Note for Simplicial Galois representations

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Disclaimer

Nothing in this Note is original except for the mistakes and misunderstanding. I also copied sentences and results from my references. I tried to covered the content of[G9] and I mainly follow[G9] and Chapter 4.7, 5, 9 and Appendix B of[GV]. My main reference for homotopy theory involved is[GJ] and I also got help from Roy Magen for tons of problems. I knew almost nothing about the content before starting preparing this.

1 Local system

Recall that if $\mathcal{F} : \mathrm{sArt}_k \to \mathrm{sSet}$ is formally cohesive, namely, $\mathcal{F}$ is homotopy invariant, preserves homotopy pullback, and $\mathcal{F}(k)$ is contractible (weakly equivalent to Δ^0), the tangent complex $\mathfrak{t}\mathcal{F} \in \mathrm{Ch}(k)$ is constructed. It is characterized in Prop 3.2,[G8] as follows: Let $DK : \mathrm{Ch}(k)_{\geq 0} \to s\mathrm{Vect}/k$ be the Dold-Kan correspondence, $\mathrm{Ch}(k)_{\geq 0}^{fd} \subset \mathrm{Ch}_{\geq 0}(k)$ be the full subcategory of objects V with $\dim_k(\pi_*(V)) < \infty$, $k \oplus V := k \oplus DK(V)$, equipped with the simplicial ring structure by $(a, u)(b, v) := (ab, av + bu)$, which is in sArt_k. $\mathfrak{t}\mathcal{F} \in \mathrm{Ch}(k)$ is the unique chain complex, up to quasi-isomorphism, together with a naturally weak equivalence

$$\mathcal{F}(k \oplus V) \to DK(\tau_{\geq 0}(\mathfrak{t}\mathcal{F} \otimes_k V)).$$

In particular, $H_{i-n}(\mathfrak{t}\mathcal{F}) \cong \pi_i \mathcal{F}(k \oplus k[n])$ for $i, n \geq 0$.

Lemma 1 (Lemma 4.29, 4.30 (iv) of[GV])**.** If G is defined by the homotopy pullback square

$$G = F_0 \times^h_{F_{01}} F_1$$

and F_0, F_{01}, F_1 are all formally cohesive, then G is also formally cohesive and $\mathfrak{t}G$ is quasi-isomorphic to the mapping cone of $\mathfrak{t}F_0 \oplus \mathfrak{t}F_1 \to \mathfrak{t}F_{01}$.

If $\mathcal{F} : \mathrm{sArt}_k \to \mathrm{sSet}$ is homotopic invariant, preserves homotopy pull-back, but $\mathcal{F}(k)$ is not contractible, we can construct $\mathfrak{t}\mathcal{F}$ as a local system on $\mathcal{F}(k)$.

Definition 1. The category $\mathrm{Simp}(Z)$ has objects pairs $([p], \sigma : \Delta^p \to Z)$, and morphisms are morphisms of simplicial sets that commute with the maps to Z.

A local system on a simplicial set Z is a functor

$$\mathcal{L} : \mathrm{Simp}(Z) \to \mathrm{Ch}(k)$$

sending all morphisms to quasi-isomorphisms.

Let $\mathcal{L}$ be a local system on Z. We define the cochain $C^*(Z, \mathcal{L})$ as follows: We first consider

$$\prod_{\sigma \in \mathrm{Simp}(Z)} \mathcal{L}_\sigma .$$

This is a bi-graded k-vector space with commuting boundary maps: The first one is that inherited from $\mathrm{Ch}(k)$, the second one is given by the degree of σ and the alternating sum of the boundary maps $d_i\sigma \to \sigma$. Hence we get a double complex. $C^*(Z, \mathcal{L})$ is defined as the total complex of the double complex.

