SEMINAR IN NUMBER THEORY: THE TOPIC PROPOSAL
MIT 18.784, SPRING 2025

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1. Assignment

To culminate the undergraduate seminar on arithmetic dynamics, you will write a 8-to-12 page term paper in the style of a research or expository mathematical journal article. The goal of the term paper assignment is for you to learn and synthesize a topic of your choosing and present it in a way that is accessible to your classmates. The writing process will be broken down into stages to help you fine-tune every aspect of the paper.

First, you will write a topic proposal. Choose a focus for your term paper that interests you and prepare a shortlist of main academic references (e.g. journal articles, arXiv preprints, books, theses). Plan how you will use each reference for your paper and what additional perspective you will provide yourself.

Write your topic proposal in LaTeX using the *amsart* class (a template is available on the course website). It should be 1 to 3 pages long and contain the following:

- the proposed topic in arithmetic dynamics;
- a list of *at least two* main academic references;
- a description of the information that you plan to use from each reference;
- a description of the perspective that you plan to provide yourself.

Upload the PDF of your topic proposal to your course Google Drive folder (or email it to me) by the due date listed on the course website.

The following sections explain how to pick a good topic. At the end is a list of possible ideas for topics.

2. Focus and breadth

When choosing your topic should be, the following points on breadth and scope can be helpful.

- **Choose a topic that aligns with your interests.** A topic that you are passionate about will make your writing more engaging.
- **Choose a topic that aligns with your knowledge.** As you may have experience already when preparing your talks this semester, some of your sources will assume background with which you or your audience may be unfamiliar. You can dedicate parts of your paper to reviewing this background (in fact, you probably should), but don't underestimate the task of understanding your sources *and* distilling the key ideas down into an accessible short paper.
- **A broader topic generally requires a longer paper to cover it adequately.** Since your paper is limited to roughly 10 pages, a narrower,

more focused topic will allow for deeper exploration. Also consider the space you will need to dedicate to introducing background material.
- **Decide on the *purpose* of your paper, as this affects its scope.** Is it a comprehensive overview of a specific conjecture or theorem? Is it a detailed investigation on how a particular technique (e.g. from topology or analytic number theory) can be applied to arithmetic dynamics?
- **Consider the perspective that you will provide.** Your paper should have value for your readers beyond what's in the original sources. You contributing something to the greater body of mathematical literature!

Example 1 (An interesting theorem)**.** You might choose to focus on a single theorem or two closely related theorems that you want to understand. You could gain an understanding of the theorem by reading two or more sources that present different proofs of the theorem, examples, and/or applications. Choose the proofs and examples that you think are the clearest and most helpful. Then in your own words, explain those proofs, examples, and applications to your audience (your peers) so that they can understand the theorem too, taking care to "fill in the gaps" for your audience (your peers) on the background that the original sources might assume from their intended audience (researchers). Even though you use your own words, be sure to cite the source of each proof, example, and application you present.

Warning 1 (Summarizing a reference)**.** You might choose to focus on a single theorem, such as the classification of rational periodic points of quadratic polynomials of period 3. You may be tempted to read about the theorem from the reference [WR94] and then simply summarize its presentation in your paper. This is a poor choice because if you simply summarize, then you have not added your own perspective; there is no reason for readers to read your paper instead of reading the original source [WR94]. Furthermore, if your presentation is close enough to the original, then you may be committing plagiarism (even if you are paraphrasing in your own words).

Warning 2 (Overly broad focus)**.** You might choose to write about a general object or concept, such as "rational functions in dynamics". You may be tempted to write a paper that presents everything you know about rational functions. It's certainly possible to write about several results and applications in a cohesive way (e.g. [BH99]), but if there isn't a pervasive central theme then your paper may benefit from a more narrow focus. If you include several different theorems and applications that are largely *unrelated to each other*, then it can be very difficult to incorporate a logical structure and you may lose your readers in the disconnected areas.

3. Examples of similar papers

Consider reading a few examples of similar articles before finalizing your choice. MathSciNet is a great resource for finding published papers on specific topics. If you need access to MathSciNet off-campus, find it through MIT Libraries.

The following journals specialize in publishing well-written articles that are typically 5 to 20 pages long.

- *The American Mathematical Monthly*
- *The Mathematical Intelligencer*
- *Mathematics Magazine*

Here are a few examples related to arithmetic dynamics: [Bea15, Ben01, BH99, DE95, Dev99, FJS00, Gra18, Ves23, WR94].

Undergraduate, master's, and doctoral theses can also be good examples, but they are typically much longer and can vary greatly in intended audience, mathematical correctness, and quality of writing.

4. List of possible topics

Here is a list of several possible topics. You can choose a topic not on this list, so long as it is about arithmetic dynamics. These are intended to serve as inspiration; for instance, you write about one of these topics as-is, write about a specific aspect of one of these topics or write about a specific theorem/example/application within these topics.

Some specific results and conjectures:

- The *abc* conjecture in arithmetic dynamics
- The Artin primitive root conjecture
- The dynamical Mordell–Lang conjecture
- The dynamical uniform boundedness conjecture
- A dynamical view of the Collatz conjecture
- The genus of dynatomic modular curves
- The irreducibility of dynatomic modular curves
- Odoni's conjecture on arboreal Galois groups

Some specific objects:

- The Berkovich projective line and its p-adic dynamics
- Dynatomic Galois representations
- Dynamical moduli spaces
- Dynatomic units
- p-adic Mandelbrot sets
- Periodic points of continued fractions
- Portraits of preperiodic points
- Post-critically finite polynomials
- The Ruelle zeta function

Some general areas of study:

- Good and bad reduction in arithmetic dynamics
- Integers in orbits of rational functions
- Periodic points of quadratic polynomials in number fields
- Primes and prime divisors in orbits of rational functions
- Randomness of quadratic polynomials over finite fields
- Stability of iterates of rational functions
- Unlikely intersections in arithmetic dynamics

5. Acknowledgments

This assignment guide has been loosely adapted from one written by Michel Goemans and Peter Shor for the Fall 2010 course "18.310: Principles of Discrete Applied Mathematics" at MIT.

References

[Bea15] A. F. Beardon. Möbius maps and periodic continued fractions. *Math. Mag.*, 88(4):272–277, 2015.

[Ben01] Robert L. Benedetto. An elementary product identity in polynomial dynamics. *Amer. Math. Monthly*, 108(9):860–864, 2001.

[BH99] Edward Bertram and Peter Horák. Some applications of graph theory to other parts of mathematics. *Math. Intelligencer*, 21(3):6–11, 1999.

[DE95] William Derrick and Jack Eidswick. Continued fractions, Chebychev polynomials, and chaos. *Amer. Math. Monthly*, 102(4):337–344, 1995.

[Dev99] Robert L. Devaney. The Mandelbrot set, the Farey tree, and the Fibonacci sequence. *Amer. Math. Monthly*, 106(4):289–302, 1999.

[FJS00] Michael Frame, Brenda Johnson, and Jim Sauerberg. Fixed points and Fermat: a dynamical systems approach to number theory. *Amer. Math. Monthly*, 107(5):422–428, 2000.

[Gra18] Andrew Granville. Using dynamical systems to construct infinitely many primes. *Amer. Math. Monthly*, 125(6):483–496, 2018.

[Ves23] Alexander P. Veselov. Conway's light on the shadow of Mordell. *Math. Intelligencer*, 45(4):371–378, 2023.

[WR94] Ralph Walde and Paula Russo. Rational periodic points of the quadratic function $Q_c(x) = x^2 + c$. *Amer. Math. Monthly*, 101(4):318–331, 1994.