POSTCRITICALLY FINITE PARAMETERS OF QUADRATIC POLYNOMIALS

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ABSTRACT. We introduce the notion of postcritically finite (PCF) maps and their corresponding PCF parameters in the moduli space. We state and prove some fundamental results regarding their multiplicity and their geometric properties. We state the characterization of algebraic curves containing infinitely many PCF parameters, which is a special case of the Dynamical André–Oort conjecture, and give a proof of a special case of the former. We apply the theorem to quadratic PCF parameters.

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1. INTRODUCTION

In order to introduce postcritically finite maps, let us first recall the following definitions.

DEFINITION 1.1. Let $f \in \mathbb{C}(z)$ be a rational function of degree $d \geq 2$. Consider the iterates of f :

$$f^n(z) := \underbrace{(f \circ \cdots \circ f)}_{n}(z),$$

$$f^0(z) := z.$$

The *orbit* of a point $\alpha \in \mathbb{C}$ is the set $\mathcal{O}(\alpha) = \{f^n(\alpha) : n \geq 0\}$.

DEFINITION 1.2. Let $f \in \mathbb{C}(z)$ be a rational function of degree $d \geq 2$. A point $\alpha \in \mathbb{C}$ is said to be a *critical point* if

$$f'(\alpha) = 0.$$

A rational map f is said to be *postcritically finite* (PCF) if the orbit of each critical point is finite.

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EXAMPLE 1.3. Consider the map $z \mapsto z^2 + i$. The only critical point of this map is 0. Observe that

$$\begin{aligned} f(0) &= i, \\ f(i) &= -1 + i, \\ f(-1+i) &= -i, \\ f(-i) &= -1 + i, \\ &\vdots \end{aligned}$$

hence the orbit of 0 is the set $\{0, +i, -i, -1+i\}$, implying that $z \mapsto z^2 + i$ must be postcritically finite.

In this paper we consider the case where $f \in \mathbb{C}[x]$ is a polynomial of the form

$$f_c(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_1 z + c, \quad a_i \in \mathbb{R}, c \in \mathbb{C}.$$

In particular, we shall treat $a_1, \ldots a_d$ as fixed and parametrize the function by its constant term c. If the function f_c is postcritically finite, we call the parameter c a *PCF parameter*. In the previous example, we saw that $c = i$ is a PCF parameter of the map $z \mapsto z^2 + c$. Using a similar computation, we can determine that $c = -i$ is also a PCF parameter. As we shall see in the following section this is no coincidence.

In this paper, we will be interested primarily in PCF parameters of the quadratic polynomial $f_c(z) = z^2 + c$. One can get an insight of the geometric properties of PCF parameters by finding them numerically and plotting them, as seen in 1. A qualitative similarity between the set of PCF parameters and the Mandelbrot set is readily apparent from the figure.

In particular, we will be studying the geometric properties through algebraic curves in $\mathbb{C}$ containing PCF parameters.

QUESTION 1.4. *What algebraic curves in $\mathbb{C}$ contain an infinite number of PCF parameters of the map $z \mapsto z^2 + c$?*

In Section 2 we take a look at some fundamental properties of PCF parameters, paying special attention to the case $d = 2$. We introduce the problem of finding the number of PCF parameters lying on an algebraic curve. In Section 3 we give the main theorem, a special case of the Dynamical André–Oort conjecture, which characterizes algebraic curves containing infinitely many PCF parameter pairs of $z \mapsto z^d + c$. In Section 4 we return to the quadratic case and investigate the special case of lines in $\mathbb{C}$.

2. FUNDAMENTAL RESULTS

In the following section, we develop an intuition for the geometry and distribution of PCF parameters of quadratic functions $z \mapsto z^2 + c$, which we call *quadratic PCFs*. We relate the concept of quadratic PCF parameters to the Mandelbrot set and provide bounds for the number of parameters conditioned on the properties of the orbit of the critical point.

