THE LOCAL-GLOBAL PRINCIPLE IN DYNAMICS OF QUADRATIC POLYNOMIALS

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ABSTRACT. In this paper we will describe progress that has been made on relating the structure of periodic and preperiodic points of quadratics in $\mathbb{Q}$ to this structure in $\mathbb{Q}_p$ for primes p.

CONTENTS

1. INTRODUCTION

Let $f_c(X)$ be the quadratic polynomial $X^2 + c$, for $c \in \mathbb{Q}$. Note that $f_c(X)$ can represent the dynamics of any quadratic $aX^2 + bX + c$ by using a linear conjugation on X, which preserves dynamical properties. Define $f_c^n(X)$ to be $f \circ f \circ \cdots \circ f(X)$, with the composition performed n times.

DEFINITION 1.1. We say that x is *periodic* if there exists $m \in \mathbb{N}$ such that $f_c^m(x) = x$, and x has *period* n if n is the minimal natural number such that $f_c^n(x) = x$.[1] We say that x is *preperiodic* if $f_c^i(x)$ is periodic for some $i \in \mathbb{N}$.

In $\mathbb{Q}$, there is a main open conjecture about the periodic points of f_c:

CONJECTURE 1.2 (Flynn–Poonen–Schaefer). [FPS97] There are no period n points of f_c in $\mathbb{Q}$ with $n > 3$.

In Section 2, we will explore what progress has been made towards resolving Conjecture 1.2; in particular Conjecture 1.2 has been solved with $n \leq 6$.[2] We will also look at some of the implications of Conjecture 1.2 for the structure of preperiodic points in $\mathbb{Q}$. While Conjecture 1.2 remains open, attempts at solving

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[1] Note that this is sometimes referred to as *exact period* or *primitive period* in other texts.

[2] With $n = 6$, this solution is contingent on the Birch and Swinnerton-Dyer conjecture.

or generalizing cases of it have motivated much of the field of arithmetic dynamics in the 21st century.

Then, in Section 3 we will give some necessary background about the p-adic numbers $\mathbb{Q}_p$ and the local-global principle. This allows us to state the following conjecture:

Conjecture 1.3 (The Quadratic Dynamic Local-Global Principle). [Kru16] The polynomial f_c has a period n point in $\mathbb{Q}$ if and only if it has a period n point in $\mathbb{Q}_p$ for all primes p, and in $\mathbb{R}$.

In Section 3 we will see that the 'only if' direction is clearly true; in contrast, the 'if' direction remains open.

In Section 4, we will use the 'only if' direction of Conjecture 1.3 to prove that there are finitely many preperiodic points of f_c in $\mathbb{Q}$, which is what Theorem 4.1 states.

In Section 5, we will explore periodic points in $\mathbb{Q}_2$. Using the 'only if' direction of Conjecture 1.3, this work in $\mathbb{Q}_2$ will allow us to resolve Conjecture 1.2 in the case where $c = m/n$ for n odd. In particular, we will show that for these values of c, f_c has only periodic points of period 1 or 2. All of the work in Sections 4 and 5 is done by elementary methods.

In Section 6, we will describe some more powerful results about Conjecture 1.3 which require more machinery. In particular, we will describe Krumm's proof of Conjecture 1.3 for $n < 6$ and his formulation of a more general conjecture, Conjecture 6.2.

2. What is known in $\mathbb{Q}$?

First, Walde and Russo [WR94, Theorem 1 and 3] studied periodic points of f_c in $\mathbb{Q}$ of periods 1, 2, and 3. They found that f_c has a fixed point in $\mathbb{Q}$ exactly when

$$c = \lambda - \lambda^2, \text{ for } \lambda \in \mathbb{Q},$$

a point of period 2 exactly when

$$c = -1 - \mu - \mu^2 \text{ for } \mu \in \mathbb{Q},$$

and a point of period 3 exactly when

$$c = -\frac{\tau^6 + 2\tau^5 + 4\tau^4 + 8\tau^3 + 9\tau^2 + 4\tau + 1}{4\tau^2(\tau+1)^2}, \text{ for } \tau \in \mathbb{Q}, \tau \neq 0, \tau \neq -1.$$

Using elementary methods, this paper solved the case of understanding the dynamics of period 1, 2, and 3 points of f_c in $\mathbb{Q}$. Examples of all of these are seen in Figure 1: With $\lambda = 1/2$, then $c = 1/4$, and $1/2$ is a fixed point of f_c; with $\mu = 1/2$, then $c = -7/4$, and $1/2$ is a period 2 point of f_c, and with $\tau = 1$, then $c = -29/16$, and $-1/4$ is a period 3 point of f_c.

