Derived Deformation Theory

The patching part

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Disclaimer

Nothing in this document is original except for my mistakes and misunderstandings. I have also liberally copied verbatim sentences/paragraphs from [GV] and a few other sources including the notes [I] by A. Iyengar. I knew almost nothing about the materials before starting preparing. Please let me know if you have any questions, comments, corrections, etc.!

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Outline

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Goals of the talk

Goals of the talk is to explain the following things

1 (Descending from T-W level)

Given allowable T-W datum Q_n with deformation rings $\mathcal{R}_n$, $\mathcal{S}_n$, $\mathcal{S}_n^{\mathrm{ur}}$, we can recover the original deformation ring by

$$\mathcal{R}_S \simeq \mathcal{R}_n \underline{\otimes}_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{ur}}$$

Moreover if given quotients

$$\pi_0 \mathcal{R}_n \twoheadrightarrow \overline{\mathrm{R}}_n, \pi_0 \mathcal{S}_n \twoheadrightarrow \overline{\mathrm{S}}_n, \pi_0 \mathcal{S}_n^{\mathrm{ur}} \twoheadrightarrow \overline{\mathrm{S}}_n^{\mathrm{ur}}$$

each inducing isomorphism on $\mathfrak{t}^0$, then the composite map $\mathcal{R}_S \to \mathcal{R}_n \underline{\otimes}_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{ur}} \to \overline{\mathrm{R}}_n \underline{\otimes}_{\overline{\mathrm{S}}_n} \overline{\mathrm{S}}_n^{\mathrm{ur}}$ induces isomorphism on $\mathfrak{t}^0$ and surjection on $\mathfrak{t}^1$.

2 (The patching theorem)

Let $\iota : \mathrm{S}_\infty^\circ \to \mathrm{R}_\infty$ be a continuous map where

$$\mathrm{S}_\infty^\circ = W(k)\left[\left[X_1, \ldots, X_s\right]\right], \mathrm{R}_\infty = W(k)\left[\left[X_1, \ldots, X_{s-\delta}\right]\right]$$

making R_∞ a finite S_∞°-module.
Let $\mathfrak{a}_n = (p^n, (1+X_i)^{p^n} - 1)$ ideals of S_∞° and

$$\mathcal{C}_n = \mathrm{R}_\infty/\mathfrak{a}_n \underline{\otimes}_{\mathrm{S}_\infty^\circ/\mathfrak{a}_n} W_n$$

Then given $\mathcal{R}_0 \in$ pro-Art$_k$ with a collection of maps $f_n : \mathcal{R}_0 \to \mathcal{C}_n$ satisfying some conditions on tangent complex (e.g. inducing isomorphism on $\mathfrak{t}^0$ and surjection on $\mathfrak{t}^1$)
Then

$$\pi_* \mathcal{R}_0 \cong \mathrm{Tor}_*^{\mathrm{S}_\infty^\circ}(\mathrm{R}_\infty, W(k))$$

as graded rings. (The left side is defined as the inverse limit $\lim_{j \in J} \pi_*(\mathcal{R}_0)_j$)

3 (Main theorem)

There is an isomorphism

$$\pi_* \mathcal{R}_S \cong \mathrm{Tor}_*^{\mathrm{S}^\circ_\infty}(\mathrm{R}_\infty, W)$$

of graded rings hence the homology $H_*(Y_0, W)_\mathfrak{m}$ carries the structure of a free graded module over the graded ring $\pi_* \mathcal{R}_S$.

Brief motivation for the main theorem

1 For suitable Galois representations, the Langlands Program predicts that the Mazur's deformation ring should act on the homology of an arithmetic group. But we have already seen $\pi_0\mathcal{R}_S$ recovers the underived deformation ring, so such an action might be upgraded to a graded action of $\pi_*\mathcal{R}_S$.

2 Numerology of the Betti number for an arithmetic group. For instance if $\Gamma \leq \mathrm{SL}_n(\mathbb{Z})$ then the exterior algebra of a vector space of dimension $\delta = \left[\frac{n-1}{2}\right]$ acts on $H^*(\Gamma, \overline{\mathbb{Q}}_p)_\chi$. Now let $\mathcal{O}$ be the ring of integers of $\overline{\mathbb{Q}}_p$ and $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathcal{O})$ the Galois representation attached to χ and $\bar{\rho}$ its mod p reduction. Suppose the standard conjecture that ρ does not have characteristic zero crystalline deformations. Then $\pi_* \mathcal{R}_{\bar{\rho}} \otimes_{\pi_0 \mathcal{R}_{\bar{\rho}}} \overline{\mathbb{Q}}_p$ is isomorphic to an exterior algebra on δ generators. Thus it naturally suggests that $\pi_* \mathcal{R}_{\bar{\rho}}$ might in fact act freely on $H_*(\Gamma, \mathcal{O})_{\bar{\rho}}$.

