Derived Hecke algebras

Columbia Student Number Theory Seminar

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Motivation: Cohomology of arithmetic groups

$\mathbf{G}$ – semisimple algebraic group over $\mathbf{Q}$
K_∞ - choice of maximal compact
$X := \mathbf{G}(\mathbf{R})/K_\infty$ - the symmetric space for $\mathbf{G}$

Goal

Given an arithmetic subgroup $\Gamma \subseteq \mathbf{G}(\mathbf{Q})$, understand

$$H^*(\Gamma, R) \simeq H^*(\Gamma\backslash X, R)$$

for $R = \mathbf{C}, \mathbf{Q}, \mathbf{Z}, \mathbf{Q}_p, \mathbf{Z}_p, \ldots$

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- Generalizes the theory of modular forms (Eichler-Shimura, Matsushima, Franke, etc.)
- Admits natural integral structure
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We restrict ourselves to the *tempered part* $H^*(\Gamma, R)_{\mathrm{temp}} \subseteq H^*(\Gamma, R)$, which can be computed using $(\mathfrak{g}, K_\infty)$-cohomology.

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Let $\delta := \mathrm{rk}\,\mathbf{G} - \mathrm{rk}\,K_\infty$ and $j_0 := \frac{\dim(\Gamma\backslash X)-\delta}{2}$.

Borel

In the notation from above,

$$\dim H^{j_0+j}(\Gamma, \mathbb{C})_{\text{temp}} = \binom{\delta}{j} \dim H^{j_0}(\Gamma, \mathbb{C})_{\text{temp}}$$

for $j \in [0, \delta]$ and vanishes outside.

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Let $\delta := \operatorname{rk} \mathbf{G}(\mathbf{R}) - \operatorname{rk} K_\infty$ and $j_0 := \frac{\dim(\Gamma \backslash X) - \delta}{2}$.

Borel's theorem

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$$\dim H^{j_0+j}(\Gamma, \mathbf{Q}(\chi))_\chi = \binom{\delta}{j} \dim H^{j_0}(\Gamma, \mathbf{Q}(\chi))_\chi$$

for $j \in [j_0, j_0 + \delta]$ and vanishes outside. Moreover, this equality respects "eigenspaces" with respect to Hecke characters $\chi : \mathbb{T} \to \overline{\mathbf{Q}}$.

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Venkatesh conjectures that this "spectral degeneration" is explained by an action of a motivic cohomology group over $\mathbf{Q}$!

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Main goal

Give a satisfactory explanation for the redundancies in $H^*(\Gamma, \overline{\mathbf{Q}})_{\text{temp}}$.

Idea: Construct a "natural" δ-dimensional $\mathbf{Q}$-space V with $\bigwedge^* \mathrm{V} \circlearrowleft H^*(\Gamma, \overline{\mathbf{Q}})_\chi$ such that $H^*(\Gamma, \overline{\mathbf{Q}})_\chi$ is freely generated in dimension j_0.

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Venkatesh's conjecture

We may take $\mathrm{V} = H^1_{\text{mot}}((M_{\text{coad}})_{\mathbf{Z}}, \mathbf{Q}(1))$, where M_{coad} is the coadjoint motive corresponding to

$$\mathrm{Ad}^* \rho_\chi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to {}^L\hat{G}(\overline{\mathbf{Q}}_p) \to \mathrm{GL}(\hat{\mathfrak{g}}^* \otimes \overline{\mathbf{Q}}_p).$$

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The action of $\mathrm{V} \otimes \mathbf{Q}_p$ can be explicated via **derived Hecke operators**.

