

# THE LONELY RUNNER CONJECTURE: CONNECTIONS TO DIOPHANTINE APPROXIMATION, GEOMETRY, AND COMBINATORICS IN SMALL CASES

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ABSTRACT. The lonely runner conjecture by Wills is a well-known problem studying  $n$  runners with distinct constant speeds on a unit circle. In this paper, we explain its significance to Diophantine approximation and study known proofs of the conjecture in the small cases of three and four runners, highlighting their geometric, combinatorial, and number-theoretic ideas. We discuss why these ideas are difficult to extend to larger  $n$ , and briefly survey recent computational progress on larger cases.

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## 1. INTRODUCTION

Consider  $n$  runners on a circular track with circumference 1, starting from the origin and running at distinct constant speeds. At any point in this process, a runner may become *lonely*, if he/she is at least  $\frac{1}{n}$  away from any other runner along the circumference. The lonely runner conjecture, proposed by Wills in 1968 [Wil68], states that regardless of the speeds, every runner will at some point be *lonely*.

CONJECTURE 1.1 (Lonely runner conjecture). For every  $n \in \mathbb{N}$ , every  $i \in [n]$ , and every set of distinct real numbers  $\{v_1, \dots, v_n\}$ , there exists a real number  $t$  such that

$$\min_{j \neq i} \|t(v_j - v_i)\| \geq \frac{1}{n}.$$

In the above,  $[n]$  denotes the set  $\{1, 2, \dots, n\}$  while  $\|x\|$  denotes the distance from any real  $x$  to the nearest integer to  $x$ .

Notice the statement is invariant under constant shifts in the speeds  $\{v_1, \dots, v_n\}$ . Indeed, subtracting one runner's speed from all speeds preserves all relative distances between runners, while making that chosen runner stationary at the origin. Hence we can reformulate the conjecture with respect to any runner staying at the

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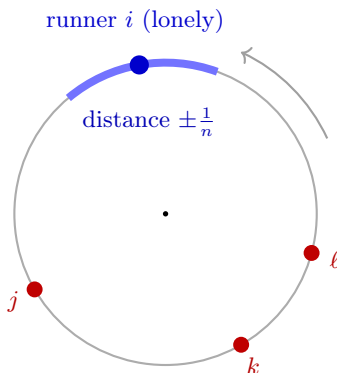


FIGURE 1. A lonely runner configuration. Runner  $i$  is at circular distance at least  $\frac{1}{n}$  from every other runner.

origin, while other runners move relative to said runner. We can further assume the runner speeds are positive, since flipping the sign of the speed of a runner does not change the distance between said runner and the stationary runner.

CONJECTURE 1.2 (Lonely runner conjecture, alternate formulation). For every  $k \in \mathbb{N}$  and every set of positive real numbers  $\{v_1, \dots, v_k\}$ , there exists a real number  $t$  such that we have

$$\min_{i \in [k]} \|tv_i\| \geq \frac{1}{k+1}.$$

In Conjecture 1.2, the variable  $k$  denotes the number of runners moving relative to the fixed runner at the origin. Thus Conjecture 1.2 is the  $n = k + 1$  total-runner version of Conjecture 1.1. For the rest of the paper, we make the same distinction between the use of variables  $n$  and  $k$ .

While the statement of the conjecture appears elementary, the complexity of the problem escalates as the number of runners increases. The conjecture exhibits rich connections to Diophantine approximation, geometry, and combinatorics, tools of which can be used to tackle small cases with few runners, yet mathematicians have only been able to prove the conjecture up to  $n = 7$  total runners without computer assistance.

In this paper, we present a focused exposition of the conjecture via known proofs for the three- and four-runner cases. In Section 2, we explain its significance to the study of Diophantine approximation. In Section 3, we examine its equivalent reformulation as a view obstruction problem, and its subsequent proof using geometric arguments in the three-runner case. In Section 4, we study the combinatorial argument in the proof of the four-runner case. We give motivation on why these ideas are difficult to extend to larger  $n$ , and finally briefly survey recent progress made using computer assistance.

## 2. CONNECTIONS TO DIOPHANTINE APPROXIMATION

**2.1. Reduction to integer speeds.** While Conjecture 1.1 presented in the previous section is defined for any distinct real speeds, the initial version of the lonely runner conjecture proposed by Wills was only for integer speeds.

CONJECTURE 2.1 (Lonely runner conjecture, integer formulation). For every  $k \in \mathbb{N}$  and every set of nonzero integers  $\{v_1, \dots, v_k\}$ , there exists a real number  $t$  such that we have

$$\min_{i \in [k]} \|tv_i\| \geq \frac{1}{k+1}.$$

Here, as in Conjecture 1.2,  $k$  denotes the number of moving runners after fixing one runner at the origin; thus this is still the  $n = k + 1$  total-runner version of the conjecture.