3 While in C-G method it is already shown that the homology $H_*(\Gamma, \mathcal{O})_{\bar{\rho}}$ has the structure of a free module under a certain Tor-algebra arising in T-W limit process. So a simple way is to identify $\pi_*\mathcal{R}_S$ with the Tor-algebra.

Such identification is reasonable and natural. Note the underived descending gives $\mathrm{R} = \mathrm{R}_\infty \otimes_{\mathrm{S}_\infty} W$.
As to be shown in the patching theorem, one way for the Tor-algebra to come into the story is by the computation of homotopy groups of the tensor product $\mathrm{R}_n \otimes_{\mathrm{S}_n} W$ (and take a limit). So in derived case there should exist some identification between $\pi_*\mathcal{R}_S$ and the Tor-algebra coming from T-W limit. Also this Tor-algebra is *a priori* obscure since it depends on all the choices made in the limit process. And it would be nice if it could be identified with a more intrinsic object.

A few background

1. We are always talking about functors $\mathcal{F} : \mathsf{Art}_k \to \mathsf{sSet}$ and we have defined almost everything in a homotopy/weak equivalence sense, e.g. homotopy (co)limits, homotopy pullback, (pro-)representability.
2. In previous talks we have seen that there is a well-defined derived deformation problem/functor to deal with. In particular the deformation functor is pro-representable by Lurie's derived Schlessinger criterion. Also (underived) local conditions can be imposed to this functor, and in this talk we always assume a crystalline condition at p.

3 Almost all the notations and assumptions are given in [GV] section 6, 7, 10 and 13.1. In particular Conjecture 6.1 in [GV] assumes the existence of some Galois representation associated to certain Hecke algebra.

4 For objects in pro-Art_k, the derived tensor product really is viewed as the representing ring of the homotopy pullback of the representing functors, hence as long as they have the same index category, the derived tensor product has functorial properties.

Notations on Galois representation

Fix p and a finite field k of characteristic p with Witt vectors $W = W(k)$, $W_n = W(k)/p^n$.

S a finite set of primes containing p, G a split semisimple algebraic group over $W(k)$, e.g. PGL_n. $T \subset G$ a maximal k-split torus.

$\mathbf{G}$ the split reductive $\mathbb{Q}$-group whose root datum is dual to that of G and $\mathbf{G}$ admits a smooth reductive model over $\mathbb{Z}\left[\frac{1}{S-\{p\}}\right]$.

(More is coming...)

'Minimal level' Galois representation

Assumptions on ρ:

a $H^0\left(\mathbb{Q}_p, \mathrm{Ad}\rho_{\mathbb{Q}_p}\right) = H^2\left(\mathbb{Q}_p, \mathrm{Ad}\rho_{\mathbb{Q}_p}\right) = 0$ (local deformation at p is representable and formally smooth)

b For $v \in S - \{p\}$, the local cohomology $H^j\left(\mathbb{Q}_v, \mathrm{Ad}\rho_{\mathbb{Q}_v}\right) = 0$ for $j = 0, 1, 2$ (local deformation ring at $S - \{p\}$ is W)

c ρ has big image: the image of ρ restricted to $\mathbb{Q}\left(\zeta_{p^\infty}\right)$ contains the image of $G^{\mathrm{sc}}(k)$ in $G(k)$ here G^{sc} is the simply connected cover. (existence of allowable T-W data[GV] remark after 6.2)

Allowable T-W datum

For $\rho : \pi_1 \mathbb{Z}\left[\frac{1}{S}\right] \to G(k)$, a set of (T-W) primes $Q = \{\ell_1, ..., \ell_n\}$ disjoint from S and each $\rho(\mathrm{Frob}_{\ell_i})$ is conjugate to some $t_{\ell_i} \in T(k)$ whose centralizer in G is T, satisfying further conditions

a p^n divides each $\ell_i - 1$

b $H^2_{\mathfrak{l}}\left(\mathbb{Z}\left[\frac{1}{SQ}\right], \mathrm{Ad}\rho\right) = 0$ where local conditions $\mathfrak{l}_v$ at $v \in S \cup Q$ namely

$$\mathfrak{l}_v = \begin{cases} H^1_f, & v = p \\ H^1, & v \in Q \\ 0, & v \in S - \{p\} \end{cases}$$

Combined with the assumptions on ρ, the vanishing condition b means that in the sequence

$$H^1\left(\mathbb{Z}\left[\frac{1}{SQ}\right], \mathrm{Ad}\,\rho\right) \xrightarrow{A} \frac{H^1(\mathbb{Q}_p,\mathrm{Ad}\rho)}{H^1_f(\mathbb{Q}_p,\mathrm{Ad}\rho)}$$
$$\to \underbrace{H^2_{\mathfrak{l}}}_{0} \to H^2\left(\mathbb{Z}\left[\frac{1}{SQ}\right], \mathrm{Ad}\rho\right) \xrightarrow{B} \bigoplus_{Q} H^2\left(\mathbb{Q}_v, \mathrm{Ad}\rho\right)$$

A is surjective and B is injective.