Setup

Let $\mathbf{G}$ be a split reductive group over a number field F. Fix a place v of F of characteristic p_v and let $q_v = \#k_v$. Set $G_v := \mathbf{G}(F_v)$. Let

$$V_v \subseteq U_v \subseteq G_v$$

with V_v - pro-p_v, normal of finite index inside U_v. Let S be a finite ring with q_v invertible. (e.g. U_v-maximal compact/Iwahori, $S = \mathbf{Z}/\ell^n$)

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with V_v - pro-p_v, normal of finite index inside U_v. Let S be a finite ring with q_v invertible. (e.g. U_v-maximal compact/Iwahori, $S = \mathbf{Z}/\ell^n$) The classical Hecke algebra is given by

$$H_S(G_v, U_v) := S[G_v//U_v] = S[G_v]^{U_v \times U_v} \simeq \mathrm{Hom}_{S[G_v]}(S[G_v/U_v], S[G_v/U_v])$$
$$\varphi([eU_v]) \leftarrow\!\shortmid \varphi$$

Derived Hecke algebra

Derived Hecke algebra

In the above notation, the *derived Hecke algebra for* (G_v, U_v) *over* S is

$$\mathcal{H}_S(G_v, U_v) := \mathrm{Ext}^*_{S[G_v]}(S[G_v/U_v], S[G_v/U_v])$$

Alternatively, if $\mathbf{P}^\bullet \to S[G_v/U_v]$ is a projective resolution of G_v-modules, then

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Note that $\mathcal{H}^0(G_v, U_v)$ recovers the classical Hecke algebra.

Derived invariants of a complex

- If P is projective U_v/V_v-module, then P is also projective as U_v-module.
- If $\mathbf{R}^\bullet \to T$ is projective resolution in $S[W]$-modules, then $H^*(\mathrm{Hom}_{W'}(\mathbf{R}^\bullet, \mathbf{R}^\bullet)) = H^*(W', T)$ for $W' \subseteq W$ of finite index.

Let $\mathbf{Q}^\bullet \to S$ in $\mathrm{Rep}^{\mathrm{sm}}(S[U_v/V_v])$, then $\mathbf{P}^\bullet = \mathrm{Ind}_{U_v}^{G_v}\mathbf{Q}^\bullet \to S[G_v/V_v]$ in $\mathrm{Rep}^{\mathrm{sm}}(S[G_v])$.
Let $\mathbf{M}^\bullet$ be a complex of G_v-modules. Define

$$\text{derived } U_v\text{-invariants of } \mathbf{M}^\bullet := \underline{\mathrm{Hom}}_{G_v}(\mathbf{P}^\bullet, \mathbf{M}^\bullet) = \underline{\mathrm{Hom}}_{U_v}(\mathbf{Q}^\bullet, (\mathbf{M}^\bullet)^{V_v})$$

Clearly,

$$\underline{\mathrm{End}}_{G_v}(\mathbf{P}^\bullet) \circlearrowleft \underline{\mathrm{Hom}}_{G_v}(\mathbf{P}^\bullet, \mathbf{M}^\bullet) \rightsquigarrow \mathcal{H}_S(G_v, U_v) \circlearrowleft H^*(\text{derived } U_v\text{-inv. of } \mathbf{M}^\bullet)$$

Arithmetic manifolds

For $K \subseteq \mathbf{G}(\mathbf{A}_f)$, fix $K^{(v)} = \prod_{w \neq v} K_w$. Recall that

$$Y(K) = \mathbf{G}(F) \backslash X_\infty \times \mathbf{G}(\mathbf{A}_f)/K$$

is the corresponding arithmetic manifold (possibly orbifold). Denote $\mathcal{H}_{v,S} = \mathcal{H}_v(G_v, K_v)$. For $U_v \subseteq G_v$, set

$$\mathbf{C}^\bullet(U_v) = \text{chain complex of } Y(K^{(v)} \times U_v) \text{ with coefficients in } S$$

If $\mathbf{M}^\bullet := \varprojlim_{U_v} \mathbf{C}^\bullet(U_v)$ and $V_v \subseteq K_v$ as before, we have

$$(\mathbf{M}^\bullet)^{V_v} \simeq \mathbf{C}^\bullet(V_v)$$

Furthermore, one can show that the natural map

$$\mathbf{C}^\bullet(K_v) = \mathbf{C}^\bullet(U_v)^{K_v/V_v} \to \underline{\mathrm{Hom}}_{S[K_v/V_v]}(S, \mathbf{C}^\bullet(V_v)).$$

is a quasi-isomorphism in $\mathbf{D}(\mathrm{Mod}_S)$, so that we obtain a quasi-isomophism

$$\mathbf{C}^\bullet(Y(K)) = \mathbf{C}^\bullet(K_v) \simeq \text{derived } K_v\text{-inv. of } \mathbf{M}^\bullet.$$

Passing to homology, this yields $\mathcal{H}_{v,S} \circlearrowleft H^*(Y(K), S)$.

