ARBOREAL REPRESENTATIONS AND ODONI'S CONJECTURE

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Abstract. We define arboreal representations corresponding to polynomials $f(x) \in K[x]$, and explain Odoni's conjecture, as well as partial progress towards the proof of the conjecture. We also briefly discuss some related results and conjectures regarding the behavior of iterated quadratics.

Contents

1. Introduction

Consider a field K with $\mathrm{char}(K) = 0$ and a rational map $f\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree d. It is a common question to consider the orbits resulting from iterating a rational function, i.e. the set $\{a, f(a), f^2(a), \dots\}$ for $a \in \mathbb{P}^1$. In this paper, we instead consider the preimages, that is, the sets

$$f^{-n}(a) = \{x \in \overline{K} \mid f^n(x) = a\}.$$

We focus on the case where f is a polynomial, although similar results may be found for rational f. For a "generic" choice of f, $f^n(x) - a$ has no repeated roots, and thus $|f^{-n}(a)| = d^n$. Throughout this paper, unless otherwise stated, we assume that $|f^{-n}(a)| = d^n$ for all $n \geq 0$.

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We are interested in the action of the Galois group $\mathrm{Gal}(\overline{K}/K)$ on the sets $f^{-n}(a)$. In particular, we may arrange the sets $f^{-n}(a)$ into a complete d-ary tree, with an edge $\alpha \to \beta$ if and only if $f(\alpha) = \beta$. The Galois group then acts by tree automorphisms on this tree, which define *arboreal representations*. Odoni's conjecture [Odo85a] asks about cases where this action induces all possible tree automorphisms.

We formalize these definitions in the following sections, and describe some partial progress towards Odoni's conjecture, such as the resolution over $\mathbb{Q}$. We also state some other related open questions about arboreal representations.

These results also have applications to various other questions in number theory, some of which may not seem immediately related. For instance, using ideas related to arboreal representations, Odoni in [Odo85b] resolves a question about the density prime divisors of the sequence given by $w_1 = 2$, $w_{n+1} = 1 + w_1 \cdots w_n$.

2. Arboreal Representations

Much of the content of this section is derived from the discussion in [BIJ+19, Section 5].

2.1. Preimages of Iterated Maps.

As in the previous section, take a polynomial $f \in K[x]$ over a field K of characteristic 0. Then, there is a natural graph structure on the sets of preimages $f^{-n}(a)$, where for each $x \in f^{-n}(a)$, we draw a directed edge from x to $f(x) \in f^{-n+1}(a)$.

Adapting the notation from [BIJ+19, Section 5], define the sets

$$T_{f,n}(a) := \{a\} \cup f^{-1}(a) \cup f^{-2}(a) \cup \cdots \cup f^{-n}(a),$$

and $T_{f,\infty} := \cup_{n\geq 0} T_{f,n}(a)$. Thus, for each $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the set $T_{f,n}(a)$ acquires the structure of a directed tree, with root a, and all edges directed towards a.

Recall from above that we assume $|f^{-n}(a)| = d^n$ for all $n \geq 0$. (This is equivalent to $f^n(x) - a$ separable over K, for all $n \geq 0$.) Furthermore, with our assumption that $|f^{-n}(a)| = d^n$, this tree is in fact a complete d-ary tree, since every $x \in T_{f,\infty}(a)$ has d distinct pre-images under f.

We now give a concrete example of such a tree.

Example 2.1. Take $f(x) = x^2 - x + 1$, and $a = 0$. Then, we have the sets

$$f^{-1}(0) = \left\{\omega, \omega^2\right\},$$
$$f^{-2}(0) = \left\{\frac{1 \pm \sqrt{4\omega - 3}}{2}, \frac{1 \pm \sqrt{4\omega^2 - 3}}{2}\right\},$$

where $\omega = e^{2\pi i/3}$.

Visually, we may arrange the elements of this tree as shown in Fig. 1.

2.2. Tree Automorphisms and Arboreal Representations.

We first give the definition of a graph automorphism.

Definition 2.2 (Graph Automorphism)**.** Given a graph $G = (V, E)$, an automorphism of G is a bijection $\varphi \colon V \to V$ such that $\varphi(u)$ and $\varphi(v)$ are adjacent if and only if u and v are adjacent.

The set of all graph automorphisms form a group, denoted $\mathrm{Aut}(G)$, where the group operation is composition.

