ALTERNATE JULIA SETS

RAY WANG

Abstract. The study of dynamical systems often involves analyzing how functions behave under iteration in different mathematical spaces. In the context of complex dynamics, tools such as Julia sets and filled Julia sets are used to understand the long-term behavior of functions in complex Euclidean space. In this paper, we will present a review of Julia sets and filled Julia sets, provide an overview of the mathematical formulation of the alternate Julia set introduced in [DRP09, Section 2], extend it to the p-adic setting, and propose a tool that can potentially be used to study the arithmetic dynamics of various types of functions. Additionally, we will summarize key results on connectivity properties and visualization techniques as discussed in [DBR13, Section 2 and 3] and provide a visualization algorithm and pseudocode that enable the visualization of alternate Julia sets with various connectivity properties.

Contents

1. Introduction

In complex dynamics, the study of iterated rational functions gives rise to rich and intricate structures known as Julia sets. For a polynomial $f : \mathbb{C} \to \mathbb{C}$, the **filled Julia set** is the set of points whose orbits under iteration of f remain bounded, while the **Julia set** itself is typically defined as the boundary of the filled Julia set. Variations such as the **alternate Julia set** modify the iteration process by alternating between different polynomials at each step, leading to new dynamical behaviors and fractal structures. Extending these ideas beyond the complex numbers, one can study dynamical systems over $\mathbb{C}_p$, the field of p-adic complex numbers. In this setting, one can define **p-adic Julia sets**, **p-adic alternate Julia sets**, and

Date: May 13, 2025.

***p*-adic Mandelbrot sets** analogously, using the p-adic norm to measure boundedness of orbits. The resulting p-adic sets display behaviors that are different from their complex counterparts, reflecting the unique topology and geometry of p-adic spaces. This paper will explore the definitions, properties, and examples of filled Julia sets, alternate Julia sets, p-adic Mandelbrot sets, and p-adic alternate Julia sets, highlighting the similarities and differences across these contexts.

2. Julia sets, filled Julia sets, and Mandelbrot sets

Julia sets and filled Julia sets are common tools for studying the complex-dynamical behavior of a given function. For a given function $F : \mathbb{C} \to \mathbb{C}$, the filled Julia set K is the set of points where their corresponding orbits under F are bounded. In other words, we can define the filled Julia set as follows:

Definition 2.1. The filled Julia set $K(F)$ of a given function F is the set of points z where the orbit $\{F^n(z)\}$ is bounded.

With the definition of the filled Julia set, we can also define the Julia set $J(F)$ that can be used to describe the dynamical property of a given function.

Definition 2.2. The Julia set $J(F)$ is the boundary of the filled Julia set $K(F)$. In mathematical terms, $J(F) = \partial K(F)$.

The points in the Julia set exhibit chaotic behavior. More specifically, points lying outside the Julia set diverge to infinity under repeated iterations of the function F, while points in the interior of the filled Julia set remain bounded. In contrast, the dynamical behavior of points in the Julia set is highly sensitive to perturbations and lacks stability.

Notably, the Julia set and the filled Julia set associated with a given function share the same connectivity properties. For a polynomial in the quadratic family $f : \mathbb{C} \to \mathbb{C}, f(z) = z^2 + c$, the corresponding Julia set and filled Julia set are either path-connected (there is a path within the set that connects any two given points in the set) or path-disconnected (for any two given points in the set, it is impossible to find a path within the set that connects the pair). The set of complex numbers c that can form a path-connected Julia set is called the Mandelbrot set. Different components of the Mandelbrot set correspond to different dynamical behaviors of the corresponding filled Julia sets, resulting in different fractal patterns that can be used to generate beautiful visualizations.

2.1. Examples.

The visualization of the Mandelbrot set and the corresponding path-connected filled Julia sets of different points in the Mandelbrot set are provided in Figure 1[Bou01, Figure 1]. Different components of the Mandelbrot set correspond to different dynamical behaviors of the corresponding filled Julia sets. For example, the largest bulb of the Mandelbrot set corresponds to the filled Julia sets that have attractive fixed points (i.e., there exists an x such that $|f'(x)| < 1$), while the second-largest bulb corresponds to the filled Julia sets that have attractive 2-cycles (i.e., there exists an x such that $|(f^2)'(x)| < 1$).