Tangent complex

For a formally cohesive functor we can have associated tangent complex and if

$$\mathcal{F} = \mathcal{F}_0 \times^h_{\mathcal{F}_{01}} \mathcal{F}_1$$

where $\mathcal{F}_0$,$\mathcal{F}_{01}$,$\mathcal{F}_1$ are all formally cohesive, then $\mathcal{F}$ is also formally cohesive and $\mathfrak{t}\mathcal{F}$ is quasi-isomorphic to the mapping cone of $\mathfrak{t}\mathcal{F}_0 \oplus \mathfrak{t}\mathcal{F}_1 \to \mathfrak{t}\mathcal{F}_{01}$.

Also for our derived deformation functor

$$\mathfrak{t}^j \mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho} = \pi_{-j} \mathfrak{t}\mathcal{F}_{\mathbb{Z}[\frac{1}{S}],\rho} \cong H_f^{j+1}\left(\mathbb{Z}\left[\frac{1}{S}\right], \mathrm{Ad}\rho\right), j \geqslant -1$$

Descending from T-W level

Consider a representation $\rho : \pi_1 \mathbb{Z}\left[\frac{1}{S}\right] \to G(k)$ satisfying conditions given above. Denote $\mathcal{R}_S$ the crystalline deformation ring of ρ. Let Q_n be an allowable T-W set. We are going to see how the derived deformation ring at base level S could be recovered from the derived deformation ring at level SQ_n and under some condition on $\mathfrak{t}^0$ and $\mathfrak{t}^1$, even well approximated just using the usual (underived) deformation ring at level SQ_n or a sufficiently deep quotient of it.

For a T-W prime q, we set

- $\rho_{\mathbb{Z}_q}$ the pullback of ρ under $\pi_1(\mathbb{Z}_q) \to \pi_1(\mathbb{Z}\left[\frac{1}{S}\right])$, $\rho_{\mathbb{Q}_q}$ pullback via $\pi_1\mathbb{Q}_q \to \pi_1\mathbb{Z}_q$
- By assumption $\rho_{\mathbb{Z}_q}$ is conjugate to a $\rho^T_{\mathbb{Z}_q} : \pi_1\mathbb{Z}_q \to T(k)$, $\rho^T_{\mathbb{Q}_q}$ pullback via $\pi_1\mathbb{Q}_q \to \pi_1\mathbb{Z}_q$
- $\mathcal{F}_{\mathbb{Z}_q}, \mathcal{F}_{\mathbb{Q}_q}$ deformation functors for $\rho_{\mathbb{Z}_q}$ and $\rho_{\mathbb{Q}_q}$, similarly for $\mathcal{F}^T_{\mathbb{Z}_q}$ and $\mathcal{F}^{T,\square}_{\mathbb{Z}_q}$ where with $\square$ means the framed versions.

Then we have diagram where all squares are object-wise homotopy pullback and s-maps are sections[GV] 8.2

$$\begin{array}{ccccccc}
\mathcal{F}_{\mathbb{Z}\left[\frac{1}{S}\right]} & \longrightarrow & \mathcal{F}_{\mathbb{Z}_q} & \xleftarrow{\sim} & \mathcal{F}^T_{\mathbb{Z}_q} & \underset{\longleftarrow}{\overset{s_{\mathbb{Z}_q}}{\longrightarrow}} & \mathcal{F}^{T,\square}_{\mathbb{Z}_q} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{F}_{\mathbb{Z}\left[\frac{1}{Sq}\right]} & \longrightarrow & \mathcal{F}_{\mathbb{Q}_q} & \xleftarrow{\sim} & \mathcal{F}^T_{\mathbb{Q}_q} & \underset{s_{\mathbb{Q}_q}}{\overset{\longleftarrow}{\longrightarrow}} & \mathcal{F}^{T,\square}_{\mathbb{Q}_q}
\end{array}$$

