p-ADIC APPROACHES TO THE DYNAMICAL MORDELL–LANG CONJECTURE

ANDREW TUNG

Abstract. In this expository paper, we introduce the Dynamical Mordell–Lang Conjecture, an important conjecture in arithmetic dynamics. After providing some general background, we state and sketch a proof of the Skolem–Mahler–Lech Theorem about linear recurrence sequences, which can be viewed as a special case of the Dynamical Mordell–Lang Conjecture. We also indicate how the techniques used to prove the Skolem–Mahler–Lech Theorem can be extended to special cases of the Dynamical Mordell–Lang Conjecture.

1. Introduction

The Dynamical Mordell–Lang Conjecture is an important conjecture in arithmetic dynamics which describes how dynamical systems interact with a fixed subvariety. Consider the following setup. Let X be a quasiprojective variety over $\mathbb{C}$, and suppose we have endomorphism Φ on X. This creates a dynamical system on X by repeatedly iterating Φ. Let Φ^n for n a nonnegative integer denote the n-th iterate of Φ. Fix some point $\alpha \in X$, and let $V \subset X$ be some subvariety. We are interested in how the orbit of α, defined to be $\{\Phi^n(\alpha) : n \geq 0\}$, interacts with V.

One particularly simple case is when V is a *periodic subvariety* of X, which means that there exists N with $\Phi^N(V) \subset V$. Furthermore, suppose there exists some m such that $\Phi^m(\alpha) \in V$. For example, we could have

$$\Phi : \mathbb{A}^2 \to \mathbb{A}^2$$
$$(x, y) \mapsto (y, x^2)$$

with $V = \{x = 0\}$ and $\alpha = (1, 0)$, in which case $N = 2$ and $m = 0$. Then since V is a periodic subvariety, $\Phi^{Nk+m}(\alpha) = (\Phi^N \circ \cdots \circ \Phi^N)(\Phi^m(\alpha)) \in V$ for all k. In other words, there is an arithmetic progression $\{Nk + m : k \in \mathbb{N}\}$ of indices n for which $\Phi^n(\alpha) \in V$. By imposing some constraint on the variety V such that it has a well-behaved structure with respect to Φ, we have found that the set of indices for which $\Phi^n(\alpha) \in V$ also has a nice structure. The Dynamical Mordell–Lang Conjecture is that the converse holds.

Conjecture 1.1 (Dynamical Mordell–Lang). [BGKT12] Let X be a quasiprojective variety over $\mathbb{C}$, and let Φ be an endomorphism of X, $V \subset X$ be a closed subvariety, and $\alpha \in X$. Then the set of integers $n \geq 0$ such that $\Phi^n(\alpha) \in V$ is the finite union of arithmetic progressions, possibly of common difference zero.

The Dynamical Mordell–Lang Conjecture gets its name from the original Mordell–Lang Conjecture, now a theorem by Faltings [Fal94]. The dynamical case specializes

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to Falting's Theorem when X is an abelian variety and Φ is the translation-by-P map for some $P \in X(\mathbb{C})$.

The Dynamical Mordell–Lang Conjecture was first proposed in 2009 by Ghioca and Tucker in [GT09], motivated by more general goals of Zhang in extending results from arithmetic geometry to dynamical systems [Zha06]. Conjecture 1.1 is known in a few special cases, like the following.

Theorem 1.2. *The Dynamical Mordell–Lang Conjecture is known if*

(1) [BGT10] *the map Φ is an étale endomorphism, or*
(2) [Xie23, Theorem 1.4] *the map Φ is an endomorphism of $\mathbb{A}^2$ defined over $\mathbb{C}$*

Although Conjecture 1.1 is stated over $\mathbb{C}$, the Dynamical Mordell–Lang Conjecture in its most general form can be formulated over any field K, for example in [BGT16, Conjecture 3.1.1.1]. Such considerations have been important as a building block for establishing results over $\mathbb{C}$. For example, in [Xie17] Xie proved a statement analogous to statement (2) in Theorem 1.2 over $\overline{\mathbb{Q}}$, which forms a large part of the proof of (2) in [Xie23]. More recently, some work has also been done over fields of positive characteristic, although the problem is significantly more difficult in this case [XY25].