While Conjecture 2.1 might appear to be a weaker statement than those in Section 1, these formulations are in fact equivalent! The equivalence hints at the deep connection between the conjecture and number theory.

The main question is what changes when the speeds are allowed to be real rather than rational. For a real speed vector  $u = (u_1, \dots, u_d)$ , the possible position vectors of the moving runners are given by the orbit  $tu$  modulo the integer lattice  $\mathbb{Z}^d$ . It is convenient to record this orbit by defining

$$(2.1) \quad M(u) = \{tu - z : t \in \mathbb{R}, z \in \mathbb{Z}^d\}.$$

The point of subtracting  $z \in \mathbb{Z}^d$  is that two real vectors which differ by an integer vector represent the same positions modulo 1. Thus  $M(u)$  is the set of all real representatives of the position vector  $tu$  modulo 1. We write  $\overline{M}(u)$  for its closure, which includes all limit points of these possible position vectors.

If the coordinates of  $u$  have no rational linear relation, this orbit is dense in the torus  $\mathbb{R}^d/\mathbb{Z}^d$ , meaning that the runners come arbitrarily close to every possible configuration. If there are rational linear relations, the orbit is not dense in the whole torus, but it is still dense in the smaller subspace cut out by those relations. Kronecker's theorem makes this principle precise, and it is exactly what allows us to reduce irrational speed vectors to rational ones.

THEOREM 2.2 (Kronecker's theorem (1884) [Kro84]). *Let  $u = (u_1, \dots, u_d) \in \mathbb{R}^d$ . Then,*

- *If  $u_1, \dots, u_d$  are linearly independent over  $\mathbb{Q}$ , then  $\overline{M}(u) = \mathbb{R}^d$ .*
- *Otherwise,  $u_1, \dots, u_d$  satisfy some linear equations of the form*

$$(2.2) \quad a_1u_1 + a_2u_2 + \dots + a_du_d = 0$$

*If we write the maximal set of linearly independent equations of the form (2.2) into the rows of a  $m \times d$  matrix  $A$  ( $A$  has rank  $m$  and  $u \in \text{Ker}(A)$ ), then  $\overline{M}(u) = \text{Ker}(A) + \mathbb{Z}^d$ .*

Note that the speeds  $v_1, \dots, v_k$  in the lonely runner conjecture are scale invariant, so the conjecture with integer speeds and rational speeds can be easily shown to be equivalent. To establish equivalence between Conjecture 1.2 and Conjecture 2.1, it suffices to prove the following lemma: after applying it with  $\delta = \frac{1}{n}$ , the rational case implies the real case.

LEMMA 2.3. [BHK01, Lemma 8] *Let  $0 < \delta < 1/2$ . Suppose that for every set  $\{v_1, \dots, v_{n-2}\} \in \mathbb{Q}$ , there exists  $t \in \mathbb{R}^+$  satisfying*

$$\{v_i t\} \in (\delta, 1 - \delta) \text{ for } i = 1, \dots, n - 2.$$

*Then for any set  $\{u_1, \dots, u_{n-1}\} \in \mathbb{R}^+$  for which there is a pair  $u_i, u_j$  such that  $\frac{u_i}{u_j} \notin \mathbb{Q}$ , there exists  $t \in \mathbb{R}^+$  satisfying*

$$\{u_i t\} \in (\delta, 1 - \delta) \text{ for } i = 1, \dots, n - 1.$$

Note that we adopt the notation  $\{x\}$  denoting the fractional part of  $x$ , for any real  $x$ .

Before entering the proof, let us outline the strategy. If the speeds  $u_1, \dots, u_{n-1}$  are  $\mathbb{Q}$ -linearly independent, Kronecker's theorem gives much more than needed: the possible runner configurations are dense in the whole torus, so they must enter the box  $(\delta, 1 - \delta)^{n-1}$ . The dependent case is subtler. There, Kronecker's theorem only gives density inside the subspace satisfying the rational linear relations among the speeds. The proof therefore constructs a rational vector  $\mathbf{w}$  in that subspace whose coordinates, up to signs and repetitions, can be handled by the rational case assumed in the lemma.

