Derived deformation theory

following closely
[**GV**] Soren Galatius and Askhay Venkatesh. *Derived Galois deformation rings.* Adv. Math. 327 (2018), 470-623.

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Columbia Student Number Theory Seminar

Disclaimer

Nothing in this document is original (except for the mistakes). I have also liberally copied verbatim sentences/paragraphs from [GV] and a few other sources, and only sometimes give attribution (to do so every time would be too distracting). I'm not an expert, my understanding of the material has large gaps, and I have made the slides to help me learn the material.

Please let me know if you have any questions, comments, corrections, etc.! Email: morey@oldwestbury.edu

Outline of the talk

- Goals of the talk
- Brief motivation for derived algebraic geometry
- Review of simplicial sets, simplicial commutative rings (GV sections 1)
- Examples of functors of interest, e.g. derived Galois deformation functor
- Homotopy colimits and limits (GV Appendix A)
- Review of representable, pro-representable functors (GV section 2, 3)
- Tangent complex of a functor (GV section 4): spectra, Dold-Kan
- Statement of Lurie's Derived Schlessinger

Goals of the talk

Goals of the talk are to explain (as best I can) the following technical statements:

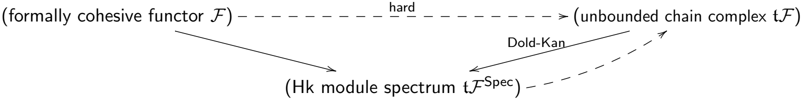
1. Let k be a field. A formally cohesive functor $\mathcal{F} : \mathsf{Art}_k \to \mathsf{sSets}$ from Artin simplicial commutative rings to simplicial sets has a **tangent complex** $\mathfrak{t}\mathcal{F} \in \mathsf{Ch}(k)$ of k-vector spaces,

$$\cdots \to \mathfrak{t}\mathcal{F}_2 \to \mathfrak{t}\mathcal{F}_1 \to \mathfrak{t}\mathcal{F}_0 \to \mathfrak{t}\mathcal{F}_{-1} \to \mathfrak{t}\mathcal{F}_{-2} \to \cdots$$

   possibly unbounded in both directions (satisfying come conditions spelled out later)

Goals continued

2. Given a formally cohesive functor $\mathcal{F}$, an simple/direct definition of $\mathfrak{t}\mathcal{F}$ is not available. The strategy Galatius-Venkatesh use is to go through homotopy theory, namely via something called Hk-module spectra:



3. (Lurie's Derived Schlessinger theorem) A formally cohesive functor $\mathcal{F} : \mathrm{sArt}_k \to \mathrm{sSets}$ is prorepresentable if and only if its tangent (chain) complex $\mathfrak{t}\mathcal{F} \in Ch(k)$ has $H_i(\mathfrak{t}\mathcal{F}) = 0$ for $i > 0$ (i.e. is coconnective).

Brief motivation for Derived Algebraic Geometry

Why consider functors $\mathcal{F} : \mathrm{SCR} \to \mathrm{sSets}$ from simplicial commutative rings to simplicial sets?

I'll follow the short overview given in:

G. Vezzosi, *What is ... a derived stack?*, p. 955-958, Notices AMS, August 2011, Volume 58, Issue 07

Fix a base commutative ring k and let $\mathrm{CommAlg}_k$ denote the category of k-algebras.
In "classical" algebraic geometry (i.e. Grothendieck style 1960s, not Italian school 1900s), there are two approaches to defining a k-scheme X, both useful in their own ways:

1. As a **ringed space**: a scheme is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X, plus some conditions (e.g. we have a cover of $(X, \mathcal{O}_X)$ by local models given by $(\mathrm{Spec}\, A, \mathcal{O}_{\mathrm{Spec}\, A})$ for some commutative k-algebra A)
2. As a **functor of points**: as a functor

   $$F : \mathrm{CommAlg}_k \to \mathrm{Sets}$$

   (plus some conditions). The functor h_X associated to a scheme X is its "functor of points" $h_X(A) := \mathrm{Hom}(\mathrm{Spec}\, A, X)$ for $A \in \mathrm{CommAlg}_k$.

Not all functors $F : \mathrm{CommAlg}_k \to \mathrm{Sets}$ are isomorphic to h_X for some scheme X; those F that said to representable (by a scheme).

Stacks

- The functor of points approach is frequently used to define *moduli problems*. These are functors $F : \mathrm{CommAlg}_k \to \mathrm{Sets}$ that assign to a k-algebra A, a set of objects *modulo isomorphisms*.
- Many such functors are not representable by scheme (usually due to non-trivial automorphisms of objects)
- So instead, keep track of the automorphisms by considering functors that assign, to a k-algebra A, objects *together with* isomorphisms between two such objects
- such a functor takes values in categories (not sets) in which every morphism is an isomorphism (such a category is called a groupoid).
- This is the notion of a *pre-stack*, and if it behaves like a sheaf for some Grothendieck topology on $\mathrm{CommAlg}_k$, it is a *stack*.

Higher Stacks

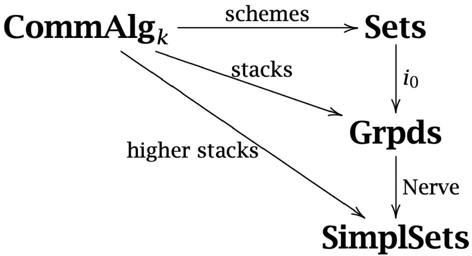
- Often we want to classify objects for which the natural notion of *equivalence* is weaker than isomorphism.
- For example, for chain complexes of A-modules, for $A \in \mathrm{CommAlg}_k$, the natural notion of equivalence is a (zig-zag of) quasi-isomorphisms.
- In such cases it is natural to once again enlarge the target category - and consider functors $F : \mathrm{CommAlg}_k \to \mathrm{sSets}$ valued in simplicial sets (or to topological spaces).

A stack $F : \mathrm{CommAlg} \to \mathrm{Grpds}$ may be viewed as a higher stack $NF : \mathrm{CommAlg}_k \to \mathsf{sSets}$ by taking the nerve of the groupoid - an n-simplex of $NF(A)$ is a list of n composible maps

$$x_0 \to x_1 \to \cdots \to x_n$$

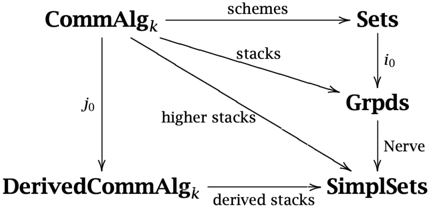
in the groupoid/category $F(A)$.

Vezzosi's article gives a nice diagram summarizing the situation discussed so far:



Derived algebraic geometry

Vezzosi: *"The main point of derived algebraic geometry is to enlarge (also) the source category, i.e., to replace commutative algebras with a more flexible notion of commutative rings servings as derived rings."*



Here $\mathbf{DerivedCommAlg}_k$ is either the category of simplicial commutative k-algebras or, when the base ring k has characteristic 0, the category of cdga's over k.

Why?

There are many reasons for expanding the source category of our functors is useful, and Vezzosi focuses on reasons arising from two classical geometric questions:

1. Derived Intersections
2. Deformation Theory

1. Derived Intersections

- Counting intersections multiplicity correctly has a long history, perhaps dating back to the number of roots of a quadratic polynomial.
- There are various approaches in classical intersection theory to deal with non-transverse intersections: moving lemmas, deformation to the normal cone, etc.
- Let V be a complex smooth projective variety, and let X, Y be two possibly singular subvarieties whose dimension add up to dim V and such that $X \cap Y$ is 0-dimensional.
- Serre's multiplicity formula says that the intersection multiplicity μ_p at a point $p \in X \cap Y$ is given by

$$\mu_p = \sum_{i \geq 0} \dim_{\mathbb{C}} \mathrm{Tor}_i^{\mathcal{O}_{V,p}}(\mathcal{O}_{X,p}, \mathcal{O}_{Y,p})$$

$$\mu_p = \sum_{i \geq 0} \dim_{\mathbb{C}} \mathrm{Tor}_i^{\mathcal{O}_{V,p}}(\mathcal{O}_{X,p}, \mathcal{O}_{Y,p})$$

The Tor groups can be computed (in theory, anyways) by

1. taking a projective resolution of $\mathcal{O}_{X,p}$ (or $\mathcal{O}_{Y,p}$) in the category of $\mathcal{O}_{V,p}$-modules,
2. tensoring this complex by $\mathcal{O}_{Y,p}$
3. and then taking the cohomology groups of the result.