3. Alternate filled Julia sets

3.1. Formulations and Boundedness Properties.

Although we can use Julia sets to model the behavior of a given dynamical system, there are multiple types of

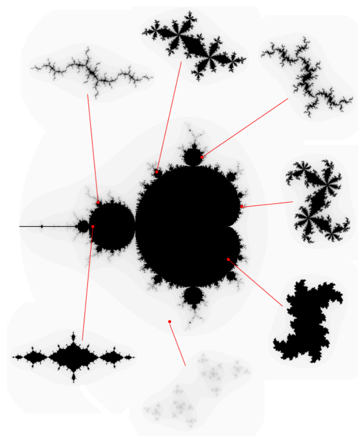

FIGURE 1. A visualization of the Mandelbrot set and the corresponding connected Julia sets of different components of the Mandelbrot set. The plot is generated by Python and is copied from [Bou01, Figure 1].

interactions in some more complex systems that cannot be described with a single function. To provide a more general framework for the problem, imagine that there are two different functions in the quadratic family, F_1 and F_2, that act on a given point z_0 iteratively. We can define the corresponding orbit of iteratively applying the two given functions in the quadratic family as follows:

DEFINITION 3.1. For two given functions

$$F_1 : \mathbb{C} \longrightarrow \mathbb{C}$$

$$z \longmapsto z^2 + c_1 \text{ and}$$

$$F_2 : \mathbb{C} \longrightarrow \mathbb{C}$$

$$z \longmapsto z^2 + c_2$$

where c_1 and c_2 are constants in $\mathbb{C}$, we define the orbit $P_{c_1 c_2}(z_0)$ of a given starting point $z_0 \in \mathbb{C}$ as the set

$$\{z_i | z_{2i} = F_2(z_{2i-1}) = z_{2i-1}^2 + c_2, z_{2i+1} = F_1(z_{2i}) = z_{2i}^2 + c_1\}$$

where the even terms of the orbits are generated by applying F_2 on the previous terms while the odd terms are generated by applying F_1 on the previous terms.

Similarly, we can also swap F_1 and F_2 to define another orbit:

$$P_{c_2c_1}(z_0) = \{z_i | z_{2i} = F_1(z_{2i-1}) = z_{2i-1}^2 + c_1, z_{2i+1} = F_2(z_{2i}) = z_{2i}^2 + c_2\}.$$

Similar to the definition of filled Julia sets, we can also define the filled alternate Julia set of two alternate dynamics by the boundedness of the orbits.

Definition 3.2. The alternate filled Julia set $K_{c_1c_2}$ corresponding to the orbit type $P_{c_1c_2}$ is defined as the set of points where their orbits under the alternate dynamics are bounded

$$K_{c_1c_2} = \{z | P_{c_1c_2}(z) \text{ is bounded}\}.$$

To analyze whether a given orbit $P_{c_1c_2}(z)$ is bounded, we need the following proposition that relates the boundedness of the odd terms and the even terms of the orbit:

Proposition 3.3. [DRP09, Proposition 2.1]

(1) If z_{2i} is bounded, then z_{2i-1} is also bounded.
(2) If z_{2i} is unbounded, z_{2i+1} is also unbounded.

Proof. $z_{2i-1} = \sqrt{z_{2i} - c_2}$ is bounded if z_{2i} is also bounded.
Similarly, $z_{2i+1} = z_{2i}^2 + c_1$ is unbounded if z_{2i} is also unbounded. □

We can formulate a conjugate orbit (which can be used to study the boundedness of the alternate orbit) of the alternate dynamics for a given starting point $z_0 \in \mathbb{C}$ as follows.

Definition 3.4. [DRP09, Equation 5]: A quartic conjugate orbit $Q_{c_1c_2}(z_0)$ of a given starting point z_0 and given parameters $c_1, c_2 \in \mathbb{C}$ can be defined as the even terms of the alternate orbit $P_{c_1c_2}(z_0)$

$$Q_{c_1c_2}(z_0) = \{z_i | z_i = (z_{i-1}^2 + c_1)^2 + c_2\}$$

With the assistance of Proposition 3.3, we can have the following theorem.

Theorem 3.5. [DRP09, Theorem 2.2 and 2.3] *The boundedness of $Q_{c_1c_2}(z_0)$ implies the boundedness of $P_{c_1c_2}$ and therefore the quartic system*

$$f(z) = (z^2 + c_1)^2 + c_2$$

and the alternating system share the same filled Julia set.

This gives us a tool to analyze the boundedness and connectedness of a given alternate filled Julia set.