The only critical point of $f_c(z) = z^2 + c$ is 0. Therefore, c is a PCF parameter if and only if the orbit of 0 is finite. This allows for a particularly tangible characterization of quadratic PCFs. In particular, 0 is postcritically finite if and only if

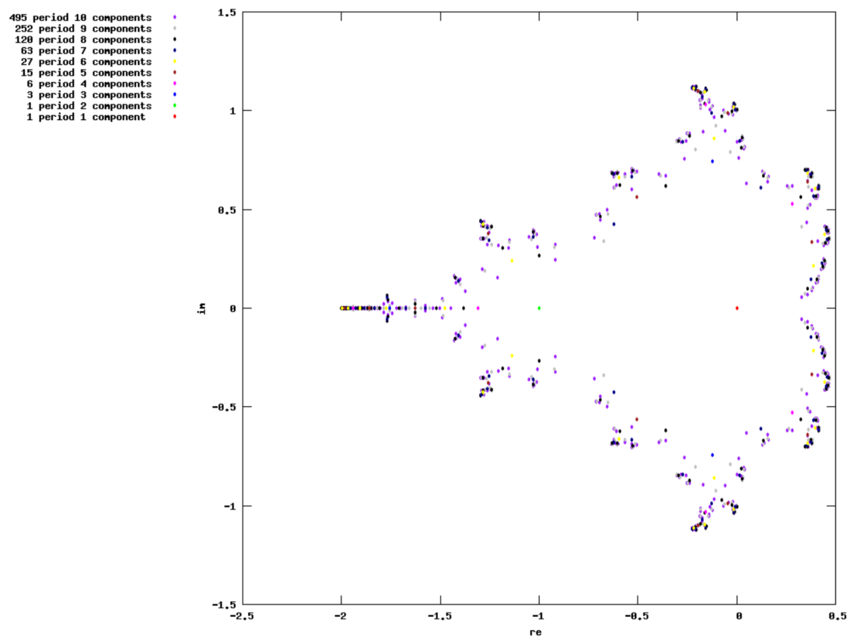

FIGURE 1. PCF parameters of the map $z \mapsto z^2 + c$, with critical orbit size ≤ 10 [Fav]

there exist $n \in \mathbb{N}, k \in \mathbb{N}_0$, such that

$$f_c^{n+k}(0) = f_c^k(0),$$

or equivalently

$$f_c^{n+k}(0) - f_c^k(0) = 0.$$

Notice that the lefthand side can be viewed as a polynomial in c, denoted $P_{n,k}(c)$. In other words, c is a PCF parameter if and only if it is a root of $P_{n,k}$ for some n and k.

The following properties follow immediately from the fact that $P_{n,k}$ is a polynomial with integer coefficients.

PROPOSITION 2.1. *The set of quadratic PCF parameters is invariant under complex conjugation.*

PROPOSITION 2.2. *The quadratic PCF parameters are algebraic integers.*

What is more, the above characterization gives a result in the number of PCF parameters.

PROPOSITION 2.3. *There are infinitely many quadratic PCF parameters. Moreover, there are infinitely many real quadratic PCF parameters.*

Proof. It suffices to consider only the polynomials $P_{n,0}$. In particular, let c be a common root of $P_{n_1,0}$ and $P_{n_2,0}$ for some distinct $n_1, n_2 \in \mathbb{N}$. Therefore, we have that $f^{n_1}(0) = f^{n_2}(0) = 0$, hence the exact period of 0 (i.e. the lowest n such that $f^n(0) = 0$) must divide both n_1 and n_2. Thus, looking at the roots of $P_{p,0}$, where p is prime, we can generate infinitely many distinct PCF parameters.

Moreover, note that polynomials $P_{p,0}$ have a zero constant term, a non-zero linear term. Indeed the first property follows from the fact that $P_{p,0}$ for any p and the second property follows from the recursive relation $P_{p,0} = (P_{p-1,0})^2 + c$. Therefore, 0 is a single root of every $P_{p,0}$. As the polynomial is of even degree 2^p, it must have another real root. This way, we can find infinitely many real roots. $\square$

Moreover, a result by [Sil12, Theorem 6.7] shows that the number of quadratic PCF parameters with an orbit of a given length must be finite.