*Proof.* Let  $\mathbf{u} = (u_1, \dots, u_{n-1})$  and define  $M(\mathbf{u})$  as in (2.1). Then, the lemma is equivalent to asserting that  $\overline{M(\mathbf{u})} \cap (\delta, 1 - \delta)^{n-1} \neq \emptyset$ , which is true by Theorem 2.2 if  $u_1, \dots, u_{n-1}$  are  $\mathbb{Q}$ -linearly independent. In our runner analogy, the density conclusion in Theorem 2.2 means that for every position configuration in  $(0, 1)^{n-1}$ , there exists times  $t$  such that the runners approximate said configuration to arbitrarily low deviation.

In the case where  $u_1, \dots, u_d$  are not  $\mathbb{Q}$ -linearly independent, we need to analyze the matrix  $A$  defined in Theorem 2.2. Since  $\text{Ker}(A)$  contains the vector  $\mathbf{u} \in \mathbb{R}^{n-1}$  and  $A$  has rational entries, there exists another vector  $\mathbf{r} \in (\mathbb{Q}_{>0})^{n-1}$  in  $\text{Ker}(A)$  which is not proportional to  $\mathbf{u}$ . The existence of two non-proportional vectors  $\mathbf{u}$  and  $\mathbf{r}$  in  $\text{Ker}(A)$  implies  $\dim \text{Ker}(A) \geq 2$ , and so there exists another rational vector  $\mathbf{s} \in \mathbb{Q}^{n-1}$  in  $\text{Ker}(A)$ .

We now construct a vector  $\mathbf{w} \in \mathbb{Q}^{n-1}$  from  $\mathbf{s}$  and  $\mathbf{r}$  as follows: Take indices  $i, j$  such that

$$\frac{s_i}{r_i} < \frac{s_j}{r_j} \text{ and } \frac{s_k}{r_k} \notin \left( \frac{s_i}{r_i}, \frac{s_j}{r_j} \right) \text{ for } k = 1, \dots, n-1.$$

Define  $\mathbf{w} = (r_i + r_j)\mathbf{s} - (s_i + s_j)\mathbf{r}$ . We note three properties of  $\mathbf{w}$ :

- $\mathbf{w}$  is rational and belongs to  $\text{Ker}(A)$ , since both  $\mathbf{s}$  and  $\mathbf{r}$  are rational vectors in  $\text{Ker}(A)$ .
- We have  $w_i = -w_j$ , since  $w_i = r_j s_i - s_j r_i = -(r_i s_j - s_i r_j) = -w_j$ .
- All entries of  $\mathbf{w}$  are non-zero. Otherwise, if  $w_k = 0$  for some index  $k$ , then  $\frac{s_k}{r_k} = \frac{s_i + s_j}{r_i + r_j}$ , contradicting  $\frac{s_k}{r_k} \notin \left( \frac{s_i}{r_i}, \frac{s_j}{r_j} \right)$ .

With this construction of  $\mathbf{w}$ , we are ready to complete the induction step. Let  $v_1, \dots, v_{n-2}$  be a collection of positive rationals containing  $|w_1|, \dots, |w_{n-1}|$ ; such a collection must exist since  $|w_i| = |w_j|$  as proven previously. By the assumption of the lemma, there exists  $t$  such that  $\{t\mathbf{v}\} \in (\delta, 1 - \delta)^{n-2}$ . Since each  $|w_i|$  appears among the  $v_j$ , the same time  $t$  gives  $\{t\mathbf{w}\} \in (\delta, 1 - \delta)^{n-1}$ . Since  $t\mathbf{w} \in \text{Ker}(A)$ , we also have  $\{t\mathbf{w}\} \in \text{Ker}(A) + \mathbb{Z}^{n-1} = \overline{M(u)}$  by Theorem 2.2, so the lemma is proven.  $\square$

*Remark 2.4.* The lemma shows that the main difficulty of the conjecture lies in cases of rational speeds, or common real multiples of such speeds. This is why the integer formulation is equivalent to the general formulation of the conjecture. The reduction to integer speeds is the first hint that the conjecture is fundamentally a problem about the multiplicative and additive structure of integers, rather than about analysis.

**2.2. Origin and Dirichlet's Theorem.** One natural question to ask is the origin of the  $\frac{1}{n}$  bound in Conjecture 1.1, or equivalently, the bound  $\frac{1}{k+1}$  in Conjecture 1.2. Now that we have reduced the conjecture to integer speeds, we may answer this question drawing from a well-known result in Diophantine approximation:

**THEOREM 2.5** (Dirichlet's Approximation Theorem (1842)). *For every  $t \in \mathbb{R}$  and  $k \in \mathbb{N}$ , there exists  $q \in \{1, 2, \dots, k\}$  such that  $\|tq\| \leq \frac{1}{k+1}$ .*

Dirichlet's theorem states, that for every real number  $t$ , we can produce an integer  $q$  in the provided range that can be used to approximate  $t$ , in the sense that  $tq$  lies close to an integer. In 1967, Wills asked a converse of the question: given a set of integer approximants  $\{v_1, \dots, v_k\}$ , does there exist a real number  $t$  that cannot be well-approximated by any of these approximants? Conjecture 1.2 asserts the optimal lower bound on  $\min_i \|t v_i\|$  is  $\frac{1}{k+1}$ , which matches the bound presented above in Dirichlet's approximation theorem.