Apply to Q_n we get

$$\mathcal{F}'_{S\coprod Q_n} \times^h_{\mathcal{F}_n^{\mathrm{loc}\prime}} \mathcal{F}_n^{\mathrm{loc,ur}\prime} \overset{\sim}{\dashrightarrow} \mathcal{F}_S$$

where $\mathcal{F}_n^{\mathrm{loc},(\mathrm{ur})} = \prod_{q\in Q_n} \mathcal{F}_q^{T,(\mathrm{ur}),\square}$ and a prime denotes a weakly equivalent functor. Then on deformation rings the equivalence above gives

$$\mathcal{R}_S \simeq \mathcal{R}_n \underline{\otimes}_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{ur}}$$

where here $\simeq$ means that the functors they represent are naturally weakly equivalent. Roughly speaking, 'a representation of $\pi_1\mathbb{Z}\left[\frac{1}{SQ}\right]$ unramified at Q is actually a representation of $\pi_1\mathbb{Z}\left[\frac{1}{S}\right]$'.

Suppose given pro-Artinian quotients

$$\pi_0 \mathcal{R}_n \twoheadrightarrow \overline{\mathrm{R}}_n, \pi_0 \mathcal{S}_n \twoheadrightarrow \overline{\mathrm{S}}_n, \pi_0 \mathcal{S}_n^{\mathrm{ur}} \twoheadrightarrow \overline{\mathrm{S}}_n^{\mathrm{ur}}$$

with a diagram $\overline{\mathrm{R}}_n \leftarrow \overline{\mathrm{S}}_n \rightarrow \overline{\mathrm{S}}_n^{\mathrm{ur}}$ commuting with the same diagram for the π_0 rings. Then we get

$$\mathcal{R}_S \rightarrow \mathcal{R}_n \underline{\otimes}_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{ur}} \rightarrow \overline{\mathrm{R}}_n \underline{\otimes}_{\overline{\mathrm{S}}_n} \overline{\mathrm{S}}_n^{\mathrm{ur}}$$

Theorem

If the quotient maps all induce isomorphisms on $\mathfrak{t}^0$ then the map $\mathcal{R}_S \rightarrow \overline{\mathrm{R}}_n \underline{\otimes}_{\overline{\mathrm{S}}_n} \overline{\mathrm{S}}_n^{\mathrm{ur}}$ induces isomorphism on $\mathfrak{t}^0$ and surjection on $\mathfrak{t}^1$.

Proof.

By properties of tangent complex, we have commutative diagram

$$
\begin{array}{ccccccc}
0 & \longrightarrow & \mathfrak{t}^0(\overline{\mathrm{R}}_n \underline{\otimes}_{\overline{\mathrm{S}}_n} \overline{\mathrm{S}}_n^{\mathrm{ur}}) & \longrightarrow & \mathfrak{t}^0(\overline{\mathrm{R}}_n) \oplus \mathfrak{t}^0(\overline{\mathrm{S}}_n^{\mathrm{ur}}) & \longrightarrow & \mathfrak{t}^0(\overline{\mathrm{S}}_n) \\
\downarrow & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\
0 & \longrightarrow & \mathfrak{t}^0(\mathcal{R}_n \underline{\otimes}_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{ur}}) & \longrightarrow & \mathfrak{t}^0(\mathcal{R}_n) \oplus \mathfrak{t}^0(\mathcal{S}_n^{\mathrm{ur}}) & \longrightarrow & \mathfrak{t}^0(\mathcal{S}_n)
\end{array}
$$

where by assumption f and g are all isomorphisms hence so is j_1.
Continue the commutative diagram

$$
\begin{array}{ccccccc}
\mathfrak{t}^0(\overline{\mathrm{S}}_n) & \longrightarrow & \mathfrak{t}^1(\overline{\mathrm{R}}_n \underline{\otimes}_{\overline{\mathrm{S}}_n} \overline{\mathrm{S}}_n^{\mathrm{ur}}) & \longrightarrow & \mathfrak{t}^1(\overline{\mathrm{R}}_n) \oplus \mathfrak{t}^1(\overline{\mathrm{S}}_n) & & \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle j_2} & & \downarrow & & \\
\mathfrak{t}^0(\mathcal{S}_n) & \xrightarrow{\beta} & \mathfrak{t}^1(\mathcal{R}_n \underline{\otimes}_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{ur}}) & \xrightarrow{\gamma} & \mathfrak{t}^1(\mathcal{R}_n) \oplus \mathfrak{t}^1(\mathcal{S}_n^{\mathrm{ur}}) & \xrightarrow{h} & \mathfrak{t}^1(\mathcal{S}_n)
\end{array}
$$

$\square$

Note $\mathfrak{t}^1\mathcal{S}_n^{\mathrm{ur}} = 0$ since $\mathcal{S}_n^{\mathrm{ur}}$ is formally smooth.

