n, v_2 - v_n, ..., v_n - v_n = 0) = (v_1', v_2', ..., v_{n-1}', 0),$$

with $v_i' = v_i - v_n$ for all $1 \leq i \leq n$. Moreover, as we only care about the net distances to this chosen runner n, we can assume that $v_i' > 0$ for all $1 \leq i \leq n-1$, so that v_i' can take on any positive real value. We would like to prove that n, which we will call *stationary*, is lonely at some point. This gives us the following conjecture, which seemingly reduces the number of runners by 1 (as the stationary one is now at the origin, with speed 0). It also reduces the dimension of the speed vector by 1: from now on, we will only use the speed vector $\mathbf{v} := (v_1, v_2, ..., v_{n-1})$ and omit the 0.

Conjecture 2.2. [Cus72] For all $v_1, v_2, ..., v_{n-1} \in \mathbb{R}_{>0}$, there exists a $t \in \mathbb{R}_{>0}$ such that $\{v_i t\} \in [\frac{1}{n}, \frac{n-1}{n}]$ for all $1 \leq i \leq n-1$.

We can now define R to be the set of all non-stationary runners,

$$R := \{1, 2, ..., n-1\}.$$

We would like to keep track of the position of all these runners at any given time, and of whether they are leaving the stationary runner alone (they are at a distance of at least $\frac{1}{n}$ from the origin, in which case we say they are *distant*) or not (they are closer to the origin). Ultimately, we would like to find a time where none of them are close to the origin. We then consider, for each runner $i \in R$, the set of times at which they are **not** *distant*, i.e., their distance to the origin is strictly less than $\frac{1}{n}$. We call these sets B_i :

$$B_i := \{t \in \mathbb{R}_{>0} : \{v_i t\} \in [0, \frac{1}{n}) \cup (\frac{n-1}{n}, 1)\}, \tag{2.1}$$

for all $i \in R$, and note that they are open. Their union $\mathcal{B} := \bigcup_{i=1}^{n-1} B_i$ is also open, and represents all the times in which at least one runner is close to the origin. Proving the conjecture then amounts to proving that there is at least one time t not contained in $\mathcal{B}$; equivalently, $\mathcal{B} \neq \mathbb{R}$. For this reason, we say t is *good* if $t \notin \mathcal{B}$. Otherwise, if $t \in \mathcal{B}$, we say that t is not *good*; we want to prove that there exists at least one *good* time. [BHK01, Introduction]

Finally, we can focus instead on the possible positions the runners can collectively reach. The question becomes: can they simultaneously get far from the stationary runner? Given t and $\mathbf{v}$, we define the position vector of the runners at time t:

$$x(t) := (\{v_1 t\}, \{v_2 t\}, ..., \{v_{n-1} t\}) \tag{2.2}$$

As we are interested in all possibilities for $x(t)$, we take these positions over all possible times:

$$X := \{x(t) : t \in \mathbb{R}_{>0}\}. \tag{2.3}$$

Our goal is to show that $X \cap [\frac{1}{n}, \frac{n-1}{n}]^{n-1} \neq \varnothing$. Note that both x and X depend on $\mathbf{v}$, but it is generally clear from context which $\mathbf{v}$ that is, as we fix it early on.

3. Rational and Irrational cases

So far, we have introduced different perspectives on this problem and even simplified it with the idea of the stationary runner (as in Conjecture 2.2). However, it remains very broad, and so our next step is to restrict our attention to rational speeds and to justify why this still proves Conjecture 2.2. As scaling the velocity vector $\mathbf{v}$ does not change the reachable positions X (2.3)(it only scales the time needed to reach said positions), reducing to rational speeds is equivalent to restricting to the case where all speeds are natural numbers. The irrational case is then one where no scalar multiple of $\mathbf{v}$ is rational, while the rational case is one where there is a scalar multiple of $\mathbf{v}$ consisting of natural numbers. In this section, we will show that proving the rational cases for both $n-1$ and n proves the conjecture for n.

Luckily, there is a theorem that characterizes the set $X(\mathbf{v})$ of reachable positions, depending on whether the entries of $\mathbf{v}$ are linearly independent over $\mathbb{Q}$ (that is, if there are rational numbers $q_1, q_2, ..., q_{n-1}$ such that $q_1v_1 + q_2v_2 + ... + q_{n-1}v_{n-1} = 0$) or not. This theorem, a generalization of Kronecker's theorem, comes in handy, as it aligns with our goal to show that $X(\mathbf{v})$ contains a vector whose every entry is in the interval $[\frac{1}{n}, \frac{n-1}{n}]$ of distant positions.

More precisely, the theorem characterizes the closure of $X(\mathbf{v})$, $\bar{X}(\mathbf{v})$. We can think of it as being the the closure in $\mathbb{R}^{n-1}$, but 0 is identified with 1 in every dimension. However, we will not worry too much about this: if the closure $\bar{X}$ of a set X (in this case $X(\mathbf{v})$) intersects an open set Y (which we will encounter soon), then X also intersects Y.