That the bound  $\frac{1}{k+1}$  in Conjecture 1.2 cannot be improved is easy to prove. Consider speeds  $\{v_1, \dots, v_k\} = [k]$  and consider any candidate time  $t$ . Then Dirichlet's theorem states that for some speed  $v_i$ , we have  $\|v_i t\| \leq \frac{1}{k+1}$ . So for any time  $t$ ,  $\min_i \|v_i t\| \leq \frac{1}{k+1}$ , and so the bound cannot be replaced by anything greater than  $\frac{1}{k+1}$ .

Dirichlet's theorem therefore explains why the constant  $\frac{1}{k+1}$  is the natural scale for the lonely runner conjecture, but it does not by itself prove Conjecture 1.2. The theorem guarantees that some integer approximant can be close to an integer at each fixed time; the lonely runner conjecture asks, conversely, for a time at which all of the prescribed quantities  $t v_1, \dots, t v_k$  simultaneously avoid the integers by a gap of size  $\frac{1}{k+1}$ . This notion of avoiding lattices motivates the geometric reformulation in the next section, where the problem becomes one of forcing every ray to hit an obstruction.

### 3. GEOMETRIC APPROACH

**3.1. The view-obstruction reformulation.** Independently of Wills, Cusick in 1974 [Cus74, Section 2] arrived at, surprisingly, an equivalent problem by studying  $k$ -dimensional geometry - the view-obstruction problem. The idea is as follows: imagine an observer at the origin looking out along a ray  $\{t(v_1, \dots, v_k) : t \geq 0\}$  with positive direction vector  $(v_1, \dots, v_k)$ . Let  $K$  be a convex body containing the origin in  $\mathbb{R}^k$ , and let  $\lambda K$  be its dilation by  $\lambda \in \mathbb{R}^+$  centered from the origin. Place  $\lambda K$  at each lattice point  $(l_1 + \frac{1}{2}, l_2 + \frac{1}{2}, \dots, l_k + \frac{1}{2})$  as translations of the original  $\lambda K$  by positive half-integer coordinates. Cusick asked, for which  $\lambda$  would the observer's view be always obstructed?

If we let  $K$  be the unit hypercube  $[-\frac{1}{2}, \frac{1}{2}]^k$  centered at the origin, the connection between the view-obstruction problem and the lonely runner conjecture appears. A ray  $t v$  avoids all hypercubes if and only if for every  $t$ , there exists a dimension  $i$  where  $\{v_i t\}$  lies outside the interval  $(\frac{1-\lambda}{2}, \frac{1+\lambda}{2})$  modulo 1. Taking the variable transformation  $\lambda = \frac{k-1}{k+1}$ , the avoidance condition says that a ray is unobstructed if and only if for every  $t$ , there exists  $i$  such that  $\|v_i t\| \leq \frac{1}{k+1}$ . The preceding equivalence proves the connection between an obstructed ray and a lonely runner.

CONJECTURE 3.1 (Lonely runner conjecture, view-obstruction formulation). For every  $k \in \mathbb{N}$  and every set of positive real numbers  $\{v_1, \dots, v_k\}$ , the intersection

$$\{t(v_1, \dots, v_k) : t \geq 0\} \cap \left( \lambda K + \mathbb{N}^k - \frac{1}{2} \mathbb{1} \right)$$

is non-empty if  $\lambda = \frac{k-1}{k+1}$  and  $K$  is the unit cube  $[-\frac{1}{2}, \frac{1}{2}]^k$ .

The view-obstruction reformulation makes the conjecture geometrically vivid, and in the case of three runners, gives rise to a simple picture proof.

**3.2. An elementary proof for three runners.** We now prove the lonely runner conjecture in the case  $k = 2$ , i.e. for three runners (two moving and one stationary). This is essentially the proof of Cusick, presented in elementary geometry.

