PERIODIC CONTINUED FRACTIONS

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Abstract. Lagrange first analyzed continued fractions and periodic continued fractions in $\mathbb{R}$. His results have since been generalized to many other fields including $\mathbb{C}$ and $\mathbb{Q}_p$, where they have been studied extensively. In this paper we give an overview of constructions for continued fractions in these fields as well as theorems for when these continued fractions are periodic.

1. Introduction

Definition 1.1. Working in a field K with some subset $S \subset K$ a **continued fraction** is an object of the form

$$a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}}$$

where a_i and b_i are in S. If we ever have $b_t = 0$, future a_i and b_i no longer matter, and we say that the fraction **terminates**. Any continued fraction is called **simple** if $b_i = 1$ for all $i \in \mathbb{N}$ except for b_t, which we allow to be 0.

Definition 1.2. Given a continued function defined by a_i and b_i, define the nth **convergent** to be

$$\frac{A_n}{B_n} = F_n = a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots + \frac{b_n}{a_n}}}}.$$

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Definition 1.3. A continued fraction **converges** to x if and only if its sequence $F_0, F_1, ...$ of convergents converges to x.

Lagrange proved that when $K = \mathbb{R}$ and $S = \mathbb{Z}$, one can construct a simple continued fraction that converges to any real number x in K. Hurwitz and Browkin proved similar results for $\mathbb{C}$ and $\mathbb{Q}_p$.

Theorem 1.4. [Lag70], [Hur23], [Bro17, Sections 3-4] *Let $K = \mathbb{R}, \mathbb{C}$, or $\mathbb{Q}_p$. Then, there exists $S \subset K$ and a function $f : K \to S$ such that all x in K can be written as a simple continued fraction*

$$x = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

where we define $a_i \in S$ and $x_i \in K$ by taking the following recurrence relations, starting with $x_0 = x$:

$$\begin{cases} a_i = f(x_i) \\ x_{i+1} = \frac{1}{x_i - a_i}. \end{cases}$$

Definition 1.5. A **periodic continued fraction** is one where a_i and b_i eventually repeat. That is, there exist positive integers m, k such that for all integers $n > m, a_{n+k} = a_n$ and $b_{n+k} = b_n$.

A **purely periodic continued fraction** is one that repeats right away; we let $m = 0$ and $a_0 = 0$ in the above definition.

1.4 states that all x can be written as continued fractions, which begs the question: what x can be written as *periodic* continued fractions?

Theorem 1.6 ([Lag70], [Hur23]). *Let $(K, S) = (\mathbb{R}, \mathbb{Z})$ or $(\mathbb{C}, \mathbb{Z}[i])$. Then, there exist $k, m, a_i, b_i \in S$ such that for all $i > m, a_{i+k} = a_i$ and $b_{i+k} = b_i$ and*

$$a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}}$$

converges to x, if and only if x is the root of some quadratic polynomial $ax^2 + bx + c$ with coefficients in S.

In this paper, we will first examine convergents to better understand what it means for a continued fraction to converge. Then, we will go through classical proofs of convergence and periodicity for $\mathbb{R}$ and $\mathbb{C}$. Finally, we will examine the p-adics and

2. Convergents

We start by developing tools for evaluating convergents, defined in 1.2.

Lemma 2.1. *For the continued fraction*

$$a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}}$$

we can recursively calculate the convergents as:

$$\begin{cases} A_0 = a_0 \\ A_1 = a_1 a_0 + b_1 \\ A_n = a_n A_{n-1} + b_n A_{n-2} \end{cases} \qquad \begin{cases} B_0 = 1 \\ B_1 = a_1 \\ B_n = a_n B_{n-1} + b_n B_{n-2} \end{cases}$$

Proof. We proceed by induction on n, with the hypothesis that the equations for A_n and B_n are true for all continued fractions. This is easily verified by hand when $n = 0, 1, 2$.