For every $\sigma : \Delta^p \to \mathrm{sSet} \in \mathrm{Simp}(Z)$, we define $\mathcal{F}_\sigma : \mathrm{sArt}_k \to \mathrm{sSet}$ by the homotopy pull-back square

$$\begin{array}{ccc} \mathcal{F}_\sigma(A) & \longrightarrow & \mathcal{F}(A) \\ \downarrow & & \downarrow \\ \Delta^p & \xrightarrow{\sigma} & \mathcal{F}(k). \end{array}$$

By definition, $\mathcal{F}_\sigma$ is homotopic invariant and preserves homotopy pull-back. $\mathcal{F}_\sigma(k) = \Delta^p$, which is contractible. Hence we have the tangent complex $\mathfrak{t}\mathcal{F}_\sigma$ and the functor $\mathfrak{t}\mathcal{F} : \sigma \mapsto \mathfrak{t}\mathcal{F}_\sigma$ from $\mathrm{Simp}(Z)$ to $\mathrm{Ch}(k)$. Since every morphism τ in $\mathrm{Simp}(Z)$ induces isomorphism on π_*, the homotopy pull-back square

$$\begin{array}{ccccc} \mathcal{F}_{\sigma'}(A) & \xrightarrow{\tau_*} & \mathcal{F}_\sigma(A) & \longrightarrow & \mathcal{F}(A) \\ \downarrow & & \downarrow & & \downarrow \\ \Delta^{p'} & \xrightarrow{\tau} & \Delta^p & \xrightarrow{\sigma} & \mathcal{F}(k). \end{array}$$

gives that $\pi_*(\mathcal{F}_{\sigma'}(A)) \cong \pi_*(\mathcal{F}_\sigma(A))$. In particular, τ_* induces an quasi-isomorphism $\mathfrak{t}\mathcal{F}_{\sigma'} \cong \mathfrak{t}\mathcal{F}_{\sigma'}$. Hence $\mathfrak{t}\mathcal{F} : \sigma \mapsto \mathfrak{t}\mathcal{F}_\sigma$ is a local system.

Let $Z = \mathcal{F}(k)$, X be any simplicial set, and $\overline{\rho} \in \mathrm{Hom}_{\mathrm{sSet}}(X, Z)$. We define

$$\mathcal{F}_{X,\overline{\rho}}(A) := \underline{\mathrm{Hom}}_{\mathrm{sSet}}(X, \mathcal{F}(A)) \times^h_{\underline{\mathrm{Hom}}_{\mathrm{sSet}}(X, \mathcal{F}(k))} (\overline{\rho} : \{*\} \to \underline{\mathrm{Hom}}_{\mathrm{sSet}}(X, \mathcal{F}(k))).$$

Then $\mathcal{F}_{X,\overline{\rho}}(A)$ is formally cohesive.

Proposition 1. The tangent complex of $\mathcal{F}_{X,\overline{\rho}}$ is

$$\mathfrak{t}\mathcal{F}_{X,\overline{\rho}}(A) \cong C^*(X, \overline{\rho}^* \,\mathfrak{t}\mathcal{F}),$$

where $\overline{\rho}^* \,\mathfrak{t}\mathcal{F}$ is the local system

$$\mathrm{Simp}(X) \xrightarrow{\overline{\rho}} \mathrm{Simp}(Z) \xrightarrow{\mathfrak{t}\mathcal{F}} \mathrm{Ch}(k).$$

A proof is given in [G8] and I sketch the proof here: If X is Δ^p and $X \to Z$ is σ, then for every simplicial k-vector space V,

$$\mathcal{F}_{X,\overline{\rho}}(k \oplus V) = \mathcal{F}_\sigma(k \oplus V)$$

by definition. In this case, $\mathfrak{t}\mathcal{F}_{X,\overline{\rho}}$ is simply $\mathfrak{t}\mathcal{F}_\sigma$. Since X is the homotopy colimit of all its simplices, $\mathcal{F}_{X,\overline{\rho}}(k \oplus V)$ is the homotopy limit of $\{\mathcal{F}_\sigma(k \oplus V)\}_{\sigma \in \mathrm{Simp}(X)}$. $\mathfrak{t}\mathcal{F}_{X,\overline{\rho}}$ is the homotopy limit of $\{\mathfrak{t}\mathcal{F}_\sigma\}_{\sigma \in \mathrm{Simp}(X)}$, which is $C^*(X, \overline{\rho}^* \,\mathfrak{t}\mathcal{F})$.