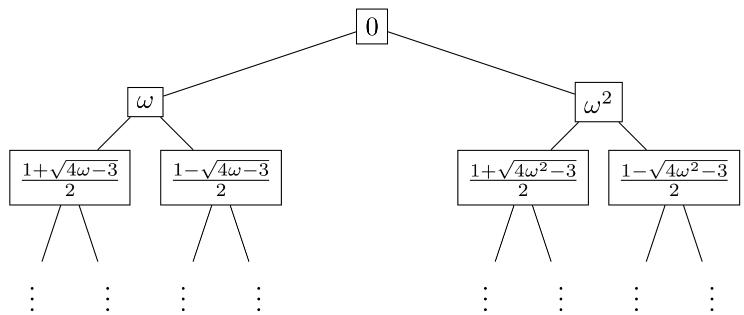

FIGURE 1. The first three layers of the tree $T_{f,\infty}$ for $f(x) = x^2 - x + 1$.

When the graph is a tree, we use the term *tree automorphism* with the same meaning as above. In particular, throughout this paper, we will only be concerned with automorphisms of trees.

In the case from before where the graph is a complete, rooted d-ary tree, it is not hard to see that any graph automorphism must permute the elements in the kth layer, for any $k \geq 0$, by induction upwards from the root.

Consider now the Galois group $G = \mathrm{Gal}(\overline{K}/K)$, where $\overline{K}$ is an algebraic closure of K. Clearly, $T_{f,\infty}(a) \subseteq \overline{K}$. We show in the following proposition that G acts on $T_{f,\infty}(a)$ by permuting the labels, which induces a tree automorphism.

Proposition 2.3. *Take $\sigma \in G = \mathrm{Gal}(\overline{K}/K)$.*

(1) For $\alpha \in T_{f,\infty}(a)$, $\sigma(\alpha) \in T_{f,\infty}(a)$, and σ is a bijection on $T_{f,\infty}(a)$.
(2) Suppose $\alpha \to \beta$ is an edge in $T_{f,\infty}(a)$. Then, for any $\sigma \in G$, $\sigma(\alpha) \to \sigma(\beta)$ is also an edge in $T_{f,\infty}(a)$.

Proof. Since $f \in K[x]$, the coefficients of f are fixed under the action of G, so f commutes with σ. So, since $a \in K$, if $f^n(\alpha) = a$, we have $f^n(\sigma(\alpha)) = \sigma(f^n(\alpha)) = a$, so $\sigma(\alpha) \in T_{f,\infty}(a)$. Since σ is an automorphism of $\overline{K}$, it follows that σ is a bijection on $T_{f,\infty}(a)$.

Similarly, if $\alpha \to \beta$ is an edge, we have $f(\alpha) = \beta$, so

$$f(\sigma(\alpha)) = \sigma(f(\alpha)) = \sigma(\beta) = \beta.$$

Hence, $\sigma(\alpha) \to \sigma(\beta)$ in $T_{f,\infty}(a)$. ∎

Example 2.4. Returning to Example 2.1, if we take $\sigma \in \mathrm{Gal}(\overline{K}/K)$ to be complex conjugation, then the image of the tree under σ is shown in Fig. 2.

Now, again taking the notation of [BIJ+19, Section 5], define

$$K_{f,n}(a) := K(f^{-n}(a)),$$

where we denote by $K(S)$ the extension of K generated by the set S. Thus, $K_{f,n}(a)$ is simply the field extension generated by the elements in the nth layer of the tree $T_{f,\infty}(a)$. As with the definition of the tree $T_{f,\infty}(a) = \cup_{n \geq 0} T_{f,n}(a)$, we define $K_{f,\infty}(a) := \cup_{n \geq 0} K_{f,n}(a)$.

By Proposition 2.3, for each $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the action of G on $T_{f,n}(a)$ induces a tree automorphism.

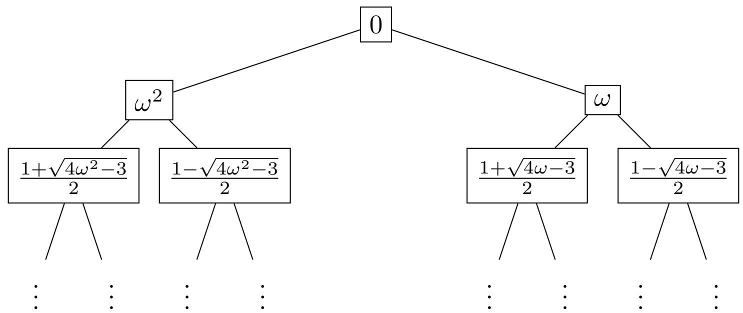

FIGURE 2. The tree $T_{f,\infty}$ from Fig. 1 after complex conjugation, where again $f(x) = x^2 - x + 1$.

It turns out that the parameter a is in some sense arbitrary, and is not strictly necessary, as we now explain.