Theorem 3.1. *(Kronecker)* [Per51, Theorem 61]

Let $n > 1$ be a positive integer, $\mathbf{v} = (v_1, v_2, ..., v_{n-1}) \in \mathbb{R}^{n-1}$ be an n-1- dimensional speed vector, and let $X(\mathbf{v}) = \{(\{v_1t\}, \{v_2t\}, ..., \{v_{n-1}t\}) \in [0,1)^{n-1} \mid t \in \mathbb{R}_{>0}\}$ (as previously defined) be the set of reachable positions. Then one of the following holds:

- *If the entries of $\mathbf{v}$, $v_1, v_2, ..., v_{n-1}$, are linearly independent over $\mathbb{Q}$, then $\bar{X}(\mathbf{v}) = [0,1)^{n-1}$, where $\bar{X}$ denotes the closure of X if we consider the topology of $[0,1)^{n-1}$ to be that of $\mathbb{R}^{n-1}/\mathbb{Z}^{n-1}$.*
- *Otherwise, there exist equations $a_1v_1 + a_2v_2 + ... + a_{n-1}v_{n-1} = 0$ where $a_j \in \mathbb{Q}$ for $1 \leq j \leq n-1$. Taking any maximal set of linearly independent vectors $\mathbf{a^i} = (a_1^i, a_2^i, ..., a_{n-1}^i)$ such that $\mathbf{a^i} \cdot \mathbf{v} = 0$, yields a matrix $A = [(\mathbf{a^1})^\mathbf{T}, (\mathbf{a^2})^\mathbf{T}, ...(\mathbf{a^m})^\mathbf{T}]^\mathbf{T}$ (where each $\mathbf{a^i}$ is a row of the matrix; note that by definition, $A\mathbf{v} = 0$). In this case, $\bar{X}(\mathbf{v}) = \mathrm{Ker}(A)$.*

We can already observe that in the first case, when the entries of $\mathbf{v}$ are linearly independent over $\mathbb{Q}$, Theorem 3.1 proves exactly what we wanted. Indeed, $\bar{X} = [0,1)^{n-1}$ implies that $X \cap (\frac{1}{n}, \frac{n-1}{n})^{n-1} \neq \varnothing$. We will therefore focus on the case where $v_1, v_2, ..., v_{n-1}$ are not linearly independent over $\mathbb{Q}$ (which includes the case of rational speeds). By exploiting the fact that a system of linear equations with rational coefficients always has a rational solution if it has any solutions, we are able to prove the following result, which allows us to link the rational case of $n-1$ to the irrational one of n :

Theorem 3.2. [BHK01, Lemma 8]
Let $n \in \mathbb{N}_{>2}$, and suppose that for any rational-valued speed vector $\mathbf{u}= (u_1, u_2, ..., u_{n-2}) \in \mathbb{Q}_{>0}^{n-2}$, *there exists a time $t \in \mathbb{R}_{>0}$ such that for all $1 \leq i \leq n-2$, $\{u_i t\} \in (\delta, 1-\delta)$ for some fixed $\delta \in (0, \frac{1}{2})$ (that is, $X(\mathbf{u}) \cap (\delta, 1-\delta)^{n-1} \neq \varnothing$.)*

Then, for any vector $\mathbf{v}= (v_1, v_2, ..., v_{n-1}) \in \mathbb{R}_{>0}^{n-1}$ in the irrational case (equivalently, such that there exists $1 \leq i < j \leq n-1$ with $\frac{v_i}{v_j} \notin \mathbb{Q}$), there exists $t \in \mathbb{R}_{>0}$ such that for all $1 \leq i \leq n-1$, $\{v_i t\} \in (\delta, 1-\delta)$, i.e., $X(\mathbf{v}) \cap (\delta, 1-\delta)^{n-1} \neq \varnothing$.

Proof. As we mentioned earlier in Theorem (3.1), if $\mathbf{v}$ is such that its entries are linearly independent over $\mathbb{Q}$, then this is true (even without the assumption of the Theorem (3.2)).

Otherwise, we construct A as in Theorem 3.1: it is a matrix with rational entries that satisfies $A\mathbf{v} = 0$ and has maximal rank among such matrices. Moreover, its rank is equal to the number of rows it has. We are interested in studying $\text{Ker}(A)$, as it is equal to $\bar{X}(\mathbf{v})$ (by Theorem 3.1). Our strategy will be to find a rational-valued vector in $\text{Ker}(A)$ that has repeated entries (in terms of absolute value) so that we can effectively reduce what we wish to prove to the assumption we are making.