Let us now return to the geometric properties of quadratic PCF parameters. Looking at the plot of a finite subset of quadratic PCF in Figure 1, we see that the parameters accumulate on the boundary of the Mandelbrot set. Recall that the Mandelbrot set is defined as:

$$\mathcal{M}_2 = \{c \in \mathbb{C} : \text{ the orbit of 0 under } z \mapsto z^2 + c \text{ is bounded}\},$$

therefore, PCF parameters must lie in the Mandelbrot set. Carleson–Gamelin [CG96] proved the following density result for quadratic PCF parameters.

Proposition 2.4 (by [CG96, Section VIII, Theorem 1.5]). *The quadratic PCF parameters are dense in $\partial\mathcal{M}_2$, the boundary of the Mandelbrot set $\mathcal{M}_2$.*

A natural question to pose is whether there exists some geometric relationship between the PCF parameters in the moduli space. Stated differently, suppose that we are constructing an algebraic curve in the moduli space of a given degree, what can be said about the number of PCF parameters lying on the curve? As we will see in the followup discussion, in most cases only a finite number of such parameters can be contained on a particular curve.

Theorem 2.5 (by [DM23, Theorem 1.3]). *Let $d \geq 2$. There exists a constant $M(d) < \infty$, such that the number of PCF paramaters on any algebraic curve of degree d is at most $M(d)$, provided that the curve has no special components.*

The special components refer to the lines containing infinitely many PCF parameters, as characterized by Theorem 3.4, which we are yet to state.

3. The dynamical André–Oort conjecture

In this section, we state the dynamical André–Oort conjecture. We explain its significance for polynomials of the form $z \mapsto z^d + c$ by providing a theorem, which considers a special case of the conjecture, and doing so characterizes the algebraic curves containing infinitely many PCF parameters. We provide a sketch of the proof of a special case, where the algebraic curve can be identified with the graph of some polynomial.

The intricate correspondence between concepts in arithmetic geometry and in dynamical systems allows for a formulation of problems surrounding PCF rational maps in terms of abelian varieties. If A_g is a moduli space of principally polarized abelian varieties, then the points A_g corresponding to varieties with complex multiplication, which we call CM points, are analogous to points parametrizing PCF maps. A more detailed treatement of the relevant definitions and the correspondence can be found in [Sil12, Section 6.5].

Before diving into the dynamical formulation of the André–Oort conjecture, let us recall some key definitions of algebraic geometry. For more details see [Har97, Section 1].

DEFINITION 3.1. The *affine n-space* over $\mathbb{C}$, denoted $\mathbb{A}^n_{\mathbb{C}}$ is the set of all n-sutples of elements of $\mathbb{C}$. A subset $Y \subset \mathbb{A}^n_{\mathbb{C}}$ is called an *algebraic set* if there exists a subset T of the polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$, such that

$$Y = \{P \in \mathbb{A}^n_{\mathbb{C}} : f(P) = 0 \ \ \forall f \in T\}.$$

If an algebraic set cannot be expressed as a union of two proper algebraic subsets, we call it an *irreducible algebraic variety*.

In the case $n = 2$ and T consisting of a single non-constant polynomial f, we call the corresponding algebraic set Y an *irreducible algebraic plane curve*.

The classical André–Oort conjecture characterizes irreducible varieties containing a Zariski closure of CM points (see [PST+24] for more details). Recall that a set is Zariski closed if and only if it is the set of common zeros of a collection of polynomials. Relevant to our discussion, the recent findings by Baker and DeMarco [DM23] allowed us to formulate an analogous conjecture regarding PCF parameters on algebraic varieties.

Before we state the dynamical analogue of the conjecture, we need to familiarize ourselves with the following definition.