2 Derived Galois deformation functor

Our goal is to define a derived version of Galois deformation functor. Let S be a finite set of rational primes, $\mathbb{Q}_S / \mathbb{Q}$ be the maximal unramifield extension outside S, and G_S be the Galois group $\mathrm{Gal}(\mathbb{Q}_S / \mathbb{Q})$. Let k be a perfect field, G be an affine algebraic group over $W(k)$, and $\mathcal{O}_G$ be the affine coordinate ring of G. We want to define a simplicial version of representation functor in this settings. We should first define the simplicial version of the representation functor $A \mapsto (\rho : G_S \to G(A))$. The functor should be homotopy invariant.

In the ordinary case, $G(A) = \mathrm{Hom}_{\mathrm{CR}}(\mathcal{O}_G, A)$. Let $A \in \mathrm{sArt}_k$. We may define the functor $A \mapsto \underline{\mathrm{Hom}}_{\mathrm{sCR}}(c(\mathcal{O}_G), A)$, where $c(\mathcal{O}_G)$ is the cofibrant replecement of $\mathcal{O}_G$. However, we do not have the Hopf algebra structure on $c(\mathcal{O}_G)$, so we do not have the simplicial group structure on $\underline{\mathrm{Hom}}_{\mathrm{sCR}}(c(\mathcal{O}_G), A)$.

Note that $G_S = \pi_1^{\text{ét}}(\mathbb{Z}[1/S], *)$, and a group homomorphism from the fundamental group of a space X to $G(A)$ should corresponds to $\mathrm{Hom}(X, BG(A))$ in the ordinary case. We can first define $BG : \mathrm{sArt}_k \to \mathrm{sSet}$ serving as the classifying space.

In the ordinary case, $G(A)$ is a group and can be regarded as the groupoid with only 1 object with automorphism $G(A)$. The classifying space is $NG(A)$, where $N_p(G(A)) = G(A)^{\times p} = \mathrm{Hom}_{\mathrm{CR}}(\mathcal{O}_{N_pG}, A)$. In the simplicial case, $[p] \mapsto \underline{\mathrm{Hom}}_{\mathrm{sCR}}(c(\mathcal{O}_{N_pG}), A)$ is a bisimplicial set, i.e. a simplicial object in sSet.

We follow[GJ] to give the definition of the geometric realization of a bisimplicial set, which is a simplicial set.

Definition 2. Let X be a bisimplicial set, $X_n := X([n])$. For every morphism $\theta \in \mathrm{Hom}_\Delta([m], [n])$, there are two associated morphisms

$$1 \times \theta_* : X_n \times \Delta^m \to X_n \times \Delta^n, \ \theta^* \times 1 : X_n \times \Delta^m \to X_m \times \Delta^m.$$

We define the geometric realization of X as the coequalizer of the diagram

$$\coprod_\theta X_n \times \Delta^m \rightrightarrows \coprod_n X_n \times \Delta^n.$$

The geometric realization is a simplicial set.

A bisimplicial set X can be seen as a functor

$$X : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathrm{Set}\,.$$

The diagonal simplicial set $d(X)$ is defined as the composition

$$\Delta^{\mathrm{op}} \xrightarrow{d} \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \xrightarrow{X} \mathrm{Set}\,.$$

Proposition 2 (1.4, Ch.3,[GJ])**.** $d(X)$ is the geometric realization of X.

Proposition 3 (1.7, Ch.3,[GJ])**.** If $f : X \to Y$ is a morphism of bisimplicial sets such that for each n, $f_n : X_n \to Y_n$ is a weak equivalence. Then the induced map $f_* : d(X) \to d(Y)$ is a weak equivalence.

Definition 3. $BG(A)$ is defined as EX^∞ (a functorial assignment of fibrant replacement) of the geometric realization of the bisimplicial set

$$[p] \mapsto \underline{\mathrm{Hom}}_{\mathrm{sCR}}(c(\mathcal{O}_{N_pG}), A).$$

BG is functorial at G.

Example 1. If A is a discrete simplicial ring, then $BG(A)$ is weak equivalent to $NG(A\pi_0(A))$. In fact, since all simplicial sets are cofibrant and all simplical rings are fibrant, it suffices to show that the diagonal of the bisimplicial set

$$[p] \mapsto \underline{\mathrm{Hom}}_{\mathrm{sCR}}(\mathcal{O}_{N_pG}, \pi_0(A))_p = \mathrm{Hom}_{\mathrm{sCR}}(\mathcal{O}_{N_pG} \times \Delta^p, \pi_0(A))$$

is $NG(\pi_0(A))$, which is clear.