Remark 2.5 ([Jon13, Section 1.1]). Note that for a function $f(x)$, if we define the shifted function $g(x) = f(x+c) - c$, we have $g(x) + c = f(x+c)$, so $g^{-1}(a-c) = f^{-1}(a) - c$. (Recall that g^{-1} and f^{-1} are sets, so this is a statement about shifting sets.) Hence, for all $n \geq 0$, we have

$$g^{-n}(0) = f^{-n}(c) - c.$$

Since we focus on the action of the Galois group, which does not change upon shifting by a constant, it is enough to consider the trees $T_{f,\infty}(0)$ to understand the general case. From here, we will restrict to the case $a = 0$, and take the shorthand notations $T_{f,n} := T_{f,n}(0)$, and $K_{f,n} := K_{f,n}(0)$.

Definition 2.6 ([BIJ⁺19, Section 5])**.** For $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let

$$\rho^{\text{arb}}_{f,n} \colon \operatorname{Gal}(\overline{K}/K) \to \operatorname{Aut}(T_{f,n})$$

be the map induced by the action of the Galois group $\operatorname{Gal}(\overline{K}/K)$ on $T_{f,n}$. Furthermore, define the *arboreal representation* associated with f to be the homomorphism $\rho^{\text{arb}}_f := \rho^{\text{arb}}_{f,\infty}$.

Remark 2.7. The word *representation* in the term *arboreal representation* comes from the fact that these are closely related to linear representations of a group G: for a linear representation, we consider homomorphisms $G \to \operatorname{GL}(V)$, for a vector space V. Here, we instead have $G \to \operatorname{Aut}(T)$, for a tree T. If we let $\{v_i\}_{i\in\mathbb{N}}$ be the set of vertices of the tree T, since the tree automorphism permutes vertices, any tree automorphism can be viewed as a map $G \to \operatorname{GL}(\mathbb{C}[v_1, v_2, \ldots])$, which gives a linear representation of G (although infinite-dimensional).

Proposition 2.8. *For $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we have*

$$\operatorname{im} \rho^{arb}_{f,n} \simeq \operatorname{Gal}(K_{f,n}/K).$$

Furthermore, we have

$$\operatorname{Gal}(K_{f,n}/K) = \operatorname{Gal}(f^n(x)/K).$$

Proof. To prove the second statement, note that $K_{f,n}$ is generated by the roots of $f^n(x) = 0$. Thus, by definition, $\operatorname{Gal}(K_{f,n}/K) = \operatorname{Gal}(f^n(x)/K)$.

For the first part, note that by the fundamental theorem of Galois theory, we have

$$\mathrm{Gal}(K_{f,n}/K) \simeq \mathrm{Gal}(\overline{K}/K)/\,\mathrm{Gal}(\overline{K}/K_{f,n}).$$

Now, for any $\sigma \in \ker \rho_{f,n}^{\mathrm{arb}}$, σ must fix all elements of the nth layer of $T_{f,n}$. Thus, since $f^{-n}(0)$ generates $K_{f,n}$, σ must fix all of $K_{f,n}$. Hence,

$$\mathrm{im}\,\rho_{f,n}^{\mathrm{arb}} \simeq \mathrm{Gal}(\overline{K}/K)/\ker \rho_{f,n}^{\mathrm{arb}} = \mathrm{Gal}(\overline{K}/K)/\,\mathrm{Gal}(\overline{K}/K_{f,n}) \simeq \mathrm{Gal}(K_{f,n}/K),$$

as claimed. ■

3. Odoni's Conjecture

In light of the above definitions and Proposition 2.8, it is natural to consider $\mathrm{im}\,\rho_f^{\mathrm{arb}}$, i.e. the set of tree automorphisms that are realized by the action of the Galois group on the $T_{f,\infty}$. This motivates the following definition.

Definition 3.1 (Odoni index)**.** For $f \in K[x]$, define the *Odoni index* of f over K to be

$$\iota_K(f) := [\mathrm{Aut}(T_{f,\infty}) : \mathrm{Gal}(K_{f,n}/K)].$$

3.1. Statement.

We first give a modern version of the statement of Odoni's conjecture. The difference is simply that Odoni's original formulation of the conjecture, in [Odo85a], does not refer to tree automorphisms, but rather gives a direct characterization of the Galois groups $\mathrm{Gal}(K_{f,n}/K)$.