DEFINITION 3.2. An n-tuple of critical points $\{c_1, \ldots, c_n\}$ of a rational map f is said to have *dynamically dependent orbits* if there exists a non-zero algebraic relation P such that

$$P(c_1, \ldots, c_n) = 0,$$

and such that P is invariant under the map $(f, \ldots, f)$. Otherwise, we say that the critical points of f are *dynamically independent*. If $\{f_v | \ v \in V\}$ is a family of functions, parametrized by points on an algebraic variety, we say that an n-tuple of critical points is *dynamically independent on* V if it is dynamically independent for every f_v.

On an intuitive level, the orbits of critical points are dynamically independent if no two of them exhibit "similar" behavior (e.g. same preperiod or period length) under a given family of functions.

CONJECTURE 3.3 (The Dynamical André–Oort Conjecture by [BDM13, Conjecture 1.4]). Let $\{f_t : t \in V\}$, where V is an irreducible algebraic variety over $\mathbb{C}$, be an N-dimensional algebraic family of rational maps of degree $d \geq 2$. Then f_τ is PCF for a Zariski-dense subset of V if and only there are at most N dynamically independent critical points on V.

Note that a subset of a variety V is Zariski dense if its closure under the Zariski topology is V.

Let us consider the case where $N = 1$ and V is an algebraic curve in $\mathbb{C}^2$. We define the family of functions as $\left\{f_{(a,b)} = (z^d + a, z^d + b) | (a, b) \in V\right\}$. If we desire that V contains infinitely many pairs of PCF parameters, the conjecture suggests that the orbits of the critical point $(0, 0)$ in the first and the second coordinate must be dynamically dependent. Indeed, as we see in the following theorem, this must indeed hold, and the set of relations between a and b is remarkably sparse.

THEOREM 3.4 ([GKNY17, Theorem 1.1]). *Let C be an irreducible algebraic plane curve in $\mathbb{C}^2$, and let $d \geq 2$ be an integer. There exist infinitely many pairs of parameters $(a, b) \in C$, such that both $z \mapsto z^d + a$ and $z \mapsto z^d + b$ are postcritically finite, if and only if one of the following conditions holds:*

(1) C is the curve $\{x = \alpha\}$, where α is a PCF parameter,
(2) C is the curve $\{y = \beta\}$, where β is a PCF parameter,
(3) C is the locus of the equation $y - \zeta x = 0$, where ζ is a $(d-1)$-st root of unity.

In the case of quadratic polynomials, we see that the third condition corresponds to the diagonal $\{x = y\}$ in $\mathbb{C}$, whereas in the cubic case, we also allow for the diagonal $\{x = -y\}$.

The if direction of Theorem 3.4 can be verified briefly. Note that for the cases (1) and (2) the theorem obviously holds, as the number of PCF parameters of maps of the form $z \mapsto z^d + c$ is infinite by Proposition 2.3. For curves of the form (3), we note that if $f_c(z) = z^d + c$ is PCF, then also $f_{\zeta c}(z) = z^d + \zeta c$ is PCF, as

$$\begin{aligned} f_{\zeta c}(z) &= z^d + \zeta c \\ &= (z/\zeta^{d-1})^d + \zeta c \\ &= \zeta\big((z/\zeta)^d + c\big) \\ &= \zeta f_c(z). \end{aligned}$$

Here, we made use of the fact that $\zeta^{d-1} = 1$ and the fact that there exists infinitely many PCF parameters c for a given d.

For the only if direction of the theorem, we show the proof of the special case, where the curve C is the graph of a polynomial $h \in \mathbb{C}[z]$ as given by [GKN16, Theorem 1.2]. First, let us consider the definition of the d-th *multibrot set* $\mathcal{M}_d$, which is a generalization of the Mandelbrot set, and the filled-in Julia set.

Definition 3.5. The *multibrot set* $\mathcal{M}_d$ which is the set of all $c \in \mathbb{C}$, such that the (possibly infinite) orbit of 0 under the map $z \mapsto z^d + c$ remains bounded, i.e.

$$\sup_{n\in\mathbb{N}} |f^n(0)| < \infty.$$

Definition 3.6. The so-called *filled-in Julia set* of a function $f : \mathbb{C} \to \mathbb{C}$ is the set of points z_0, such that $f^n(z_0)$ remains bounded, i.e.

$$\{z_0 \in \mathbb{C} : \sup_{n\in\mathbb{N}} |f^n(z_0)| < \infty\}.$$

The boundary of the filled-in Julia set is called the *Julia set.*

The proof relies on the following two properties of the multibrot set.