As A is a matrix of rational entries, the rank of A over $\mathbb{Q}$ is the same as that over $\mathbb{R}$: both equal the number of rows of A. This, in turn, tells us that the dimension of $\text{Ker}(A)$ is the same when it is taken over $\mathbb{Q}$ than when it is taken over $\mathbb{R}$. Since $\text{Ker}(A)$ already contains $\mathbf{v}$, $\text{Ker}(A)$ must also contain vectors with rational entries, say $\mathbf{r} = (r_1, r_2, ..., r_{n-1})$. Moreover, as we are working in the irrational case, $\mathbf{v}$ is not proportional to $\mathbf{r}$. Thus, $\text{Ker}(A)$ must have dimension at least 2. This is great: it means, in turn, that $\text{Ker}(A)$ must have dimension at least two over $\mathbb{Q}$. In this kernel, we now have infinitely many more vectors, and in particular, infinitely many more rational-valued vectors.

Together with our assumption, we will now be able to construct a vector which lies in both $\bar{X}(\mathbf{v}) = \text{Ker}(A)$ and $(\delta, 1-\delta)^{n-1}$. The assumption of the theorem deals with vectors of dimension $n-2$ instead of $n-1$. To overcome this problem, we will want the vector we construct to have at least two repeated entries, so that for our purposes, we essentially reduce its dimension by 1. Finally, as $(\delta, 1-\delta)^{n-1}$ is an open set, this will conclude our proof.

First, from $\text{Ker}(A)$ having the same dimension over $\mathbb{R}$ than over $\mathbb{Q}$, there must exist a vector, say $\mathbf{s} = (s_1, s_2, ..., s_{n-1})$, that has rational entries and is arbitrarily close to $\mathbf{v}$. In particular, as all the entries of $\mathbf{v}$ are positive, we can impose the same constraint on $\mathbf{s}$. We can moreover assume that $\mathbf{r}$ and $\mathbf{s}$ are not proportional to each other as $\dim(\text{Ker}(A)) \geq 2$. We will now construct $\mathbf{w}$ such that it is a linear combination of $\mathbf{r}$ and $\mathbf{s}$, has repeated entries in terms of their absolute values, and no 0 entries. We care for the absolute value because we only care for runners' net distance to the origin (and we do not want anyone's distance to always be 0, as would happen if an entry were 0). In this case, we have that for any real number, $\{a\} \in (\delta, 1-\delta) \iff \{|a|\} \in (\delta, 1-\delta)$ because δ and $1-\delta$ are symmetric with respect to $\frac{1}{2}$.

The way in which we construct $\mathbf{w}$ is simple: we take

$$\mathbf{w} := (r_i + r_j)\mathbf{s} - (s_i + s_j)\mathbf{r} \tag{3.1}$$

for some fixed $1 \leq i < j \leq n-1$. However, we cannot just take any i and j : in order to guarantee that no entries of $\mathbf{w}$ are 0, we must have that

$$(r_i + r_j)s_k - (s_i + s_j)r_k \neq 0 \text{ for all } 1 \leq k \leq n-1. \tag{3.2}$$

This is not hard. We first order the set $\{\frac{s_i}{r_i} : 1 \leq i \leq n-1\}$ in increasing order. It must have cardinality at least 2 as $\mathbf{r}$ and $\mathbf{s}$ are not proportional. We can then define i and j by fixing any two consecutive elements $\frac{s_i}{r_i}$ and $\frac{s_j}{r_j}$ (that is, so that they satisfy $\nexists 1 \leq k \leq n-1$ with $\frac{s_k}{r_k} \in (\frac{s_i}{r_i}, \frac{s_j}{r_j})$).

The vector $\mathbf{w}$ must lie in $\mathrm{Ker}(A)$ and have rational entries. Moreover,

$$|w_i| = |w_j| = |r_i s_j - s_j r_i|. \tag{3.3}$$

Finally, for all $1 \leq k \leq n-1$, $w_k \neq 0$ as $w_k = 0$ would mean that

$$w_k = (r_i + r_j)s_k - (s_i + s_j)r_k = 0 \qquad \text{from our definition of } \mathbf{w}$$
$$\Rightarrow \frac{s_k}{r_k} = \frac{s_i + s_j}{r_i + r_j}$$

However, $\frac{s_i+s_j}{r_i+r_j} \in (\frac{s_i}{r_i}, \frac{s_j}{r_j})$, which is not the case for $\frac{s_k}{r_k}$.

Thus, the set $\{|w_k| : 1 \leq k \leq n-1\} \subset \mathbb{Q}_{>0}$ has cardinality at most $n-2$, which means that we can apply our original assumption: there must exist $t \in \mathbb{R}_{>0}$ such that $\{|w_i|t\} \in (\delta, 1-\delta)$ for all $1 \leq i \leq n-1$. The vector $t\mathbf{w}$ thus satisfies that $t\mathbf{w} \in \mathrm{Ker}(A)$ and $t\mathbf{w} \in (\delta, 1-\delta)^{n-1}$, as we wanted. Thus, $X(\mathbf{v}) \cap (\delta, 1-\delta)^{n-1} \neq \varnothing$ by Theorem (3.1). □

From this Theorem(3.2), it is straightforward to conclude that the rational case of Conjecture 2.2 for $n-1$ implies the irrational case of Conjecture 2.2 for n as follows.