Conjecture 3.2 (Odoni's Conjecture, [BIJ+19, Conjecture 5.4])**.** *Let $d \geq 2$ be an integer. Then, there exists a monic polynomial $f(x) \in \mathbb{Q}[x]$ with $\deg f = d$ such that $\iota_{\mathbb{Q}}(f) = 1$. Equivalently, the map ρ_f^{arb} is surjective.*

Remark 3.3. Odoni's original statement of the conjecture takes K to be any *Hilbertian field* (which we will not define here). However, this more general conjecture was disproved in [DK22]. Here, we mostly focus on the case of $K = \mathbb{Q}$, which has in fact been resolved, as we will see in Theorem 4.4. However, most of the definitions and results given in this paper should still hold when $\mathbb{Q}$ is replaced with a general field K, and there are some results in Section 5 that relate to Odoni's conjecture specifically over $\mathbb{Q}$.

We discuss some progress towards Odoni's conjecture in Section 4. In addition to Odoni's conjecture, there are also some interesting questions regarding when $\iota_K(f)$ is finite, which we briefly address in Section 5.

3.2. Wreath Products.

We now review the definition of the wreath product given in [Odo85a, Section 4], in preparation for Odoni's original statement. We use Odoni's notation of $G[H]$ for the wreath product of G and H. Note that this is written in the opposite order of some other notations — the wreath product is sometimes also denoted $H \wr G$ or $H \,\mathrm{Wr}\, G$.

Definition 3.4 (Wreath Product)**.** Let G and H be groups acting on sets A, B, respectively, and let H^A denote the set of all functions $A \to H$. Then $G[H]$ is a group where the elements are from the set $G \times H^A$, and which acts on the set $A \times B$.

For $g \in G$, $\eta \in H^A$, define the action $[g; \eta]\colon A \times B \to A \times B$ such that

$$(a, b) \mapsto (g(a), \eta(a)(b)).$$

For multiplication, $[g_1, \eta_1] \cdot [g_2, \eta_2]$ is the map

$$(a, b) \mapsto (g_1(g_2(a)), \eta_1(g_2(a))(\eta_2(a)(b))).$$

Then, define the iterated wreath product by

$$[G]^1 = G,\ [G]^{n+1} = G[[G]^n].$$

Given this definition, we can now state Odoni's formulation of the conjecture.

Conjecture 3.5 ([Odo85a, Conjecture 7.5])**.** *For all $d \geq 2$, there exists a monic polynomial $f(x) \in \mathbb{Q}[x]$ of degree d such that for all $n \geq 2$, $\operatorname{Gal}(f^n(x)/\mathbb{Q}) \simeq [S_d]^n$, where S_d is the symmetric group of degree d.*

Remark 3.6. Odoni shows in [Odo85a, Corollary 7.4] that for fixed d, for each choice of n, there exists some monic $f(x) \in \mathbb{Q}[x]$ with $\operatorname{Gal}(f^n(x)/\mathbb{Q}) \simeq [S_d]^n$. However, the argument is not sufficient to prove that the same choice of f works for all d.

In fact, Odoni's argument works over all Hilbertian K, and as noted in Remark 3.3, the conjecture does not hold in general for Hilbertian fields.

3.3. **Combinatorics of Tree Automorphisms.** It is perhaps not immediately clear that the versions of Odoni's conjecture given in Conjecture 3.2 and Conjecture 3.5 are equivalent. In this section, we explain this connection, using a combinatorial description of tree automorphisms.

Lemma 3.7. *Let T_n be a complete d-ary tree with n layers after the root, that is, T_n has $1 + d + \cdots + d^n$ vertices. Then, $\operatorname{Aut}(T_n) \simeq [S_d]^n$.*

Proof. We induct on n. For $n = 1$, this is clear, as a tree automorphism permutes the d leaves.

Suppose this holds for some n. Consider T_{n+1}. By the induction hypothesis, $\operatorname{Aut}(T_n) \simeq [S_d]^n$. Then, an automorphism of T_{n+1} can be viewed in the following way: permute the vertices adjacent to the root of T_{n+1}, and then permute each subtree, which is isomorphic to T_n.

This is exactly the description of the wreath product of $G = S_d$ and $H = [S_d]^n$, where G acts on the set A of depth-1 vertices. Then, enumerating the leaves under each depth-1 vertex, we get a labelling of the leaves of T_n with pairs (a, b) with $a \in [d]$ and $b \in [d]^n$, where $[d] = \{1, \dots, d\}$.

Now, picking a permutation of each T_n-subtree corresponds exactly to picking a function $A \to H$. It is not hard to check that the map $(a, b) \mapsto (g(a), \eta(a)(b))$, for $g \in S_d$, $\eta \in [S_d]^n$ preserves edges, since it acts on each subtree independently, and sends each subtree rooted at a depth-1 vertex to another subtree rooted at a depth-1 vertex. (The latter point comes from the fact that for all b, (a, b) goes to $(g(a), b')$ for some b'.)