Theorem 3.7 ([GKN16, Theorem 1.2]). *For each $d \geq 2$, there does not exist a polynomials $h(z)$, whose filled-in Julia set is $\mathcal{M}_d$.*

Proposition 3.8 ([GKN16, Proposition 2.2]). *Let $\mu(z) = Az + B$ be an automorphism of $\mathbb{C}$, such that $\mu(\mathcal{M}_d) = \mathcal{M}_d$. Then $A = \zeta$ and $B = 0$, where ζ is a $(d-1)$-st root of unity.*

Proof of Theorem 3.4 when C is a graph of a polynomial. It suffices to show that h can be at most linear, and if so, it must be of the form (3).

The boundary of the multibrot set $\mathcal{M}_d$ corresponds to points $c \in \mathbb{C}$, whose every open neighborhood contains infinitely many distinct PCF parameters. Therefore, the closure of the set of PCF parameters must contain $\mathcal{M}_d$. A result by Ghioca–Krieger–Nguyen [GKN16, Theorem 2.1] shows that $t \in \mathbb{C}$ is a PCF parameter if and only if $h(t)$ is a PCF parameter. Therefore, by continuity of h and its nonsingularity away from the origin, the open mapping theorem gives that the boundary $\partial\mathcal{M}_d$ is totally invariant under h, i.e. $h^{-1}(\partial\mathcal{M}_d) = \partial\mathcal{M}_d$. Applying the same reasoning for $\mathbb{C} \setminus \mathcal{M}_d$, which is an open neighborhood of ∞, we see that the set $\mathbb{C} \setminus \mathcal{M}_d$, must also be totally invariant. Therefore, $h^{-1}(\mathcal{M}_d) = \mathcal{M}_d$.

If h is linear, it must therefore be an affine symmetry of $\mathbb{C}$. Hence, Proposition 3.8 implies that h must be of the form (3).

Assume, towards a contradiction, that h has degree $d \geq 2$ and let K_h and J_h denote its filled-in Julia set and its Julia set, respectively. By Montel's theorem, J_h is the minimal closed set containing at least 3 points that are totally invariant under h. Therefore, $J_h \subset \partial\mathcal{M}_d$ and $K_h \subset \mathcal{M}_d$. On the other hand, every points of $\mathcal{M}_d$ is bounded by iteration, therefore, by definition, $\mathcal{M}_d \subset K_h$. We conclude that $\mathcal{M}_d = K_h$, which contradicts Theorem 3.7. □

4. Quadratic polynomials

We would like to apply Theorem 3.4 to the case of real algebraic curves in $\mathbb{R}^2$ passing through complex PCF parameters c of the map $z \mapsto z^2 + c$. Assume an algebraic curve is given by the set of zeros of a polynomial $P(x, y) \in \mathbb{R}[x, y]$. Substituting

$$\begin{aligned} x &\mapsto \frac{1}{2}(z + w), \\ y &\mapsto \frac{1}{2i}(z - w) \end{aligned} \tag{4.1}$$

gives rise to a polynomial in z and w of the same degree as the original polynomial, but with complex coefficients. This procedure allows us to identify the algebraic curves in $\mathbb{R}^2$ (which we can identify with $\mathbb{C}$) with algebraic curves in $\mathbb{C}^2$.

Moreover, the substitution preserves the PCF parameters on the curve. In fact, if a PCF parameter c lies on the real algebraic curve, then the corresponding complex algebraic curve must contain $(c, \bar{c})$, which is a pair of PCF parameters by Proposition 2.1. The converse likewise holds.

We have now developed a sufficient amount of material to answer the following question 1.4.