Explicit description of $\mathcal{H}(G, U)$: invariant functions

For $x, y \in G/U$, denote by $G_{xy} \subseteq G$ the pointwise stabilizer of (x, y). In this model, elements of $\mathcal{H}(G, U)$ are assignments,

$$\mathcal{H}(G, U) \ni h \rightsquigarrow (x, y) \mapsto h(x, y) \in H^*(G_{xy}, S),$$

satisfying

- h is G-invariant,
- h has finite support modulo G.

with product

$$h_1 * h_2(x, y) = \sum_{z \in G/U} h_1(x, z) \cup h_2(z, y),$$

where cup-products are computed inside $H^*(G_{xyz}, S)$.

Explicit description of $\mathcal{H}(G, U)$: double cosets

Given $x \in G/U$, represented by $x = g_x U$, set $U_x := U \cap g_x U g_x^{-1}$. We show that

$$\bigoplus_{x \in [U \backslash G/U]} H^*(U_x, S) \xrightarrow{\sim} \mathcal{H}_{v,S}.$$

as follows: given $z \in [U \backslash G/U]$ and $\alpha \in H^*(U_z, S)$, associate the function $h_{z,\alpha}$ on $G/U \times G/U$ satisfying

- $h_{z,\alpha}(x, y) = 0$ if $(z, eU) \notin G \cdot (x, y)$,
- $h_{z,\alpha}(z, eU) = \alpha$.

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$$\mathcal{H}_{v,S} = \bigoplus_{\lambda \in X_*(T)^+} H^*(M_\lambda(k_v), S)$$

with $\# M_\lambda(k_v) = (q_v - 1)^{r_\lambda}$ for dominant λ. These vanish if $(q_v - 1) \in S^\times$.

Assume $q_v = 1$ **in** S. If $S = \mathbf{Z}/\ell^n$ this is reminiscent of Taylor-Wiles primes of level n. The analogy is explained by *Koszul duality*.

Describing the $\mathcal{H}_v$-action

The action $\mathcal{H}_{v,S} \circlearrowleft H^*(Y(K), S)$ should arise from action on $H^*(K_v, S)$.
Pick $\alpha \in H^*(K_v, S)$, then $\alpha = \mathrm{Inf}(\beta)$ for $\beta \in H^*(K_v/K_{v,1}, S)$.
If $K_1 = K^{(v)} \times K_{v,1}$, we have

$$\begin{array}{c} Y(K_1) \\ \Big\downarrow{\scriptstyle K_v/K_{v,1}} \\ Y(K) \end{array} \rightsquigarrow Y(K) \to BK_v/K_{v,1}$$

Pulling back along the map to the classifiying space, we get

$$H^*(K_v, S) \to H^*(Y(K), S), \quad \alpha \to \langle \alpha \rangle$$

The images of these classes are "Hecke-trivial." Indeed, for any $\langle\alpha\rangle$ as above and any Hecke operator T supported at w, not dividing $|S|$ or K, we have

$$T\langle\alpha\rangle = \deg(T)\langle\alpha\rangle.$$

For that reason these commute with the Hecke operators.