In this expository paper, we hope to give an idea of some of the techniques currently used to approach the Dynamical Mordell–Lang Conjecture. One of these, known as the "p-adic arc lemma" uses so-called p-adic analytic parameterizations of the orbits $\{\Phi^n(\alpha)\}$. In Section 2, we introduce this technique by stating and sketching a proof of the Skolem–Mahler–Lech Theorem, a classical theorem about arithmetic progressions. Surprisingly, the techniques used to prove the Skolem–Mahler–Lech Theorem are broadly the same as those used to prove case (1) in Theorem 1.2. In Section 3 we show how to translate the Skolem–Mahler–Lech Theorem into more geometric language and indicate how the method of proof generalizes to more general settings (e.g. when X is affine and Φ is an automorphism).

2. The Skolem–Mahler–Lech Theorem

The Skolem–Mahler–Lech Theorem is a classical result about linear recurrence sequences, which we will see in Section 3 are a special case of the setting of the Dynamical Mordell–Lang Conjecture. Moreover, the method of proof of the Skolem–Mahler–Lech Theorem is very closely related to current approaches to the Dynamical Mordell–Lang Conjecture. In this section, we explain the result and give a brief outline of the proof.

Definition 2.1. A linear recurrence sequence over a field K is a sequence $a_0, a_1, \ldots$ with each $a_i \in K$ such that there exists $b_i \in K$ so that the sequence satisfies a recurrence

$$a_n = \sum_{i=1}^{m} b_i a_{n-i}$$

for $n \geq m$.

Example 2.2. The Fibonacci sequence ($F_0 = 0$, $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, and so on) is a linear recurrence sequence over $\mathbb{C}$. It satisfies the recurrence $F_n = F_{n-1} + F_{n-2}$.

One natural question to ask about linear recurrence sequences is to examine their dynamical behavior. For example, some sequences like the Fibonacci sequence do not repeat at all after the first few terms. Others are periodic, like the following example.

Example 2.3. Consider the sequence defined by $a_0 = 0$ and $a_1 = 1$ and the recurrence $a_n = a_{n-1} - a_{n-2}$. The first few values of this sequence are:

n	1	2	3	4	5	6	7	8	9	$\cdots$
a_n	0	1	1	0	-1	-1	0	1	1	$\cdots$

As we can see, $a_7 = a_1$ and $a_8 = a_2$, so since the value of a_n only depends on the previous two values, the sequence repeats after the 6th entry.

One natural question to ask is the following.

Question 2.4. *Given a linear recurrence sequence a_n and a constant c, for which n does $a_n = c$?*

For example, for the Fibonacci sequence we might ask when $F_n = 1$, in which case the answer is $n = 1$ and $n = 2$ only. For the sequence in Example 2.3, we might ask when $a_n = 1$, in which case the answer is all n for which $n \equiv 2, 3 \bmod 6$. From these examples, we might conjecture that in general, the set of n for which $a_n = c$ has a nice structure: the set can be divided into a finite number of pieces, each of which is a single index or an arithmetic progression. As it turns out, this is correct.

Theorem 2.5 (Skolem–Mahler–Lech Theorem). [BGT16, Theorem 2.5.4.1] *Let a_n be a linear recurrence sequence over $\mathbb{C}$. Then the set of n for which $a_n = 0$ is a union, possibly empty, of finitely many arithmetic progressions, possibly of common difference zero.*

Remark 2.6. Technically, Theorem 2.5 does not quite answer Question 2.4: in Question 2.4 above we considered the question of when $a_n = c$ for arbitrary c, but Theorem 2.5 only answers the question for $c = 0$. As it turns out, these are equivalent, because of the following fact.