THEOREM 3.2. [Cus74, Section 3] *Let  $v_1, v_2$  be distinct real numbers. Then there exists  $t \in \mathbb{R}$  such that*

$$\|v_1 t\| \geq \frac{1}{3} \text{ and } \|v_2 t\| \geq \frac{1}{3}.$$

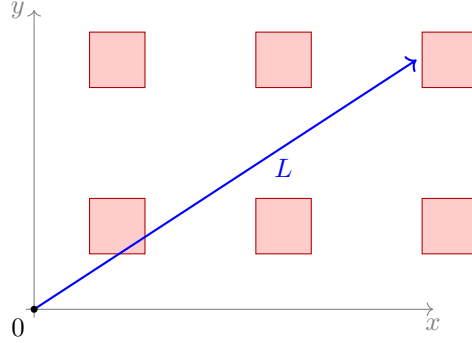


FIGURE 2. The view-obstruction picture for  $k = 2$ . The red squares of side  $1/3$  are centered at half-integer lattice points; the lonely runner conjecture for two moving runners asserts that any ray  $L$  from the origin with positive, distinct direction components must hit at least one square.

*Proof.* As explained in Section 3.1, it suffices to prove that all rays starting from the origin in  $\mathbb{R}^2$  pointing in positive directions  $(v_1, v_2) \in \mathbb{R}^{+2}$  must eventually hit one of the squares with side length  $\frac{1}{3}$  and centered at a half-integer lattice point  $(l_1 - \frac{1}{2}, l_2 - \frac{1}{2}) \in \mathbb{N}^2 - \frac{1}{2} \mathbb{1}$ . By symmetry, it suffices to prove so for rays of slope less than or equal to 1, i.e.  $v_1 \geq v_2$ .

Instead of thinking about rays hitting squares, we consider projections of squares and prove that the squares cover all possible ray directions. In fact, we claim that the squares centered at  $y = \frac{1}{2}$  are sufficient to cover all rays of slope  $\leq 1$ . Since a square centered at  $(l_1 - \frac{1}{2}, l_2 - \frac{1}{2})$  is the region  $[l_1 - \frac{1+\lambda}{2}, l_1 - \frac{1-\lambda}{2}] \times [l_2 - \frac{1+\lambda}{2}, l_2 - \frac{1-\lambda}{2}]$ , it covers rays of slope within the interval

$$\left[ \frac{l_2 - \frac{1+\lambda}{2}}{l_1 - \frac{1-\lambda}{2}}, \frac{l_2 - \frac{1-\lambda}{2}}{l_1 - \frac{1+\lambda}{2}} \right].$$

It suffices to prove that if  $l_2 = 1$ , the intervals for each  $l_1 \in \mathbb{N}$  overlap for adjacent values of  $l_1$ , i.e.

$$\frac{\left(1 - \frac{1+\lambda}{2}\right)}{\left(x + 1 - \frac{1-\lambda}{2}\right)} \leq \frac{\left(1 - \frac{1-\lambda}{2}\right)}{\left(x - \frac{1+\lambda}{2}\right)}.$$

Substituting  $\lambda = \frac{1}{3}$ , this reduces to  $\frac{1}{3}(x - \frac{1}{3}) \leq \frac{2}{3}(x + \frac{2}{3})$  and is equivalent to  $0 \leq \frac{1}{3}x + \frac{5}{9}$ , which is trivially true. Hence, the slopes of rays obstructed by all the squares collectively cover  $(0, \infty)$ , and so all rays from the origin must be obstructed. This completes the proof.  $\square$

**3.3. Why the geometric approach struggles in higher dimensions.** The view-obstruction picture remains valid in any dimension; indeed, this is one of the standard geometric formulations of the lonely runner conjecture [PS24, Section 3.2]. The difficulty is more specifically that the elementary proof of Theorem 3.2 uses a one-dimensional feature of the two-dimensional setting. After restricting to slopes at most 1, each square obstructs an interval of possible slopes, and the proof only has to check that adjacent intervals overlap.

For  $k > 2$ , the space of ray directions is  $(k - 1)$ -dimensional after normalization, and each translated cube cuts out a region in this higher-dimensional direction space. Proving that these regions cover all possible directions is therefore no longer an interval-overlap argument; it becomes a higher-dimensional covering problem with many interacting faces and no natural linear order on directions. Thus the geometric formulation does generalize, but the simple two-dimensional proof does not scale directly.

Modern geometric approaches can be viewed as more sophisticated ways of handling this higher-dimensional covering problem. For example, Malikiosis, Santos, and Schymura [MSS25, Theorem A] derived a linearly exponential bound on counterexamples to the lonely runner conjecture, with an argument involving zonotopes and their properties in  $n$ -dimensional geometry. This was later utilized by Rosenfeld [Ros25, Theorem 1] to prove the 8 runner case of the conjecture via computer assistance.