- It is possible to choose a resolution of say $\mathcal{O}_{X,p}$ by **commutative differential graded** $\mathcal{O}_{V,p}$**- algebras** (cdga for short).
- Tensoring the resolution of $\mathcal{O}_{X,p}$ with $\mathcal{O}_{Y,p}$ gives a complex denoted $\mathcal{O}_{X,p} \otimes^{\mathbb{L}}_{\mathcal{O}_{V,p}} \mathcal{O}_{X,p}$, and we view this as a cdga.
- There is a famous fact that if we are in characteristic 0 (e.g complex algebraic varieties), cdga's are equivalent to simplicial commutative rings (specifically there is a Quillen equivalence between appropriate model structures).

Galatius-Venkatesh and Galois representations

Take

- V = moduli space of all p-adic representations of Gal($\overline{\mathbb{Q}_p}/\mathbb{Q}_p$)
- X = moduli space of geometric p-adic representations of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$
- Y= moduli space of Gal($\mathbb{Q}^{(S)}/\mathbb{Q}$) p-adic representations

[**GV**]: "*It is unsurprising that a derived version [of Mazur's Galois deformation ring] should play a role ...the intersection of X and Y is not, in general, transverse.*"

2. Deformation theory: the cotangent complex -1960s, 1970s

- For $A \in \mathrm{CommAlg}_k$, find a resolution of A by a simplicial commutative k-algebra $P_\bullet$ such that each level P_n that is a polynomial k-algebra.
- For example $P_0 = k[A]$, $P_1 = k[P_0]$, etc.
- Then consider simplicial module of Kahler differentials $\Omega_{P_n/k} \otimes_k A$.
- We can turn this simplicial module $\Omega_{P_\bullet/k} \otimes A$ into a chain complex (the non-normalized one) by taking differentials the alternating sum of the face maps.
- Switching to cohomological indexing gives the cotangent complex $\mathbb{L}_{A/k}$.

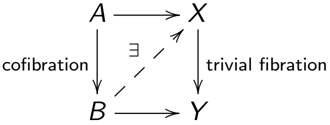
- Classical deformation theory studies only a small two term truncation of this complex, giving rise to tangent and obstruction spaces.
- In derived algebraic geometry, the full cotangent complex is studied; this is related to Andre-Quillen cohomology of rings, etc.

Review of [**GV**] sections 1,2,3

Now we turn to the [**GV**] paper.

Simplicial Sets

- Let Δ be the simplex category (objects are $[n] = \{0, 1, \ldots, n\}$ and maps are non-decreasing functions).
- Then the category of simplicial sets is sSets $= \mathrm{Fun}(\Delta^{\mathsf{op}}, \mathrm{Sets})$ (so a simplicial set X is sequence of sets $X_0, X_1, \ldots, X_n$ with a bunch of maps between them, including face and degeneracy maps).
- We equip sSets with the standard/Quillen model category structure in which
  - fibrations are the Kan fibrations (every horn can be filled in),
  - the weak equivalences $X \to Y$ are maps that induce isomorphisms on homotopy groups of the geometric realizations $|X| \to |Y,$
  - cofibrations have the left lifting property with respect to all trivial fibrations (trivial fibrations = fibrations that are also weak equivalence).



Hom's in sSets are not just sets but simplicial sets

- sSets is enriched over itself: for $X, Y \in$ sSets the *n*-simplices of $\underline{\text{Hom}}(X, Y) \in$ sSets are given by

$$\underline{\text{Hom}}(X, Y)_n = \text{Hom}_{\text{sSets}}(X \times \Delta^n, Y)$$

- For example, think of an element $H \in \underline{\text{Hom}}(X, Y)_1$ as a homotopy $H : X \times \Delta^1 \to Y$ between the two maps $f_0 : X \to Y$ and $f_1 : X \to Y$ that are the restriction of H to $X \times \Delta^0$ along the two face maps $d_0, d_1 : \Delta^0 \rightrightarrows \Delta^1$.

$$f_0, f_1 : X = X \times \Delta^0 \rightrightarrows X \times \Delta^1 \xrightarrow{H} Y$$

- Following Galatius-Venkatesh, we'll drop the underline and **always regard** $\text{Hom}(X, Y)$ **as a simplicial set**. So for example we have a functor $\text{Hom}(X, -) : \text{sSets} \to \text{sSets}$.

Simplicial commutative rings

- Let SCR be the category of simplicial commutative rings, enriched in sSets as follows: for $R, S \in \mathrm{SCR}$, $\mathrm{Hom}_{\mathrm{SCR}}(R, S) \in \mathrm{sSets}$ is the sub-simplicial set of $\mathrm{Hom}_{\mathrm{sSets}}(R, S)$ consisting of level-wise ring homomorphisms, i.e. $R_n \to S_n$ is a ring homomorphism
- SCR is a model category - the fibrations and weak equivalences are fibrations and weak equivalences of the underlying simplicial sets; the cofibrations then are maps in SCR satisfying left lifting property with respect to trivial fibrations.

Artin local simplicial rings Art_k

- Let k be a field (usually a finite field), considered as a discrete simplicial ring.
- Art_k is the full subcategory of SCR_k (simplicial commutative rings R along with the data of a map $R \to k$) such that
    1. the discrete ring $\pi_0(R)$ is an Artin local ring with residue field k
    2. The associated graded ring $\pi_*(R)$ is finitely generated $\pi_0(R)$ module (in particular, $\pi_n(R) = 0$ for $n >> 0$).
- Art_k is too small to be a model category as it doesn't have enough limits and colimits (e.g. it doesn't have an initial object), but we study its homotopy theory via the forgetful functors

  $$\mathrm{Art}_k \to \mathrm{SCR}_k \to \mathrm{SCR}$$

  so for example $R \in \mathrm{Art}_k$ is cofibrant means that it's image in SCR is cofibrant.

Examples of functors $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$

Before diving into [**GV**] section 2,3 on various conditions on functors $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$, let's give some examples of such functors:

- (Representable) $\mathcal{F}(-) := \mathrm{Hom}(R, -)$ for fixed $R \in \mathrm{Art}_k$ (we will typically want R to be cofibrant)
- (Pro-representable) $\mathcal{F}(-) := \mathrm{colim}_{\, j \in J} \mathrm{Hom}(R_j, -)$ where $(R_j)_{j \in J^{\mathrm{op}}}$ is a pro-object in Art_k
- The terminal functor $\mathcal{F}(-) = \{*\}$ This will not be representable for k a field of characteristic p, but will be pro-representable.
- $\mathcal{F}(A) := \ker(A \to k)$. This functor is denoted in [**GV**] by $\mathfrak{m} : \mathrm{Art}_k \to \mathrm{Sets}$.

more examples

- If $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_{01}$ are functors related by a diagram

$$\begin{array}{ccc} & & \mathcal{F}_1 \\ & & \downarrow \\ \mathcal{F}_0 & \longrightarrow & \mathcal{F}_{01} \end{array}$$

  then we can form a new functor that is the component wise homotopy pullback:

$$\mathcal{F}(A) := \mathcal{F}_0(A) \times^h_{\mathcal{F}_{01}(A)} \mathcal{F}_1(A)$$

  for all $A \in \mathrm{Art}_k$.
- Given a based functor $* \to \mathcal{F}$, a special case of the previous construction is the loop space functor $\Omega\mathcal{F} := * \times^h_{\mathcal{F}} *$.

The real example: Derived Galois deformations functors

- Fix a prime p and a finite field k of characteristic p.
- Let S be a set of primes containing p.
- Fix a Galois representation

$$\rho : \mathrm{Gal}(\mathbb{Q}^{(S)}/\mathbb{Q}) \to G(k)$$

notice it is 'mod p' representation (in one of the senses of the term), and a natural question is if it can be lifted to a Galois representation with coefficients in a p-adic field like $W(k)$.