The action of $h_{z,\alpha}$ on $H^*(Y(K), S)$ is given by

$$H^*(Y(K)) \xrightarrow{\pi_1^*} H^*(Y(K_z)) \xrightarrow{\cup\langle\alpha\rangle} H^*(Y(K_z)) \xrightarrow{[g_z]\circ\pi_{2,*}} H^*(Y(K))$$

Summary

- $\mathcal{H}_S(G,U) := \mathrm{Ext}^*_{S[G]}(S[G/U], S[G/U]) = H^*\left(\underline{\mathrm{Hom}}_{S[G]}(\mathbf{P}^\bullet, \mathbf{P}^\bullet)\right)$
- $\mathcal{H}_S(G,U) \circlearrowleft H^*$(derived U-invariants of $\mathbf{M}^\bullet$)
- $\mathbf{M}^\bullet := \varprojlim_U \mathbf{C}^\bullet(U_v) \rightsquigarrow \mathcal{H}_{v,S} \circlearrowleft H^*(Y(K), S)$
- $\oplus_{x \in [U \backslash G/U]} H^*(U_x, S) \xrightarrow{\sim} \mathcal{H}_{v,S}$, so elements are indexed $h_{z,\alpha}$ with $z \in [U \backslash G/U]$, and $\alpha \in H^*(U_x, S)$
- The element $h_{z,\alpha}$ acts on $H^*(Y(K), S)$ via

$$H^*(Y(K)) \xrightarrow{\pi_1^*} H^*(Y(K_z)) \xrightarrow{\cup\langle\alpha\rangle} H^*(Y(K_z)) \xrightarrow{[g_z]\circ\pi_{2,*}} H^*(Y(K))$$

Derived Satake isomorphism

Suppose $S = \mathbf{Z}/\ell^r$ and $q_v = 1$ in S. Assume $\mathbf{G}$ is *split* over F_v. Let $\mathbf{A}$ be a torus and $W = W(\mathbf{G}, \mathbf{A})$ the associated Weyl group. Assume $\ell \nmid |W|$.

Theorem (Derived Satake isomorphism)

In the above notation, we have

$$\mathcal{H}_{v,\mathbf{Z}/\ell^r}(G_v, K_v) \xrightarrow{\sim} \mathcal{H}_{v,\mathbf{Z}/\ell^r}(A_v, A_v \cap K_v)^W$$

given by restriction.

Example

Suppose $q \equiv 1 \ (mod\ \ell)$. Then

$$\mathcal{H}_{q,\mathbf{Z}/\ell}(\mathrm{PGL}_2(\mathbf{Q}_q), \mathrm{PO}_2(\mathbf{Q}_q)) = \mathbf{Z}/\ell[x_0^{\pm 1}, y_1, z_2]^{\mathbf{Z}/2}$$

with x_0, y_1, z_2 of degrees $0, 1, 2$, respectively, and $\mathbf{Z}/2$-action permuting $x_0^{\pm 1}$ and negating y_1 and z_2.

Example

Pick $\mathbf{G} = \mathrm{Res}_{F/\mathbf{Q}}\mathrm{PGL}_2$ over imaginary quadratic field $F/\mathbf{Q}$. Let $\mathfrak{q} \lhd \mathcal{O}_F$ be relatively prime to ℓ, and set $k_\mathfrak{q} = \mathcal{O}_F/\mathfrak{q}$. Let $\alpha : k_\mathfrak{q}^\times \to \mathbf{Z}/\ell^m$ be a homomorphism. Pulling it back via

$$\Gamma_0(\mathfrak{q}) \to k_\mathfrak{q}^\times, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a/d,$$

we obtain $\langle\alpha\rangle \in H^1(\Gamma_0(\mathfrak{q}), \mathbf{Z}/\ell^m)$. Construct a derived Hecke operator

$$T_{\mathfrak{q},\alpha} : H^1(\mathrm{PGL}_2(\mathcal{O}_F)) \xrightarrow{\pi_1^*} H^1(\Gamma_0(\mathfrak{q})) \xrightarrow{\cup\langle\alpha\rangle} H^2(\Gamma_0(\mathfrak{q})) \xrightarrow{\pi_{2,*}} H^2(\mathrm{PGL}_2(\mathcal{O}_F)).$$

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We need to use torsion coefficients – there are **no** homomorphisms $\alpha : k_\mathfrak{q}^\times \to \mathbf{Z}$!