Fact 2.7. [BGT16, Lemma 2.5.1.3] *Let a_n and b_n be two linear recurrence sequences. Then $c_n := a_n + b_n$ is also a linear recurrence sequence.*

The question of when a linear recurrence sequence a_n equals c can be reduced to when the sum of the sequences a_n and the constant sequence $b_n := -c$ equals zero. (The constant sequence is, of course, a linear recurrence sequence with the recurrence relation $b_n = b_{n-1}$.) By Fact 2.7, this sum is another linear recurrence sequence, so by the Skolem–Mahler–Lech Theorem, the set of n for which it equals zero is the finite union of arithmetic progressions.

Surprisingly, the proof of the Skolem–Mahler–Lech Theorem uses methods from p-adic analysis. This ultimately stems from the fact that $\mathbb{Z}$ embeds into $\mathbb{Q}_p$ as a compact set. We will sketch the proof below; a full proof can be found in the literature in Section 2.5 [BGT16], with the original proof from a paper of Hansel's [Han85]. The proof consists of three main steps:

(1) Find a closed formula for the nth term a_n, of the form

$$a_n = \sum_{i=1}^{m} f_i(n) r_i^n,$$

for some polynomials $f_i \in \mathbb{C}[x]$ and complex numbers $r_i \in \mathbb{C}$.

(2) Find a suitable prime p so that the entire formula in step 1 embeds "nicely" in $\mathbb{Z}_p$, in the sense that the coefficients of the f_i and all of the r_i lie in $\mathbb{Z}_p$, together with some other conditions.

(3) Consider a suitable p-adic analytic function $g_k(z)$ using the a_n. Then use results from p-adic analysis to analyze the zeros of $g_k(z)$.

In the three subsections that follow, we introduce the tools necessary for each step, and in Subsection 2.4 we provide a complete proof sketch of the Skolem–Mahler–Lech Theorem putting all the results together.

2.1. **Formula for a Linear Recurrence Sequence.** For the first step of the proof, we illustrate how one finds a closed formula for a linear recurrence sequence with the example of the closed form for the Fibonacci sequence, which may perhaps already be familiar to the reader.

Example 2.8 (closed form for the Fibonacci sequence). Using the recurrence $F_{n+2} = F_{n+1} + F_n$, we see that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix} = \begin{pmatrix} F_{n+2} \\ F_{n+1} \end{pmatrix},$$

so using the initial values $F_1 = 1$ and $F_0 = 0$, it follows that

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}.$$

In particular, to find a formula for the nth term F_n, it suffices to analyze how the powers of a certain matrix behave. As it turns out, the matrix in this case is diagonalizable, as follows:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{pmatrix} \begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}^{-1},$$

so

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \varphi & 0 \\ 0 & \overline{\varphi} \end{pmatrix}^n \begin{pmatrix} \varphi & \overline{\varphi} \\ 1 & 1 \end{pmatrix}^{-1}.$$

Expanding this, we get a closed form for the Fibonacci numbers, namely

$$F_n = \frac{1}{\sqrt{5}}(\varphi^n - \overline{\varphi}^n).$$

This takes the form given in step 1 of the proof, where the r_i are φ and $\overline{\varphi}$, and the polynomials f_i are constants $f_1 = \frac{1}{\sqrt{5}}$ and $f_2 = -\frac{1}{\sqrt{5}}$.

However, in general things need not be so simple. In particular, the characteristic polynomial of the relevant matrix might have repeated roots, and hence the matrix might not be diagonalizable. But as the following example illustrates, it is still possible to find a closed form.