Mazur and underived Galois deformations

- Mazur initiated to study of (underived) deformation theory of ρ, that is functors $F : \mathsf{discArt}_k \to \mathrm{Sets}$
- He showed that under certain hypotheses, various deformation functors associated to ρ were pro-representable by $R = (R_\alpha)$ with $R_\alpha \in \mathsf{discArt}_k$
- He did this by checking that the (underived) Schlessinger conditions were satisfied.

Derived Galois deformations

- Analogously, the main number theoretic functors of interest in [**GV**] will be various *derived* deformation functors $\mathcal{F}_\rho : \mathrm{Art}_k \to \mathrm{sSets}$ associated to $\rho : \mathrm{Gal}(\mathbb{Q}^{(S)}/\mathbb{Q}) \to G(k)$.
- Their definition is slightly involved so we are not going to define them in this talk
- But very informally speaking, $\mathcal{F}_\rho$ sends an Artinian simplicial ring $A \in \mathrm{Art}_k$ to a simplicial set of "conjugacy classes of deformations of ρ to A" [GV]
- [**GV**] is show that some of these derived deformations functors are pro-representable, using Lurie's Derived Schlessinger theorem.
- Lurie's theorem says' that a necessary and sufficient condition for pro-representability is vanishing of the homology groups of the tangent complex in positive degree.

Very sketchy definition of derived deformation functor

- Galatius-Venkatesh define the derived deformation functor $\mathcal{F}_\rho : \mathrm{Art}_k \to \mathrm{sSets}$ associated to ρ roughly as follows.
- First there exists a pro-simplicial set $X_S = (X_\alpha)$ (a projective system of simplicial sets) ... it will be the etale homotopy type (whatever that means) of some ring of integers. Also there is a functor $BG : \mathrm{Art}_k \to \mathrm{sSets}$.
- Define, for any $A \in \mathrm{Art}_k$,

$$\mathcal{F}_\rho(A) := \text{homotopy fiber over } \rho \text{ of}$$
$$\mathrm{Hom}_{\mathrm{sSets}}(X_S, BG(A)) \to \mathrm{Hom}_{\mathrm{sSets}}(X_S, BG(k))$$

$$\begin{array}{ccc} \mathcal{F}_\rho(A) & \longrightarrow & \mathrm{Hom}(X_S, BG(A)) \\ \downarrow & & \downarrow \\ * & \xrightarrow[\{\rho\}]{} & \mathrm{Hom}(X_S, BG(k)) \end{array}$$

- These functors are homotopy invariant, preserve homotopy pullback, and are formally cohesive (these terms are defined later in the talk).

Homotopy colimits and homotopy limits

GV Appendix A

Why homotopy (co)limits?

- Ordinary colimits and limits in a model category C are not homotopy invariant.
- In other words, if we have two diagrams $F : I \to C$ and $G : I \to C$ and a natural transformation $T : F \to G$ such that

  $$T(i) : F(i) \xrightarrow{\sim} G(i)$$

  is a weak equivalence for all $i \in I$ (i.e. T is a level-wise weak equivalence and hence will be a weak equivalence in an appropriate model category structure on C^I), then $\operatorname{colim}_{i \in I} T : \operatorname{colim}_{i \in I} F \to \operatorname{colim}_{i \in I} G$ need not be a weak equivalence.

Example

- For a simple example in the category of (nice) topological spaces, consider S^n as formed by gluing two hemispheres D^n along the equatorial S^{n-1}:
- The pushout/colimit of a diagram F below is colim $F = S^n$.

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & D^n \\ \downarrow & & \\ D^n & & \end{array}$$

- Replacing each contractible D^n by pt we get the diagram G below whose pushout is colim $G = \mathrm{pt}$.

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & \mathrm{pt} \\ \downarrow & & \\ \mathrm{pt} & & \end{array}$$

- We have an obvious level-wise weak equivalence $F \to G$, but clearly the map $S^n \to \mathrm{pt}$ of colimits is not a weak equivalence.
- In this example, ~~the~~ a homotopy colimit (of both diagrams) is S^n

ambiguity of homotopy (co)limit

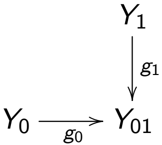
- The ordinary (co)-limit of a diagram is unique up to unique isomorphism, so it's common to say *the* (co)-limit.
- However, the *homotopy* (co)-limit of a diagram is not defined up to unique isomorphism, so there are several possible "choices" or models for an object in a model category C to be *a* homotopy (co)limit of a diagram $F : I \to C$.
- The most important property if homotopy (co)limits if $F, G : I \to C$ are two diagrams and there is a natural transformation $T : F \to G$ that such that $T(i) : F(i) \to G(i)$ is a weak equivalence, then the induced map (I guess for a functorial choice of models for homotopy (co)limit)

  $$(\mathrm{co})\lim F \to (\mathrm{co})\lim G$$

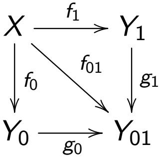
  is a weak equivalence.

homotopy limit example

Let I^{op} be the three object category $0 \to 01 \leftarrow 1$ (here $0, 1$ and 01 are just arbitrary names of the objects) and so a functor $Y : I^{\mathrm{op}} \to \mathrm{sSets}$ is a diagram



Assume Y takes values in Kan complexes. We will see that to give that to give a map $X \to Y_0 \times^h_{Y_{01}} Y_1$ from a simplicial set X to the homotopy limit $Y_0 \times^h_{Y_{01}} Y_1$ and tuples $(f_0, f_1, f_{01}, h_0, h_1)$ of maps



with $h_i : X \times \Delta^1 \to Y_{01}$ is a homotopy between f_{01} and $g_i \circ f_i$ $(i = 0, 1)$. The diagram does not commute, rather it "commutes up to homotopy".

Two approaches to defining a homotopy (co)limit

Two approaches to defining the homotopy (co)limit of a diagram $F : I \to C$

1. Abstract homotopy theory approach (via model categories or $(\infty, 1)$-categories): put a model structure (if possible) on C^I and take derived functors of usual (co)lim - which by definition means take the (co)limit of a (co)fibrant replacement of $F \in C^I$.
2. Explicit formulas e.g Bousfield-Kan, in which we 'thicken' if necessary each object $F(i)$ and take the usual (co)limit of the new diagram.

Quote from professionals

[AO] Sergey Arkhipova and Sebastian Orsted. *Homotopy (co)limits via homotopy (co)ends in general combinatorial model categories*. arXiv:1807.03266v6.

From [**AO**]: *"For many purposes, the abstract existence of homotopy limits is all you need. However, there are also many cases where a concrete, minimalistic realization of them is useful for working with abstract notions."*

1. Abstract approach: model structure(s) on C^I

- Let C be a model category and I a small category (the index category of our functors).
- Various naive (i.e. defined level-wise) classes of cofibrations, weak equivalence, and fibrations may not give rise to model structures on C^I.
- However if C is a combinatorial model category (such as sSets with its standard model structure), then the following two are indeed model structures on C^I:
  1. **Projective model structure** C^I_{Proj} where weak equivalences and fibrations are calculated componentwise
  2. **Injective model structure** C^I_{Inj} where weak equivalences and cofibrations are calculated componentwise

Abstract approach continued

- Let C be a combinatorial model category. Then we have two pairs of Quillen adjunctions:

$$\mathsf{colim}\ : C^I_{\mathsf{Proj}} \to C : \mathrm{const}$$

$$\mathsf{const} : C \to C^I_{\mathsf{Inj}} : \mathsf{lim}$$

- One (not very concrete since it involves cofibrant replacement) way to define the homotopy colimit hocolim : $\mathsf{Ho}(C^I_{\mathsf{Proj}}) \to \mathsf{Ho}(C)$ is as the left derived functor of colim $: C^I_{\mathsf{Proj}} \to C$.
- By definition, this means that if $Q : C^I_{\mathsf{Proj}} \to C^I_{\mathsf{Proj}}$ is functorial cofibrant replacement $QF \overset{\sim}{\twoheadrightarrow} F$, then the homotopy colimit of a diagram $F \in C^I$ is the colimit in the homotopy category $\mathsf{Ho}(C^I)$ of QF:

$$\mathsf{hocolim} : \mathsf{Ho}(C^I_{\mathsf{Proj}}) \xrightarrow{\mathsf{Ho}(Q)} \mathsf{Ho}(C^I_{\mathsf{Proj}}) \xrightarrow{\mathsf{Ho}(\mathsf{colim}\ )} \mathsf{Ho}(C)$$

$$\mathsf{hocolim}(F) := \mathsf{colim}\ \ QF$$

Abstract approach, finished

- The functorial cofibrant replacement map $QF \to F$ induces a map colim $QF \to$ colim F, i.e map

$$\operatorname{hocolim} F \to \operatorname{colim} \; F$$

(which need not be a weak equivalence in C, although [GV] Lemma A.2 says that if C is filtered then the map is weak equivalence).