Corollary 3.3. [BHK01, section 4] *Let $n \geq 3$ be a natural number and suppose that for any vector $\mathbf{u} = (u_1, u_2, ..., u_{n-2}) \in \mathbb{N}^{n-2}$, there exists $t \in \mathbb{R}_{>0}$ such that for all $1 \leq i \leq n-2$, $\{u_i t\} \in [\frac{1}{n-1}, \frac{n-2}{n-1}]$.*

Then, for any vector $\mathbf{v} = (v_1, v_2, ..., v_{n-1}) \in \mathbb{R}^{n-1}_{>0}$ such that there exists $1 \leq i, j \leq n-1$ with $\frac{v_i}{v_j} \notin \mathbb{Q}$, there exists $t \in \mathbb{R}_{>0}$ such that for all $1 \leq i \leq n-1$, $\{v_i t\} \in [\frac{1}{n}, \frac{n-1}{n}]$.

Proof. First, as already mentioned, all positive rational-valued vectors are proportional to some natural-valued ones. As

$$\frac{1}{n} < \frac{1}{n-1} < \frac{n-2}{n-1} < \frac{n-1}{n},$$

we can take any δ such that

$$\frac{1}{n} < \delta < \frac{1}{n-1} < \frac{n-2}{n-1} < 1 - \delta < \frac{n-1}{n}.$$

We conclude using the above theorem, as $(\delta, 1-\delta) \subset [\frac{1}{n}, \frac{n-1}{n}]$. In fact, $(\delta, 1-\delta) \subset (\frac{1}{n}, \frac{n-1}{n})$. □

This concludes our discussion of how the rational case for $n-1$ implies the irrational case for n. We end this section by stating another corollary to Theorem 3.1, which concerns the edge cases.

Corollary 3.4. *Suppose that Conjecture 2.2 holds for $n-1$. Then, if there exists a vector* $\mathbf{v} \in \mathbb{R}_{>0}^{n-1}$ *such that for all* $t \in \mathbb{R}_{>0}$, *there exists* $1 \leq i \leq n-1$ *with* $\{v_i t\} \in [0, \frac{1}{n}] \cup [\frac{n-1}{n}, 1)$, *then there is a vector* $\mathbf{r} \in \mathbb{N}_{>0}^{n-1}$ *and a constant* $c \in \mathbb{R}$ *such that* $c\mathbf{v} = \mathbf{r}$.

Proof. If the conjecture holds for $n-1$, we can take δ as in Corollary 3.3: that is, $\frac{1}{n} < \delta < \frac{1}{n-1}$. This excludes the possibility of extremal cases in the irrational case, as $([0, \frac{1}{n}] \cup [\frac{n-1}{n}, 1)) \cap (\delta, 1-\delta) = \varnothing$. □

What the corollary tells us is that if the lonely runner conjecture holds for n-1, then if there are any extremal cases for n- those cases where the only times that a runner is lonely, their distance to the nearest runner is exactly $\frac{1}{n}$– there must be an extremal case where all the speeds are integers. In fact, in all extremal cases, the speed vector will be proportional to a vector of natural numbers. Moreover, if the conjecture were to not hold for n, then there has to be a set of integer speeds so that the conjecture does not hold for that set. This will come back in our discussion of extremal cases.

4. The irrational case for $n = 6$: the rational case for $n = 5$

We have finally significantly reduced the problem at hand and are now ready to prove the irrational case for $n = 6$, which as we have seen, can be concluded from the rational case for $n = 5$. We will do this recursively, and we assume the conjecture for $n < 5$, whose proofs can be found here [Bie98, Section 3]. Although not exactly the same, they use techniques and ideas similar to those presented here.

In this proof and the following , we will come back to some core ideas that will allow us to construct t given the possible positions the runners find themselves in. First of all, we will look at the circle as being separated into n open segments of length $\frac{1}{n}$, as well as the n endpoints of those intervals, positioned at a distance of $\frac{1}{n}$ from one another. We would like the runners to be in one of the $n-2$ *distant* intervals, as opposed to the two intervals $(0, \frac{1}{n})$ and $(\frac{n-1}{n}, 1)$, which we call *friendly*, as they are adjacent to our stationary runner (we also say 0 is *friendly*). We can moreover notice that out of the n endpoints of the intervals, $0, \frac{1}{n}, \ldots, \frac{n-1}{n}$, only 0, is in a *friendly* position, while the rest are *distant*.

One strategy we will use for manipulating runners will be the following. If the runners are not all *distant*, we will make all of them run distances that are integer multiples of $\frac{1}{n}$. If they are in one of the intervals, they will end up in another interval (of which there are $n-1$), whereas if they are in one of the endpoints, they will end up in another endpoint (of which there are also $n-1$). We thus introduce the following sets C_i, such that $x \in C_i \iff \{x + \frac{1}{n}, x + \frac{2}{n}, \ldots, x + \frac{n-1}{n}\}$ has i friendly elements and $n-i-1$ distant ones (see Figure 1).