## 4. COMBINATORIAL APPROACH

Notice that in the geometric approach, we do not utilize the integer formulation of the conjecture stated in Conjecture 2.1 due to the continuous nature of geometry. In order to take advantage of the discreteness of the integer formulation, one proof approach is to perform case analysis based on modular properties of the speeds.

In this section we present the proof by Renault [Ren04, Appendix A] for the lonely runner conjecture with four runners. The four-runner case was first proved earlier by Cusick and Pomerance [CP84]; we use Renault's later proof because it gives a short elementary illustration of the combinatorial method. While Renault uses an argument of case analysis around congruence classes to prove the six-runner case, we present the simpler four-runner case to illustrate the main ideas of the combinatorial method.

### 4.1. An elementary proof for four runners.

THEOREM 4.1. [Ren04, Appendix A] *Let  $v_1, v_2, v_3$  be positive integers. Then there exists  $t \in \mathbb{R}$  such that*

$$\|v_1 t\| \geq \frac{1}{4}, \|v_2 t\| \geq \frac{1}{4} \text{ and } \|v_3 t\| \geq \frac{1}{4}.$$

Note that here we use the integer formulation of the lonely runner conjecture, where  $v_1, v_2, v_3 \in \mathbb{N}$ . In the notation of Conjecture 1.2, this is the case  $k = 3$ , corresponding to  $n = 4$  total runners. We take runner 0 to be stationary at the origin, and we let runners 1, 2, and 3 have speeds  $v_1, v_2$ , and  $v_3$ , respectively. We can further assume

$$(4.1) \quad \gcd(v_1, v_2, v_3) = 1,$$

since the conjecture is scale-invariant in the speeds, as explained previously. If none of  $v_1, v_2, v_3$  is a multiple of 4, then taking  $t = \frac{1}{4}$  works: each  $v_i$  is congruent to 1, 2, or 3 modulo 4, so  $x_i(\frac{1}{4}) = \{\frac{v_i}{4}\}$  lies in  $\{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ , and the theorem statement holds true in this case. So we may assume, after relabeling the runners if necessary, that

$$(4.2) \quad 4 \mid v_1.$$

Given these assumptions and using the result of Theorem 3.2, we can now apply our combinatorial arguments.

Denote the position of runner  $i$  at time  $t$  by  $x_i(t) = \{v_i t\}$ , and say that runner  $i$  is safe at time  $t$  if  $x_i(t) \in [\frac{1}{4}, \frac{3}{4}]$ . Define

$$(4.3) \quad T = \left\{ t \in \mathbb{R} : x_i(t) \in \left[ \frac{1}{4}, \frac{3}{4} \right] \forall i \in \{2, 3\} \right\}$$

to be the set of times at which both runners 2 and 3 are safe. By Theorem 3.2,  $T$  contains an interval of positive length, and we can consider  $t_0 \in T$  such that

$$(4.4) \quad x_1(t_0) > 0 \quad \text{and} \quad \|x_1(t_0)\| = \max_{t \in T, x_1(t) > 0} \|x_1(t)\|.$$

If runner 1 is safe at time  $t_0$ , we are done. Hence we assume, for the sake of contradiction, that  $x_1(t_0) \notin [\frac{1}{4}, \frac{3}{4}]$ . By symmetry (between  $t$  and  $-t$ ), we further assume without loss of generality that

$$(4.5) \quad x_1(t_0) \in \left( 0, \frac{1}{4} \right).$$

Since  $t_0$  maximizes  $x_1(t)$  among the safe times in (4.4), it follows it must be the case that at least one runner is at position  $\frac{3}{4}$ , say runner 2 with

$$(4.6) \quad x_2(t_0) = \frac{3}{4}.$$

An important lemma repeatedly used in combinatorial proofs to the conjecture is as follows:

LEMMA 4.2. [BGG<sup>+</sup>98, Section 3] *For any runner index  $i$ , any times  $t_1, t_2 \in \mathbb{R}$ , and any  $\lambda \in \mathbb{R}$ , we have  $x_i(t_1 + t_2) = \{x_i(t_1) + x_i(t_2)\}$  and  $x_i(\lambda t_1) = \{\lambda x_i(t_1)\}$ .*

The lemma follows simply from expanding the definition of  $x_i(t) = \{v_i t\}$ . As seen below, and in other combinatorial approaches to the conjecture, the lemma allows “adding” of different runner configurations and the clever manipulation of