- There is a dual story for homotopy limits of diagrams $F : I \to C$, using C^I_{Inj}. In this situation we use functorial fibrant replacement $F \xrightarrow{\sim} RF$, and set $\operatorname{hocolim} F := \lim RF$ as our model for a homotopy limit, and hence taking ordinary limits gives a map

$$\varprojlim_I F \to \operatorname{holim}_I F$$

Explicit formulas for homotopy (co)limits

Following appendix A of Galatius-Venkatesh, we give specific models of a homotopy (co)limits.

Homotopy colimits

- Let I be a small category and $X : I \to \mathrm{sSets}$ a functor. The **homotopy colimit** $\mathrm{hocolim}_{i \in I} X \in \mathrm{sSets}$ is a simplicial set with the following universal property:
- to specify a map $f : \mathrm{hocolim}_{i \in I} X \to Y$ amounts to specifying maps of simplicial sets

$$f_i : X(i) \times N(i \downarrow I) \to Y$$

in a way compatible with morphisms $i \to i'$ in I, i.e. the following diagram commutes:

$$\begin{array}{ccc} X(i) \times N(i' \downarrow I) & \longrightarrow & X(i) \times N(i \downarrow I) \\ \big\downarrow & & \big\downarrow{\scriptstyle f_i} \\ X(i') \times N(i' \downarrow I) & \xrightarrow{\;f_{i'}\;} & Y \end{array}$$

-

Homotopy colimits

- Hence we see that we can construct a specific simplicial set hocolim X with this property by take to be the coequalizer (in sSets) of the diagram

$$\coprod_{i_0 \to i_1 \in \mathrm{Mor}(I)} X(i) \times N(i' \downarrow I) \rightrightarrows \coprod_{i \in C} X(i) \times N(i \downarrow I)$$

- Taking $Y = \text{colim}_{\,i \in I} X$ to be the ordinary colimit and $f_i(\mathrm{can}_i, \mathrm{term}) : X(i) \times N(i \downarrow I) \to (\text{colim } X) \times \Delta^0 \simeq X$, we get a canonical map of simplicial sets from the homotopy colimit to the ordinary colimit:

$$\text{hocolim}_{i \in I} X \to \underset{i \in I}{\text{colim}} \; X$$

Homotopy limits

- Let $Y : I^{\mathrm{op}} \to \mathrm{sSets}$ be a functor. The **homotopy limit** $\mathrm{holim}_{i \in I} Y$ is a simplicial set with the universal property that to give a map $f : X \to \mathrm{holim}\, Y$ from a simplicial set X amounts to
- for all $i \in I$, giving a map of simplicial sets $f_i : X \times N(i \downarrow I^{\mathrm{op}}) \to Y(i)$ compatible with maps $i_1 \to i_0$ in I^{op}, i.e. such that the following diagram commutes

$$\begin{array}{ccc} X \times N(i_1 \downarrow I^{\mathrm{op}}) & \xrightarrow{f_{i_1}} & Y(i_1) \\ \uparrow & & \downarrow \\ X \times N(i_0 \downarrow I^{\mathrm{op}}) & \xrightarrow{f_{i_0}} & Y(i_0) \end{array}$$

Homotopy limits

- Since
  $\mathrm{Hom}(X \times N(i \downarrow I^{\mathrm{op}}), Y(i)) = \mathrm{Hom}(X, \mathrm{Hom}(N(i \downarrow I^{\mathrm{op}}), Y(i)))$
  (using simplicial Hom throughout - perhaps we need Y is Kan fibrant valued functor) the above diagram is equivalent to commutativity of

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Hom}(N(i_1 \downarrow I^{\mathrm{op}}), Y(i_1)) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(N(i_0 \downarrow I^{\mathrm{op}}), Y(i_0)) & \longrightarrow & \mathrm{Hom}(N(i_1 \downarrow I^{\mathrm{op}}), Y(i_0)) \end{array}$$

- Thus holim Y is the equalizer in sSets of
  $\mathrm{holim}\, Y = \mathrm{eq}\left(\prod_{i \in I} \mathrm{Hom}(N(i \downarrow I^{\mathrm{op}}), Y(i)) \rightrightarrows \prod_{i_1 \to i_0} \mathrm{Hom}(N(i_1 \downarrow I^{\mathrm{op}}), Y(i_0))\right)$

Representable and Pro-representable functors

Review of GV sections 2, 3

Definition

Let $\mathcal{F}, \mathcal{G} : \mathrm{Art}_k \to \mathrm{sSets}$ be two functors. A natural transformation $T : F \to G$ is called a **natural weak equivalence** if it is a component-wise weak equivalence, i.e. $T(A) : \mathcal{F}(A) \xrightarrow{\sim} \mathcal{G}(A)$ for all $A \in \mathrm{Art}_k$.
Two functors are **naturally weakly equivalent** if there exists a finite *zig-zag* of natural weak equivalences between them.

example

- Any functor $\mathcal{F} : \mathsf{Art}_k \to \mathsf{sSets}$ has a natural weak equivalence to a functor valued in Kan complexes:
- it is a theorem of Milnor that the unit map $\mathcal{F}(-) \to \mathrm{Sing}|\mathcal{F}(-)|$ associated to the Quillen equivalence $(|-|, \mathsf{sSets} \to \mathrm{Top} : \mathrm{Sing})$ is a natural weak equivalence.
- Could also use the natural weak equivalence $\mathcal{F}(A) \to \mathrm{Ex}^{\infty}(A)$, where $A \to \mathsf{Ex}(A)$ is Kan's adjoint subdivision, and $A \to \mathsf{Ex}^{\infty}(A)$ is the colimit of $A \to \mathsf{Ex}(A) \to \mathsf{Ex}(\mathsf{Ex}(A)) \to \cdots$. The advantage it has is it stays within sSets (e.g. does not require passing to geometric realization) and is in some sense "smaller" than $\mathrm{Sing}\mathcal{F}(A)$.

Simplicially enriched functors

Definition

A **simplicial enrichment** of a functor $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$ is the specification of maps of simplicial sets

$$\mathcal{F}_{A,B} : \mathrm{Hom}_{\mathrm{Art}_k}(A, B) \to \mathrm{Hom}_{\mathrm{sSets}}(\mathcal{F}(A), \mathcal{F}(B))$$

for all objects $A, B \in \mathrm{Art}_k$ agreeing with $\mathcal{F}$ on 0-simplices of $\mathrm{Hom}_{\mathrm{Art}_k}(A, B)$ and compatible with composition in the sSets-enriched categories Art_k and sSets.

Simplicial Yoneda

When $\mathcal{F}$ is simplicially enriched, there is a "Simplicial Yoneda Lemma"

Lemma (Simplicial Yoneda)

Assume $\mathcal{F}$ is simplicially enriched. Then for $R \in \mathsf{Art}_k$, there is functorial bijection of sets

$$\theta_R : \mathrm{Nat}(\mathsf{Hom}(R, -), \mathcal{F}) \to \mathcal{F}(R)_0$$

(where Nat denotes the set of natural transformation of functors) given by sending $T \in \mathrm{Nat}(\mathsf{Hom}(R, -), \mathcal{F})$, to $T(\mathrm{id}_R) \in \mathcal{F}(R)_0$.