- $C_0 := \{0\}$
- $C_1 := (0, \frac{1}{n}) \cup (\frac{n-1}{n}) \cup \{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\}$
- $C_2 := (\frac{1}{n}, \frac{2}{n}) \cup (\frac{2}{n}, \frac{3}{n}) \cup \ldots \cup (\frac{n-2}{n}, \frac{n-1}{n})$.

We have not yet explained how we will make all the runners run integer multiples of $\frac{1}{n}$, but given that they are all going at integer speeds, this will be simple, using the following equality, where we define $\{\mathbf{x}\} = \{(x_1, x_2, \ldots, x_{n-1})\} := (\{x_1\}, \{x_2\}, \ldots, \{x_{n-1}\})$:

$$X(t + \delta t') = \{X(t) + \delta X(t')\} \tag{4.1}$$

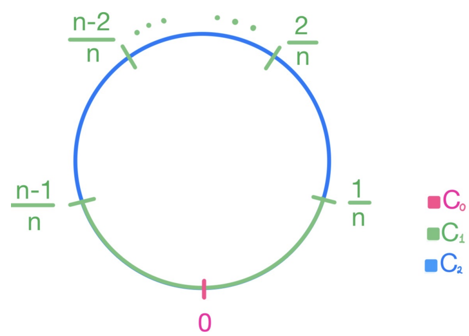

FIGURE 1. Partition of [0,1) into C_0, C_1, and C_2.

If we set t' to be an integer and δ to $\frac{1}{m}$, where m will be either n or a divisor of n for the case $n = 6$, we get:

$$X(t + \frac{t'}{m}) = \{X(t) + \frac{1}{m}X(t')\} \tag{4.2}$$

However, as all speeds are integers and so is t', $X(t') \in \mathbb{N}^{n-1}$, giving us a way to make runners run integer multiples of $\frac{1}{n}$. If $m \mid v_i$ for some $1 \leq i \leq n-1$, runner i will simply go back to where they were at time t after a time of t' has elapsed.

Because we are using an inductive argument and $[\frac{1}{n}, \frac{n-1}{n}] \subset [\frac{1}{m}, \frac{m-1}{m}]$ for $m < n$, we can assume that for any proper subset $S \subset R$ of the runners, there is a time t for which all the runners in S are at distant positions: $\{\{v_i t\} \mid i \in S\} \subset [\frac{1}{n}, \frac{n-1}{n}]$.

Finally, if we are to make the runners run for lapses of time of $\frac{1}{m}$ (which in turn makes them run distances that are integer multiples of $\frac{1}{m}$) for some integer m, it will be useful to determine which of the runners, if any, are actually returning to the same position every time. That is, we would like to know if any of the speeds v_i is divisible by m, and if so, how many. A first step to that is the following theorem:

LEMMA 4.1. [Ren04, Lemma 2.1] *Assume that the lonely runner conjecture (2.2) does not hold for n: there exist $\mathbf{v}= (v_1, v_2, ..., v_{n-1}) \in \mathbb{N}^{n-1}$ such that $X(\mathbf{v}) \cap [\frac{1}{n}, \frac{n-1}{n}] = \varnothing$. Then, for each $1 \leq m \leq n$, there exists $i : 1 \leq i \leq n-1$ such that $m \mid v_i$.*

Proof. Suppose this was not the case, and $m \nmid v_i$ for all $1 \leq i \leq n-1$. At time $t = \frac{1}{m}$, all runners will be at a distance of at least $\frac{1}{m}$ from the origin: $\frac{1}{m} \leq \{\frac{v_i}{m}\} \leq \frac{m-1}{m}$ when $m \nmid v_i$. This yields a contradiction, as we have now explicitly found a time t that makes the conjecture hold. □

We now get into the proof of the conjecture for the rational case of $n = 5$:

THEOREM 4.2. [Bie98] *The rational case of the lonely runner conjecture (1.1) holds for $n = 5$. That is, for any vector of positive integers $\mathbf{v}= (v_1, v_2, ..., v_4) \in \mathbb{N}^4$, there exists a time $t \in \mathbb{R}_{>0}$ such that for all $1 \leq i \leq 4$, $\{v_i t\} \in [\frac{1}{5}, \frac{4}{5}]$.*

Proof. We will suppose, by contradiction, that this is not the case. Fix any $\mathbf{v}= (v_1, v_2, ..., v_4) \in \mathbb{N}^4$, and without loss of generality, assume that there is no prime number p dividing all of $v_1, v_2, ..., v_4$. Our assumption will be that there is no time $t \in \mathbb{R}_{>0}$ for which all $1 \leq i \leq 4$, $\{v_i t\} \in [\frac{1}{5}, \frac{4}{5}]$.