Let a line in $\mathbb{C} \cong \mathbb{R}^2$ be given by

$$\{(x, y) \in \mathbb{R}^2 | ax + by = c\}$$

for some $a, b, c \in \mathbb{R}$. Using the substitution (4.1), we acquire the following line in $\mathbb{C}^2$

$$\left\{(z, w) \in \mathbb{C}^2 : \frac{1}{2}a(z + w) + \frac{1}{2i}b(z - w) = c\right\},$$

or equivalently

$$\left\{(z, w) \in \mathbb{C}^2 : \frac{1}{2}z(a - ib) + \frac{1}{2}w(a + ib) = c\right\}.$$

Theorem 3.4 shows that the corresponding $\mathbb{C}^2$ lines are vertical/horizontal lines through a PCF parameter and the diagonal $\{z = w\}$. Observe that since a and b are real parameters, no lines in $\mathbb{R}^2$ correspond to vertical/horizontal lines in $\mathbb{C}^2$. On the other hand, having $a = c = 0$ gives rise to the diagonal $\{z = w\}$, therefore, the real axis in $\mathbb{C}$ contains infinitely many PCF parameters.

Having dealt with the infinite case, the set of lines containing at least as little as 3 PCF points appears to be surprisingly sparse. In fact, the only known line containing more than 2 PCF points is the imaginary axis containing the $\{0, +i, -i\}$, which we have shown to be PCF parameters. The imaginary axis in $\mathbb{C}$ corresponds to the complex line $w = -z$ in $\mathbb{C}^2$.

A particularly interesting case left to consider are horizontal lines in $\mathbb{C}$ that contain at least two PCF parameters, a trivial example of which is the real axis uncovered above.

In other cases, where a horizontal line contains two distinct PCF parameters c_1 and c_2, they must have the same nonzero imaginary part. The complex line defined by $z + w = c := c_1 + \overline{c_2} \in \mathbb{R}$, thus contains the four PCF points

$$(c_1, \overline{c_2}), (\overline{c_1}, c_2), (\overline{c_2}, c_1), (c_2, \overline{c_1}).$$

The result by [DM23, Theorem 1.6] shows that in the space $\mathbb{C}^2$ there are at most finitely many lines not of the type described in Theorem 3.4 that contain more than two PCF parameters. Therefore, returning to our line in $\mathbb{C}$, there can exist at most finitely many horizontal lines containing two or more PCF parameters. However, at the present time, we do not know any examples other than the real line.

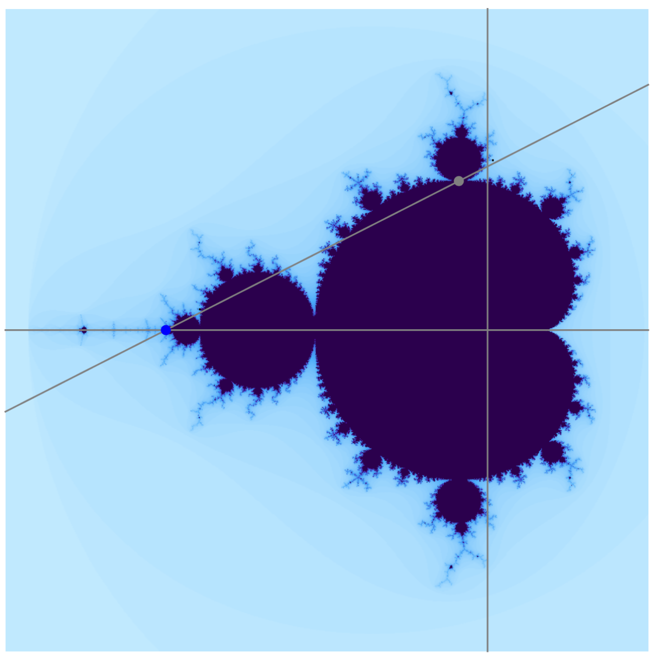

FIGURE 2. Mandelbrot set with the marked real and imaginary axis and a line going through a pair of PCF points $c_1 \approx -1.401155$ and $c_2 \approx -0.125 + 0.649519i$.

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