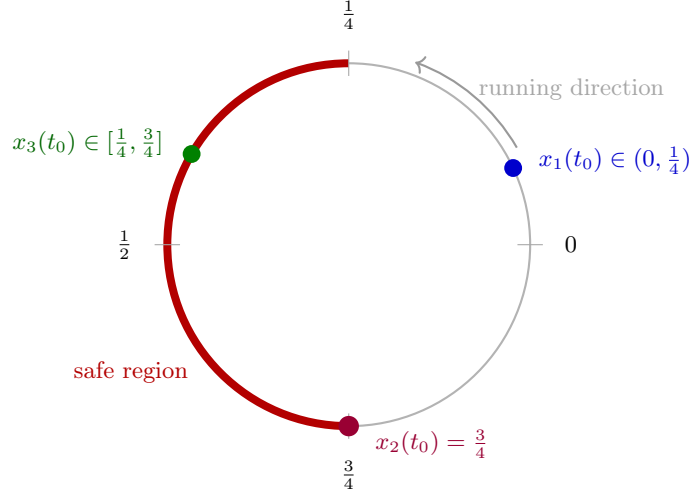


FIGURE 3. Illustration of contradiction setup in the four-runner proof at time  $t_0$ . Runners 2 and 3 are safe, runner 1 is assumed not to be safe, and after symmetry and relabeling we may take  $x_1(t_0) \in (0, \frac{1}{4})$  and  $x_2(t_0) = \frac{3}{4}$ .

rational positions. In this proof, (4.2) implies that for any  $\lambda \in \{1, 2, 3\}$  and any  $\alpha \in \{0, 1, 2, 3\}$ , we have

$$(4.7) \quad x_1\left(\lambda t_0 + \frac{\alpha}{4}\right) = \{\lambda x_1(t_0)\}.$$

Together with (4.5), this means that if  $\lambda \geq 2$ , then

$$(4.8) \quad \left\|x_1\left(\lambda t_0 + \frac{\alpha}{4}\right)\right\| > \|x_1(t_0)\|.$$

We divide into two cases based on the parity of  $v_3$ :

- If  $v_3$  is even, then  $v_2$  must be odd by (4.1) and (4.2). Then, using the lemma and (4.6), we have

$$x_1\left(t_0 + \frac{1}{2}\right) = x_1(t_0), \quad x_3\left(t_0 + \frac{1}{2}\right) = x_3(t_0), \quad x_2\left(t_0 + \frac{1}{2}\right) = \left\{x_2(t_0) + \frac{1}{2}\right\} = \frac{1}{4}.$$

– If  $x_3(t_0) < \frac{3}{4}$ , we can find  $\epsilon > 0$  such that time  $t' = t_0 + \frac{1}{2} + \epsilon$  contradicts the choice of  $t_0$  in (4.4), since  $\|x_1(t')\| > \|x_1(t_0)\|$  while  $x_2(t')$  and  $x_3(t') \in (\frac{1}{4}, \frac{3}{4})$ .

– If  $x_3(t_0) = \frac{3}{4}$ , then time  $t' = 2t_0$  contradicts the choice of  $t_0$  in (4.4), since (4.6) gives  $x_2(t') = x_3(t') = \frac{1}{2}$  while (4.8) gives  $\|x_1(t')\| > \|x_1(t_0)\|$ .

- If  $v_3$  is odd, consider times  $t_1 = 3t_0$  and  $t_2 = 3t_0 + \frac{1}{2}$ . By (4.6),  $x_2(t_1) = \frac{1}{4}$ , and  $x_2(t_2)$  is either  $\frac{1}{4}$  or  $\frac{3}{4}$  depending on the parity of  $v_2$ ; in either case, runner 2 is safe at both times. Also, (4.7) and (4.8) give  $\|x_1(t_1)\| = \|x_1(t_2)\| > \|x_1(t_0)\|$ . Finally, since  $v_3$  is odd, the lemma gives  $x_3(t_2) = \{x_3(t_1) + \frac{1}{2}\}$ , so at least one of  $x_3(t_1)$  and  $x_3(t_2)$  must fall within  $[\frac{1}{4}, \frac{3}{4}]$  and runner 3 at the corresponding time will be safe. Hence, the corresponding time contradicts the choice of  $t_0$  in (4.4).

After exhausting all cases, the proof is complete.