Example 2.9. Consider the linear recurrence sequence a_n defined by $a_0 = 1$, $a_1 = 2$, $a_2 = 2$, and $a_n = 3a_{n-1} - 4a_{n-3}$. As before we can write down a matrix relation for the coefficients:

$$\begin{pmatrix} 3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_{n-2} \\ a_{n-3} \end{pmatrix} = \begin{pmatrix} a_n \\ a_{n-1} \\ a_{n-2} \end{pmatrix}.$$

This time, however, the matrix $\left(\begin{smallmatrix} 3 & 0 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix}\right)$ is not diagonalizable: its characteristic polynomial is $x^3 - 3x^2 + 4$, which has a multiple root at $x = 2$ and a single root at $x = -1$. However, it is still possible to find a closed form: we guess a solution of the form

$$a_n = (An + B)2^n + C(-1)^n,$$

for some $A, B, C \in \mathbb{C}$. Since 2 is a double root of the characteristic polynomial, instead of having a constant coefficient in front of 2^n, we use a linear polynomial in n. (In general, for a root r of multiplicity k, we would have a degree $k-1$ polynomial in n as the coefficient of r^n.) It is not hard to check that a_n of this form satisfy the recurrence relation; in general this follows from the fact that a double root of a polynomial is also a root of its derivative. To find a closed form for a_n it suffices to solve for A, B, and C such that $a_0 = 1$, $a_1 = 2$, and $a_2 = 2$. This gives $A = -\frac{1}{3}$, $B = \frac{11}{9}$, and $C = -\frac{2}{9}$, so our closed form is

$$a_n = \left(-\frac{1}{3}n + \frac{11}{9}\right)2^n - \frac{2}{9}(-1)^n.$$

For general linear recurrence sequences, a similar procedure works: find the characteristic polynomial and its roots, and guess a solution which is a sum of exponential functions of the roots, possibly with polynomial coefficients in the case of a multiple root. The general statement is the following.

Theorem 2.10. [BGT16, Proposition 2.5.1.4] *Let a_n be a linear recurrence sequence over $\mathbb{C}$. Then there exist polynomials $f_i \in \mathbb{C}[z]$ and $r_i \in \mathbb{C}$ such that $a_n = \sum_{i=1}^m f_i(n) r_i^n$.*

2.2. **An Embedding Lemma.** Having found a closed form for the terms of the sequence, the next two steps in the proof view this formula as an analytic function. To motivate this, consider the simplest case where a_n is some linear recurrence sequence over $\mathbb{R}$ and the roots of the corresponding characteristic polynomial are all real and distinct. Then we can write $a_n = \sum_{i=1}^m c_i r_i^n$ for some $c_i, r_i \in \mathbb{R}$. As a function in n, this is real analytic, since it is a finite linear combination of exponential functions. We would like to find $x \in \mathbb{Z}$ such that the function

$$f(x) = \sum_{i=1}^{m} c_i r_i^x$$

is zero. However we do not have enough control over the zeros, since $\mathbb{Z}$ is not compact in $\mathbb{R}$. One can imagine many real analytic functions with infinitely many zeros, scattered in arbitrary ways.

To gain more control over the roots of analytic functions, we move to the p-adics $\mathbb{Q}_p$. The key property is that, unlike the integers in $\mathbb{R}$, the integers in $\mathbb{Q}_p$ are compact, since they embed into $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$. We will see in the third step in the proof that we can control the roots of analytic functions over $\mathbb{C}_p$, the algebraic closure of $\overline{\mathbb{Q}_p}$. In particular, the zeros are isolated, just as in $\mathbb{C}$, so over a compact set there can only be finitely many of them.

As suggested earlier, the second step of the proof of the Skolem–Mahler–Lech Theorem relies on the following lemma, first proven by Lech in [Lec53]. Since the proof is rather technical, we omit it here.

Lemma 2.11. [BGT16, Proposition 2.5.3.1] *Suppose K is a field which is finitely generated over $\mathbb{Q}$, and let $S \subset K$ be a finite subset. Then there exist infinitely many primes p so that there exists an embedding $\sigma : K \to \mathbb{Q}_p$, and moreover $\sigma(S) \subset \mathbb{Z}_p$.*

2.3. **Some p-adic Analysis.** The third step in the proof the Skolem–Mahler–Lech Theorem relies on the following facts from p-adic analysis about analytic functions.