The problem becomes much more complicated once the period becomes larger than 3, with the main conjecture being Conjecture 1.2. To study points of period n, it is necessary to introduce the dynatomic polynomial

$$\Phi_n(X, c) = \prod_{d|n} (f_c^d(X) - X)^{\mu(n/d)} .$$

Here, the product is taken over all divisors $d \in \mathbb{N}$ which divide n. In stating this formula, we introduce the Möbius function $\mu(m)$ for a natural number m. Define

$$\mu(m) := \begin{cases} 0 & \text{if } m \text{ is divisible by the square of some prime} \\ 1 & \text{if } m \text{ is the product of an even number of distinct primes} \\ -1 & \text{if } m \text{ is the product of an odd number of distinct primes} \end{cases} .$$

Then, it is not difficult to show that if x is a point of period n of f_c, then $\Phi_n(x,c) = 0$, as discussed in [Sil07, Section 4.1]. This is because for a period n point of f_c, $f_c^n(x) - x = 0$, but $f_c^m(x) - x \neq 0$ for $m < n$, which implies that $\Phi_n(x,c) = 0$. So, to show that there are no rational points of period n of any polynomials f_c, it suffices to show that Φ_n has no rational roots (x,c) for $x, c \in \mathbb{Q}$. This reformulates the dynamical problem of Conjecture 1.2 into a (very difficult) algebraic geometry problem about rational points on an algebraic curve. For small n, Conjecture 1.2 has been solved:

Theorem 2.1. [Mor98][FPS97][Sto08] *There are no rational points on Φ_n for $n \in \{4,5,6\}$. With $n = 6$, it is necessary to assume the Birch and Swinnerton-Dyer conjecture to prove this.*

Theorem 2.1 was shown for $n = 4$ by Morton in 1998, for $n = 5$ by Flynn and Poonen and Schaefer in 1997, and for $n = 6$ by Stoll in 2008, assuming the Birch and Swinnerton-Dyer conjecture (BSD). Thus, assuming BSD, there are no rational points of period 4, 5, or 6 for any f_c. All other cases of Conjecture 1.2 remain open.

Assuming Conjecture 1.2, Poonen [Poo98] also found that the preperiodic points of f_c will form a directed graph isomorphic to one of the 12 possibilities shown in Figure 1. These have anywhere from 0 preperiodic points, such as with $c = 1$, to 8 preperiodic points, with $c = -29/16$. Hence, the preperiodic rational points of f_c for all values of c are also well-understood assuming the Flynn–Poonen–Schaefer Conjecture.

3. Background about p-adic numbers

All of the material in this section can be found in an introductory text on the p-adic numbers, such as [Gou20]. For $a \in \mathbb{Z}$, define the p-adic valuation $v_p(a)$ to be the maximum $n \in \mathbb{N} \cup \{0\}$ such that $p^n | a$, with $v_p(0) = \infty$. For a/b in $\mathbb{Q}$, define $v_p(a/b) = v_p(a) - v_p(b)$. Then, define the p-adic absolute value $|x|_p = p^{-v_p(x)}$ for $x \in \mathbb{Q}$. Then, the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$ are the p-adic numbers $\mathbb{Q}_p$. We can write $\mathbb{Q}_p$ as the set of series

$$\mathbb{Q}_p = \left\{ \sum_{i=k}^{\infty} c_i p^i \,\middle|\, k \in \mathbb{Z}, c_i \in \{0, 1, \dots, p-1\} \right\} . \tag{3.1}$$

For an element x in $\mathbb{Q}_p$ represented as above, $v_p(x)$ is the minimum i such that c_i is nonzero. The set of elements x of $\mathbb{Q}_p$ with $v_p(x)$ non-negative is called the p-adic integers $\mathbb{Z}_p$. We can readily check that for $x, y \in \mathbb{Q}_p$,

$$v_p(xy) = v_p(x) + v_p(y), \text{ so } |xy|_p = |x|_p |y|_p . \tag{3.2}$$

Also, since adding two numbers cannot give a sum with less factors of p than both numbers, for $x, y \in \mathbb{Q}_p$,

$$v_p(x+y) \geq \min(v_p(x), v_p(y)), \text{ so } |x+y|_p \leq \max(|x|_p, |y|_p) . \tag{3.3}$$