In particular, a morphism from the fundamental groupoid of X to $G(k)$ gives a 0-simplex of $\underline{\mathrm{Hom}}_{\mathrm{sSet}}(X, G(k))$. If X is path connected and pointed, a group homomorphism $\mathrm{Hom}(\pi_1(X,*), G(k))$ gives a 0-simplex of $\underline{\mathrm{Hom}}_{\mathrm{sSet}}((X,*), (NG(k),*))$. If X is not path connected, it also gives a map $\overline{\rho} \in \mathrm{Hom}_{\mathrm{sSet}}((X,*), (NG(k),*))$ sending all other components to the base point.

Let $X = (X_\alpha)$ be a pro-object in sSet. We set

$$\mathrm{Hom}_{\text{pro-sSet}}(X, \cdot) := \varinjlim_\alpha \mathrm{Hom}_{\mathrm{sSet}}(X_\alpha, \cdot).$$

We define

$$\mathcal{F}_{X,G}(A) := \underline{\mathrm{Hom}}_{\text{pro-sSet}}(X, BG(A)).$$

Let $\overline{\rho}$ be a morphism from the fundamental groupoid of X to $G(k)$. We define

$$\mathcal{F}_{X,\overline{\rho}}(A) := \underline{\mathrm{Hom}}_{\text{pro-sSet}}(X, BG(A)) \times^h_{\underline{\mathrm{Hom}}_{\text{pro-sSet}}(X, BG(k))} \overline{\rho}$$

The framed deformation functor is defined similarly. In the framed version, X is a pro-object in sSet_* with base point x, BG is pointed, $\mathcal{F}^\square_{X,G}(A)$ is defined by the homotopy pullback square

$$\begin{array}{ccc} \mathcal{F}^\square_{X,G}(A) & \longrightarrow & \Delta^0 \\ \downarrow & & \downarrow * \\ \underline{\mathrm{Hom}}_{\text{pro-sSet}}(X, BG(A)) & \xrightarrow{x^*} & BG(A) \end{array}.$$

$\mathcal{F}^\square_{X,G}(A)$ is weak equivalent to the ordinary pullback, which is $\underline{\mathrm{Hom}}_{\text{pro-sSet}_*}(X, BG(A))$. Let $\overline{\rho} : \pi_1(X,x) \to G(k)$. be a group homomorphism. We define

$$\mathcal{F}^\square_{X,\overline{\rho}}(A) := \mathcal{F}^\square_{X,G}(A) \times^h_{\mathcal{F}^\square_{X,G}(k)} \overline{\rho}.$$

Back to $G_S = \pi_1^{\text{ét}}(\mathbb{Z}[1/S], *)$. G_S is a profinite group

$$G_S = \varprojlim_\alpha G_\alpha$$

where each α corresponds to the Galois group of a finite Galois subextension of $\mathbb{Q}_S\,/\,\mathbb{Q}$. We define $X = (X_\alpha)_\alpha$ where $X_\alpha := NG_\alpha$ and deformation functors

$$\mathcal{F}^{\square}_{\mathbb{Z}[1/S],\overline{\rho}} := \mathcal{F}^{\square}_{X,\overline{\rho}},\ \ \mathcal{F}_{\mathbb{Z}[1/S],\overline{\rho}} := \mathcal{F}_{X,\overline{\rho}}$$

Similarly for given finite place v of $\mathbb{Q}$, we can define $\mathcal{F}^{\square}_{\mathbb{Q}_v,\overline{\rho}}$, $\mathcal{F}_{\mathbb{Q}_v,\overline{\rho}}$ in the same way.

Lemma 2. $\pi_0\,\mathcal{F}^{\square}_{\mathbb{Z}[1/S],\overline{\rho}} : \mathrm{Art}_k \to \mathrm{Set}$ is the usual framed deformation functor. If the centralizer of the image of $\overline{\rho}$ is trivial, then $\pi_0\,\mathcal{F}_{\mathbb{Z}[1/S],\overline{\rho}} : \mathrm{Art}_k \to \mathrm{Set}$ is the usual unframed deformation functor.