Homotopy invariant functors

Most functors we will be interested in will be homotopy invariant:

Definition

A functor $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$ is **homotopy invariant** if it preserves weak equivalences: if $\phi : A \to B$ is a weak equivalence in Art_k, then $\mathcal{F}(\phi) : \mathcal{F}(A) \to \mathcal{F}(B)$ is a weak equivalence.

Proposition. For any homotopy invariant functor $\mathcal{F}$, there exists a natural weak equivalence $T : \mathcal{F} \to \mathcal{F}'$ where $\mathcal{F}'$ is simplicially enriched and Kan valued.

See [GV] for the proof, they give a relatively simple construction for $\mathcal{F}'$.

So when proving a property about a *homotopy invariant* functor $\mathcal{F}$ that depends only on the naturally weakly equivalence class of the functor, we may assume $\mathcal{F}$ is simplicially enriched and Kan valued. For instance, we will assume this in the proof of Lurie's derived Schlessinger theorem.

Representable functors

Definition

A functor $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$ is **representable** if it is naturally weakly equivalent (so, potentially via some zig-zags) to $\mathrm{Hom}(R, -) : \mathrm{Art}_k \to \mathrm{sSets}$ for some cofibrant $R \in \mathrm{Art}_k$.

- Proposition. A representable functor $\mathcal{F}$ is homotopy invariant.
- Proof: since homotopy invariance is just a question about weak equivalences, we can assume $\mathcal{F} = \mathrm{Hom}(R, -)$ for some cofibrant R.
- We need to show that if $A \to B$ is a weak equivalence between simplical rings (which are automatically fibrant), then $\mathrm{Hom}(R, A) \to \mathrm{Hom}(R, B)$ is a weak equivalence of simplicial sets.
- Ken Brown's lemma for model categories says we can assume additionally $A \to B$ is a trivial fibration.
- Then use R is cofibrant (and so $\mathrm{pt} \times \partial\Delta^n \to R \times \Delta^n$ is cofibration) and left lifting property of cofibrations with respect to trivial fibrations.

Pro-representability

A pro-object of a category C is a functor $R : J^{op} \to C$, where J is some small filtered category.

Definition

A functor $\mathcal{F} : \mathrm{sArt}_k \to \mathrm{sSets}$ is **pro-representable** if there exists a functor $R : J^{op} \to \mathrm{sArt}$, also written as $R = (j \to R_j)$, indexed by a filtered category J, and with all $R_j \in \mathrm{sArt}$ cofibrant, such that $\mathcal{F}$ is naturally weakly equivalent (recall this means via zig-zag) to

$$\operatorname*{colim}_{j \in J} \mathrm{Hom}(R_j, -)$$

Any pro-representable functor is homotopy invariant, since

1. For cofibrant $R_j \in \mathrm{Art}_k$, $\mathrm{Hom}(R_j, -) : \mathrm{Art}_k \to \mathrm{sSets}$ is homotopy invariant
2. filtered colimits of simplicial sets commute with homotopy groups.

example of a pro-repr but not representable functor

- Recall that k is a finite field of characteristic p, and simplicial rings $R \in \mathrm{Art}_k$ come with maps $R \to k$.
- The terminal functor $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$ given by $\mathcal{F}(A) = \{*\}$ is pro-representable, but not representable.
- The functor $\mathcal{F}$ is not representable because suppose it was represented by $R \in \mathrm{Art}_k$, and let n be such that $p^n = 0 \in \pi_0$ (recall $\pi_0(R)$ is Artin local and p is in the maximal ideal).
- Choose some ring $A \in \mathrm{Art}_k$ such that $p^n \neq 0 \in \pi_0(A)$ (for example the discrete ring $W(k)/p^{n+1}$, then $\mathrm{Hom}(R, A) = \emptyset$ which is not equivalent to $\mathcal{F}(A) = \{*\}$.

example, continued

- To show $\mathcal{F} = *$ is prorepresentable, let R_n be the ring obtained by freely adjoining to the discrete simplicial ring $\mathbb{Z}$ a generator y in degree 1 from 0 to p^n, i.e. $d_0 y = 0$ and $d_1 y_1 = p^n$.

$$0 \xrightarrow{y} p^n$$

- Note that $p \cdot y$ is an edge $0 \to p^{n+1}$ in $(R_n)_1$.
- This R_n is the cofibrant approximation to $W(k)/p^n$.
- We have map $R_{n+1} \to R_n$ that in degree 1 adds to $(R_{n+1})_1$ the generator $0 \to p^n$ of $(R_n)_1$. Then $\mathcal{F}$ is pro-represented by the projective system $n \mapsto R_n$ ... see [**GV**] Prop 3.4 for details of the proof.

This definition of $X \times_S^h Y$ is well defined up to weak equivalence in sSets, but not up to isomorphism (the isomorphism class depends on the choice of factorization). Regardless of the model we choose, the following definition makes sense:

Definition

We say that a commutative diagram in sSets

$$\begin{array}{ccc} Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

is a **homotopy pullback square** (or **homotopy Cartesian**) if the composite map

$$Y \to T \times_S X \to T \times_S^h X$$

is a weak equivalence in sSets.

Definition

Let $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$ be homotopy invariant (i.e. preserves weak equivalences).

We say $\mathcal{F}$ **preserves homotopy pullback** if $\mathcal{F}$ for every *strictly* cartesian diagram

$$\begin{CD} A @>>> B \\ @VVV @VVV \\ C @>>> D \end{CD}$$

in Art_k with $B \to D$ surjective in each simplicial degree, applying $\mathcal{F}$ gives a homotopy cartesian square,

$$\begin{CD} \mathcal{F}(A) @>>> \mathcal{F}(B) \\ @VVV @VVV \\ \mathcal{F}(C) @>>> \mathcal{F}(D) \end{CD}$$

i.e. the natural map $\mathcal{F}(A) \to \mathcal{F}(C) \times_{\mathcal{F}(D)} \mathcal{F}(F) \to \mathcal{F}(C) \times^h_{\mathcal{F}(D)} \mathcal{F}(F)$ is a weak equivalence in sSets.

Remark

- Easy fact: a map of simplicial rings $B \to D$ is surjective in all simplicial degrees if and only if it is a fibration and induces a surjection in π_0. (To prove this use $\Delta^0 \to \Delta^n$ is trivial cofibration).
- If we have a strict fiber square in Art_k
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$
then since $B \to D$ is a fibration between fibrant objects (recall that all simplicial abelian groups are Kan complexes), the strict Cartesian fiber $A = B \times_D C$ is also a (model for) a homotopy fiber.

Formally cohesive functor

Definition

$\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$ is

- **reduced** if $\mathcal{F}(k)$ is contractible (and non-reduced otherwise)
- **formally cohesive** if its homotopy invariant, preserves homotopy pullback, and is reduced.

Why formally cohesive functors?

Recall Schlessinger's theorem. Let $\mathrm{discArt}_k$ be the category of ordinary Artin local rings along with a map to k.

Theorem (Schlessinger)

Let $F : \mathrm{discArt}_k \to \mathrm{Sets}$ *be a functor. For any fiber square diagram in* $\mathrm{discArt}_k$

$$\begin{array}{ccc} A' \times_A A'' & \longrightarrow & A'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

consider the canonical map

$$(*) \qquad F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

$$\begin{array}{cccc} F(A' \times_A A'') & & & \\ & \searrow^{*} & & \\ & F(A') \times_{F(A)} F(A'') & \longrightarrow & F(A'') \\ & \downarrow & & \downarrow \\ & F(A') & \longrightarrow & F(A) \end{array}$$

statement of Schlessinger's theorem continued

$$(*) \qquad F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

Assume $F(k)$ has exactly one element. Then F is prorepresentable if and only if all of the following properties hold:

(H1) $(*)$ is a surjection whenever $A'' \to A$ is a small extension

(H2) $(*)$ is a bijection when $A'' \to A$ is $k[\epsilon] \to k$.

(H3) the tangent space $F(k[\epsilon])$ is finite dimensional as a k-vector space

(H4) $(*)$ is a bijection whenever $A' = A''$ is a small extension of A and the maps from A' and A'' to A are the same.