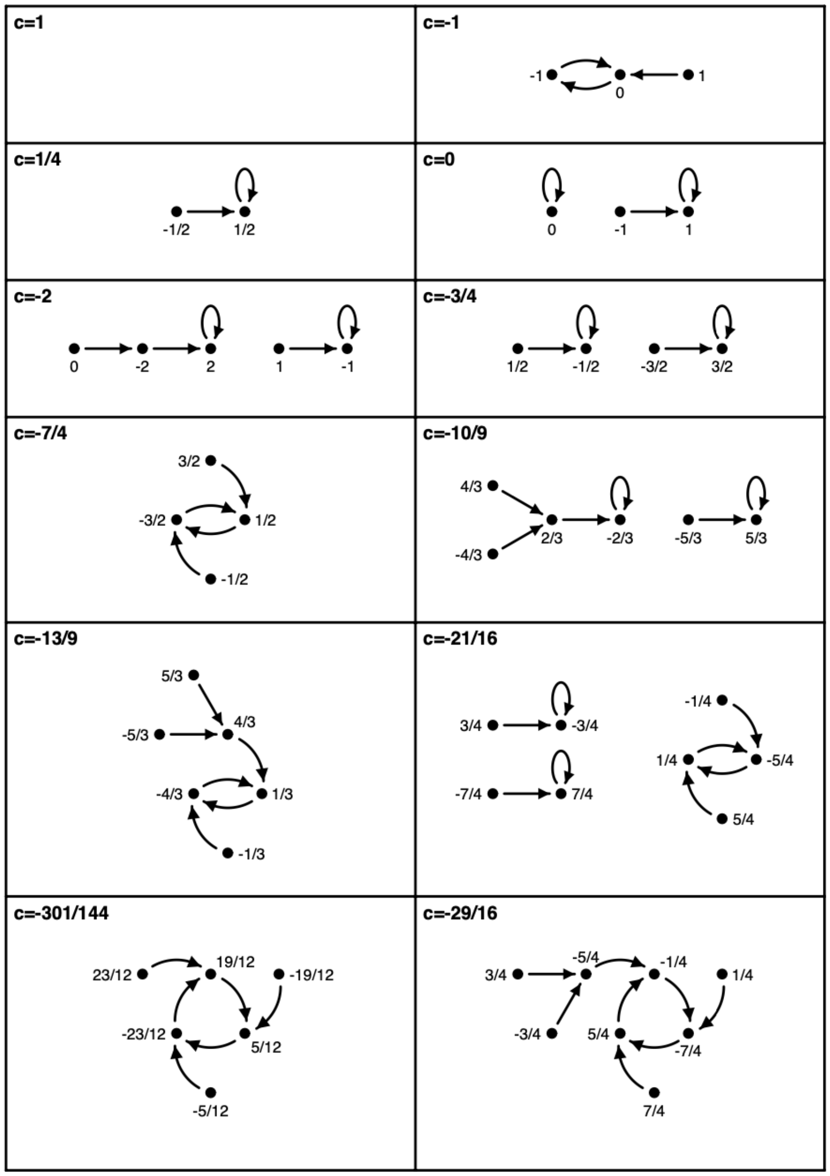

FIGURE 1. The graph of the rational periodic points of f_c for various values of c, due to [Poo98].

Whenever $|x|_p \neq |y|_p$, we will have equality in Equation (3.3).

We now describe the solutions to quadratic equations in $\mathbb{Q}_2$, which will be useful in Section 5.

LEMMA 3.1. *Let $a \in \mathbb{Z}$ be odd. Then, a has a root in $\mathbb{Q}_2$, meaning that there is a solution to the equation $X^2 - a = 0$ in $\mathbb{Q}_2$, if and only if* $a \equiv 1 \pmod 8$.

To help prove this, we state a strong form of Hensel's Lemma:

LEMMA 3.2 (Strong Form Of Hensel's Lemma). [Gou20, Theorem 4.5.3, Problem 120] *For $g(X) \in \mathbb{Z}[X]$, if there exists $\alpha \in \mathbb{Z}_p$ such that $|g(\alpha)|_p < |g'(\alpha)|_p^2$, then there exists a unique $x \in \mathbb{Z}_p$ such that $g(x) = 0$, and $|x - \alpha|_p < 1$.*

We can think of α as an approximate root of g, and x as an exact root of g. This ability to turn approximate roots into exact roots will be exactly what we need to prove Lemma 3.1:

Proof of Lemma 3.1. Let $g(X) = X^2 - a \in \mathbb{Z}[x]$, for $a \in \mathbb{Z}$. We want to know for which odd a there will exist a root x of g in $\mathbb{Q}_2$. By looking at squares modulo 8, only 1 is a square, so there will be no root of g if $a \equiv 3, 5, 7 \pmod 8$. With $a \equiv 1 \pmod 8$, then $|g'(1)|_2 = \frac{1}{2}$ and $|g(1)|_2 \leq \frac{1}{8}$. So, by Lemma 3.2, since $1/8 < (1/2)^2$, g has a root x in $\mathbb{Q}_2$, a square root of a. ☐

Now, we can state Hasse's local-global principle, for $f(X_1, \ldots, X_m)$ in $\mathbb{Q}[X_1, \ldots, X_m]$:

$$\text{(3.4)} \qquad \begin{array}{r} f(X_1,\ldots,X_m) = 0 \text{ has} \\ \text{a solution in } \mathbb{Q} \end{array} \quad \dashleftarrow\!\dashrightarrow \quad \begin{array}{l} f(X_1,\ldots,X_m) = 0 \text{ has} \\ \text{a solution in } \mathbb{Q}_p \text{ for all} \\ \text{primes } p \text{ and in } \mathbb{R} \end{array} .$$
[3]

This principle gets its name because $\mathbb{Q}_p$ is called a local field, and $\mathbb{Q}$ is called a global field, and this principle relates 'local data' about $\mathbb{Q}_p$ to 'global data' about $\mathbb{Q}$. The $\dashrightarrow$ direction follows directly from the inclusions $\mathbb{Q} \subseteq \mathbb{Q}_p$ and $\mathbb{Q} \subseteq \mathbb{R}$. The $\dashleftarrow$ direction is much more powerful, since solving the equations on the right is typically easier using different forms of Hensel's Lemma and the Intermediate Value Theorem. Unfortunately, it is only sometimes true: for example, it is true for quadratic forms by Hasse–Minkowski's Theorem, but not true for general cubic equations; this is discussed in [Gou20, Section 4.8]. We will also see in Theorem 6.3 that when viewing $\Phi_n(X, c)$ as a two-variable polynomial, we get a contradiction to (3.4).