Now, we write the nth convergent

$$\frac{A_n}{B_n} = a_0 + \frac{b_1}{a_1 + \ldots + \frac{b_{n-1}}{a_{n-1} + \frac{b_n}{a_n}}} = a_0 + \frac{b_1}{a_1 + \ldots + \frac{a_n b_{n-1}}{a_n a_{n-1} + b_n}}.$$

This would be the $(n-1)$st convergent for the continued fraction where

$$a'_{n-1} = a_n a_{n-1} + b_n$$

$$b'_{n-1} = a_n b_{n-1}$$

and all other a_i and b_i are the same. Note that this alternate continued fraction has the same convergents as the original up to $n-2$.

Using the inductive hypothesis, we can write

$$\begin{aligned}
\frac{A_n}{B_n} = \frac{A'_{n-1}}{B'_{n-1}} &= \frac{a'_{n-1}A_{n-2} + b'_{n-1}A_{n-3}}{a'_{n-1}B_{n-2} + b'_n B_{n-3}} \\
&= \frac{a_n a_{n-1}A_{n-2} + a_n b_{n-1}A_{n-3} + b_n A_{n-2}}{a_n a_{n-1}B_{n-2} + a_n b_{n-1}B_{n-3} + b_n B_{n-2}} \\
&= \frac{a_n(A_{n-1}) + b_n A_{n-2}}{a_n(B_{n-1}) + b_n B_{n-2}} \qquad (2.1)
\end{aligned}$$

which gives us the desired result. □

COROLLARY 2.2.

$$A_n B_{n-1} - A_{n-1}B_n = (-1)^{n-1} b_1 b_2 \ldots b_n \qquad (2.2)$$

Proof. We induct on n. The base case is trivial. Using the recurrence relations from Lemma 2.1,

$$\begin{aligned}
A_n B_{n-1} - A_{n-1}B_n &= (a_n A_{n-1} + b_n A_{n-2})B_{n-1} - A_{n-1}(a_n B_{n-1} + b_n B_{n-2}) \\
&= -b_n(A_{n-2}B_{n-1} - A_{n-1}B_{n-2}) \\
&= -b_n(-1)^{n-2} b_1 b_2 \ldots b_{n-1}.
\end{aligned}$$

□

3. CONSTRUCTIONS

In this section, we will demonstrate Lagrange and Hurwitz's constructions of continued fractions for $\mathbb{R}$ and $\mathbb{C}$, and then we will prove collectively that they converge.

3.1. **Real Numbers.** We take $K = \mathbb{R}$ and $S = \mathbb{Z}$. We want to construct simple continued fractions for $x \in \mathbb{R}$.

EXAMPLE 3.1. Here are some real numbers and possible continued fractions.

$$\begin{aligned} \frac{5}{3} &= 1+\frac{2}{3} = 1+\frac{1}{1.5} = 1+\frac{1}{1+\frac{1}{2}} \\ \pi &= 3+0.1415... = 3+\frac{1}{7.0625...} = 3+\frac{1}{7+\frac{1}{15+\frac{1}{\cdots}}} \\ \sqrt{2} &= 1+0.4142... = 1+\frac{1}{2.4142...} = 1+\frac{1}{2+\frac{1}{2+\frac{1}{\cdots}}}. \end{aligned}$$

Here, we use the natural method of taking $a_0 = \lfloor x \rfloor$ and "filling in the rest" with $\frac{1}{a_1+\frac{1}{a_2+\dots}}$ and recursing. Intuitively, each step should get us closer and closer to the desired result. We express this more formally in the following Theorem, which motivates the general construction in 1.4.

THEOREM 3.2 (Lagrange). [Lag70] *Any real number x can be uniquely written as a simple continued fraction*

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

where $a_i \in \mathbb{Z}$ and $a_i > 0$ for $i > 0$ by taking $x_0 = x$ and

$$\begin{cases} a_i = \lfloor x_i \rfloor \\ x_{i+1} = \frac{1}{x_i - a_i} \end{cases}$$

for all i.

3.2. **Complex Numbers.** We now move to simple continued fractions where the a_i are Gaussian integers. Here, we obtain an analogous construction to that of real continued fractions.

THEOREM 3.3. [Hur23] *Any complex number z can be written as a simple continued fraction*

$$z = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

where $a_i \in \mathbb{Z}[i]$ by taking $z_0 = z$ and

$$\begin{cases} a_i = [z_i] \\ z_{i+1} = \frac{1}{z_i - a_i} \end{cases}$$

where $[z_i]$ denotes the closest Gaussian integer to z_i, with ties broken by rounding up.