This completes the induction, so we have $\operatorname{Aut}(T_n) \simeq [S_d]^n$. ∎

Now, recall from before that $\mathbb{Q}_{f,n}$ is the field generated by the roots of $f^n(x)$ over $\mathbb{Q}$. Hence, $\operatorname{Gal}(\mathbb{Q}_{f,n}/\mathbb{Q}) = \operatorname{Gal}(f^n(x)/\mathbb{Q})$. Thus, Lemma 3.7 shows that Odoni's formulation is equivalent to the statement that

$$\operatorname{Gal}(\mathbb{Q}_{f,n}/\mathbb{Q}) \simeq \operatorname{Aut}(T_{f,n})$$

for all positive integers n, which is the version given in Conjecture 3.2, for all finite $n \in \mathbb{Z}_{\geq 0}$.

To get to $n = \infty$, we first give an informal definition of an inverse limit. The inverse limit of a sequence of objects (such as groups or sets) is the set of all sequences contained in the Cartesian product, where the elements are consistent with "projection maps" $\varphi_{i,j}$ between coordinates, for $i < j$.

Example 3.8 (Inverse Limits)**.** As an explicit example, consider the sequence of rings $\mathbb{Z}/p^n\mathbb{Z}$ for $n \in \mathbb{N}$, and p prime. Then, taking the projection maps to be the usual maps $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ for $m > n$, the inverse limit consists of all sequences

$$\{(x_i)_{i\in\mathbb{N}} \mid x_i \in \mathbb{Z}/p^i\mathbb{Z}, \quad x_i \equiv x_j \pmod{p^i} \quad \forall i < j\},$$

which gives exactly the p-adic integers $\mathbb{Z}_p$.

Using this definition, we can view $\mathrm{Gal}(\mathbb{Q}_{f,\infty}/\mathbb{Q})$ as the inverse limit of the groups $\mathrm{Gal}(\mathbb{Q}_{f,n}/\mathbb{Q})$, where the projection maps $\varphi_{i,j}$ are given by the action of an element $\sigma \in \mathrm{Gal}(\mathbb{Q}_{f,j}/\mathbb{Q})$ on $\mathbb{Q}_{f,i}$. Likewise, $\mathrm{Aut}(T_{f,\infty})$ is the inverse limit of the groups $\mathrm{Aut}(T_{f,n})$, where the projections come from restricting an automorphism of $T_{f,j}$ to $T_{f,i}$.

Hence, these two forms of Odoni's conjecture are equivalent, as surjectivity for each of the components implies surjectivity for the inverse limits as well.

In some situations, it may be desirable to work with Odoni's original statement, as it gives an explicit description of the Galois groups, which lends itself to reinterpretation or direct arguments, such as in [Odo85b] or [Sto92], the latter of which we discuss in Section 5.

4. Partial Results

Odoni's conjecture has been proven over all number fields K, with explicit constructions for prime degrees over $\mathbb{Q}$. We give a brief review of the history of such results.

4.1. **Quadratics.** The first progress is by Odoni [Odo85b], which resolves the case $d = 2$.

Theorem 4.1 ([Odo85b, Theorem 1])**.** *Let $f(x) = x^2 - x + 1$. Then, for all $n \geq 1$, $f^n(x)$ is irreducible in $\mathbb{Z}[x]$, and $\mathrm{Gal}(\mathbb{Q}_{f,n}/\mathbb{Q})$ is isomorphic to a Sylow 2-subgroup of S_{2^n}.*

Note that the wreath product $G[H]$ has size $|G|\,|H|^{|A|}$, where G acts on A, so in this case,

$$|[S_2]^n| = 2 \cdot \left|[S_2]^{n-1}\right|^2,$$

and induction gives $|[S_2]^n| = 2^{2^n-1}$. It is not hard to check that $\nu_2(2^n!) = 2^{2^n-1}$, so $[S_2]^n$ is isomorphic to a Sylow 2-subgroup of S_{2^n}. (In particular, $[S_2]^n$ is a subgroup of S_{2^n} since it acts on the leaves of the complete binary tree with n layers below the root.) Hence, Theorem 4.1 proves Odoni's conjecture for $d = 2$.

We will not prove this theorem, but the proof of irreducibility follows the lines of looking at the discriminant of $f^n(x)$. Using this, one can understand $\mathbb{Q}_{f,n}$ and show that $f^n(x)$ is irreducible.