3.2. **Connectivities.** As stated in the previous section, every alternate filled Julia set has an equivalent quartic filled Julia set conjugate. We can hence investigate the connectivity of a given filled Julia set by analyzing the connectivity of the quartic conjugate. To explore the connectivity of a given (filled) Julia set, we need a conjecture that was proven by Scott Sutherland.

Theorem 3.6. [Sut14, Section 6]

(1) A given Julia set is connected iff all the orbits of the critical points are bounded.

(2) A given Julia set is totally disconnected (which is called Cantor dust) iff all the orbits of the critical points are unbounded. This kind of Julia sets is referred to as Fatou dust.

As we have shown in Theorem 3.5, the Julia set of the quartic system $F(z) = (z^2 + c_1)^2 + c_2$ is equivalent to the alternate Julia set. We can analyze the connectivity of the alternate filled Julia set by analyzing F. By solving the following equation

$$F'(z) = 2(z^2 + c_1) * 2z = 0,$$

we can get three critical points of F to be 0 and $\pm\sqrt{-c_1}$. By discussing the properties of the corresponding critical orbits, we can determine the connectivity of the alternate Julia set we are interested in.

THEOREM 3.7. [DRP09, Section 3.1]

(1) The alternate Julia set is connected if the orbits of 0 *and* $\pm\sqrt{-c_1}$ *are bounded.*
(2) The alternate Julia set is totally disconnected if the orbits of 0 *and* $\pm\sqrt{-c_1}$ *are bounded.*
(3) The alternate Julia set is disconnected if the orbit of either 0 *and* $\pm\sqrt{-c_1}$ *is bounded.*

The boundedness of a given alternate Julia set, therefore, can be determined by simulating the alternating dynamics in a computer program.

3.3. **Visualization algorithms and examples.** In this section, we move from the theoretical formulations of the Alternate Julia set to its visual representation. By leveraging computer programs, we can transform the mathematical concepts into vivid and intuitive images, offering a new perspective on the structure and complexity of the set. Since the behaviors of a filled Julia set can be modeled by the corresponding quartic function, it seems like the visualizations of the desired set can be easily shown with the JuliaSetPlot command in Mathematica. However, due to the lack of computational power, the command cannot be directly plotted in an online Mathematica notebook. As a result, we decided to provide a visualization algorithm that can be run in a Google Colab notebook. To simulate the dynamics, we apply the alternating functions to a set of initial points within a given range (in our case, we choose $a + bi$ for $a, b \in R$, $|a| < 1.5$, $|b| < 1.5$). We set a boundary of absolute distances between $f^n(z_0)$ and the origin for a given z_0. After that, we plot the number of iterations needed for a given point to go beyond the threshold. The pseudocode for the visualization algorithm can be found in Algorithm 1. We choose three different pairs of c_1 and c_2, representing different types of connectivities of the alternate Julia set. The visualization can be found in Figure 2, 3, and 4.

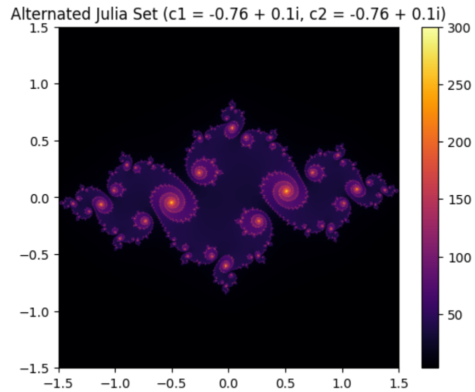

FIGURE 2. A totally disconnected alternate Julia set with $c_1 = -0.76 + 0.1i$, $c_2 = -0.76 + 0.1i$.

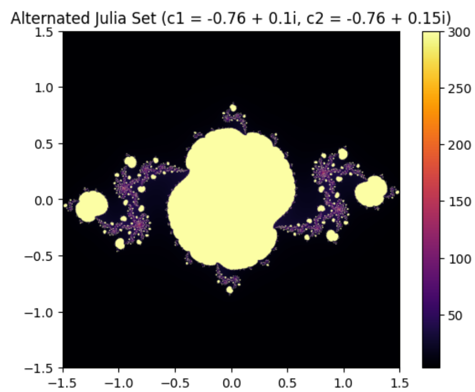

FIGURE 3. A disconnected alternate Julia set with $c_1 = -0.76 + 0.1i$, $c_2 = -0.76 + 0.15i$.