Moreover by our assumption on ρ and Q_n

$$\begin{array}{ccc}
\mathfrak{t}^1\mathcal{R}_n & \xrightarrow{\;=\;} & H^2_f\left(\mathbb{Z}\left[\frac{1}{SQ_n}\right], \mathrm{Ad}\rho\right) \\
 & & \big\downarrow \simeq \\
\Big\downarrow h & & H^2\left(\mathbb{Z}\left[\frac{1}{SQ_n}\right], \mathrm{Ad}\rho\right) \\
 & & \big\downarrow B \\
\mathfrak{t}^1\mathcal{S}_n & \xrightarrow{\;=\;} & \bigoplus_{v \in Q_n} H^2(\mathbb{Q}_v, \mathrm{Ad}\rho^{(T)})
\end{array}$$

where the first isomorphism comes from the long exact sequence for f-cohomology and use the fact that A is surjective and second local cohomology at p is 0. The last term is by [GV] 8.3. Hence in the above diagram h is injective, γ is 0, β is surjective. Since g is isomorphism, j_2 is surjective. This finishes the proof.

For future use, we also want to understand $\mathcal{S}_q$ and $\mathcal{S}_q^{\mathrm{ur}}$ better.

Lemma

The pro-rings $\mathcal{S}_q^{(\mathrm{ur})}$ are homotopy discrete, i.e. the map $\mathcal{S}_q^{(\mathrm{ur})} \to \pi_0 \mathcal{S}_q^{(\mathrm{ur})} = \mathrm{S}_q^{(\mathrm{ur})}$ induces weak equivalence of represented functors[GV] 8.6. And the commutative diagram is a homotopy pullback

$$\begin{array}{ccc} \mathrm{S}_q^{\circ} & \longrightarrow & \mathrm{S}_q \\ \downarrow & & \downarrow \\ W(k) & \longrightarrow & \mathrm{S}_q^{\mathrm{ur}} \end{array}$$

where $\mathrm{S}_q^{\circ} = W\left[[Y_1, \ldots, Y_r]\right] / ((1+Y_i)^{p^N} - 1)$ for $r = \mathrm{rank}(G)$, $N = \mathrm{ord}_p(q-1)$ and the diagram is still a homotopy pullback after they all quotient by p^n.

Remark

With the same notations, actually in the above lemma[GV] 8.14

$$\mathrm{S}_q = W\left[\left[X_1, \ldots, X_r, Y_1, \ldots, Y_r\right]\right] / ((1+Y_i)^{p^N} - 1)$$
$$\mathrm{S}_q^{\mathrm{ur}} = W\left[\left[X_1, \ldots, X_r\right]\right]$$

Canonically these rings can be identified as

$$\mathrm{S}_q^{\circ} = \text{ completed group algebra of } \mathbf{T}\left(\mathbb{F}_q\right) = W\left[\mathbf{T}(\mathbb{F}_q)_p\right]$$

$$\mathrm{S}_q = \text{ completed group algebra of } \mathbf{T}\left(\mathbb{Q}_q\right)^{\text{tame}}$$

$$\mathrm{S}_q^{\mathrm{ur}} = \text{ completed group algebra of } \mathbf{T}\left(\mathbb{Q}_q\right)^{\text{ur}}$$

where $\mathbf{T}$ is the dual torus to T in $\mathbf{G}$ and $(\ )_p$ denotes the p-part.

Patching

Theorem

Let continuous map

$$\mathrm{S}_\infty^\circ = W(k)[[X_1,\dots,X_s]] \xrightarrow{\iota} \mathrm{R}_\infty = W(k)[[X_1,\dots,X_{s-\delta}]]$$

making R_∞ *a finite* S_∞°*-module. Let* $\mathfrak{a}_n = (p^n, (1+X_i)^{p^n}-1)$ *and*

$$\mathcal{C}_n = \mathrm{R}_\infty/\mathfrak{a}_n \underline{\otimes}_{\mathrm{S}_\infty^\circ/\mathfrak{a}_n} W_n$$

We have natural maps $e_{n,m} : \mathcal{C}_n \to \mathcal{C}_m$ *for* $n > m$.