Remark 4.2. Odoni notes that even if $f(x)$ is irreducible, it is not true in general that $f^n(x)$ will be irreducible, with the explicit example $f(x) = x^2 + 10x + 17$, for which

$$f(f(x)) = x^4 + 20x^3 + 144x^2 + 440x + 476 = (x^2 + 12x + 34)(x^2 + 8x + 14).$$

4.2. **Higher Degrees.** For higher degrees, i.e. $d > 2$, a number of results have been proven in recent years. For example, Looper [Loo19] proves Odoni's conjecture over $\mathbb{Q}$ for all prime degrees, and in fact, gives an explicit family of functions f satisfying the condition $\iota_{\mathbb{Q}}(f) = 1$.

Theorem 4.3 ([Loo19, Theorem 1.1]). *Let $f_{p,k}(x) = x^p + kpx^{p-1} - kp$, for $k \in \mathbb{N}$ and p an odd prime not dividing k. Then, $\iota_{\mathbb{Q}}(f_{3,2}(x)) = 1$, and if $p \geq 5$ and $k \equiv 1 \pmod 3$, $\iota_{\mathbb{Q}}(f_{p,k}(x)) = 1$.*

We briefly sketch some of the ideas in the proof. Looper considers a trinomial (i.e. sum of three monomials) $f(x) \in \mathbb{Z}[x]$, and looks at the primitive divisors of $f^n(a)$, for $a \in \mathbb{Z}$. The idea is then to consider the Galois groups

$$\mathrm{Gal}(M_i/\mathbb{Q}(\alpha_i)),$$

where $\{\alpha_i\}$ are the roots of $f^{n-1}(x)$, and M_i is the splitting field of $f(x) - \alpha_i$ over $\mathbb{Q}(\alpha_i)$. Under certain conditions, given in [Loo19, Proposition 2.3], it is then enough to show that these Galois groups are all isomorphic to S_d in order to conclude that

$$\mathrm{Gal}(\mathbb{Q}_{f,n}/\mathbb{Q}_{f,n-1}) \simeq S_d^{\deg f^{n-1}(x)}.$$

The remainder of the proof verifies that the claimed trinomials do in fact satisfy these conditions, and uses this to show that these choices of f satisfy Odoni's conjecture.

In a similar vein, Benedetto and Juul [BJ19] give a non-constructive result over general number fields K.

Theorem 4.4 ([BJ19, Theorem 1.1]). *Let K be a number field. Suppose $d \geq 2$, and either d is even or at least one of d and $d - 2$ is not a square in K. Then, there exists some monic polynomial $f(x) \in K[x]$ such that $\iota_K(f) = 1$.*

In particular, since it is never the case that d and $d-2$ are simultaneously squares in $\mathbb{Q}$ when d is an odd integer, this result proves Odoni's conjecture over $\mathbb{Q}$.

Finally, [Spe18] proves Odoni's conjecture for any number field K, which generalizes the above Theorem 4.4.

5. Related Questions and Applications

5.1. **A Problem on Primes dividing a Sequence.** In Odoni's paper [Odo85b] proving the $d = 2$ case, Theorem 4.1 is used to prove the following result about divisibility.

Theorem 5.1 ([Odo85b, Theorem 2]). *Consider the sequence defined by $w_1 = 2$, $w_{n+1} = 1 + w_1 \cdots w_n$ for $n \geq 1$, and let $\mathcal{P}_x$ be the set of primes at most x that divide w_k for some k. Then,*

$$|\mathcal{P}_x| = O\left(\frac{x}{(\log x)(\log\log\log x)}\right)$$

as $x \to \infty$, and in particular, has density zero in the set of all primes.

This sequence is familiar from the classic proof that there are infinitely many primes. To understand the connection with Theorem 4.1, note that rearranging the condition gives

$$w_{n+1} = 1 + \prod_{i=1}^{n} w_i = 1 + w_n(w_1 \cdots w_{n-1}) = 1 + w_n(w_n - 1) = f(w_n),$$

where we again take $f(x) = x^2 - x + 1$. Thus, we have $w_{n+1} = f^n(2)$ for all $n \geq 0$. As with Theorem 4.1, the proof is somewhat involved, and again uses the discriminant of $f^n(x)$; we will not prove this theorem here.

5.2. **Quadratic Polynomials.** For the case of quadratic polynomials, there are some more specific known results. For example, Stoll [Sto92] describes classes of polynomials $f(x) = x^2 + a$, with $a \in \mathbb{Z}$, that satisfy Odoni's conjecture, i.e. for which $\iota_{\mathbb{Q}}(f) = 1$.