The proposition says that the derived Galois deformation functors are generalizations of usual Galois deformation functors. Proofs of unframed/framed version are given in Lemma 7.2,[GV] / Lemma 2.5,[G9], respectively. The essential thing inside the proofs is computation of

$$\pi_0\underline{\mathrm{Hom}}_{\mathrm{sSet}}(NG_\alpha, NG(A)),\ \ \pi_0\underline{\mathrm{Hom}}_{\mathrm{sSet}\,*}(NG_\alpha, NG(A)).$$

Note that for groups G_1, G_2, also considered as groupoids of 1 object,

$$\underline{\mathrm{Hom}}_{\mathrm{sSet}}(NG_\alpha, NG(A)) = N\,\mathrm{Fun}(G_1, G_2).$$

$N\,\mathrm{Fun}(G_1, G_2)_0 = \mathrm{Hom}(G_1, G_2)$. Let $\psi_1, \psi_2 \in \mathrm{Hom}(G_1, G_2)$. The set of natural transformations from ψ_1 to ψ_2 is

$$\{\eta \in G_2 \mid \eta\psi_1\eta^{-1} = \psi_2\}.$$

Hence

$$\pi_0\underline{\mathrm{Hom}}_{\mathrm{sSet}}(NG_\alpha, NG(A)) = \mathrm{Hom}(G_\alpha, G(A))/\text{conjugation by } G(A).$$

In the framed (pointed) case, we only have trivial conjugations Hence

$$\pi_0\underline{\mathrm{Hom}}_{\mathrm{sSet}\,*}(NG_\alpha, NG(A)) = \mathrm{Hom}(G_\alpha, G(A)).$$

Unfolding the definition and permuting π_0 and colimit give that

$$\pi_0\underline{\mathrm{Hom}}_{\text{pro-sSet}}(X, BG(A)) = \mathrm{Hom}(G_S, G(A))/\text{conjugation by } G(A),$$

$$\pi_0\underline{\mathrm{Hom}}_{\text{pro-sSet}_*}(X, BG(A)) = \mathrm{Hom}(G_S, G(A)).$$

$\pi_0\,\mathcal{F}_{X,\overline{\rho}}(A)$ ($\pi_0\,\mathcal{F}^{\square}_{X,\overline{\rho}}(A)$) should be the fiber over the point at $\pi_0\,\mathcal{F}_{X,\overline{\rho}}(k)$ ($\pi_0\,\mathcal{F}^{\square}_{X,\overline{\rho}}(k)$) corresponding to $\overline{\rho}$ (I am not sure why), which gives the usual unframed/framed deformation functor.

3 Pro-representability

We should prove that $\mathcal{F}_{X,\overline{\rho}}$ is formally cohesive and compute the tangent complex in order to make use of Lurie's derived Schlessinger criterion. $\mathcal{F}_{X,\overline{\rho}}(k)$ is reduced and homotopic invariant by construction. It suffices to show that it preserves homotopy pullback, which is reduced to that BG preserves homotopy pullback.

The following two propositions are used in the proof that BG preserves homotopy pullback.

Proposition 4. If $F : \mathrm{sArt}_k \to \mathrm{sSet}$ is homotopy invariant, $F(A)$ is path-connected and $A \to \Omega F(A)$ (loop space) preserves homotopy pullbacks for all $A \in \mathrm{sArt}_k$, and $\pi_0 \Omega F(A) \to \pi_0 \Omega F(B)$ is surjective whenver $\pi_0 F(A) \to \pi_0 F(B)$ is surjective, then Fpreserves homotopy pullback.

Proposition 5. $G(A) := \underline{\mathrm{Hom}}_{\mathrm{sCR}}(c(\mathcal{O}_G), A) \to \Omega BG(A)$ is weak equivalent.

Lemma 3. $\mathfrak{t}\, BG$ is a local system with the only homology $\mathfrak{g} := \mathrm{Lie}(G)$ at degree 1, with the adjoint $G(k)$-action.

Proposition 6. $\mathfrak{t}\, \mathcal{F}_{\mathbb{Z}[1/S],\overline{\rho}}$ is quasi-isomorphic to $C^{*+1}(X, \overline{\rho}^* \, \mathfrak{g})$. Hence

$$H_* \, \mathfrak{t}\, \mathcal{F}_{\mathbb{Z}[1/S],\overline{\rho}} \cong H^{*+1}(X, \overline{\rho}^* \, \mathfrak{g}) = H^{*+1}_{\text{ét}}(\mathbb{Z}[1/S], \mathrm{Ad}\, \overline{\rho}).$$

This is explained in page 75 of[GV]. One should be able to directly identify the cohomology of the local system on X with the étale cohomology of $\mathrm{Spec}(\mathbb{Z}[1/S])$ (I don't know how yet).