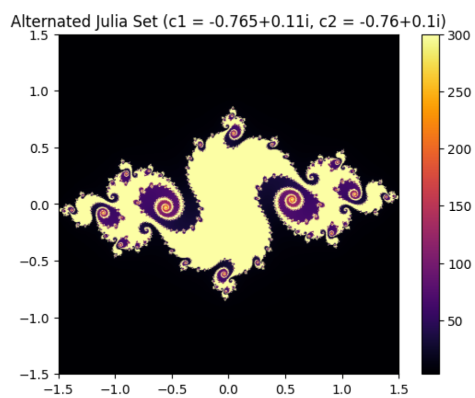

FIGURE 4. A connected alternate Julia set with $c_1 = -0.765 + 0.11i$, $c_2 = -0.76 + 0.1i$.

Algorithm 1 Alternate Julia sets visualization

Input: *width*, *height*, *zoom*, *center_x*, *center_y*, c_1, c_2, *max_iter*, *threshold*
Output: *image*[height][width]
for $y = 0$ to $height - 1$ **do**
  for $x = 0$ to $width - 1$ **do**
    $zx \leftarrow (x - \frac{width}{2}) \cdot zoom + center\_x$
    $zy \leftarrow (y - \frac{height}{2}) \cdot zoom + center\_y$
    $z \leftarrow zx + i \cdot zy$
    $iter \leftarrow 0$
    while $iter < max\_iter$ **and** $|z| < threshold$ **do**
      if *iter* is odd **then**
        $z \leftarrow z^2 + c_1$
      else
        $z \leftarrow z^2 + c_2$
      end if
      $iter \leftarrow iter + 1$
    end while
    $image[y][x] \leftarrow iter$
  end for
end for
return *image*

4. p-adic Mandelbrot sets and p-adic Julia sets

Having concluded our examination of the Alternate Julia set within the framework of complex dynamics, we now proceed to consider analogous constructions in the context of p-adic Mandelbrot and Julia sets. This transition from the complex to the p-adic setting highlights profound differences in topology and analytic behavior, offering new perspectives on dynamical systems. In the complex dynamical setting, the Mandelbrot set is defined by the connectivities of Julia sets of functions $f(z) = z^2 + c$ in the quadratic family. We can therefore formulate a definition of p-adic Mandelbrot sets of quadratic families with boundedness of orbits of the origin. To begin with, we need to define the p-adic analog of complex field.

Definition 4.1. The p-adic analog $\mathbb{C}_p$ of the complex field $\mathbb{C}$ is defined as

$$\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}_p}$$

where $\overline{\mathbb{Q}}_p$ is the algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$, and $\widehat{\overline{\mathbb{Q}}_p}$ denotes the completion of $\overline{\mathbb{Q}}_p$ with respect to the p-adic norm $|\cdot|_p$.

We can then define the p-adic Mandelbrot set in the field.

Definition 4.2. The Mandelbrot set $\mathcal{M}_p^2$ for quadratic families in $\mathbb{C}_p$ consists of the points where the sequence $\{f^n(0)\}$, generated by iterating the function

$$f : \mathbb{C}_p \to \mathbb{C}_p,\ f(z) = z^2 + c \text{ for a given } c \in \mathbb{C}_p$$

remains bounded.

However, the structure of $\mathcal{M}_p^2$ that corresponds to the quadratic family is simple in contrast to its complex-dynamical counterpart.

THEOREM 4.3. [Sil13, Page 2] $\mathcal{M}_p^2$ *is the unit disk around the origin.*

Proof. For the case $|c|_p \leq 1$, $|f(0)|_p = |c|_p \leq 1$. Suppose that for all $n \leq N \in \mathbb{N}$, there is a positive $r \in \mathbb{R}$ such that $|f^n(0)|_p \leq r$. Then,

$$|f^{n+1}(0)|_p = |f^n(0)^2 + c|_p$$

is bounded by $max(|c|_p, |f^n(0)|_p)$. Therefore, by induction, for all $|c|_p \leq 1$, c is in the Mandelbrot set. For $|c|_p > 1$, $|f^n(0)|_p$ grows exponentially with n, so any point outside the unit disk is not in the p-adic Mandelbrot set. $\square$

We want to study the more complex dynamics in the p-adic space by expanding the definition of Mandelbrot sets to a more general case. The reason why we only consider a single type of orbit $\{f^n(0)\}$ is that each function in the quadratic family only has one critical point at the origin, while higher order polynomials have multiple critical points and, therefore, more diverse types of dynamical and connectivity behaviors determined by Theorem 3.6. With this in mind, we can define the Mandelbrot set of higher-order polynomials in $\mathbb{C}_p$.