Theorem 5.2 ([Sto92, Section 3])**.** *Let $f(x) = x^2 + a$. If $a \in \mathbb{Z}$ is such that:*

- $a > 0$ *and* $a \equiv 1, 2 \pmod 4$*, or*
- $a < 0$ *and* $a \equiv 0 \pmod 4$*, and* $-a$ *is not a square,*

then $\mathrm{Gal}(f^n(x)/\mathbb{Q}) \simeq [S_2]^n$ *for all* $n \geq 1$*, i.e.* $\iota_{\mathbb{Q}}(f) = 1$*.*

Remark 5.3. It is not known whether this characterizes all f of the form $x^2 + a$ for which $\iota_{\mathbb{Q}}(f) = 1$.

The proof of Theorem 5.2 relatively short, and we give a sketch of the main ideas here, following the proof in [Sto92].

We first prove an elementary property of the sequence defined by $c_1 = -a$, $c_{n+1} = f(c_n)$. Equivalently, since $f(0) = a$, and $f(-a) = f(a)$, we have $c_n = f^n(0)$ for all $n \geq 1$.

Lemma 5.4 ([Sto92, Lemma 1.1])**.** *There exists a sequence $\{b_n\}_{n\geq 1}$ such that for all $n \geq 1$,*

$$c_n = \prod_{d|n} b_d,$$

and the b_n are pairwise coprime.

Sketch of Proof. Using Möbius inversion, we have

$$b_n = \prod_{d|n} c_d^{\mu(n/d)},$$

where μ is the Möbius function[1]. Now, if $p \mid c_k$ for some k, we may consider the smallest n such that $p \mid c_n$.

Letting $m = \nu_p(c_n)$, we can then show, with a periodicity argument, that $p \mid c_k$ if and only if $n \mid k$, and furthermore, that $\nu_p(c_k) = m$ whenever $p \mid c_k$. (To get that the ν_p is exactly m, note that modulo p^{m+1}, squaring a multiple of p^m gives zero.)

Then, using the inversion formula from above, $p \mid b_k$ if and only if $k = n$, so all the b_n are coprime. ■

Definition ([Sto92])**.** We say that a set of nonzero rationals $\{a_1, \ldots, a_n\}$ is 2-*independent* if for any non-empty subset $S \subseteq \{1, \ldots, n\}$, the product

$$\prod_{s\in S} a_s$$

is not a perfect square.

The key claim is then as follows:

[1]The Möbius function is 1 on the product of an even number of distinct primes, -1 on the product of an odd number of distinct primes, and 0 otherwise.

Theorem 5.5 ([Sto92])**.** *For all n, $\mathrm{Gal}(\mathbb{Q}_{f,n}/\mathbb{Q}) \simeq [S_2]^n$ if and only if $c_1, \ldots, c_n$ are 2-independent. This is also equivalent to $b_1, \ldots, b_n$ being 2-independent.*

Note that the equivalence between the c_i and b_i being 2-independent follows from the two relations

$$c_n = \prod_{d|n} b_d,$$
$$b_n = \prod_{d|n} c_d^{\mu(n/d)}.$$

The other part is harder, and involves looking at the properties of 2-Kummer extensions of $\mathbb{Q}$, which are Galois extensions where every element in the Galois group has order at most 2. In particular, the extension $\mathbb{Q}_{f,n+1}/\mathbb{Q}_{f,n}$ is a 2-Kummer extension, since it is obtained by adjoining square roots of elements in $\mathbb{Q}_{f,n}$.

Now, since the b_i are coprime, and $b_1 = -a$ is not a square, by assumption, it is enough to check that none of $|b_2|, \ldots, |b_n|$ is a square.

With Theorem 5.5 in hand, the rest of the proof of Theorem 5.2 consists of describing some even polynomials for which none of the $|b_i|$ are a square, and showing that the desired quadratics have this property. We omit the details here, but they can be found in [Sto92, Section 2].

In addition to Odoni's conjecture, it is interesting to consider the polynomials $f \in K[x]$ for which $\iota_K(f) < \infty$. In fact, if we again consider quadratics of the form $f(x) = x^2 + a$ as in Theorem 5.2, there is a general conjecture characterizing all f with $\iota_{\mathbb{Q}}(f) < \infty$.

Conjecture 5.6 ([Jon13, Conjecture 3.11])**.** *Suppose $f \in \mathbb{Q}[x]$ is a quadratic polynomial. Then, $\iota_{\mathbb{Q}}(f) = \infty$ if and only if one of the following is true:*

(1) f is post-critically finite[2]*,*
(2) for some $r \geq 2$, the two critical points γ_1 and γ_2 of f satisfy $f^r(\gamma_1) = f^r(\gamma_2)$,
(3) $f^k(0) = 0$ for some $k \geq 1$,
(4) for some Möbius transformation[3] *$\mu(x)$ fixing 0, $\mu \circ f = f \circ \mu$.*

For an explicit example, consider the following example given by [Sto92].