Langlands-Fontaine-Mazur

$$\left\{\begin{matrix}\text{algebraic automorphic}\\ \text{representations of } G\end{matrix}\right\} \quad \text{———} \quad \left\{\begin{matrix}\text{algebraic admissible}\\ \text{hom's } G \to {}^L G(\overline{\mathbf{Q}}_\ell)\end{matrix}\right\}$$

$$\{\hat{G} \text{ motives over } \mathbf{Q}\}$$

Galois cohomology and reciprocity laws

Fix $\chi : \mathbb{T} \to \mathbf{Z}_p$ (no congruences) at level $Y(K)$. Conjecturally, we may attach a Galois representation

$$\rho_\chi : \underbrace{\mathrm{Gal}(\overline{F}/F)}_{=: G_F} \to \mathrm{GL}_2(\mathbf{Z}_p)$$

unramified away from a set of primes T containing all primes above p.
Assume ρ is crystalline at all primes above $p > 2$. Set $\rho_m := \rho_\chi \ (mod\ p^m)$.
Denote by $\mathrm{Ad}^*\rho$ the $\mathbf{Z}_p$-linear dual to $\mathrm{Ad}\ \rho$.
For $\mathfrak{q} \notin T$, let $F_\mathfrak{q}$ be the completion of F, and embed

$$\mathbf{Z}_p \text{ with trivial } G_{F_\mathfrak{q}} \text{ action} \hookrightarrow \mathrm{Ad}\ \rho\big|_{G_{F_\mathfrak{q}}}$$
$$1 \mapsto 2\rho(\mathrm{Frob}_\mathfrak{q}) - \mathrm{tr}\big(\rho(\mathrm{Frob}_\mathfrak{q})\big)$$

Similarly, the embedding $\mathbf{Z}/p^m \hookrightarrow \mathrm{Ad}\ \rho_m\big|_{G_{F_{\mathfrak{q}}}}$ yields

$$\mathbf{Z}/p^m \times \mathrm{Ad}^*\rho(1) \to \mu_{p^m}.$$

By local reciprocity,

$$H^1(G_{F_{\mathfrak{q}}}, \mathbf{Z}/p^m) \times H^1(G_{F_{\mathfrak{q}}}, \mathrm{Ad}^*\rho(1)) \to \mathbf{Z}/p^m,$$

and restricting the second argument to classes unramified away from T and crystalline at p, we get a pairing

$$H^1(G_{F_{\mathfrak{q}}}, \mathbf{Z}/p^m) \times H^1_f(\mathcal{O}_{\mathfrak{q}}[\frac{1}{T}], \mathrm{Ad}^*\rho(1)) \to \mathbf{Z}/p^m.$$

$$H^1(G_{F_{\mathfrak{q}}}, \mathbf{Z}/p^m) \times H^1_f(\mathcal{O}[\frac{1}{T}], \mathrm{Ad}^*\rho(1)) \to \mathbf{Z}/p^m.$$

Let $\alpha : k_{\mathfrak{q}}^{\times} \to \mathbf{Z}/p^m$, and extend arbitrarily to $\tilde{\alpha} : F_{\mathfrak{q}}^{\times}/(1+\mathfrak{q}) \to \mathbf{Z}/p^m$. Up to unramified classes, this yields $\tilde{\alpha} \in H^1(G_{F_{\mathfrak{q}}}, \mathbf{Z}/p^m)$. The pairing with $H^1_f(\mathcal{O}_{\mathfrak{q}}[\frac{1}{T}], \mathrm{Ad}^*\rho(1))$ is independent of the choice of lift $\tilde{\alpha}$, so we obtain a well-defined homomorphism

$$[\mathfrak{q}, \alpha] : H^1_f(\mathcal{O}_{\mathfrak{q}}[\frac{1}{T}], \mathrm{Ad}^*\rho(1)) \to \mathbf{Z}/p^m,$$

which we relate to $T_{\mathfrak{q},\alpha}$.