From Lemma 4.1, we know that there is at least one runner $i \in R$ such that $5 \mid v_i$. We will distinguish all such runners from the rest, as they are the only ones who run full laps in $\frac{1}{5}$ time. They are thus the ones that we cannot move around the circle when making the rest run for intervals of length $\frac{1}{5}$.

We will divide the proof into several steps. First, we will prove that there is exactly one runner whose speed is divisible by 5. Then, we will find a time t in which said runner is *distant,* and at most one other runner is as well (so at least two other runners are not). For this, we will do several cases. Finally, we will use our runner manipulation strategy to prove that at least one of the times $t+\frac{1}{5}, t+\frac{2}{5}, t+\frac{3}{5}$ and $t+\frac{4}{5}$ is *good* (it makes all runners distant).

Definition 4.3. The set $D \subset R$ of runners is the following:

$$D := \{i \in R : 5 \mid v_i\} \tag{4.3}$$

Moreover, since 5 is prime, $5 \nmid v_i$, implies $\{v_i, 2v_i, 3v_i, 4v_i\} = \{1,2,3,4\} \bmod 5$, so we have:

$$\Big\{\{\frac{1}{5}v_i\}, \{\frac{2}{5}v_i\}, \{\frac{3}{5}v_i\}, \{\frac{4}{5}v_i\}\Big\} = \Big\{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\Big\} \quad \text{for all } i \in R \setminus D. \tag{4.4}$$

Meanwhile,

$$\Big\{\{\frac{1}{5}v_i\}, \{\frac{2}{5}v_i\}, \{\frac{3}{5}v_i\}, \{\frac{4}{5}v_i\}\Big\} = \{0\} \quad \text{for all } i \in D. \tag{4.5}$$

We defined the sets C_0, C_1 and C_2 by partitioning the interval $[0,1)$ by number of *distant* positions exactly $\frac{k}{5}$ away, where k ranged from 1 to 4. We similarly define, for every t, the following partition of $R : R = D \sqcup R_0 \sqcup R_1 \sqcup R_2 :$

$$R_0(t) := \{i \in R \setminus D : \{v_i t\} \in C_0\} = \{i \in R \setminus D : v_i t \in \mathbb{N}\} \tag{4.6}$$

$$R_1(t) := \{i \in R \setminus D : \{v_i t\} \in C_1\} \tag{4.7}$$

$$R_2(t) := \{i \in R \setminus D : \{v_i t\} \in C_2\} \tag{4.8}$$

Notice that for a given t, all runners in *friendly* positions are either in D, in $R_0(t)$, or in $R_1(t)$. Likewise, all runners in $R_2(t)$ are *distant.*

As mentioned earlier, the set D will be of particular use to us, and we would first like to determine its size, as this will allow us to more concretely work with it. We have already seen that it contains at least one element, and as no prime number divides all velocities, we can suppose that it does not contain all 4 runners. We will now prove that $|D| = 1$:

Lemma 4.4. [Bie98] *Suppose Conjecture 2.2 does not hold for $n = 5$ for a given $\mathbf{v} \in \mathbb{N}^{n-1}$ and no prime divides all speeds v_i with $1 \leq i \leq n-1$. Then D, the set of runners whose speed is divisible by 5, cannot contain more than 1 element.*

Proof. By contradiction, suppose $|D| \geq 2$. As $|D| < 4$, our inductive step tells us that there is a time t for which all runners in D are *distant.* Moreover, we are assuming that there is then at least one runner that is not *distant.* That is, $|R_0(t) \cup R_1(t)| \geq 1$. There can thus be at most one element in the set of distant runners not in D, which is $R_2(t)$.

This is where we go back to our definitions of R_0, R_1, and R_2 and make the runners run distances that are multiples of $\frac{1}{5}$. We consider the set of times

$$T(t) = \Big\{t+\frac{1}{5}, t+\frac{2}{5}, t+\frac{3}{5}, t+\frac{4}{5}\Big\}.$$

Putting Equations (4.2), (4.4) and (4.5) together, we get the following, which is at the heart of all the arguments we will make in this proof. The runners in both $R_0(t)$ and D find themselves in *friendly* positions at zero of the times in T (those in D come back to the position they were in at time t, and those in $R_0(t)$ cycle through the endpoints $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}$ and $\frac{4}{5}$, which are all distant). The runners in $R_1(t)$ are in a *friendly* position exactly once (they go through each interval exactly once, except the one they started out in, which is *friendly*, so there is exactly one other *friendly* interval left to go to) . Finally, those in $R_2(t)$ are *friendly* at exactly two times (one for each *friendly* interval). Summing these up, we get that there are at most

$$0(|D| + |R_0(t)|) + |R_1(t)| + 2|R_2(t)| = |R_1(t)| + 2|R_2(t)| \tag{4.9}$$

times that are **not** good in $T(t)$. However, as $|R_2(t)| \leq 1 < |D|$, we have that

$$|R_1(t)| + 2|R_2(t)| < |R_1(t)| + |R_2(t)| + |D| + |R_0(t)| = |R| = 4,$$

so there must be at least one time that is *good*: a time where all the runners are *distant*, and this yields a contradiction, so there is only one runner in D. □

Having shown this, it will be useful to know if the rest of the speeds can have prime factors in common, and if so how many of them can be divisible by the same prime. From now on, we will suppose that runner 1 is the one whose speed is divisible by 5: $5 \mid v_1$. First, we will see that for any prime p, p can divide at most one of v_2, v_3, and v_4.