Example 5.7. Take $f(x) = x^2 - 2$. Then, 0 is the only critical point of f. Since

$$f^{k+2}(0) = f^k(f(f(0))) = f^k(f(-2)) = f^k(2) = 2$$

for all $k \geq 0$, f is post-critically finite.

Thus, according to Conjecture 5.6, we should have $\iota_{\mathbb{Q}}(f) = \infty$. In fact, we can explicitly state that

$$\mathbb{Q}_{f,n} = \mathbb{Q}(\zeta_{2^{n+2}}) \cap \mathbb{R},$$

where $\zeta_{2^{n+2}}$ is a primitive (2^{n+2})-nd root of unity. First, note that if $f^n(x) = 0$, we must have $|x| \leq 2$, else

$$|f(x)| = \left|x^2 - 2\right| \geq |x|^2 - 2 > |x|,$$

and iterating can never reach 0. Thus, by induction, we have $f^{-n}(x) \subset \mathbb{R}$ for all n. In particular, we have $f^{-n}(x) \subset [-2, 2]$.

[2]A map f is post-critically finite if the orbits of its critical points under f are finite.
[3]Möbius transformations are maps of the form $x \mapsto \frac{ax+b}{cx+d}$ with $ad - bc \neq 0$.

Now, making the substitution $x = 2\cos\theta$, we have

$$f(2\cos\theta) = 4\cos^2\theta - 2 = 2(2\cos^2\theta - 1) = 2\cos(2\theta),$$

so $f^n(x) = 2\cos(2^n\theta)$. Thus, we have

$$f^{-n}(0) = \left\{ 2\cos\theta \;\middle|\; \theta = \frac{2k+1}{2^{n+1}}\pi, k \in \{0, \dots, 2^n - 1\} \right\}.$$

Now, it is clear that $K_{f,n} \subseteq \mathbb{Q}(\zeta_{2^{n+2}}) \cap \mathbb{R}$, since $\mathrm{Re}(\zeta_{2^{n+2}}) = \cos\frac{\pi}{2^{n+1}}$. For the other inclusion, note the powers of $\zeta_{2^{n+2}}$ form a $\mathbb{Q}$-basis of $\mathbb{Q}(\zeta_{2^{n+2}})$, so since the real part of every power is contained in the set $f^{-n}(0)$, any real element is generated by $f^{-n}(0)$ over $\mathbb{Q}$.

This already gives

$$|\mathrm{Gal}(\mathbb{Q}_{f,n}/\mathbb{Q})| \leq \deg\zeta_{2^{n+2}} = 2^{n+1},$$

so $\iota_{\mathbb{Q}}(f) = \infty$. It is not hard to check that we in fact have $\mathrm{Gal}(\mathbb{Q}_{f,n}/\mathbb{Q}) = \mathbb{Z}/2^n\mathbb{Z}$.

More generally, one direction of Conjecture 5.6 is known — if any of these conditions hold, $\iota_{\mathbb{Q}}(f) = \infty$.

Proposition 5.8 ([Jon13, Section 3])**.** *If any of the conditions in Conjecture 5.6 hold, then $\iota_{\mathbb{Q}}(f) = \infty$.*

Proof. We give prove only the third case, which is simplest, using ideas from [Jon13, Section 3]. The proofs of the other three cases are more involved, and we omit them here.

For the third case, If $f^k(0) = 0$ for some $k > 1$, then 0 appears arbitrarily high in the tree. Since 0 is always fixed by any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, if 0 appears in the kth layer, the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is contained in the stabilizer of that vertex of the tree, which has index d^k in $\mathrm{Aut}(T_{f,\infty})$, by orbit-stabilizer. Thus, for an unbounded sequence of k, we have $\iota_{\mathbb{Q}}(f) \geq d^k$, so $\iota_{\mathbb{Q}}(f) = \infty$. ∎

5.3. **Summary and Further Questions.** As noted above, [BJ19] proves Odoni's conjecture over $\mathbb{Q}$, and [Spe18] proves Odoni's conjecture over all number fields. However, the more general cases are still open. In particular, it is interesting to ask whether Odoni's conjecture holds over some class of fields more general than number fields, although not so general as the Hilbertian fields for which Odoni originally made the conjecture.

The questions about finite Odoni index for quadratic f are also interesting, specifically the classification of all polynomials $x^2 + a$ with Odoni index equal to 1, to complement the result in Theorem 5.2.

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