Slogan

The Selmer group $H^1_f(\mathcal{O}_{\mathfrak{q}}[\frac{1}{T}], \mathrm{Ad}^*\rho(1))$ provides indexing of the derived Hecke operators via the homomorphisms $[\mathfrak{q}, \alpha]$ with $\mathfrak{q} \notin T$ and $\alpha : k_v^{\times} \to \mathbf{Z}/p^m$.

Reciprocity laws

It is believed that

$$[\mathfrak{q}, \alpha] = [\mathfrak{q}', \alpha'] \overset{?}{\Rightarrow} T_{\mathfrak{q},\alpha} = T_{\mathfrak{q}',\alpha'}$$

Currently, we only know this is ''asymptotically" true.

Lemma (Venkatesh)

There is $N_0(m)$ such that for primes $\mathfrak{q}$ and $\mathfrak{q}'$ satisfying

1. $\mathrm{Nm}(\mathfrak{q}) = \mathrm{Nm}(\mathfrak{q}') \equiv 1 \pmod{p^{N_0(m)}}$,
2. *the eigenvalues of $\rho(\mathrm{Frob}_{\mathfrak{q}})$ (resp. $\rho(\mathrm{Frob}_{\mathfrak{q}'})$) modulo p are distinct elements of $\mathbf{Z}/p$, and*
3. $[\mathfrak{q}, \alpha] = [\mathfrak{q}', \alpha']$,

the actions of $T_{\mathfrak{q},\alpha}$ and $T_{\mathfrak{q}',\alpha'}$ on $H^*(Y(K), \mathbf{Z}/p^m)$ are the same.

Selmer groups as p-adic avatars of motivic cohomology

Let:

- p –prime,
- $\mathbf{G}$ – semisimple algebraic group over $\mathbf{Q}$,
- Π – tempered cohomological cuspidal representation for $\mathbf{G}$ with $\Pi^{K_0} \neq 0$,
- $\chi : \mathbb{T}_{K_0} \to \mathbf{Q}$ – Hecke character corresponding to Π,
- $\rho_\chi : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to {}^L\hat{G}(\mathbf{Q}_p)$ – Galois representation associated to χ,
- M_{coad} – coadjoint (Chow) motive over $\mathbf{Q}$ corresponding to $\mathrm{Ad}^*\ \rho_\chi$, i.e.

$$H^0_{et}((M_{coad})_{\overline{\mathbf{Q}}}, \mathbf{Q}_p) \simeq \mathrm{Ad}^*\ \rho_\chi.$$

By work of Voevodsky, one may define motivic cohomology group $H^1_{mot}(M_{coad}, \mathbf{Q}(1))$, admitting comparison map

$$H^1_{mot}(M_{coad}, \mathbf{Q}(1)) \otimes_{\mathbf{Q}} \mathbf{Q}_p \to H^1(G_{\mathbf{Q}}, \mathrm{Ad}^* \rho_\chi(1)).$$

Scholl constructs subspace $H^1_{mot}((M_{coad})_{\mathbf{Z}}, \mathbf{Q}(1)) \subseteq H^1_{mot}(M_{coad}, \mathbf{Q}(1))$ of "classes with good integral models." The restriction of the comparison map is conjectured to land in a Bloch-Kato cohomology

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and, furthermore, to be an isomorphism.

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and, furthermore, to be an isomorphism. Beilinson's conjecture yields

$$\dim_{\mathbf{Q}} \mathrm{V} = \mathrm{ord}_{s=0} \ L(s, \mathrm{Ad}^* \ \rho_\chi(1)) = \delta.$$

Conjecture

Conjecture (Venkatesh)

In the above notation, let

$$\wedge^* \mathrm{V}_{\mathbf{Q}_p} \circlearrowleft H^*(Y(K_0), \mathbf{Q}_p)_\Pi$$

be the action furnished by the comparison map with $H^1_f(\mathbf{Z}[\frac{1}{T}], \mathrm{Ad}^* \rho_\chi(1))$. Then the action of $\wedge^* \mathrm{V}$ preserves the rational structure on $H^*(Y(K_0), \mathbf{Q})_\Pi$