Lemma 2.12. [BGT16, Lemma 2.3.4.2] *Let $a \in \mathbb{C}_p$ with $|a-1|_p < p^{-1/(p-1)}$. Then the function $f(z) = a^z$ defined by $\exp(\log(1+(a-1))z)$ is analytic on $|z|_p < 1$.*

Proof (sketch). We recall the following facts about the p-adic logarithm and the p-adic exponential, which can be found in, for example, [Gou20, Section 5.7 & 7.1]. The p-adic exponential function $\log(z)$ can be defined using a power series expansion around $z = 1$, as follows:

$$\log(1+z) = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{z^i}{i}.$$

It turns out that $\log(1+z)$ is analytic for $|z|_p < 1$, or in other words $\log(z)$ is analytic for $|z-1|_p < 1$. Similarly we can define the p-adic exponential function

$$\exp(z) := \sum_{i=0}^{\infty} \frac{z^i}{i!}$$

which is analytic for all z with $|z|_p < p^{-1/(p-1)}$. Then the lemma follows easily using the construction given in the statement and facts about composition of analytic functions. □

Lemma 2.13. [BGT16, Lemma 2.3.6.1] *Suppose $F(z) = \sum_{i=0}^{\infty} a_i z^i$ is a power series in $\mathbb{C}_p$ which is convergent around some open disc around $z = 0$. If F is not identically zero, then the zeros of F are isolated.*

We omit the proof of Lemma 2.13 for the sake of brevity. By applying Lemma 2.13 to the compact region $|z|_p \leq 1$, we find that the zeros of F which lie in $\mathbb{Z}_p$ are isolated. This is more commonly known as Strassmann's Theorem [Str28].

Theorem 2.14 (Strassmann). [Cas78, Chapter 3, Exercise 21] *Suppose $F(z) = \sum_{i=0}^{\infty} a_i z^i$ is a power series with $a_i \in \mathbb{Q}_p$, and suppose $a_i \to 0$. If F is not identically zero, then there are only finitely many $a \in \mathbb{Z}_p$ with $F(a) = 0$.*

2.4. **Proof of the Theorem.** With all these tools in place, we can finally sketch the proof of the Skolem–Mahler–Lech Theorem.

Proof sketch (of Theorem 2.5). **Step 1:** From Theorem 2.10, we can write

$$a_n = \sum_{i=1}^{n} f_i(n) r_i^n$$

for some $f_i \in \mathbb{C}[z]$ and $r_i \in \mathbb{C}$.

Step 2: From Lemma 2.11, we can choose a prime p so that the coefficients of the f_i and the r_i lie in $\mathbb{Z}_p$, and moreover, each of the r_i is nonzero mod p (i.e. $|r_i|_p = 1$). This is because there are infinitely many p satisfying the first constraint, and the second constraint only removes finitely many p.

Step 3: Now observe that by Fermat's little theorem,

$$|r_i^{p-1} - 1|_p \leq 1/p$$

for all i. Let

$$N = \begin{cases} 2 & \text{if } p = 2 \\ p-1 & \text{if } p \neq 0 \end{cases}.$$

Then

$$|r_i^N - 1|_p < p^{-1/(p-1)},$$

since for p odd, $p^{-1/(p-1)} < \frac{1}{p}$, while for p even, $|r_i^2 - 1|_2 \leq \frac{1}{4} < \frac{1}{2} = p^{-1/(p-1)}$.