Lemma 4.5. [Bie98, Section 3, (4)]
Assume that there is a prime p such that $p|v_i$ and $p|v_j$ for $2 \leq i < j \leq 4$. Then, there is a time t that is good.

Proof. We will split this into two cases: one where $p|v_1$ and another where $p \nmid v_1$. In both, we can do something similar to what we did before, but this time taking time intervals of $\frac{1}{p}$.

We first do the case where $p \nmid v_1$. We can define $D_p = \{k \in R : p \mid v_k\}$ and consider the set of times $T_p(0) = \{\frac{1}{p}, \frac{2}{p}, ..., \frac{p-1}{p}\}$.

In all those times, the runners in D_p will be at the origin: D_p will be a subset of $R_0(\frac{a}{p})$ for all a. Meanwhile, for the other runner $l \notin D_p, D$ in at least one of the times, say $t = \frac{b}{p}$, they will be *distant* (from the equivalent formulation of (4.4)). Without taking l into consideration, we already know that at this time t, $|R_0(t)| \geq 2$ and $|D| = 1$. As in the proof of Lemma 4.4, out of the set of times

$$T(t) = \Big\{t + \frac{1}{5}, t + \frac{2}{5}, t + \frac{3}{5}, t + \frac{4}{5}\Big\},$$

at most

$$|R_1(t)| + 2|R_2(t)|$$

are **not** *good*. However, there is at most one element left, l, which could be in either $R_1(t)$ or $R_2(t)$. As a result,

$$|R_1(t)| + 2|R_2(t)| \leq 2,$$

which is enough to let us conclude that there is at least one *good* time in $T(t)$.

Now, for the case where $p \mid v_1$, we have that the number of speeds p divides is 3 (but not 4). Again, consider the set $D_p = \{k \in R : p \mid v_k\} = \{1, i, j\}$. By our inductive step, there exists a time t where all runners in D_p are distant. If we consider the set of times

$$T_p(t) = \Big\{ t + \frac{a}{p} : 1 \leq a \leq p-1 \Big\},$$

All runners in D_p will still be *distant* at one of those times as they will be back at their position from time t, while the one runner left will be *distant* at at least one of the times, so there will exist at least one *good* time. ☐

Finally, we will find a time t for which 1 is distant and $|R_1(t)| + 2|R_2(t)| \leq 3$. This is achieved, for example, if only runner 1 is distant. We will divide this into two cases. The first is when v_1 is the largest of the speeds, and the second is when it is not.

When 1 is the fastest runner, and $v_1 > v_2, v_3, v_4$, it is easy to find such a time: we just take $t = \frac{1}{5v_1}$. Then, all runners except 1 still have not reached $\frac{1}{5}$ (which is where 1 is), so 1 is the only distant runner.

On the other hand, if 1 is not the fastest, there exists a fastest runner k, say $k = 4$, such that $k \neq 1$. We will take a time t such that this runner is at 0, so of the form $\frac{a}{v_4}$, and we will furthermore want 1 to be distant, and finally at least one of 2 or 3 to be friendly.

For this, we first notice that only one of v_2 or v_3 could be equal to $v_4 - v_1$, and we can assume that $v_2 \neq v_4 - v_1$. Runner 2 will be friendly at the time t we will choose. As $v_4 > v_1, v_2, v_3$, $v_2 \neq v_4 - v_1$ means that $v_2 \not\equiv \pm v_1 \bmod v_4$. By Lemma 4.5, v_2 and v_4 are relatively prime, so we can take the inverse of $v_2 \bmod v_4$. We thus set $a := v_2^{-1} \bmod v_4$ and take time $t' = \frac{a}{v_4}$.