Then given $\mathcal{R}_0 \in$ pro-Art$_k$, $\mathfrak{t}^i\mathcal{R}_0$ supported in degree 0, 1 and satisfying the 'Euler characteristic' relation

$$\dim \mathfrak{t}^0\mathcal{R}_0 - \dim \mathfrak{t}^1\mathcal{R}_0 = \dim(\mathrm{R}_\infty) - \dim(\mathrm{S}^\circ_\infty)$$

with a collection of maps $f_n : \mathcal{R}_0 \to \mathcal{C}_n$ such that for every $n > m$, the composite $f_{n,m} : \mathcal{R}_0 \to \mathcal{C}_n \stackrel{e_{n,m}}{\to} \mathcal{C}_m$ induces isomorphism on $\mathfrak{t}^0$ and surjection on $\mathfrak{t}^1$. Then as graded rings

$$\pi_*\mathcal{R}_0 \cong \mathrm{Tor}_*^{\mathrm{S}^\circ_\infty}(\mathrm{R}_\infty, W(k))$$

Proof.

First for $A = (j \mapsto A_j)$ and $B = (i \mapsto B_i)$, $A, B \in$pro-Art$_k$ let

$$\text{pro-Art}_k(A, B) = \lim_i \operatorname{colim}_j \text{Art}_k(A_j, B_i)$$

and $[A, B] = \pi_0 (\text{pro-Art}_k(A, B))$. Then for our discussion where $A = \mathcal{R}_0$ and $B = \mathcal{C}_n$, we have $[\mathcal{R}_0, \mathcal{C}_n]$ is finite since $\pi_* \mathfrak{t}\mathcal{R}_0$ vanishes beside 0, -1.

Then by a compactness argument we could replace f_n by g_n with same condition but additional compatibility g_n and $e_{n+1,n} \circ g_{n+1}$ are homotopic. Then we can glue the natural transformations of functors induced by g_n

$$\text{Hom}\,(\mathcal{C}_n, -) \to \text{Hom}\,(\mathcal{R}_0, -)$$

to get

$$\text{hocolim}_n \,\text{Hom}\,(\mathcal{C}_n, -) \to \text{Hom}\,(\mathcal{R}_0, -)$$

which is also an isomorphism on $\mathfrak{t}^0$ and epimorphism on $\mathfrak{t}^1$.

Claim actually it is an isomorphism on $\mathfrak{t}^i$ for all i by showing both sides vanish besides 0, 1 and have the same "Euler characteristic". We have an exact triangle in the derived category of k-modules

$$\mathfrak{t}\mathcal{C}_n \to \mathfrak{t}\left(\mathrm{R}_\infty/\mathfrak{a}_n\right) \oplus \mathfrak{t}\left(W_n\right) \to \mathfrak{t}\left(\mathrm{S}^\circ_\infty/\mathfrak{a}_n\right) \xrightarrow{[1]}$$

and taking cohomology and taking direct limit gives

$$\varinjlim \mathfrak{t}^i\mathcal{C}_n \to \mathfrak{t}^i\mathrm{R}_\infty \oplus \underbrace{\mathfrak{t}^i W(k)}_{0} \to \mathfrak{t}^i\mathrm{S}^\circ_\infty \xrightarrow{[1]}$$

and $\mathfrak{t}^i(\mathrm{R}_\infty) = \mathfrak{t}^i(\mathrm{S}^\circ_\infty) = 0$ since these are power series rings thus formally smooth.
Then by exactness we have

$$\begin{aligned}\dim \mathfrak{t}^1\mathcal{C} - \dim \mathfrak{t}^0\mathcal{C} &= \dim \operatorname{colim}_n \mathfrak{t}^1\mathcal{C}_n - \dim \operatorname{colim}_n \mathfrak{t}^0\mathcal{C}_n \\ &= \dim \mathfrak{t}^0 S^\circ_\infty - \dim \mathfrak{t}^0 R_\infty \\ &= \dim \mathfrak{t}^1\mathcal{R}_0 - \dim \mathfrak{t}^0\mathcal{R}_0\end{aligned}$$

Hence the pro-objects $\mathcal{R}_0$ and $(n \mapsto \mathcal{C}_n)$ represent equivalent functors and hence the induced map of homotopy groups

$$\pi_* \mathcal{R}_0 = \varprojlim \pi_* (\mathcal{R}_0)_j \to \varprojlim \pi_* \mathcal{C}_n$$

is also an isomorphism. This concludes the proof since by computation of homotopy groups of a tensor product[Q] theorem 6 and [GV] 7.6 we have

$$\varprojlim \pi_i \mathcal{C}_n = \varprojlim \mathrm{Tor}_i^{\mathrm{S}^\circ_\infty/\mathfrak{a}_n} (\mathrm{R}_\infty/\mathfrak{a}_n, W_n) \cong \mathrm{Tor}_i^{\mathrm{S}^\circ_\infty} (\mathrm{R}_\infty, W)$$