Now let k be a nonnegative integer at most N. Define the function

$$g_k(z) := \sum_{i=1}^{m} f_i(k+Nz)(r_i)^{k+Nz} = \sum_{i=1}^{m} f_i(k+Nz) r_i^k (r_i^N)^z,$$

which is just a_{k+Nz} by the closed formula from step 1. By Lemma 2.12, the exponential function $(r_i^N)^z$ is analytic. Since the f_i are just polynomials and r_i^k is a constant, it follows that $g_k(z)$ is analytic. But by Theorem 2.14, $g_k(z)$ is either identically zero or has finitely many zeros in the disc $\{z \in \mathbb{C}_p : |z|_p \leq 1\}$.

In the former case, this means $g_k(z) = 0$ for all $z \in \mathbb{C}_p$, so certainly $g_k(z) = 0$ for all integers z. This means $a_{k+Nz} = 0$ for all $z \geq 0$, or in other words we have an arithmetic progression of indices corresponding to zeros of a. In the latter case, since every integer $z \in \mathbb{Z}$ has p-adic absolute value at most 1, we have only finitely many integers z so that $a_{k+Nz} = 0$. In either case, the indices i of the form $k + Nz$ with $a_i = 0$ are a finite union of arithmetic progressions, possibly of common difference zero.

Every index $i \geq 0$ can be written as $k + Nz$ for some $0 \leq k < N$ and integer $z \geq 0$. Applying the same analysis as above for every k, since there are only finitely many possible k, we obtain that the indices i with $a_i = 0$ are a finite union of arithmetic progressions. □

3. Generalizations of Skolem's Method

At first glance it is not clear how the Skolem–Mahler–Lech Theorem has anything to do with the Dynamical Mordell–Lang Conjecture. The relationship between the two becomes clearer after we reformulate the Skolem–Mahler–Lech Theorem in more geometric language, as follows. Let $(b_1, \ldots, b_m) \in \mathbb{C}^m$ be a m-tuple of complex numbers. Consider the affine space $\mathbb{A}^m$ with the Zariski topology and the function

$$\Phi : \mathbb{A}^m \to \mathbb{A}^m$$
$$(x_1, \ldots, x_m) \mapsto (x_2, \ldots, x_m, b_1 x_m + \cdots + b_m x_1)$$

Let $V = \{x_1 = 0\}$ be the hyperplane in $\mathbb{A}^m$ with first coordinate zero, and let $\alpha \in \mathbb{A}^m$. The Dynamical Mordell–Lang Conjecture in this case states that the set of n such that $\Phi^n(\alpha) \in V$ should be the finite union of arithmetic progressions. In fact, this statement is exactly the Skolem–Mahler–Lech Theorem. Let a_n be a linear recurrence sequence, with recurrence relation $a_n = b_1 a_{n-1} + \cdots + b_m a_{n-m}$. Suppose that $(a_0, \ldots, a_{m-1}) = \alpha$ are the starting values for the recurrence. Then $\Phi^n(\alpha) = (a_n, \ldots, a_{n+m-1})$, so the question of when $\Phi^n(\alpha) \in V$ is just the question of when $a_n = 0$. By the Skolem–Mahler–Lech Theorem, we know that the answer is a finite union of arithmetic progressions.

In fact, the connection between the Skolem–Mahler–Lech Theorem and the Dynamical Mordell–Lang Conjecture is much deeper than just its translation into dynamical language, which is perhaps unimpressive. Consider the following result, which is the Dynamical Mordell–Lang Conjecture in the case of linear maps.

THEOREM 3.1. [BGT16, Theorem 4.1.0.4] *Suppose $\Phi : \mathbb{A}^m \to \mathbb{A}^m$ is a linear automorphism, in the sense that it is given by*

$$\Phi((x_1, \ldots, x_m)) = (f_1(x_1), \ldots, f_m(x_m))$$

for some linear functions $f_i \in \mathbb{C}[z]$. Let $V \subset \mathbb{A}^m$ be some subvariety, and let $\alpha \in \mathbb{A}^m$. Then the set of $n \geq 0$ so that $\Phi^n(\alpha) \in V$ is the finite union of arithmetic progressions.