DEFINITION 4.4. For a function family

$$f(z) = z^n + c_1 z^{n-2} + c_2 z^{n-3} + ... + c_{n-2} z + c_{n-1}$$

(there is a conjugate of every single order n polynomial in the family), we can define the Mandelbrot set $\mathcal{M}_p^n$ as a set of vectors $v \in \mathbb{C}_p^{n-1}$ such that all critical orbits of f are bounded in $\mathbb{C}_p$.

To know whether the structure of a given p-adic Mandelbrot set is simple, we can use the following theorem to conclude that some Mandelbrot sets share the same geometric property.

THEOREM 4.5. [Sil13, Theorem 1] *If* $p \geq n$, *then* $\mathcal{M}_p^n$ *is a poly-disk*

$$\{(c_1, c_2, ..., c_{n-1}) \text{ where } |c_k|_p \leq 1 \text{ for } 1 \leq k \leq n-1\}.$$

This theorem can also be used to explain why the Mandelbrot set of the quadratic family (where $n = 2$) is a unit disk. Some calculations and examples in [Sil13, Page 2 and 3] and [And13, Figure 1, 2, and 3] also show that the complexity of some $\mathcal{M}_p^n$ with $n > p$ can be comparable to its counterpart in the complex-dynamical setting.

4.1. **Examples.**

[Sil13, Example 2] Consider the function

$$f : \mathbb{C}_2 \to \mathbb{C}_2,\ f(z) = z^3 - \frac{3}{4}z - \frac{3}{4}.$$

By solving $f'(z) = 3z^2 - \frac{3}{4} = 0$, we find that the critical points of f are $\pm\frac{1}{2}$. The critical orbit at $\frac{1}{2}$ is bounded:

$$\tfrac{1}{2} \xmapsto{f} -1 \xmapsto{f} -1\ .$$

We can also know that the critical orbit at $-\frac{1}{2}$ is also bounded:

$$-\tfrac{1}{2} \xmapsto{f} -\tfrac{1}{2}.$$

Therefore, the pair $(-\frac{3}{4}, -\frac{3}{4}) \in \mathcal{M}_2^3$.

Since $|\frac{3}{4}|_2 = 4$, the Mandelbrot set $\mathcal{M}_2^3$ is not a unit polydisk.
Similarly, we can define the p-adic analog of the filled Julia set in $\mathbb{C}_p$.

Definition 4.6. For a given function $F : \mathbb{C}_p \to \mathbb{C}_p$, we can define the p-adic filled Julia set $Q_p(F)$ as the set of points z where the orbit of z under F is bounded.

The p-adic filled Julia set is a useful tool for studying the arithmetic dynamics of a given function in the p-adic field.

5. p-adic alternate Julia sets

In this section, we extend the definitions and results from [DRP09], [And13], and [Sil13] to propose the definition of p-adic alternate filled Julia sets and some basic properties of them. Based on the previous discussion, we can formulate the definition of the orbits of two given alternating functions

$$F_1 : \mathbb{C}_p \to \mathbb{C}_p$$

and

$$F_2 : \mathbb{C}_p \to \mathbb{C}_p.$$

For our interest, we only discuss the system consisting of the quadratic family.

Definition 5.1. For two given functions

$$F_1 : \mathbb{C}_p \longrightarrow \mathbb{C}_p$$

$$z \longmapsto z^2 + c_1 \text{ and}$$

$$F_2 : \mathbb{C}_p \longrightarrow \mathbb{C}_p$$

$$z \longmapsto z^2 + c_2$$

where c_1 and c_2 are constants in $\mathbb{C}_p$, we define the orbit $P_p^{c_1c_2}(z_0)$ of a given starting point $z_0 \in \mathbb{C}_p$ as the set $\{z_i | z_0 = z_0, z_{2i} = F_2(z_{2i-1}), z_{2i+1} = F_1(z_{2i})\}$.

We can also define the p-adic alternate Julia set and Mandelbrot set in this manner.

Definition 5.2. The p-adic alternate Julia set $K_p^{c_1c_2}$ of F_1 and F_2 is the set of points z where its orbit $P_p^{c_1c_2}(z)$ is bounded under the p-adic distance metric.

Definition 5.3. The p-adic Mandelbrot set of alternate Julia sets is the set of (c_1, c_2) such that $K_p^{c_1c_2}$ is connected.