Example: representable functors ... almost formally cohesive

For $R \in \mathrm{Art}_k$ cofibrant, consider the functor $\mathrm{Hom}(R, -) : \mathrm{Art}_k \to \mathrm{sSets}$ represented by R

1. $\mathrm{Hom}(R, -)$ is homotopy invariant (saw this earlier;[**GV**] claim R cofibrant is required)
2. $\mathrm{Hom}(R, -)$ preserves homotopy pullback (proof: $\mathrm{Hom}(R, -)$ sends strict Cartesian diagram
$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$
to the strict Cartesian diagram
$$\begin{array}{ccc} \mathrm{Hom}(R, A) & \longrightarrow & \mathrm{Hom}(R, B) \\ \downarrow & & \downarrow \text{fibration} \\ \mathrm{Hom}(R, C) & \longrightarrow & \mathrm{Hom}(R, D) \end{array}$$

Since R is cofibrant, a lifting diagram argument shows $\mathrm{Hom}(R, -)$ preserves fibrations. Since $B \twoheadrightarrow D$ is a fibration, and $\mathrm{Hom}(R, B) \to \mathrm{Hom}(R, D)$ is a fibration. Hence the last Cartesian diagram is also a homotopy pullback square.

representable functors may not be reduced

However $\mathrm{Hom}(R, k)$ need not be contractible, so a representable functor need not be formally cohesive.

Dealing with non-reduced functors

In general, in the number theory applications in [GV] the Galois deformation functors will not be reduced, and there are two ways to proceed in cases of homotopy invariant homotopy pullback functors $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$ with $\mathcal{F}(k)$ not contractible:

1. Pick any 0-simplex $x \in \mathcal{F}(k)_0$ and then homotopy fiber $\mathcal{F}_x$ over x will be a formally cohesive functor (see next example).
2. Replace the target category sSets of $\mathcal{F}$ by $\mathrm{sSets}_{/Z}$ where $Z = \mathcal{F}(k)$, so $\mathcal{F}(k)$ is terminal, but we also will need to assume it is homotopy terminal i.e. the simplicial set $\mathrm{Hom}(X, Z)$ is contractible for any cofibrant $X \in \mathrm{sSets}_{/Z}$.

The following construction shows how to modify a homotopy pullback preserving functor to get reduced (hence formally cohesive) functors:
If $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$ is homotopy invariant and preserves pullbacks, and $\overline{\rho} \in \mathcal{F}(k)$ is a zero-simplex, define $\mathcal{F}_{\overline{\rho}} : \mathrm{Art}_k \to \mathrm{sSets}$ by

$$\mathcal{F}_{\overline{\rho}}(A) = \{*\} \times^h_{\mathcal{F}(k)} F(A)$$

This idea will come up later when we make "local systems" of functors $\mathcal{F}_\sigma$ as $\sigma \in \mathcal{F}(k)$.

Homotopy limits of formally cohesive functors are formally cohesive

The class of formally cohesive functors $\mathrm{Art}_k \to \mathrm{sSets}$ is closed under taking objectwise homotopy limits, i.e, if $\mathcal{F} : I^{op} \to \mathrm{sSets}$ is a diagram of formally cohesive functors, then the functor $\mathcal{F}$ defined by

$$\mathcal{F}(A) := \mathrm{holim}_{i \in I^{op}} \mathcal{F}_i(A)$$

is formally cohesive.

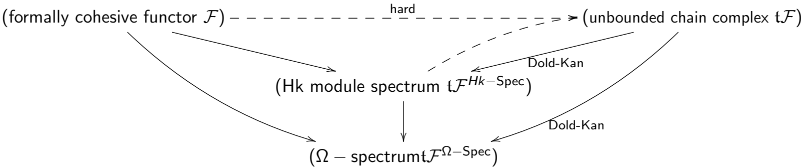
Tangent complex of a formally cohesive functor

GV section 4

Let $\mathcal{F} : \mathsf{Art}_k \to \mathsf{sSets}$ be a formally cohesive functor. The tangent complex of $\mathfrak{t}\mathcal{F} \in \mathsf{Ch}(k)$ will be a chain complex of k-vector spaces, possibly unbounded in both directions.

$$\cdots \to \mathfrak{t}\mathcal{F}_2 \to \mathfrak{t}\mathcal{F}_1 \to \mathfrak{t}\mathcal{F}_0 \to \mathfrak{t}\mathcal{F}_{-1} \to$$

[**GV**] *"It seems difficult to directly define a chain complex $\mathfrak{t}\mathcal{F}$ from a formally cohesive functor $\mathcal{F}$. Instead, we construct an essentially equivalent incarnation of it, as a spectrum with the structure of a module spectrum over the Eilenberg-MacLane spectrum Hk.* Then they use the Dold-Kan correspondence, which provides a bridge between spectra and chain complexes.



Spectrum (plural: spectra)

First, a definition of a spectrum from stable homotopy theory; GV start with the most elementary version:

Definition

1. A **spectrum** $E = (E_n)_{n \in \mathbb{Z}_{\geq 0}}$ is a sequence of based simplicial sets with maps $\epsilon_n : E_n \to \Omega E_{n+1}$ (recall the loop space is the based simpicial set $\Omega E_{n+1} := \mathrm{Hom}_*(S^1, E_{n+1})$.
2. Recall on sSets there is a functorial fibrant replacement $r_X : X \xrightarrow{\sim} \mathrm{Ex}^\infty X$, due to Kan.
3. The spectrum (E_n) is an Ω-**spectrum** if for all n the composition $E_n \xrightarrow{\epsilon_n} \Omega E_{n+1} \to \Omega \mathrm{Ex}^\infty E_{n+1}$ is a weak equivalence.

Homotopy groups of a spectrum

The **homotopy groups** of a spectrum E are defined for $k \in \mathbb{Z}$ (so negative homotopy groups can exist)

$$\pi_k(E) = \operatorname*{colim}_{n \to \infty} \pi_{n+k} E_n$$

where the colimit is along the maps $\pi_{n+k}(E_n) \to \pi_{n+k+1}(E_{n+1})$ given by

$$\pi_{n+k}(E_n) = [S^{n+k}, E_n] \xrightarrow{S^1 \wedge -} [S^1 \wedge S^{n+k}, S^1 \wedge E_n] \to [S^{n+k+1}, E_{n+1}] = \pi_{n+k+1}(E_{n+1})$$

where [,] denotes (based) homotopy classes of maps, and in the last step we used the map $S^1 \wedge E_n \to E_{n+1}$ that is adjoint to $\epsilon_n : E_n \to \Omega E_{n+1}$.

Just like the classical tangent space $F(k[\epsilon])$ of a functor $\mathrm{discArt}_k \to \mathrm{Sets}$ is defined by evaluating F at the specific Artin ring $k[\epsilon]$, the tangent object associated to a functor $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$ will be given by evaluating $\mathcal{F}$ at some specific Artin simplicial rings $k \oplus k[n]$ for $n \geq 0$.

$$\mathcal{F}(k \oplus k[n])$$

which we recall next.

$k[n]$

- For V a simplicial k-module, let $k \oplus V \in \mathrm{SCR}_{/k}$ defined by square-zero extension in each simplicial degree. If $k \oplus V \in \mathrm{Art}_k$ if and only if $\dim_k \pi_*(V) < \infty$.
- $k[n]$ is free simplicial k-module generated by the pointed simplicial set $S^n = \Delta^n/\partial\Delta^n$ (i.e. the p-simplices of $k[n]$ the free k-module on the p-simplices of S^n modulo the span of the basepoint). (So if I understand correctly, the $k[n]_p = 0$ for $p < n$, and for $p = n$ we have $k[n]_n = k$.)

$$\pi_i k[n] = \begin{cases} k & i = n \\ 0 & \text{otherwise} \end{cases}$$

An alternative way to construct $k[n]$ is to apply Dold-Kan functor (see below) to the $\Sigma^n k$, the chain complex with k in (homological) degree n and zeroes elsewhere.