In the context of this paper, we can define a quadratic dynamic local-global principle, stated in Conjecture 1.3:

$$\text{(3.5)} \qquad \begin{array}{r} f_c \text{ has a period} \\ n \text{ point in } \mathbb{Q} \end{array} \quad \dashleftarrow\!\dashrightarrow \quad \begin{array}{l} f_c \text{ has a period} \\ n \text{ point in } \mathbb{Q}_p \text{ for all} \\ \text{primes } p \end{array} .$$

This is a special case of Hasse's principle, using that period n points are characterized by roots of the dynatomic equation $\Phi_n(X, c)$ for fixed c. Again, the $\dashrightarrow$ direction follows from the inclusions $\mathbb{Q} \subseteq \mathbb{Q}_p$ and $\mathbb{Q} \subseteq \mathbb{R}$. In Sections 4 and 5, we will explore the contrapositive of this, seeing how showing nonexistence of periodic points in $\mathbb{Q}_p$ for some p will give nonexistence in $\mathbb{Q}$. The $\dashleftarrow$ direction remains unsolved in the general case, but in Section 6 we will see that it is solved for $m < 6$.

4. Finiteness of Preperiodic Points

In this section, we will examine an elementary argument that uses local data to prove the finiteness of preperiodic points in $\mathbb{Q}$. This section is a slight generalization of [WR94, Theorem 6, Corollaries 4 and 5], who make the same arguments for periodic points. While there are many proofs of this theorem, this argument is

[3]The choice of $\mathbb{Q}_p$ and $\mathbb{R}$ on the right is not arbitrary; by Ostrowski's Theorem, discussed in [Gou20, Theorem 3.1.4], these are the completions of $\mathbb{Q}$ with respect to each nontrivial absolute value.

particularly elegant because of how elementary it is. For example, there is another proof due to Northcott [Nor50], which uses fancy tools such as height functions. (See [Sil07, Theorem 3.12] for more details.)

THEOREM 4.1 (Finiteness of Preperiodic Points). *There are finitely many preperiodic points of f_c in $\mathbb{Q}$.*

To prove this theorem, we use the following lemma about local data:

LEMMA 4.2. *Let λ be a preperiodic point of f_c in $\mathbb{Q}_p$, with $|\lambda|_p > 1$. Then $|\lambda|_p^2 = |c|_p$.*

Proof. Let λ be a preperiodic point of f_c in $\mathbb{Q}_p$ with $|\lambda|_p > 1$.

Case 1: First, assume that $|\lambda|_p^2 > |c|_p$. Then, using (3.2) and (3.3),

$$|f_c(\lambda)|_p = |\lambda^2 + c|_p = |\lambda^2|_p = |\lambda|_p^2 > |\lambda|_p \ .$$

Notice that $f_c(\lambda)$ will also be preperiodic with $|f_c(\lambda)|_p > 1$ and $|f_c(\lambda)^2|_p > |c|_p$, so we can repeatedly apply this result to get

$$|\lambda|_p < |f_c(\lambda)|_p < |f_c^2(\lambda)|_p < \cdots \ .$$

Therefore, λ cannot be preperiodic, since iterating f_c on it causes its p-adic absolute value to grow infinitely.

Case 2: Now, assume that $|\lambda|_p^2 < |c|_p$. Then, using (3.2) and (3.3),

$$|f_c(\lambda)|_p = |\lambda^2 + c|_p = |c|_p \ .$$

Now, notice that

$$|c|_p > |\lambda|_p^2 > |\lambda|_p > 1, \text{ so } |f_c(\lambda)|_p^2 = |c|_p^2 > |c|_p \ .$$

Notice that $f_c(\lambda)$ will also have $|f_c(\lambda)|_p > 1$, so we can apply Case 1 to say that $f_c(\lambda)$ is not a preperiodic point of f_c. Therefore, λ is not preperiodic.

Therefore, combining Case 1 and Case 2 proves the lemma. □

Remark 4.3. For an example of a point in Case 1, take $c = 1$ and $\lambda = \frac{1}{2}$, with $p = 2$, and then maps of f will give

$$\frac{1}{2} \xrightarrow{f_c} \frac{5}{4} \xrightarrow{f_c} \frac{41}{16} \xrightarrow{f_c} \frac{1937}{256} \xrightarrow{f_c} \cdots \ .$$

As n increases, the value of $|f_c^n(\lambda)|_2$ increases without bound, as seen by the rapidly increasing powers of 2 in the denominators.