Apply Lurie's derived Schlessinger criterion and we see that $F_{\mathbb{Z}[1/S],\overline{\rho}}$ is pro-representatble if and only if $H^0(G_S, \overline{\rho}^* \, \mathfrak{g}) = H^0_{\text{ét}}(\mathbb{Z}[1/S], \mathrm{Ad}\, \overline{\rho}) = 0$.

If G has nontrivial center, $\mathcal{F}_{X,\overline{\rho}}$ is never pro-representable. In Chapter 5.4,[GV], a modification is given in the case G has nontrivial and we give a sketch here: Let $Z \to G$ be a central algebraic subgroup and $PG := G/Z$. A functor $B^2 Z : \mathrm{sArt}_k \to \mathrm{sSet}$ such that there is a natural fibration sequence

$$BG(A) \to BPG(A) \to B^2 Z(A).$$

Let X be a pro-simplicial set. The functor $\mathcal{F}_{X,G,Z} : \mathrm{sArt}_k \to \mathrm{sSet}$ is defined by the homotopy pullback square

$$\begin{array}{ccc} \mathcal{F}_{X,G,Z}(A) & \longrightarrow & \underline{\mathrm{Hom}}_{\text{pro-sSet}}(\pi_0(X), B^2 Z(A)) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}_{\text{pro-sSet}}(X, BPG(A)) & \longrightarrow & \underline{\mathrm{Hom}}_{\text{pro-sSet}}(X, B^2 Z(A)) \end{array}.$$

$\mathcal{F}_{X,G,Z}(A)$ is homotopy invariant and preserves homotopy pullbacks, and fits in the natural fibration sequence

$$\underline{\mathrm{Hom}}_{\text{pro-sSet}}(X, BZ(A)) \to \underline{\mathrm{Hom}}_{\text{pro-sSet}}(X, BG(A)) \to \mathcal{F}_{X,G,Z}(A)$$

Let $\overline{\rho}$ be a morphism from the fundamental groupoid of X to $G(k)$. $\overline{\rho}$ gives a 0-simplex of $\underline{\mathrm{Hom}}_{\text{pro-sSet}}(X, BG(A))$ and hence a 0-simplex of $\mathcal{F}_{X,G,Z}(k)$, also denoted by $\overline{\rho}$. $\mathcal{F}_{X,\overline{\rho}}(A)$ is defined as the homotopy fiber of $\mathcal{F}_{X,G,Z}(A)$ over $\overline{\rho} \in \mathcal{F}_{X,G,Z}(k)$.

Proposition 7. $\mathcal{F}_{X,\overline{\rho}}$ is formally cohesive and the tangent complex of it fits in the cofibrant sequence

$$C^{*+1}(\pi_0(X), \mathrm{Ad}(Z)) \to C^{*+1}(X, \mathrm{Ad}(G)) \to \mathfrak{t}\, \mathcal{F}_{X,\overline{\rho}},$$

where $\mathrm{Ad}(G)$ is $\mathfrak{g}$ with adjoint action of $G(k)$, $\mathrm{Ad}(Z)$ is the Lie algebra of Z as a subspace of $\mathfrak{g}$, and $C^{*+1}(\pi_0(X), \mathrm{Ad}(Z)) \to C^{*+1}(X, \mathrm{Ad}(G))$ is induced by canonical maps $\mathrm{Ad}(Z) \to \mathrm{Ad}(G)$ and $X \to \pi_0(X)$. The framed version $\mathcal{F}^{\square}_{X,\overline{\rho}}$ is also defined similarly.

4 Local condition

We follow Appendix B of[GV] to define "cohomology classes that belong to $\mathcal{L}$". Let M be a G_S-module. For every $v \in S$, we fix an embedding $\iota_v : \mathbb{Q}_S \to \overline{\mathbb{Q}_v}$ and this induces a group homomorphism $\iota_v^* : \mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v) \to G_S$. We denote by $H^*(\mathbb{Q}_v, M)$ the group cohomology $H^*(\mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v), M)$.