$k \oplus k[n]$

- $k \oplus k[n]$ for the square zero extension (in each simplicial degree). Note $k \oplus k[0] = k[\epsilon]$ (where as usual $\epsilon^2 = 0$)
- Define $\widetilde{k[n]}$ as follows: Factor $0 \to k[n]$ into $0 \xrightarrow{\sim} \widetilde{k[n]} \twoheadrightarrow k[n]$ into a weak equivalence $0 \to \widetilde{k[n]}$ followed by a fibration $\widetilde{k[n]} \to k[n]$.
- Note $\widetilde{k[n]}$ is a model for the homotopy fiber product $\{0\} \times^h_{k[n]} k[n]$

- Apparently ([**GV**] proof of Lemma 3.11) we have a strict pullback square

$$\begin{array}{ccc} k \oplus k[n] & \longrightarrow & k \oplus \widetilde{k[n+1]} \\ \downarrow & & \downarrow \\ k & \longrightarrow & k \oplus k[n+1] \end{array}$$

A formally cohesive functor $\mathcal{F}$ will turn it into a homotopy pullback square

$$\begin{array}{ccc} \mathcal{F}(k \oplus k[n]) & \longrightarrow & \mathcal{F}(k \oplus \widetilde{k[n+1]}) \\ \downarrow & & \downarrow \\ \mathcal{F}(k) & \longrightarrow & \mathcal{F}(k \oplus k[n+1]) \end{array}$$

so we have weak equivalences (recalling that $\mathcal{F}(k) \simeq *$)

$$\begin{aligned} \mathcal{F}(k \oplus k[n]) &\xrightarrow{\sim} \mathcal{F}(k) \times^h_{\mathcal{F}(k \oplus k[n+1])} \mathcal{F}(k \oplus \widetilde{k[n+1]}) \\ &\xrightarrow{\sim} \{*\} \times^h_{\mathcal{F}(k \oplus k[n+1])} \{*\} \\ &= \Omega \mathcal{F}(k \oplus k[n+1]) \end{aligned}$$

Tangent spectrum

Definition

Let $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSets}$ be a formally cohesive functor, and suppose for convenience it is Kan valued.

The **tangent** Ω**-spectrum** $\mathfrak{t}\mathcal{F}^{\Omega-\mathrm{Spec}}$ ([**GV**] call it tangent *complex*, but I'm replacing 'complex' by Ω-spectrum to emphasize it is a Ω-spectrum) is the Ω-spectrum whose n-th space is given by

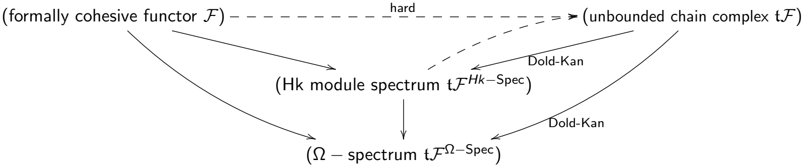
$$\mathfrak{t}\mathcal{F}_n^{\Omega-\mathrm{Spec}} := \mathcal{F}(k \oplus k[n])$$

and the structure maps

$$\mathcal{F}(k \oplus k[n]) \to \Omega\mathcal{F}(k \oplus k[n+1])$$

are the weak equivalences mentioned a couple of slides ago.

$\mathcal{F} \mapsto \mathfrak{t}\mathcal{F}^{\Omega-\mathrm{Spec}}$ is the lower left functor in the strategy below.



Next we explain the lower right functor labeled Dold-Kan

Dold-Kan

- The (classical) Dold-Kan correspondence

$$\text{Dold-Kan} : \mathsf{Ch}_+(k) \to \mathrm{sMod}_k$$

is an equivalence of categories between between the category $\mathsf{Ch}_+(k)$ non-negatively graded k-linear chain complexes and the category sMod_k. simplicial k-modules

- GV extend it to a functor

$$\text{Dold-Kan} : \mathsf{Ch}(k) \to \Omega - \mathsf{Spec}$$

from unbounded chain complexes to Ω-spectra.

- $$\text{Dold-Kan} : \mathrm{Ch}_+(k) \to \mathrm{sMod}_k$$
- The inverse functor $N : \mathrm{sMod}_k \to \mathrm{Ch}_+(k)$ is easier to define: given a simplicial k-module A, $NA \in \mathrm{Ch}_+(k)$ is it's normalized chain complex:

  $$(NA)_n = \bigcap_{i=1}^{n} \ker(d_i : A_n \to A_{n-1})$$

  and the differential $\partial : NA_n \to NA_{n-1}$ is given by $d_0|NA_n$.
- Intuitively, the k-module $(NA)_n$ consists of n-simplices of A whose boundary faces are $0 \in A_{n-1}$, except for the face opposite the initial vertex of the simplex.
- Hence an element in $\ker \partial : NA_n \to NA_{n-1}$ will be a n-simplex whose boundary faces are all 0, i.e. it will be an S^n.

- Conversely, given a non-negatively graded chain complex $(C_*, \delta) \in \mathrm{Ch}_+(k)$, the simplicial abelian group in degree n is given by

$$\text{Dold-Kan}(C)_n = \bigoplus_{[n]\twoheadrightarrow[\ell]} C_\ell$$

- if $\theta : [m] \to [n]$ is a morphism in the simplex category Δ, then induced map

$$\theta^* : \text{Dold-Kan}(C)_n \to \text{Dold-Kan}(C)_m$$

  is given, on the summand V_ℓ indexed by $[n] \twoheadrightarrow [\ell]$, by
- $d : V_\ell \to V_s$ where V_s is the summand indexed by $[m] \twoheadrightarrow [s]$ defined by first factoring $[m] \to [n] \twoheadrightarrow [\ell]$ into a surjective followed by injective map $[m] \twoheadrightarrow [s] \hookrightarrow [\ell]$.
- Intuitively, the elements of the k-module C_n are the non-degenerate simplices appearing in $\text{Dold-Kan}(C)_n$ via the summand indexed by $\mathrm{id} : [n] \to [n]$.

Dold-Kan preserves homotopy/homology groups

Composing with the forgetful functor

$$\mathsf{Ch}_+(k) \xrightarrow{\text{Dold-Kan}} \mathrm{sMod}_k \xrightarrow{\text{forget}} \mathsf{sSets}$$

sends a chain complex (C_*, ∂) to a Kan simplicial set with base point given by the 0-element, and whose homotopy groups are isomorphic to the homology groups of (C_*, ∂):

$$\pi_n(\text{Dold-Kan}(C)) = H_n(C_*)$$

Dold-Kan as a functor to spectra

- For $C \in \mathrm{Ch}(k)$ define the shifted chain complex $\Sigma C \in \mathrm{Ch}(k)$ by $(\Sigma C)_n = C_{n-1}$ for $n > 0$.
- For $C \in \mathrm{Ch}_+(k)$, we have a canonical weak equivalence of based Kan simplicial sets

$$\text{Dold-Kan}(C) \xrightarrow{\simeq} \Omega\text{Dold-Kan}(\Sigma C)$$

  from the fact that the homology groups of ΣC are the shifted homology groups of C
- So the sequence of Kan fibrant simplicial sets $(\text{Dold-Kan}(\Sigma^n C))_n$ is a Ω-spectrum.

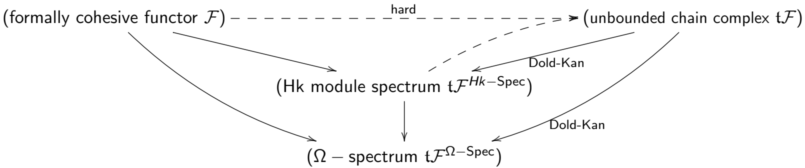
Extending Dold-Kan to unbounded chain complexes

- For any unbounded chain complex let $(C_*, \partial) \in \mathrm{Ch}(k)$ $\tau_{\geq 0} C \in \mathrm{Ch}_+(k)$ be the soft truncation: in degree 0, $(\tau_{\geq 0} C)_0 = \ker(C_0 \xrightarrow{\partial} C_{-1})$, while $(\tau_{\geq 0} C)_n = C_n$ for all $n \geq 1$.
- Applying Dold-Kan-functor to the non-negatively graded chain complex $\tau_{\geq 0}(\Sigma^n C)$ gives as before a weak equivalence of based simplicial sets

$$\text{Dold-Kan}(\tau_{\geq 0}(\Sigma^n C)) \to \Omega \text{Dold-Kan}(\tau_{\geq 0}(\Sigma^{n+1} C))$$

- Definition

  Define a functor Dold-Kan : $\mathrm{Ch}(k) \to \Omega -$ spectra by sending $(C_*, \partial) \in \mathrm{Ch}(k)$ to the spectrum $E = (\text{Dold-Kan}(\tau_{\geq 0}(\Sigma^n C)))_n$ and the weak equivalences $\epsilon_n : E_n \to \Omega E_{n+1}$ indicated above. The homotopy groups of this spectrum are canonically isomorphic to the homotopy groups of C.