Remark 3.4. Note that this construction is not quite a complex analogue of Lagrange's construction for real numbers. In fact, this construction suggests a real analogue that is distinct from Lagrange's construction that takes the nearest integer instead of the floor, and allows a_i to be negative.

3.3. **Proofs of Convergence.** In general, given K, S, and f, we attempt to write x in K as a simple continued fraction

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

by defining $a_i \in S$ and $x_i \in K$ by taking the following recurrence relations, starting with $x_0 = x$:

$$\begin{cases} a_i = f(x_i) \\ x_{i+1} = \frac{1}{x_i - a_i}. \end{cases}$$

In Lagrange's algorithm, $f(x) = \lfloor x \rfloor$. In Hurwitz, $f(x) = [x]$. To prove the convergence of their algorithms, we can make an observation about the generalized construction that helps us understand what happens to the convergents $\frac{A_n}{B_n}$ as we apply the procedure.

LEMMA 3.5. [Dan15] *For nonnegative integers n,*

$$B_n x - A_n = (-1)^n (x_1 x_2 \dots . x_{n+1})^{-1} \tag{3.1}$$

Proof. This is proven by induction on n.

For the base case, we have $F_1 = \frac{a_0 a_1 + 1}{a_1}$, so

$$B_1 x - A_1 = a_1(x - a_0) - 1 = \frac{a_1}{x_1} - 1 = -\frac{1}{x_1 x_2}$$

Now, given that $B_{n-1}x - A_{n-1} = (-1)^{n-1}(x_1 x_2 \dots x_n)$, we write

$$\begin{aligned} B_n x - A_n &= (a_n B_{n-1} + B_{n-2})x - (a_n A_{n-1} + A_{n-2}) \\ &= a_n(B_{n-1}x - A_{n-1}) + (B_{n-2}x - A_{n-2}) \\ &= a_n(-1)^{n-1}(x_1 x_2 \dots x_n)^{-1} + (-1)^{n-2}(x_1 x_2 \dots x_{n-1})^{-1} \\ &= (a_n - x_n)(-1)^{n-1}(x_1 x_2 \dots x_n)^{-1} \\ &= \left(-\frac{1}{x_{n+1}} (-1)^{n-1} (x_1 x_2 \dots x_n)^{-1} \right) \\ &= (-1)^n (x_1 x_2 \dots x_{n+1})^{-1} \end{aligned}$$

as desired.

□

Now, we will prove the following:

PROPOSITION 3.6. [Lag70] *Lagrange's construction (3.2) yields a continued fraction that converges to x.*

Proof. Lemma 3.5 gives $|\frac{A_n}{B_n} - x| = |\frac{1}{x_1 x_2 \dots x_{n+1} B_n}|$. LIn 3.2, $x_{i+1} = \frac{1}{x_i - \lfloor x_i \rfloor} \geq 1$, and B_n is a nonzero integer and thus $|B_n| \geq 1$. Thus, $\frac{A_i}{B_i}$ converges to x. □

PROPOSITION 3.7. [Hur23] *Hurwitz's construction (3.3) yields a continued fraction that converges to z.*

Proof. Lemma 3.5 gives $|\frac{A_n}{B_n} - z| = |\frac{1}{z_1 z_2 \dots z_{n+1} B_n}|$. In 3.3, $|z_{i+1}| = \left| \frac{1}{z_i - [z_i]} \right| \geq \sqrt{2}$, and B_n is a nonzero Gaussian integer and thus $|B_n| \geq 1$. Thus, $\frac{A_i}{B_i}$ converges to z.

□

Remark 3.8. In Hurwitz's algorithm (3.3), we define a_i to be $[z_i]$. [Dan15] and [DN14] discuss general continued fraction algorithms where we let a_i be any complex number with $|a_i - z_i| \leq 1$. Notably, we may restrict S to a subset other than Gaussian integers.