Obstructed T-W method

Assume associated to a given Hecke eigenclass there is a Galois representation $\rho : \pi_1 \mathbb{Z}\left[\frac{1}{S}\right] \to G(k)$ (satisfying pages of conditions), then the following data can be found (and more):

a map ι as in the patching theorem

b allowable T-W data Q_n, associated covering groups Δ_n and deformation rings $\mathcal{R}_n$

c function $K(n) \to \infty$ and commutative diagram

$$\begin{array}{ccccc}
\mathrm{S}^{\circ}_{\infty} & \xrightarrow{\iota} & \mathrm{R}_{\infty} & \longrightarrow & \pi_0 \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \\
\downarrow f_n & & \downarrow g_n & & \downarrow \\
\overline{S^{\circ}_n} & \longrightarrow & \overline{\mathrm{R}}_n & \longrightarrow & \pi_0 \mathcal{R}_{\mathbb{Z}[\frac{1}{S}]}/(p^n, \mathfrak{m}^{K(n)})
\end{array}$$

inducing isomorphisms

$$\begin{cases} \mathrm{S}_\infty^\circ/\mathfrak{a}_n \to \overline{S_n^\circ} := W_n[\Delta_n] \\ \mathrm{R}_\infty/\mathfrak{a}_n \cong \mathrm{R}_\infty \otimes_{\mathrm{S}_\infty^\circ} \overline{S_n^\circ} \to \overline{\mathrm{R}}_n := \pi_0\mathcal{R}_n/(p^n, \mathfrak{m}^{K(n)}) \end{cases}$$

d complex D_∞ of free S_∞°-modules with

$$H_* \left(D_\infty \otimes_{\mathrm{S}_\infty^\circ} W\right) \cong H_* \left(Y_0, W\right)_{\mathfrak{m}}$$

compatible with R_∞-actions and $H_*(D_\infty)$ is concentrated in degree q where $H_q(D_\infty)$ is a finite free R_∞-module

e The map $\mathrm{R}_\infty \otimes_{\mathrm{S}_\infty^\circ} W \twoheadrightarrow \pi_0\mathcal{R}_{\mathbb{Z}[\frac{1}{S}]}$ is isomorphism and the action of $\mathrm{R}_\infty \otimes_{\mathrm{S}_\infty^\circ} W$ on $H_*(Y_0, W)_{\mathfrak{m}}$ extends to a free graded action of $\mathrm{Tor}_*^{\mathrm{S}_\infty^\circ}(\mathrm{R}_\infty, W)$ on $H_*(Y_0, W)_{\mathfrak{m}}$

Remark

Let $M = H_q(D_\infty)$ then actually the map $D_\infty \to M$ is a projective resolution. Hence

$$H_*(Y_0, W)_{\mathfrak{m}} = H_*(D_\infty \otimes_{S_\infty^\circ} W) \cong \mathrm{Tor}_*^{S_\infty^\circ}(M, W)$$

and since M is free over R_∞, the whole homology carries a free action of $\mathrm{Tor}_*^{S_\infty^\circ}(R_\infty, W)$.

Sketch of the proof of main theorem

- The set-up from obstructed T-W method gives T-W sets Q_n, limit rings R_∞, S°_∞ and isomorphims $\mathrm{R}_\infty/\mathfrak{a}_n \simeq \overline{\mathrm{R}}_n$ and more. Put

$$\widetilde{\Delta_n} := \prod_{q \in Q_n} \mathbf{T}\left(\mathbb{F}_q\right)_p$$

and $\Delta_n = \widetilde{\Delta_n}/p^n$. Then the diagram

$$\overline{\mathrm{R}}_n \leftarrow \underbrace{\pi_0 \mathcal{S}_n \otimes_{W[\tilde{\Delta}_n]} W_n\left[\Delta_n\right]}_{:=\overline{\mathrm{S}}_n} \rightarrow \underbrace{\pi_0 \mathcal{S}_n^{\mathrm{ur}}/p^n}_{:=\overline{\mathrm{S}}_n^{\mathrm{ur}}}$$

admits a map from

$$\overline{\mathrm{R}}_n \leftarrow \overline{\mathrm{S}^\circ_n} \rightarrow W_n$$