If 1 is distant at time t', we have achieved what we wanted: $|R_0(t')| \geq 1$, $D = 1$, $|R_1(t')| \geq 1$ and so

$$|R_1(t')| + 2|R_2(t')| \leq 1 + 2 = 3.$$

Otherwise, if 1 is not distant, we can find a time t where it is, but 2 remains friendly and 4 stays at 0. In fact, this time will be of the form $t = 2^m t'$ for some integer m : we will exploit the fact that if at some time t', the runner's distance to the origin, d, is less than $\frac{1}{5} (< \frac{1}{4})$ (which it is in this case), then at time $2t'$, its distance to the origin will be $2d$. In order to use this, we will first see that runner $1'$s position at time t' is at least twice that of v_2. Runner 1's position at t' is $\{\frac{av_1}{v_4}\} = \frac{k}{v_4}$ for some $0 \leq k \leq v_4 - 1$. However, $v_4 \nmid av_1$ (as $v_1 < v_4$ and $gcd(a, v_4) = 1$), so $k \neq 0$. Moreover, as $v_1 \not\equiv \pm v_2 \bmod v_4$, $av_1 \not\equiv \pm 1 \bmod v_4$, so really $2 \leq k \leq v_4 - 2$. In particular,

$$1 - 2(\frac{1}{v_4}) \geq \{t'v_1\} = \frac{k}{v_4} \geq 2\Big(\frac{1}{v_4}\Big) = 2\Big(\{t'v_2\}\Big). \tag{4.10}$$

Therefore, taking the smallest power of 2, 2^m, such that $2^m\{\frac{k}{v_4}\} \geq \frac{1}{5}$ (which must exist given that $\frac{1}{5} < \frac{1}{4}$) and setting $t := 2^m t'$ guarantees that at time t, 1 will be distant, v_4 will be at 0 still, and v_2 will be friendly because of the minimality of m. In this case, at time t, $|R_0(t)| \geq 1$, $|D| = 1$ and $|R_2(t)| \leq 1$ (only 3 could be distant and not in D). Thus, $|R_1(t)| + 2|R_2(t)| \leq 3$.

Having found this t such that 1 is distant in all three cases (the one where runner 1 is the fastest, the one where it is not and it is distant at t' and the one where it is not but it is not distant at t'), we can finally conclude our proof. As in the proof of Lemma 4.5, we know that out of the set

$$T(t) = \left\{t + \frac{1}{5}, t + \frac{2}{5}, t + \frac{3}{5}, t + \frac{4}{5}\right\},$$

at most

$$|R_1(t)| + 2|R_2(t)|$$

times are not *good*. However, we have that the inequality

$$|R_1(t)| + 2|R_2(t)| \leq 3$$

for all these cases, so there must be at least one time in t which is *good*, as we wanted.

□

5. SECTION 5: EDGE CASES

Finally, we wish to find out whether there exist any *extremal* velocity vectors. Although we have alluded to this several times in the past, we now concretely define it.

DEFINITION 5.1. A speed vector $\mathbf{v}= (v_1, v_2, ..., v_n) \in \mathbb{R}^n_{>0}$ is said to be *extremal* if there is an $1 \leq k \leq n$ such that for every time t, one of $\{v_i t\}$ with $1 \leq i \neq j \leq n-1$ is such that $|\{v_i t\} - \{v_k t\}| \in [0, \frac{1}{n}] \cup [\frac{n-1}{n}, 1)$ and at the same time, there exists a time t such that for all $1 \leq i \leq n$, $|\{v_i t\} - \{v_k t\}| \in [\frac{1}{n}, \frac{n-1}{n}]$.

This means that at all *good* times t, there must be one runner at the spot $\frac{1}{n}$ and another one at $\frac{n-1}{n}$. This definition is subtle, but it is equivalent to the one we had been using before, where the only guarantee is that there is always at least one runner that is $\frac{1}{n}$ away from a given runner k. Indeed, suppose that for runner k, there are runners at exactly $\frac{1}{n}$ distance on only one side (either in front or behind), but no runners closer than that. Then, taking a time just slightly larger or slightly smaller, respectively, makes that runner be at a distance greather than $\frac{1}{n}$ from all other runners.

From Corollary 3.4, we know that in fact, all *extremal* speed vectors are multiples of vectors of positive integer speeds. However, we don't yet know if there even are any *extremal* vectors at all. As it turns out, there is at least one speed vector for every n. Namely, it is

$$\mathbf{v} = c(1, 2, ..., n) \text{ for } c \in \mathbb{R}_{>0}.$$

We will not prove this here, but the idea is simple: taking the runner with speed 1 as the stationary runner, we see that they would not be spaced out evenly enough if they were to all be at a distance strictly greater than $\frac{1}{n}$ from the origin (see [BHK01, Introduction]). For $n = 6$, which is the case that we are studying, there is in fact only one more family such vector, and it is

$$\mathbf{v} = c(1, 3, 4, 5, 9) \text{ for } c \in \mathbb{R}_{>0},$$

as seen in [BHK01, Theorem 3]. We will also refrain from proving this, but it is interesting to note that the first proof of the rational case of the lonely runner conjecture for $n = 6$, given by [BHK01] in 2001 heavily used the ideas of time intervals introduced in 2, and was able to prove that these were the only edge cases from the proof of the conjecture. However, a significantly shorter proof of 2.2 for $n = 6$ was given by [Ren04] in 2004, heavily inspired by the ideas of the proof for the irrational case for $n = 6$, but with no mention of the edge cases.

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