4. Periodicity

4.1. Real Numbers.

Theorem 4.1 (Lagrange). [Lag70] *Given a real number x, there exists a periodic continued fraction*

$$a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \cdots}}}$$

with $a_i, b_i \in \mathbb{Z}$ that converges to x if and only if x is the root of some quadratic polynomial $ax^2 + bx + c$ with integer coefficients.

We sketch the proof of the if direction given by Northshield. [Nor11].

Proof sketch. All rationals $x = \frac{p}{q}$ have finite continued fractions

$$0 + \cfrac{p}{q + \frac{0}{\cdots}}.$$

Thus, we only worry about quadratic irrationals for the if direction. We claim that the construction given by 3.2 is periodic.

Consider the following function

$$f(x) = \begin{cases} x - 1 \text{ if } x \geq 1 \\ \frac{1}{\frac{1}{x} - 1} \text{ if } x < 1. \end{cases}$$

This function always subtracts 1 from the leftmost nonzero term of the Lagrange continued fraction for x. The continued fraction must be periodic as long as $f^n(x) = x$ for some n. Furthermore, if x_i is the root of $a_i x^2 + b_i x + c_i$, then $x_{i+1} := f(x_i)$ is the root of $a_{i+1}x^2 + b_{i+1} + c_{i+1}$ where

$$\begin{aligned} a_i, a_{i+1} &> 0 \\ b_i^2 - 4a_i c_i &= b_{i+1}^2 - 4a_{i+1}c_{i+1} \\ a_{i+1} - b_{i+1} + c_{i+1} &= a_i \text{ or } c_i. \end{aligned}$$

Starting from some positive $x = x_0$ and applying f infinitely many times, we see that

1. if c_i is negative infinitely often, then b_i takes on finitely many values among these instances, so by pigeonhole, there is some triple a_i, b_i, c_i occurs at least three times, which gives us a repeated root $x_j = x_k$.
2. if c_i is positive for all large i, then b_i must be negative, since x_i is positive. Then, $a_i - b_i + c_i$ is decreasing and nonnegative, forming a contradiction.

Thus, any positive real quadratic irrational has a Lagrange continued fraction that eventually repeats. This is easily extended to all real quadratic irrationals. □

Next, we sketch a proof of the only if direction using Möebius transformations, motivated by [Lan45]. First, we restate the only if form of 4.1:

COROLLARY 4.2 (Lagrange). *Given a real number x and $k, m, a_i, b_i \in \mathbb{Z}$ such that for all $i > m, a_{i+k} = a_i$ and $b_{i+k} = b_i$ and*

$$a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}}$$

converges to x, x is the root of some quadratic polynomial $ax^2 + bx + c$ with integer coefficients.

We approach this direction by looking at how extending a continued fraction up and to the left affects what it converges to. For this, we have the following Lemma:

LEMMA 4.3. *Given fixed a_i and b_i, we define, as a function of z:*

$$S_k(z) = a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{\dots + \frac{b_k}{a_k + z}}} \tag{4.1}$$

Then,

$$S_k(z) = \frac{A_k + zA_{k-1}}{B_k + zB_{k-1}}$$

EXAMPLE 4.4. In the purely periodic continued fraction

$$\sqrt{2} - 1 = \cfrac{1}{2 + \frac{1}{2 + \dots}},$$

we have $S_1(z) = \frac{1}{2+z}$ and $S_2(z) = \frac{1}{2 + \frac{1}{2+z}} = \frac{2+z}{5+2z}$.

Proof. Observe that S_k is the $k+1$th convergent of the continued fraction with the first k terms as a_k and b_k, and the $k+1$th terms being $b'_{k+1} = z$ and $a'_{k+1} = 1$, and thus by (2.1) can be written as $\frac{A_k + zA_{k-1}}{B_k + zB_{k-1}}$. □

If a purely periodic continued fraction of period k converges to x, then suppose the continued fraction defined by $a'_i = a_{i+k}$ and $b'_i = b_{i+k}$ converges to x'. Clearly, $S_k(x') = (x)$, where k is the period. However, this continued fraction ends up being exactly the same as the original, so $S_k(x) = x$. Solving for x gives

$$\begin{aligned} xB_k + x^2B_{k-1} &= A_k + xA_{k-1} \\ B_{k-1}x^2 + (B_k - A_{k-1})x - A_k &= 0 \qquad (4.2) \end{aligned}$$

Since A_i and B_i are integers, x is the root of a quadratic equation with integral coefficients. In the above example, we have $k = 1$, so $x^2 + 2x - 1 = 0$, which can easily be verified for $x = \sqrt{2} - 1$.