Proof. First observe that we can assume $V = V(F)$ for a single polynomial $F \in \mathbb{C}[x_1, \ldots, x_m]$. This is because if $V = V((F_1, \ldots, F_k)) = \cap_i V(F_i)$ then the set of $n \geq 0$ such that $\Phi^n(\alpha) \in V$ is the intersection of the sets of $n \geq 0$ such that $\Phi^n(\alpha) \in V(F_i)$. If each of these sets is a finite union of arithmetic progressions, then since the intersection of two arithmetic progressions is an arithmetic progression (or empty), we are done.

Now we observe the following fact about iterating linear functions. If $f(z) = az + b$ is a linear function, then by simple calculation

$$f^n(z) = \begin{cases} a^n z + \left(\frac{a^n - 1}{a - 1}\right) b & \text{if } a \neq 1 \\ z + nb & \text{if } a = 1 \end{cases}.$$

In particular, in either case the formulas are sums of polynomials in n times exponentials in n. Hence we can write

$$F(f_1^n(x_1), \ldots, f_m^n(x_m)) = \sum_{i=1}^{\ell} g_i(n) r_i^n$$

for some polynomials $g_i \in \mathbb{C}[n]$ and $r_i \in \mathbb{C}$. Since this is the same form as the formula for a linear recurrence sequence, we can use the same second two steps from the proof of the Skolem–Mahler–Lech Theorem, as these did not use anything other than the form of the formula. So we conclude $\sum_{i=1}^{\ell} g_i(n) r_i^n$ is zero for n lying in a finite union of arithmetic progressions, as desired. □

A more general result was proven by Bell in [Bel06] for automorphisms of affine varieties, using very similar techniques.

THEOREM 3.2. [Bel06, Theorem 1.3] *Let X be an affine variety over $\mathbb{C}$, Φ be an automorphism of X, and $\alpha \in X$. Let V be a subvariety of X. Then the set of $n \in \mathbb{Z}$ such that $\Phi^n(\alpha) \in V$ is the finite union of arithmetic progressions.*

Proof sketch. The proof consists of four main steps:

(1) Reduce the problem to the case where $X = \mathbb{A}^n$, using a result of Srinivas [Sri91] on embedding dimensions.
(2) Find p such that the coefficients of the automorphism Φ, the defining equations of V, and point α embeds into $\mathbb{Q}_p$, with integer coefficients.
(3) Split the orbit $\{\Phi^n(\alpha)\}$ into pieces in arithmetic progression and find p-adic analytic parameterizations of each piece.
(4) For each parameterization, argue by Strassmann's theorem (Theorem 2.14) that it is either identically zero (hence an arithmetic progression of n satisfies $\Phi^n(\alpha) \in V$) or has only finitely many zeros.

□

Very broadly, the approach is the same as the proof of the Skolem–Mahler–Lech Theorem: the second step here corresponds to the second step in the previous proof, and the last step corresponds to the third step in the previous proof. However in this proof the third step is significantly more complicated. Whereas in the case of linear recurrence sequences it is possible to find a p-adic analytic function for the nth iterate of a generic point (i.e. regardless of the initial values of the recurrence), in this case it turns out that no such formula holds. This is because general automorphisms of $\mathbb{A}^n$ are quite complicated, unlike the linear automorphisms considered in Theorem 3.1. However, for fixed points α it is still possible to write down these analytic parameterizations, although it requires a lot of work.

In their paper [BGT10, Theorem 3.3], Bell, Ghioca, and Tucker use a broadly similar approach to prove the case of the Dynamical Mordell–Lang Conjecture for so-called étale (flat and unramified) maps. The key difficulty in this proof is similar to the difficult in Theorem 3.2, which is finding a p-adic analytic parameterization of the orbits. A promising future approach to proving the Dynamical Mordell–Lang Conjecture in more general cases is to find these parameterizations for larger classes of maps [BGT16, Section 1.6].

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Massachusetts Institute of Technology
Email address: `atung@mit.edu`