Remark 4.4. For an example of a point in Case 2, take $c = \frac{1}{8}$ and $\lambda = \frac{1}{2}$, with $p = 2$, and then maps of f will give

$$\frac{1}{2} \xrightarrow{f_c} \frac{3}{8} \xrightarrow{f_c} \frac{17}{64} \xrightarrow{f_c} \frac{801}{4096} \xrightarrow{f_c} \cdots \ .$$

Again, as n increases, the value of $|f_c^n(\lambda)|_2$ increases without bound.

We can now state another lemma, which turns Lemma 4.2 into a global form.

LEMMA 4.5. *Let $\lambda = j/k$ be a preperiodic point of f_c in $\mathbb{Q}$, with $c = m/n$, and both fractions in lowest terms. Then $n = k^2$.*

Proof. First, notice that $|j/k|_p > 1$ exactly when $p|k$, by the definition of $|\cdot|_p$. For p such that $p|k$, let $v_p(k) = a$, so $|j/k|_p = p^{-a}$ and $|m/n|_p = p^{-2a}$ by Lemma 4.2. This means that $v_p(n) = 2a$. So, for all primes p dividing k, which divide k a times, then they divide n $2a$ times. It now suffices to show that if $p \nmid k$, then $p \nmid n$. Assume $p|n$ but $p \nmid k$. So, $|m/n|_p > 1$ and $|j/k|_p \leq 1$. Then, using (3.2) and (3.3),

$$|f_c(j/k)|_p = |j^2/k^2 + m/n|_p = |m/n|_p \ .$$

Now, applying Lemma 4.2 shows that $f_c(j/k)$ is not preperiodic, which is a contradiction. Therefore, if $p \nmid k$, then $p \nmid n$. So, for all primes, n has twice as many factors of them as k, so $n = k^2$. □

We are now ready to prove Theorem 4.1.

Proof. By Lemma 4.5, we can let $\lambda = j/k$ be a preperiodic point of f_c, and let $c = m/k^2$. Then, if $|\lambda| > |c| + 1$, we have that

$$|f_c(\lambda)| = |\lambda^2 + c| \geq |\lambda|^2 - |c| > |\lambda|^2 - |\lambda| + 1 = (|\lambda| - 1)^2 + |\lambda| \geq |\lambda| \ .$$

So, repeating this,

$$|\lambda| < |f_c(\lambda)| < |f_c^2(\lambda)| < \cdots ,$$

so λ is not preperiodic. Therefore, for $\lambda = j/k$ to be preperiodic,

$$-1 - |c| \leq j/k \leq |c| + 1, \text{ so } (-1 - |c|)k < j < (|c| + 1)k.$$

This gives a maximum of $2(|c| + 1)k + 1$ values of j, and thus values of λ. □

5. Using $\mathbb{Q}_2$ to understand $\mathbb{Q}$

In this section we will use the $\dashrightarrow$ direction of (3.5), specifically by showing the nonexistence of periodic points in $\mathbb{Q}_2$, and then using that this implies their nonexistence in $\mathbb{Q}$. We again follow [WR94, Theorems 7 and 8, Corollaries 6 and 7]. The results in this section will use elementary methods to narrow the search for points of period > 2 of f_c to the case where $c = m/n$, with n even and m odd.

Theorem 5.1. *If $c = m/n$, with n odd and m even, then f_c has zero or two fixed points in $\mathbb{Q}$, and no other periodic points.*

We will prove this using a lemma about $\mathbb{Q}_2$.

Lemma 5.2. *If $c = m/n$, with n odd and m even, then f_c has two fixed points in $\mathbb{Q}_2$, and no other periodic points.*

It is clear that Lemma 5.2 implies Theorem 5.1 since $\mathbb{Q} \subseteq \mathbb{Q}_2$. To rule out the existence of one fixed point in $\mathbb{Q}$, we will show in the proof of Lemma 5.2 that if one fixed point is x, the other is $1 - x$, and $x \neq 1/2$.

Proof Of Lemma 5.2. Let $m = 2m'$ for $m' \in \mathbb{Z}$. First, notice that f_c has a fixed point x whenever

$$x^2 + \frac{2m'}{n} = x, \text{ so } \left(x - \frac{1}{2}\right)^2 = \frac{n^2 - 8m'n}{(2n)^2} \ .$$

This has a solution whenever $n^2 - 8m'n$ is a square in $\mathbb{Q}_2$. For n odd, $n^2 - 8m'n \equiv 1 \pmod 8$, so it is always a square by Lemma 3.1. So, f_c has a fixed point x, satisfying $x^2 + c = x$, so $c = x(1 - x)$. Now, notice that

$$(1 - x)^2 + c = 1 - 2x + x^2 + c = 1 - x \ ,$$

so $1-x$ is also a fixed point of f_c. This gives two fixed points of f_c, unless $x = 1-x$, so $x = 1/2$ and $c = 1/4$, which is not possible by the assumption $c = m/n$ for m even and n odd. Now, we use that

$$|x|_2\ |1-x|_2 = |c|_2 = \left|\frac{2m'}{n}\right|_2 < 1\ .$$

This means that either $|x|_2 < 1$ or $|1-x|_2 < 1$; without loss of generality assume the former, as we can swap x and $1-x$ to achieve this. Then,

$$|1-x|_2 = \max(|1|_2, |-x|_2) = 1\ .$$

Now, let $y \in \mathbb{Q}_2$, $y \neq x$, $y \neq 1-x$. We will show that y is not periodic by considering three cases:

Case 1: $|y|_2 < 1$. Here,

$$|y-x|_2 \leq \max(|y|_2, |-x|_2) < 1\ .$$

Let $|y-x|_2 = 2^{-k}$, with $k > 0$, so $v_2(y-x) = k$. Then, let $y - x = 2^k u$, where $v_2(u) = 0$ so $|u|_2 = 1$. Then, we will show that $f_c^n(y)$ approaches x. Notice that

$$|f_c(y) - x|_2 = |(x+2^k u)^2 + c - x|_2 = |2^{k+1}ux + 2^{2k}u^2|_2 \leq 2^{-(k+1)}\ . \tag{5.1}$$

So, $|f_c(y) - x|_2 < |y-x|_2$. Also, $|f_c(y)|_2 < 1$. Doing this repeatedly, $f_c^n(y)$ approaches x at every step, so y cannot be periodic.

Case 2: $|y|_2 = 1$. Let $z = 1-x$. Then, since $|z|_2 = 1$, we have that $v_2(z) = v_2(y) = 0$. Writing z and y as 2-adic expansions as in (3.1), both of them have $c_0 \neq 0$, so $c_0 = 1$. Therefore, $y - z$ has $c_0 = 0$, so $v_2(y-z) > 0$. Let $v_2(y-z) = k$ with $k > 0$, so $|y-z|_2 = 2^{-k}$. Then, let $y - z = 2^k u$, where $v_2(u) = 0$ so $|u|_2 = 1$. Then, we will show that $f_c^n(y)$ approaches z. Notice that

$$|f_c(y) - z|_2 = |(z+2^k u)^2 + c - z|_2 = |2^{k+1}uz + 2^{2k}u^2|_2 \leq 2^{-(k+1)}\ . \tag{5.2}$$

So, $|f_c(y) - z|_2 < |y-z|_2$. Also, $|f_c(y)|_2 = \max(|y^2|_2, |c|_2) = 1$. Doing this repeatedly, $f_c^n(y)$ approaches z at every step, so y cannot be periodic.

Case 3: $|y|_2 > 1$. In this case we can apply Lemma 4.2 to show that y is not periodic.

So, we have shown all cases, and we are done. □

Now, let us see why this proof only works in $\mathbb{Q}_2$, rather than $\mathbb{Q}_p$ for other values of p. One reason is because of the extra factor of 2 which appears in the cross term of $(x+2^k u)^2$, leading to the $k+1$ in (5.1). So, could we do a similar thing with the dynamics of $X^3 + c$ in $\mathbb{Q}_3$? In Case 1, this is probably possible, assuming there is a fixed point x with $|x|_3 < 1$. However, in Case 2, in (5.2), we rely on the fact that $k+1 \geq 2k$, so we need $k > 0$. In other words, we needed that $2|y-z$, for any y and z such that $|y|_2 = |z|_2 = 1$. While this is true in $\mathbb{Q}_2$, the analogous statement is not true in $\mathbb{Q}_3$. For example, $|1|_3 = |2|_3 = 1$, but $3 \nmid 2-1$. So, Case 2 cannot be salvaged even for cubic dynamics in $\mathbb{Q}_3$. So, the work in the proof of Lemma 5.2 relies very heavily on working in $\mathbb{Q}_2$, and is hard to generalize to other primes. We can observe this in Figure 1: we see nonfixed periodic points of f_c with $c = m/n$ where $p|m$, for many different $p \neq 2$.

Now, we consider fractions with odd numerator and denominator of c, and state a theorem which is similar in spirit to Theorem 5.1.

Theorem 5.3. *If $c = m/n$, with m and n odd, then f_c has zero or two period 2 points in $\mathbb{Q}$, and no other periodic points.*

Similarly to Theorem 5.1, this theorem is proved using a lemma in $\mathbb{Q}_2$.

LEMMA 5.4. *If $c = m/n$, with m and n odd, then f_c has two period 2 points in $\mathbb{Q}_2$, and no other periodic points.*

It is clear that Lemma 5.4 implies Theorem 5.3 since $\mathbb{Q} \subseteq \mathbb{Q}_2$. We will sketch some parts of the proof of Lemma 5.4 since it is similar to Lemma 5.2.