Then, if a periodic continued function with preperiod m converges to y, we can write $y = S_m(x)$, where x is the root of a quadratic $ax^2 + bx + c$ with integer coefficients. We can then write the inverse Moebius tranformation $x = -\frac{A_k - yB_k}{A_{k-1} - yB_{k-1}}$. Expanding the quadratic in terms of y gives

$$\begin{aligned} a\left(\frac{A_k - yB_k}{A_{k-1} - yB_{k-1}}\right)^2 - b\frac{A_k - yB_k}{A_{k-1} - yB_{k-1}} + c &= 0 \\ a(A_k - yB_k)^2 - b(A_k - yB_k)(A_{k-1} - yB_{k-1}) + c(A_{k-1} - yB_{k-1})^2 &= 0 \\ a'y^2 + b'y + c' &= 0 \end{aligned}$$

$$\text{for } \begin{cases} a' = aB_k^2 - bB_kB_{k-1} + cB_{k-1}^2 \\ b' = -2aA_kB_k + bA_kB_{k-1} + bA_{k-1}B_k - 2cA_{k-1}B_{k-1} \\ c' = aA_k^2 - bA_kA_{k-1} + cA_{k-1}^2. \end{cases}$$

In these formulas, a', b', and c' are clearly integers, so we see that y is also the root of a quadratic polynomial with integer coefficients, completing the proof.

Remark 4.5. Note that the construction in 3.2 is not unique if we allow $a_i < 0$. In particular, at each step, we can choose a_i to be either the floor or the ceiling of z_i, and the continued fraction will still converge. Thus, it is possible to construct aperiodic continued fractions for quadratic irrationals in $\mathbb{R}$. For example,

$$x = \sqrt{2} - 1 = \frac{1}{2 + x} = \frac{1}{2 + \frac{1}{2 + \dots}}$$

$$y = \sqrt{2} - 2 = \frac{1}{-2 + \frac{1}{4+y}} = \frac{1}{-2 + \frac{1}{4 + \frac{1}{-2+\dots}}}$$

Thus, we can construct a sequence of a_is for $z = \sqrt{2}$ by starting with $a_0 = 1$, then, for all $n \geq 0$,

$$\text{choose: } \begin{cases} a_{2n+1} = 2, a_{2n+2} = 2 \\ \text{add one to } a_{2n}, a_{2n+1} = -2, a_{2n+2} = 3. \end{cases}$$

If we make our choices aperiodically (e.g. according to an irrational integer's binary representation), then the resulting continued fraction will clearly be aperiodic.

4.2. **Complex Numbers.** We now state the analogue of Lagrange's Theorem for complex continued fractions.

THEOREM 4.6. [Hur23] *Given a complex number z, there exists a periodic continued fraction*

$$a_0 + \frac{b_1}{a_1 + \frac{b_2}{a_2 + \frac{b_3}{a_3 + \dots}}}$$

with $a_i, b_i \in \mathbb{Z}[i]$ that converges to z if and only if x is the root of some quadratic polynomial $ax^2 + bx + c$ with coefficients in $\mathbb{Z}[i]$.

We start with the if statement:

PROPOSITION 4.7 (Hurwitz, if). *Given a complex number z that is the root of a quadratic polynomial $az^2 + bz + c$ where $a, b,$ and c are Gaussian integers, the corresponding complex continued fraction produced by Hurwitz's algorithm (3.3) is periodic.*

We provide a sketch of the proof from [DN14, Section 4], starting with the following Lemma:

LEMMA 4.8. *For the Hurwitz construction, there exist infinitely many n such that $|B_{n+1}||A_{n+1} - zB_{n+1}| \leq C$ and $|B_n||A_n - zB_n| \leq C$, where $C = \frac{1}{\sqrt{2}-1}$*