This is because we have explicit descriptions of the objects

$$\widetilde{\Delta_n} = \prod \mathbf{T}(\mathbb{F}_q)_p \qquad \cong \prod (\mathbb{Z}/p^{n'_q})^r$$
$$\Delta_n = \prod \mathbf{T}(\mathbb{F}_q)/p^n \quad \cong (\mathbb{Z}/p^n)^{r \cdot \# Q_n}$$

where $r = \mathrm{rank}(G), n'_q = \mathrm{ord}_p(q-1)$. Then just as the lemma before we have the following homotopy pullback

$$\begin{array}{ccc} \overline{\mathrm{S}^\circ_n} & \longrightarrow & \overline{\mathrm{S}}_n \\ \downarrow & & \downarrow \\ W_n & \longrightarrow & \overline{\mathrm{S}}^{\mathrm{ur}}_n \end{array}$$

and induces a weak equivalence on derived tensor product. Thus we get

$$\mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \to \mathcal{R}_n \underline{\otimes}_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{ur}} \to \overline{\mathrm{R}}_n \underline{\otimes}_{\overline{\mathrm{S}}_n} \overline{\mathrm{S}}_n^{\mathrm{ur}} \xleftarrow{\sim} \overline{\mathrm{R}}_n \underline{\otimes}_{\overline{\mathrm{S}_n^\circ}} W_n$$

where the last equivalence is from the lemma.

But the last term is isomorphic to $\mathrm{R}_\infty/\mathfrak{a}_n \underline{\otimes}_{\mathrm{S}_\infty^\circ/\mathfrak{a}_n} W_n$ hence we get maps $\mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \longrightarrow \mathrm{R}_\infty/\mathfrak{a}_n \underline{\otimes}_{\mathrm{S}_\infty^\circ/\mathfrak{a}_n} W_n$ in pro-Art$_k$.

- Now it suffices to check the tangent complex condition to apply the patching theorem. We have to show $\mathfrak{t}^i\mathcal{R}_{\mathbb{Z}[\frac{1}{S}]}$ is supported in degree 0, 1 and have the correct 'Euler characteristic'. This is just a computation of the f-cohomology.

  Also we need to study the composite

$$\mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \to \mathrm{R}_\infty/\mathfrak{a}_n \underline{\otimes}_{\mathrm{S}_\infty^\circ/\mathfrak{a}_n} W_n \to \mathrm{R}_\infty/\mathfrak{a}_m \underline{\otimes}_{\mathrm{S}_\infty^\circ/\mathfrak{a}_m} W_m$$

  for $n > m$ and we want to show they induce isomorphism on $\mathfrak{t}^0$ and surjection on $\mathfrak{t}^1$.

To show this, we apply the descending theorem from T-W level. Then we are left to verify the quotients

$$\pi_0 \mathcal{R}_n \twoheadrightarrow \overline{\mathrm{R}}_{n,m}, \pi_0 \mathcal{S}_n \twoheadrightarrow \overline{\mathrm{S}}_{n,m}, \pi_0 \mathcal{S}_n^{\mathrm{ur}} \twoheadrightarrow \overline{\mathrm{S}}_{n,m}^{\mathrm{ur}}$$

induce isomorphisms on $\mathfrak{t}^0$ where

$$\overline{\mathrm{R}}_{n,m} = \overline{\mathrm{R}}_n/\mathfrak{a}_m, \overline{\mathrm{S}}_{n,m} = \overline{\mathrm{S}}_n/\mathfrak{a}_m, \overline{\mathrm{S}}_{n,m}^{\mathrm{ur}} = \overline{\mathrm{S}}_n^{\mathrm{ur}}/p^m$$

This is true because they are injective on tangent complex and the ideals defining these quotients all live in the square of the maximal ideal for $n > m > 1$.
Then the resulting map

$$\mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} \to \overline{\mathrm{R}}_{n,m} \underline{\otimes}_{\overline{\mathrm{S}}_{n,m}} \overline{\mathrm{S}}_{n,m}^{\mathrm{ur}}$$

is isomorphism on $\mathfrak{t}^0$ and surjection on $\mathfrak{t}^1$.

Then we have commutative diagram

$$\begin{array}{ccccc}
\mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} & \longrightarrow & \mathrm{R}_\infty/\mathfrak{a}_n \underline{\otimes}_{\mathrm{S}_\infty^\circ/\mathfrak{a}_n} W_n & \longrightarrow & \mathrm{R}_\infty/\mathfrak{a}_m \underline{\otimes}_{\mathrm{S}_\infty^\circ/\mathfrak{a}_m} W_m \\
\Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow{t} \\
\mathcal{R}_{\mathbb{Z}[\frac{1}{S}]} & \longrightarrow & \overline{\mathrm{R}}_n \underline{\otimes}_{\overline{\mathrm{S}}_n} \overline{\mathrm{S}}_n^{\mathrm{ur}} & \longrightarrow & \overline{\mathrm{R}}_{n,m} \underline{\otimes}_{\overline{\mathrm{S}}_{n,m}} \overline{\mathrm{S}}_{n,m}^{\mathrm{ur}}
\end{array}$$

Here the map t induces isomorphisms on tangent complexes. This finishes the proof.

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