In contrast to the normal quadratic p-adic set that only has one critical orbit, the structure of the alternate p-adic Julia set is more complex. To describe the structure of the alternate p-adic Julia set, we have the following theorem.

Theorem 5.4. *The even terms z_{2i} of the orbit $P_p^{c_1c_2}(z_0)$ are bounded iff the odd terms z_{2i+1} of the orbit are also bounded.*

Proof. Suppose that for all $n \in \mathbb{N}$, $|z_{2n}| \leq r$ where r is a non-negative real number. We have

$$\begin{aligned} |z_{2n+1}|_p &= |z_{2n}^2 + c_1|_p \\ &\leq max(|z_{2n}^2|_p, |c_1|_p) \\ &\leq max(r^2, |c_1|_p). \end{aligned}$$

Therefore, all odd terms are bounded. If the set of all even terms $\{z_{2n}\}$ is unbounded,

$$\begin{aligned}\sup|z_{2n+1}^2| &= \sup|z_{2n}^2 + c_1|_p \\ &= \sup|z_{2n}^2|_p \to \infty.\end{aligned}$$

As a result, the odd terms are also unbounded if the even terms are unbounded. Similarly, by symmetry we can show that if $|z_{2n-1}|$ is bounded, $|z_{2n}| = |z_{2n-1}^2 + c_2|$ is also bounded and if $|z_{2n-1}|$ is unbounded then $|z_{2n}| = |z_{2n-1}^2 + c_2|$ should also be unbounded. □

We can also form a quartic p-adic Julia set similar to what we have done in the complex case.

Definition 5.5. The quartic p-adic filled Julia set $Q_p^{c_1 c_2}$ corresponding to the p-adic alternate filled Julia set can be defined as the p-adic filled Julia set $Q_p(F)$ of the function $F = F_2 \circ F_1$.

Theorem 5.6. *$Q_p^{c_1 c_2}$ is the p-adic alternate Julia set of F_1 and F_2.*

Proof. From Theorem 5.4, we know that odd terms and even terms of the orbit at a given point have the same boundedness properties. Therefore, the set of points where the even terms of the orbits are bounded (which is $Q_p^{c_1 c_2}$) is equivalent to the p-adic alternate Julia set where both even terms and odd terms of the orbits are bounded. □

To analyze the connectivity of the alternate Julia set, we can use the formulation of the p-adic Mandelbrot set. We know from Theorem 4.5 that for all $p \geq 4$, the p-adic Mandelbrot set of order 4 polynomial is a poly-disk. On the other hand, for $p \leq 3$, the structure of the Mandelbrot set can be extremely complex, making it difficult to infer the connectivity properties of the p-adic Julia set without running computer simulations. We can construct the conjugated quartic polynomial by selecting c_1 and c_2 and we can use the polynomial to determine the connectivity of the p-adic filled Julia set with the help of the p-adic Mandelbrot set. The result can be expanded to finite order polynomial functions F_1 and F_2 and their p-adic Julia set of the conjugate function $F = F_2 \circ F_1$ by adjusting the value of p based on Theorem 4.5.

References

[And13] Jacqueline Anderson. Bounds on the radius of the p-adic Mandelbrot set. *Acta Arith.*, 158(3):253–269, 2013.

[Bou01] Paul Bourke. Julia set. https://paulbourke.net/fractals/juliaset/, 2001. Accessed: 2025-04-28.

[DBR13] Marius-F. Danca, Paul Bourke, and Miguel Romera. Graphical exploration of the connectivity sets of alternated Julia sets. *Nonlinear Dynam.*, 73(1-2):1155–1163, 2013.

[DRP09] Marius-F. Danca, M. Romera, and G. Pastor. Alternated Julia sets and connectivity properties. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 19(6):2123–2129, 2009.

[Sil13] Joseph H. Silverman. What is. . . the p-adic Mandelbrot set? *Notices Amer. Math. Soc.*, 60(8):1048–1050, 2013.

[Sut14] Scott Sutherland. An introduction to julia and fatou sets. In C. Bandt, M. Barnsley, R. Devaney, K. J. Falconer, V. Kannan, and V. Kumar, editors, *Fractals, Wavelets, and their Applications*, pages 37–60. Springer-Verlag, 2014. Based on lectures at the International Conference on Fractals and Wavelets, Rajagiri School of Engineering and Technology, Kerala, India, November 2013. Available at https://www.math.stonybrook.edu/~scott/Papers/India/Fatou-Julia.pdf.

Massachusetts Institute of Technology
Email address: rxwangtw@mit.edu