Sketch of proof. Period 2 points of f_c are solutions to the dynatomic polynomial

$$\Phi_2(X,c) = \frac{f_c^2(X) - X}{f_c(X) - X} = X^2 + X + c + 1 \ .$$

The polynomial $\Phi_2(X,c)$ has two roots in $\mathbb{Q}_2$ exactly when the discriminant

$$\Delta = 1 - 4(c+1) = \frac{-3n - 4m}{n} = \frac{-3n^2 - 4mn}{n^2}$$

is a square in $\mathbb{Q}_2$. This is true exactly if $-3n^2 - 4mn$ is a square in $\mathbb{Q}_2$. Since n and m are odd, $-3n^2 - 4mn \equiv 1 \pmod 8$ so it is a square in $\mathbb{Q}_2$ by Lemma 3.1. Therefore, $\Phi_2(X,c)$ has a rational root $x \in \mathbb{Q}$, and f_c has a period 2 point $x \in \mathbb{Q}$. Then, $f_c(x) = x^2 + c = -1 - x$ will also be a period 2 point. Let $z = -1 - x$. Now, we sketch a similar method to Lemma 5.2. First, it is possible to show that $xz = c + 1$, so $|xz|_2 < 1$. Let $|x|_2 < 1$ and $|z|_2 = 1$. Through similar methods, we can show that for $y \neq x, z$,

(5.3)

$$\text{if } |y|_2 < 1, \text{ then } |f_c(y) - z|_2 < |y - x|_2 < 1 \text{ and if } |y|_2 = 1, \text{ then } |f_c(y) - x|_2 < |y - z|_2 < 1 \ .$$

So, iterating will give that for all points with $|y|_2 \leq 1$, $f_c^n(y)$ will approach the orbit of x and z. Specifically, for $|y|_2 < 1$, $f_c^n(y)$ approaches x with even n and z with odd n, and vice versa for $|y|_2 = 1$. With $|y|_2 > 1$, we again apply Lemma 4.2 to show that y is not periodic. □

We now compare the proofs of Lemma 5.2 and Lemma 5.4. For $c = m/n$, in Lemma 5.2 we found fixed points when $n^2 - 4mn$ is a square in $\mathbb{Q}_2$, and in Lemma 5.4 we found period 2 points when $-3n^2 - 4mn$ is a square in $\mathbb{Q}_2$. Then, using Lemma 3.1, there will be fixed points with n odd and m even, and period 2 points with n and m odd. Then, in both of (5.1) and (5.3), we showed that $|f_c(y) - f_c(x)|_2 < |y - x|_2$, when $|y - x|_2 < 1$, and the same holds when replacing x by z, the second periodic point. This means that y will approach the orbit of x, as f_c is iterated on it, meaning y cannot be periodic. The only difference is that x was fixed in (5.1), and x was in a period 2 orbit with z in (5.3).

So, in this special case where n is odd, the behavior of f_c is particularly easy to understand. Together, Lemma 5.2 and Lemma 5.4 characterize the behavior of all $f_c(x)$ in $\mathbb{Q}_2$ with $v_2(c) \geq 0$, and Theorem 5.1 and 5.3 do the same in $\mathbb{Q}$.

While attacking the general case of the Flynn–Poonen–Schaefer conjecture remains very difficult, there has recently been further progress made using arithmetic and combinatorial means, combined with some deeper theoretical results. Specifically, Eliahou and Fares [EF22] found that for any $c = m/n$, where $16 \nmid n$, then f_c can only have rational periodic points of period 1 or 2. These authors also found that if n has only one or two distinct prime factors, f_c can only have periodic points of period 1, 2, and 3. So, for values of c where n has the form $n = p^a q^b$ for primes p and q and $a, b \in \mathbb{N} \cup \{0\}$, the Flynn–Poonen–Schaefer Conjecture holds.

6. Bigger Local-Global Theorems

In this section, we will explore work done to study the ⇠ direction in the quadratic dynamic local-global principle of (3.5) and Hasse's local-global principle of (3.4).

In Section 6.1, we will discuss Krumm's work on trying to resolve this for (3.5), and study progress made towards proving and strengthening Conjecture 1.3. Krumm's proofs are interesting because they use mostly Galois theory and algebraic number theory, in contrast to most other proofs in this area of arithmetic dynamics, such as the proofs of Theorem 2.1, which are done using algebraic geometry.

In Section 6.2, we will describe how Morton proved that the 2-variable dynatomic polynomial $\Phi_4(X, c)$ is a counterexample to the ⇠ direction of (3.4).

6.1. **Krumm's Theorems.** For general primes p, Krumm [Kru16] proved that the ⇠ direction in (3.5) holds true for $n < 6$. In the proof of the $n = 5$ case, he needed to assume that a certain algebraic curve has only two rational roots.

With $n = 1, n = 2$, and $n = 3$, this involves showing that for any c where f_c has no period n point in $\mathbb{Q}$, there is a prime p such that f_c has no period n point in $\mathbb{Q}_p$. With $n = 4$ and $n = 5$, this involves showing that for all $c \in \mathbb{Q}$, there is a prime p such that f_c has no period n point in $\mathbb{Q}_p$. This is because of Theorem 2.1, which says that there are no period 4 or 5 points of f_c in $\mathbb{Q}$ for any $c \in \mathbb{Q}$. In fact Krumm proved a stronger statement:

Theorem 6.1. [Kru16, Theorem 1.3 and 1.4] *For any c, there are infinitely many primes p such that f_c has no period 4 points in $\mathbb{Q}_p$. If we assume that a specific genus 11 algebraic curve has only two rational points, which agrees with numerical data, there are also infinitely many primes p such that f_c has no period 5 points in $\mathbb{Q}_p$.*