**4.2. Why the combinatorial approach struggles in higher dimensions.** As seen from the proof above, even for four runners, the case analysis approach is lengthy and involves breaking down into many nested subcases based on modular properties. In fact, in Renault’s proof for six runners, he uses the fact that  $6 + 1$  is prime, and he uses seven pages of terse casework with cases like “exactly two speeds are even”. The number of possible residue patterns grows rapidly: in the stationary-runner formulation with  $k$  moving speeds, considering residues modulo  $k + 1$  already gives  $(k + 1)^k$  possible residue vectors before using symmetries, gcd reductions, or other structural simplifications. This makes it hard to generalize the combinatorial approach to greater numbers of runners, and for cases where  $n + 1$  is not prime.

In principle, this kind of combinatorial search can be aided by a computer, but raw case analysis is not enough on its own: one first needs a finite reduction showing that only finitely many speed vectors, or finitely many structured residue configurations, have to be checked. The modern computer-assisted results discussed in Section 5 follow this philosophy, replacing hand casework by finite reductions plus exhaustive verification. Indeed, no proof is known of the conjecture for more than seven runners, without the aid of computers.

## 5. RECENT DEVELOPMENTS FOR LARGER NUMBERS OF RUNNERS

The cases of the lonely runner conjecture beyond seven total runners, equivalently beyond  $k = 6$  moving speeds in the stationary-runner formulation, have only been settled very recently, and all known proofs rely on substantial computer searches. Some of the ingredients discussed below, such as Tao’s finite reduction and the zonotope bound of Malikiosis, Santos, and Schymura, appear in peer-reviewed sources, while the newest computer verifications are currently available as arXiv preprints. The breakthroughs of the last few years all proceed by the same overall strategy: *finite reduction*, i.e. producing an explicit upper bound  $B(k)$  such that the conjecture holds for all speed vectors  $\mathbf{v}$  in the stationary-runner formulation if and only if it holds for those with  $\max_i v_i \leq B(k)$ , followed by a (usually highly nontrivial) computer search over  $\mathbf{v}$  with  $\max_i v_i \leq B(k)$ .

**5.1. Tao’s finite reduction and zonotope improvements.** The first effective finite reduction is due to Tao, who showed that the conjecture holds whenever  $\max_i v_i \leq 1.2k$  [Tao18, Proposition 1.5] (a soft bound on counterexamples), and more importantly proved a finite reduction with bound roughly  $B(k) = k^{O(k^2)}$  [Tao18, Theorem 1.3]. Tao’s bound is far too large to enable any practical computer verification - even for  $k = 7$ , the search space is astronomical.

The decisive improvement came from Malikiosis, Santos, and Schymura [MSS25, Theorem A], who reformulated the problem in terms of the *covering radii of zonotopes*, as mentioned previously in Section 3.3. A zonotope is the Minkowski sum of finitely many line segments, and the lonely runner conjecture (in its view-obstruction form) is equivalent to a statement about how a certain explicit zonotope  $Z(\mathbf{v}) \subset \mathbb{R}^k$ , built from the speed vector, must cover a target box. By analyzing the structure of these zonotopes via tools from the geometry of numbers, the authors are able to prove a linearly exponential bound on counterexamples. In the notation

of the stationary-runner formulation with  $k$  moving speeds, their theorem says that a primitive minimal counterexample must satisfy

$$\sum_{\emptyset \neq S \subseteq [k]} \gcd(\{v_i : i \in S\}) \leq \binom{k+1}{2}^{k-1}.$$

In particular, since the singleton subsets contribute the individual speeds, it is enough to check speed vectors satisfying the weaker but simpler bound

$$\max_i v_i \leq \binom{k+1}{2}^{k-1} \leq k^{2k}.$$

This enabled Rosenfeld, in a 2025 preprint, to prove the conjecture up to eight runners via computational methods [Ros25, Theorem 1].

**5.2. Computer checking beyond eight runners.** Rosenfeld’s verification for eight runners illustrates what the finite-reduction strategy looks like in practice. After the linearly exponential bound of Malikiosis, Santos, and Schymura made the search space finite but still large, Rosenfeld added arithmetic filtering, including constraints on the prime divisors of a minimal counterexample, to reduce the remaining cases enough for a computer check [Ros25, Section 2]. Thus, the computation is not merely a brute-force enumeration of all speed vectors, but the final step after substantial mathematical pruning.

Very recent preprints pushed the same strategy further. In 2025, Trakulthongchai refined Rosenfeld’s approach with an additional sieve and proved the cases of nine and ten runners [Tra25, Theorem 1.2]. Even more recently, Sungkawichai and Trakulthongchai announced a computer-assisted proof for eleven, twelve, and thirteen runners, using further sieving ideas together with a polynomial method argument [ST26]. These results suggest that the current frontier is not simply faster computation, but finding stronger structural reductions that make the finite check small enough to carry out.

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