Consider a family of subspaces $\{\mathcal{L}_v^i \subset H^i(\mathbb{Q}_v, M)\}$. We write for short that $\mathcal{L}_v \subset H^*(\mathbb{Q}_v, M)$, $\mathcal{L} \subset \prod_{v\in S} H^*(\mathbb{Q}_v, M)$. We want to define $H^*_{\mathcal{L}}(G_S, M)$ fitting the long exact sequence

$$H^*_{\mathcal{L}}(G_S, M) \to H^*(G_S, M) \to \left(\prod_{v\in S} H^*(\mathbb{Q}_v, M)\right) / \mathcal{L} \xrightarrow{+1} .$$

Let $C^*(M)$ ($C^*(\mathbb{Q}_v, M)$) be the inhomogeneous cochain complex computing $H^*(G_S, M)$ ($H^*(\mathbb{Q}_v, M)$). Suppose that for each v there is a subcomplex

$$C^*_{\mathcal{L}}(\mathbb{Q}_v, M) \subset C^*(\mathbb{Q}_v, M)$$

such that

1. $C^*_{\mathcal{L}}(\mathbb{Q}_v, M)$ is invariant under conjugacy.
2. $H^*(\mathbb{Q}_v, M) \to H^*(C^*(\mathbb{Q}_v, M)/C^*_{\mathcal{L}}(\mathbb{Q}_v, M))$ is surjective with kernel $\mathcal{L}_v$.

We define $H^*_{\mathcal{L}}(G_S, M)$ as the cohomology of the mapping cone

$$C^*(M) \oplus \left(\bigoplus_{v\in S} \frac{C^*(\mathbb{Q}_v, M)}{C^*_{\mathcal{L}}(\mathbb{Q}_v, M)}\right)[-1].$$

Then $H^*_{\mathcal{L}}(G_S, M)$ fits in the long exact sequence.

Let $\overline{\rho} : G_S \to G(k)$ be a fixed Galois representation. Let $v \in S$. $\mathcal{F}_{\mathbb{Z}[1/S],\overline{\rho}}$ is pull-backed to the functor $\mathcal{F}_{\mathbb{Q}_v,\overline{\rho}}$ by $\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Q}_v)) \to \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbb{Z}[1/S]))$.

Definition 4. A derived local condition at v is a simplicially enriched functor $D_v : \mathrm{sArt}_k \to \mathrm{sSet}$ along with a natural transformation

$$D_v \to \mathcal{F}_{\mathbb{Q}_v,\overline{\rho}} .$$

The global deformation functor corresponds to the derived local condition is defined as

$$\mathcal{F}^D_{\mathbb{Z}[1/S],\overline{\rho}} := \mathcal{F}_{\mathbb{Z}[1/S],\overline{\rho}} \times^h_{\mathcal{F}_{\mathbb{Q}_v,\overline{\rho}}} D_v$$

If D_v is formally cohesive, the tangent complex of $\mathcal{F}^D_{\mathbb{Z}[1/S],\overline{\rho}}$ is the homotopy fiber of

$$\mathfrak{t}\,\mathcal{F}_{\mathbb{Z}[1/S],\overline{\rho}} \oplus \mathfrak{t}\,D_v \to \mathfrak{t}\,\mathcal{F}_{\mathbb{Q}_v,\overline{\rho}},$$

which is represented by the mapping cone.

A derived local condition can be presented by a zig-zag or a sequence of zig-zags. If we have a zigzag

$$\begin{array}{ccc} & & \mathcal{F}_{\mathbb{Q}_v} \\ & & \downarrow \\ D'_v & \longrightarrow & \mathcal{F}'_{\mathbb{Q}_v} \\ \downarrow & & \\ D_v & & \end{array}$$

with vertical maps being naturally weak equivalence, then define D''_v by the homotopy pullback diagram

$$\begin{array}{ccc} D''_v & \longrightarrow & \mathcal{F}_{\mathbb{Q}_v} \\ \downarrow & & \downarrow \\ D'_v & \longrightarrow & \mathcal{F}'_{\mathbb{Q}_v} \end{array}.$$

$D''_v \to D_v$ is a naturally weak equivalence and $D''_v \to \mathcal{F}_{\mathbb{Q}_v}$ is a local condition. Similarly, process can be apply for more zig-zags.