In order to show these theorems, we introduce the Dirichlet density of a set S of primes in $\mathbb{Z}$, as done in [Kru16, Section 2]. Let M be the set of all primes in $\mathbb{Z}$, so $S \subseteq M$. Define

$$\delta(S) = \lim_{s \to 1^+} \frac{\sum_{p \in S} p^{-s}}{\sum_{p \in M} p^{-s}} \ . \tag{6.1}$$

Then, it is clear that $\delta(M) = 1$. With S being finite, $\delta(S) = 0$, since the denominator of (6.1) diverges. It is also clear that for S and T disjoint, $\delta(S \cup T) = \delta(S) + \delta(T)$. So, $\delta(S)$ can be seen as a measurement of the size of a set S of primes, which ranges from 0 to 1.

Let $T_{n,c}$ denote the set of primes p for which $\Phi_n(X, c)$ has no root in $\mathbb{Q}_p$, with c fixed. Then, Krumm showed that for $n = 4$ and $n = 5$, if $\Phi_n(X, c)$ has no root in $\mathbb{Q}$, then $\delta(M \backslash T_{n,c}) < 1$. This means that $\delta(T_{n,c}) > 0$, so $T_{n,c}$ is an infinite set. Therefore, using the results of [Mor98] and [FPS97], which say $\Phi_n(X, c)$ has no roots in $\mathbb{Q}$ with $n = 4, 5$, there are infinitely many primes p for which $\Phi_n(X, c)$ has no root in $\mathbb{Q}_p$. So, f_c has no period n points in $\mathbb{Q}_p$ for all of these primes.

Further, in [Kru18], Krumm provided an explicit bound that, $\delta(T_{4,c}) > 0.39$. In particular, more than 39% of primes p will have no period 4 points of f_c in $\mathbb{Q}_p$. To prove this theorem, he was able to explicitly calculate the Galois groups of $\Phi_4(X, c)$ for all c in $\mathbb{Q}$. This is a much stronger form of Morton's Theorem from Section 2 about the nonexistence of period 4 points of f_c in $\mathbb{Q}$. This motivates the following conjecture, which is a big strengthening of Conjectures 1.2 and 1.3:

Conjecture 6.2. [Kru19][Conjecture 1.1] For all $n > 3$ and all $c \in \mathbb{Q}$,

$$\delta(T_{n,c}) > 0 \,.$$

In [Kru18], Krumm proved Conjecture 6.2 for $n = 4$. In [Kru19], Krumm proved Conjecture 6.2 for $n = 5, 6, 7, 9$ to hold for all c outside a finite set, but unfortunately had no way to describe which values of c would be in such a finite set. These are the strongest results known about Conjecture 6.2.

6.2. **Morton's Theorem and Hasse's Principle.** Another result of Morton in [Mor98] is that the $\dashleftarrow$ direction in (3.4) fails for the two-variable function $\Phi_n(X, c)$ for $n = 4$. In other words, it represents a counterexample to Hasse's local-global principle.

Theorem 6.3. [Mor98, Proposition 6] *Fix a prime p. If we let c vary, the curve $\Phi_n(X, c) = 0$ has infinitely many points defined with $X, c \in \mathbb{Q}_p$ or in $\mathbb{R}$.*

To prove this, Morton showed that for

$$c = -\frac{1}{4q^2} - \frac{3}{4} \,,\ q \in \mathbb{Q}_p \,, v_p(q) \geq 1 \,, \tag{6.2}$$

then $\Phi_n(X, c) = 0$ has a root in $\mathbb{Q}_p$.

In particular, Morton's proof shows that for certain values of $c \in \mathbb{Q}$, of the form in (6.2), there exist some primes p such that $\Phi_n(X, c) = 0$ has a root in $\mathbb{Q}_p$. For example, with $q = 15$ and

$$c = -\frac{1}{4(15)^2} - \frac{3}{4} = -\left(\frac{13}{15}\right)^2 \,,$$

then there are periodic points of f_c of any period in $\mathbb{Q}_3$ and $\mathbb{Q}_5$. So, for all n, by making choices of q and c using (6.2), there are infinitely many $c \in \mathbb{Q}_p$ such that $f_c(X)$ has a period n point in $\mathbb{Q}_p$. A similar statement can be proved in $\mathbb{R}$ using the intermediate value theorem. This proves a contradiction to Hasse's local-global principle of (3.4).

In contrast, Krumm showed in Theorem 6.1 that for all fixed $c \in \mathbb{Q}$, with $n = 4, 5$, there are infinitely many primes p such that $\Phi_n(X, c) = 0$ has no roots in $\mathbb{Q}_p$. So, for these c given in (6.2), $\Phi_n(X, c) = 0$ has a root in $\mathbb{Q}_p$ for some primes p, but does not have a root for infinitely many primes p. So, Morton's contradiction to Hasse's local-global principle does not affect any of Krumm's work.

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