Proof. Rewriting

$$|A_n - zB_n| = \left| B_n \left(\frac{A_n}{B_n} - \frac{z_{n+1}A_n + A_{n-1}}{z_{n+1}B_n + B_{n-1}} \right) \right| = \frac{1}{|z_{n+1}B_n + B_{n-1}|}$$

(the last step uses Corollary 2.2) gives

$$|B_n||A_n - zB_n| = \left|\frac{B_n}{z_{n+1}B_n + B_{n+1}}\right| = \left|\frac{1}{z_{n+1} + \frac{B_{n-1}}{B_n}}\right|$$

Since $|B_n|$ is bounded below, there must be infinite values of n such that $|B_{n-1}| \leq |B_n|$, in which case the value above is at most

$$\frac{1}{|z_{n+1}| - |\frac{B_{n-1}}{B_n}|} \leq \frac{1}{\sqrt{2}-1} = C$$

We claim that all such n satisfy the condition in the Lemma. Evidently, if $|B_n| \leq |B_{n+1}|$, we are done. Otherwise, we have

$$\begin{aligned} |B_{n+1}||A_{n+1} - zB_{n+1}| &= |B_{n+1}||(z_1 z_2 ... z_{n+1})^{-1}| \\ &\leq |B_n||(z_1 z_2 ... z_n)^{-1}| \\ &= |B_n||A_n - zB_n| \\ &\leq C \end{aligned}$$

□

We now go back to proving the proposition.

Proof sketch. For Hurwitz's construction, proving that $z_i = z_j$ for some pair i, j suffices. Using (3.1), we can express

$$z = S_k\left(\frac{1}{z_{k+1}}\right) = \frac{z_{k+1}A_k + A_{k-1}}{z_{k+1}B_k + B_{k-1}},$$

which gives us z_i in terms of z.

Suppose $z_0 = z$ is a root of $az^2 + bz + c$. The sequence of z_i are then roots of $a_i z^2 + b_i z + c_i$, where we can calculate a_i, b_i, and c_i in terms of A_i and B_i, as well as a, b, and c.

After doing the computation, it turns out that for all i:

$$\begin{aligned} b_i^2 - 4a_i c_i &= b^2 - 4ac \\ |a_i| &\leq |2az + b||B_{n-1}||A_{n-1} - zB_{n-1}| \\ |c_i| &= |a_{i-1}| \end{aligned}$$

Thus, as long as $|B_{n-1}||A_{n-1} - zB_{n-1}| \leq C$ and $|B_{n-2}||A_{n-2} - zB_{n-2}| \leq C$ for some constant C, a_i and c_i are bounded by constants based on z. If this occurs infinitely many times, it implies that some triple a_i, b_i, c_i appears at least 3 times, which would give us some pair $z_i = z_j$, as desired. □

For the other direction, we prove the stronger statement:

PROPOSITION 4.9. *Given a complex number z and $k, m \in \mathbb{Z}$, $a_i \in \mathbb{C}$ such that for all $i > m, a_{i+k} = a_i$ and*

$$a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

converges to z, z is the root of some quadratic polynomial az^2+bz+c with coefficients in $\mathbb{Z}[i]$.

Proof sketch. To prove this, we can simply reuse the proof for the only if side of 4.1, which proves this for reals. The only adaptation necessary is that a Moebius transformation on a root of a quadratic polynomial with coefficients in $\mathbb{Z}[i]$ will also yield roots of a quadratic polynomial with coefficients in $\mathbb{Z}[i]$. □

5. p-ADICS

5.1. **Construction.** We begin our exploration of p-adic continued fractions with a construction by Browkin.

Remark 5.1. For the p-adic numbers, there is no pre-existing concept of an integer that works well as a restriction for a_i. We could try using the p-adic integers, but any p-adic number α can be written as $a_0 + \frac{1}{a_1}$ for p-adic integers a_0 and a_1, so this is not interesting. In this construction, a_i are restricted to finite p-adic representations $(...0000c_0.c_{-1}c_{-2}...c_{-k})$, so they are standard rational numbers in $\mathbb{Q}$.