Example 2. Let $D_v := \mathcal{F}_{\mathbb{Z}_v,\overline{\rho}}$ and $S' := S - \{v\}$. Then

$$\mathcal{F}^D_{\mathbb{Z}[1/S],\overline{\rho}} = \mathcal{F}^D_{\mathbb{Z}[1/S'],\overline{\rho}}$$

This is formula (8.5),[GV].

Let D be an undrived local condition. Suppose the unframed underived deformation functors are pro-representable with universal deformation rings

$$R_v \to R^D_v.$$

Suppose also $\mathcal{F}_{\mathbb{Z}[1/S],\overline{\rho}}$ is pro-representable by the pro-simplicial ring $\mathcal{R}_v$. Then we have morphisms

$$\mathcal{R}_v \to \pi_0\mathcal{R}_v = R_v \twoheadrightarrow R^D_v.$$

Let $D_v := \underline{\mathrm{Hom}}_{\text{pro-sArt}_k}(c(\mathcal{R}_v), \cdot)$. We have a zig-zag

$$\begin{array}{ccc} & & \underline{\mathrm{Hom}}_{\text{pro-sArt}_k}(\mathcal{R}_v, \cdot) \\ & & \downarrow \\ \underline{\mathrm{Hom}}_{\text{pro-sArt}_k}(c(R^D_v), \cdot) & \longrightarrow & D_v \end{array}$$

Let D^*_v be the homotopy limit of the diagram. Then we obtain a derived local condition $D^*_v \to \mathcal{F}_{\mathbb{Q}_v}$ and a corresponging global deformation functor, denoted by $\mathcal{F}^D_{\mathbb{Z}[1/S],\overline{\rho}}$.

The underived local condition defines a subspace

$$H^1_D \subset H^1(\mathbb{Q}_v, \mathrm{Ad}\,\overline{\rho}),$$

which is the tangent space of the functor represented by R^D_v.

Theorem 1. Suppose R_v^D is formally smooth. Equivalently, the corresponding tangent complex is nonvanishing only in degree 0. Then

$$H^* \mathcal{F}^D_{\mathbb{Z}[1/S],\overline{\rho}} \cong H^{*+1}_{\mathcal{L}}(\mathbb{Z}[1/S], \mathrm{Ad}\,\overline{\rho}),$$

here $\mathcal{L} = H^1_D \subset H^1(\mathbb{Q}_v, \mathrm{Ad}\,\overline{\rho})$.

Since $D_v^* \to D_v$ is naturally weak equivalent, $\mathfrak{t}\, D_v^* \to \mathfrak{t}\, D_v$ is quasi-isomorphic. By the condition, $\tau_{\geq 0}\, \mathfrak{t}\, D_v \to \mathfrak{t}\, D_v$ is quasi-isomorphic. The map $\tau_{\geq 0}\, \mathfrak{t}\, D_v \to \mathfrak{t}\, \mathcal{F}_{\mathbb{Q}_v,\overline{\rho}}$ factors through

$$\tau_{\geq 0}\, \mathfrak{t}\, D_v \to \tau_{\geq 0}\, \mathfrak{t}\, \mathcal{F}_{\mathbb{Q}_v,\overline{\rho}} \to \mathfrak{t}\, \mathcal{F}_{\mathbb{Q}_v,\overline{\rho}},$$

and $\tau_{\geq 0}\, \mathfrak{t}\, D_v \to \tau_{\geq 0}\, \mathfrak{t}\, \mathcal{F}_{\mathbb{Q}_v,\overline{\rho}}$ is a quasi-isomorphic onto the subcomplex corresponding to $H^1_D \subset H^1(\mathbb{Q}_v, \mathrm{Ad}\,\overline{\rho})$. Therefore, $\mathfrak{t}\, \mathcal{F}^D_{\mathbb{Z}[1/S],\overline{\rho}}$ is the complex with cohomology $H^{*+1}_{\mathcal{L}}(\mathbb{Z}[1/S], \mathrm{Ad}\,\overline{\rho})$.

References

[G8] https://nms.kcl.ac.uk/ashwin.iyengar/lntsg2019/G8.pdf

[G9] https://nms.kcl.ac.uk/ashwin.iyengar/lntsg2019/G9.pdf

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