Theorem 5.2. [Bro17, Section 3-4] [Rom24, Section 2] *Let p be an odd prime. Any p-adic number α can be expressed as a simple continued fraction by taking $\alpha_0 = \alpha$ and*

$$\begin{cases} a_i = s(\alpha_i) \\ \alpha_{i+1} = \frac{1}{\alpha_i - a_i} \end{cases}$$

where if $\alpha = \sum_{i=-r}^{\infty} c_i p^i$ with $c_i \in \{-\frac{p-1}{2}, ..., \frac{p-1}{2}\}$:

$$s(\alpha) = \sum_{i=-r}^{0} c_i p^i$$

A variation of this algorithm defines

$$t(\alpha) = \sum_{i=-r}^{-1} c_i p^i$$

and specifies

$$\begin{cases} a_i = s(\alpha_i) \text{ if i even} \\ a_i = t(\alpha_i) \text{ if i odd} \\ \alpha_{i+1} = \frac{1}{\alpha_i - a_i} \end{cases}$$

Proof. Since this construction is similar to Lagrange and Hurwitz's real and complex constructions of continued fractions, we can reference 3.3. Thus,

$$|\alpha - \frac{A_n}{B_n}| = |\frac{(-1)^n(\alpha_1\alpha_2...\alpha_{n+1})^{-1}}{B_n}|$$

For Browkin's original algorithm, all α_i take the form $\frac{1}{....c_3c_2c_1 0.000}$, so $|\frac{1}{\alpha_i}| \le \frac{1}{p}$. Additionally, since $|a_i| \ge p$, we can claim that $|B_i| = |a_i B_{i-1} + B_{i-2}| > |B_{i-1}|$ using Lemma 2.2, which is evident by induction.

The variation can be similarly handled, but instead of the strict versions that still hold when i is even, we have $|\frac{1}{\alpha_i}| \le 1$ and $|B_i| \ge |B_{i-1}|$ when i is odd. This is still clearly enough to guarantee convergence.

Thus, the convergents $\frac{A_n}{B_n}$ converge to α in a p-adic sense.

□

Remark 5.3. For any field F and subset F' of F such that $\forall x \in F, \exists x' \in F'$ such that $|x - x'| < 1$, we can analogously construct a simple continued fraction with $a_i \in F'$ that converges to x.

5.2. **Periodicity.** We can now move on to periodicity of p-adic continued fractions. In general, we do not know whether or not all quadratic irrationals have periodic representations, but we do know that periodic continued fractions still converge to quadratic irrationals.

LEMMA 5.4. *Given a p-adic rational α and $k, m \in \mathbb{Z}$, $a_i \in \mathbb{Q}$ such that $\forall i > m, a_{i+k} = a_i$ and*

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

converges to z in the p-adics, z is the p-adic root of some quadratic polynomial $az^2 + bz + c$ with coefficients in $\mathbb{Z}$.

Proof. The proof of this is analogous to that of Corollary 4.2, where the Möebius transform of a p-adic root of a quadratic polynomial is also the p-adic root of another quadratic polynomial. Note that the quadratic polynomials obtained from 4.2 is the same for reals and p-adics. □

Since the quadratics are the same, we get the following proposition from [Rom24, Proposition 1]:

PROPOSITION 5.5. *Given a quadratic polynomial with roots $\alpha^{(p)}$ in $\mathbb{Q}_p$ and $\alpha^{(r)}$ in $\mathbb{R}$, if $\alpha^{(p)}$ has p-adic continued fraction*

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \dots}}$$

that is eventually periodic, then the fraction converges to $\alpha^{(r)}$ or $\bar{\alpha}^{(r)}$ in $\mathbb{R}$.

Thus, by constructing p-adic continued fractions for quadratic irrationals that do not approach the corresponding quadratic irrationals in $\mathbb{R}$, we can show that the p-adic continued fractions do not repeat. For example, in [Rom24, Section 3], we see that $\sqrt{19}$'s Browkin continued fraction seems to converge to 1.357..., which is significantly different from $\sqrt{19}$ in $\mathbb{R}$, suggesting that the Browkin continued fraction is not periodic.

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