- 

- [**GV**] *"Dold-Kan functor from* Ch(k) *to spectra ... is a "forgetful" functor. It remembers enough about a chain complex to recover its homology groups (viz. as the homotopy groups of the spectrum) and in particular it detects quasi-isomorphisms, but it does not remember enough information to recover the k-module structure on these homology groups.*
- In other words, the homotopy groups $\pi_n(\text{Dold-Kan}(C))$ don't see the k module structure on $H_i(C)$; we only have that the two groups are isomorphic as abelian groups.

- To recover the k-module structure on $\pi_n(\text{Dold-Kan}(C))$, we need to put some type of "k-module" structure on the Ω-spectrum Dold-Kan(C)
- The incarnation of any ring R (e.g. k) in the category of Ω-spectra is the Eilenberg-Maclane spectrum $HR = (K(R,n))_n$, consisting of Eilenberg-Maclane spaces $K(R,n)$ which represent the functor (on topological spaces) given by singular cohomology with coefficients in R:

$$H^n_{\text{sing}}(X,R) = [X, K(R,n)]$$

- Suppose we have a Ω-spectrum $\mathbf{X}$. Using the intuition that a spectrum is a topologist's abelian group, and a k-module is a abelian group A with a map $k \otimes_{\mathbb{Z}} A \to A$ satisfying the usual conditions, some sort of k module structure on $\mathbf{X}$ would mean a map of spectra

$$Hk \otimes \mathbf{X} \to \mathbf{X}$$

where $\otimes$ is some sort of tensor product (typically called smash product $\wedge$) of spectra.

Smash products of spectra

- Defining a reasonable notion of smash product on the level of spectra (as opposed to homotopy category of spectra) is quite thorny/technical; and led to the introduction of other notions of spectra: spectra associated to Γ-spaces, symmetric spectra, orthogonal spectra, S-modules, ...
- For a survey from 2001 see M.A. Mandell, J.P. May, S. Schwede, B. Shipley, *Model categories of diagram spectra*, Proc. Lond. Math. Soc. (3) 82 (2) (2001) 441-512.
- GV explain in some detail Segal's Γ-spaces and the category of spectra valued in Γ-spaces.
- This category has a smash product, and hence we can define a category of Hk-module spectra. Let's take this as a blackbox.
- The Dold-Kan functor $\mathrm{Dold-Kan} : \mathsf{Ch}(k) \to \Omega - \mathsf{spectra}$ constructed above naturally takes values Hk-module spectra.

Cohomology of (pro)-Artin rings

If $B \to k$ is a map of ordinary rings, then we have a natural bijection of sets

$$\mathsf{Hom}_{Alg/k}(B, k[\epsilon]) \simeq \mathrm{Der}(B, k) \simeq \mathsf{Hom}_{B-Mod}(\Omega_{B/\mathbb{Z}}, k)$$

We can extend this levelwise to to the simplicial setting. Let $R \in \mathsf{Art}_k$ be cofibrant. We have a natural isomorphism of simplicial sets

$$\mathsf{Hom}_{\mathsf{Art}_k}(R, k \oplus k[n]) \simeq \mathsf{Hom}_{sR-mod}(\Omega_{R/\mathbb{Z}}, k[n])$$

where $\Omega_{R/\mathbb{Z}}$ is the simplicial R-module $\Omega_{R_n/\mathbb{Z}}$ of Kahler differentials taken levelwise. Here $sR - mod$ is the category of simplicial R-modules,enriched in sSets, and $k[n]$ is made into an R-module via the structure map $R \to k$.

For $R \in \mathrm{Art}_k$, let R^c be a cofibrant replacement of R, and let

$$\pi_{-n}\mathfrak{t}R := \pi_0(\mathrm{Hom}_{\mathrm{Art}_k}(R^c, k \oplus k[n])) \simeq \pi_0(\mathrm{Hom}_{sR-mod}(\Omega_{R/\mathbb{Z}}, k[n]))$$

For a pro-object $R : I^{\mathrm{op}} \to \mathrm{Art}_k$, we define

$$\pi_{-n}\mathfrak{t}R := \operatorname*{colim}_{i \in I} \ \pi_{-n}R_i$$

$\pi_{-n}\mathfrak{t}R$ is identified with the Andre-Quillen cohomology of $\mathbb{Z} \to R$ with coefficients in k

Tangent complex of a representable functor

- Let $R \in \mathrm{SCR}_k$ (so comes with a fixed map $\overline{\rho} : R \to k$, this will make $\mathcal{F}_R$ reduced) be cofibrant, and let $\mathcal{F}_R = \mathrm{Hom}_{\mathrm{SCR}\,/k}(R, -) : \mathrm{Art}_k \to \mathrm{sSets}$.
- What is the tangent complex $\mathfrak{t}\mathcal{F}_R$?
- Answer: $\mathfrak{t}\mathcal{F}_R$ is quasi-isomorphic to the chain complex $\mathrm{Hom}_{\mathrm{Ch}(k)}(\text{Dold-Kan}(sL_{R/\mathbb{Z}} \otimes_R k), k)$ where Hom is the the internal hom of chain complexes $\mathrm{Hom}_i(A, B) = \mathrm{Hom}_0(A, B[i]))$ and $sL_{R/\mathbb{Z}}$ is the simplicial R-module incarnation of the cotangent complex of $\mathbb{Z} \to R$.
- Homotopy groups of $\mathfrak{t}\mathcal{F}$ are Andre-Quillen cohomology groups

$$\pi_{-i}\mathfrak{t}\mathcal{F}_R \simeq D^i_{\mathbb{Z}}(R, k)$$

so in particular for $n > 0$, $\pi_n(\mathfrak{t}\mathcal{F}_R) = 0$ (foreshadowing of Lurie's derived Schlessinger)

Andre-Quillen cohomology

- Let $A \to B$ be a map of simplicial commutative rings (a map of discrete rings will provide an interesting enough example) and a (constant?) B module M,
- if $A \to B$ is cofibrant the Andre-Quillen cohomology groups $D^i_A(B, M)$ are the cohomology of total co-chain complex assocated to the co-simplicial abelian group $\mathrm{Der}_{A_n}(B_n, M) \simeq \mathrm{Hom}(\Omega_{B_n/A_n}, M)$ of level-wise A-linear derivations of B taking values in M.

The cotangent complex $L_{B/A}$ is the total chain complex associated to the simplicial B-module $\Omega_{P_\bullet/A} \otimes B$, where $P_\bullet \to B$ is a simplicial resolution of B by free A-algebras.
The cotangent complex can compute Andre-Quillen cohomology by taking the cohomology of the complex $\mathrm{Hom}_B(L_{B/A}, M)$

$$D_A^i(B, M) = H^i(\mathrm{Hom}_B(L_{B/A}, M)$$

GV claim

$$\pi_{-i}\mathfrak{t}\mathcal{F}_R \simeq D^i_{\mathbb{Z}}(R,k)$$

and say it's because the tangent complex of $\mathcal{F}_{R,\overline{\rho}}$ is quasi-isomorphic to the complex

$$\mathrm{Hom}_{\mathrm{Ch}(k)}(L_{R/\mathbb{Z}} \otimes_R k, k)$$

Here's my attempt at a justification:

$$\begin{aligned}
\pi_{-i}\mathfrak{t}\mathcal{F}_R &:= \operatorname*{colim}_{j\in\mathbb{N}} \pi_{j-i}((\mathfrak{t}\mathcal{F}_R)_j) \text{ defn of homotopy groups of a spectrum} \\
&\overset{?}{=} \pi_0((\mathfrak{t}\mathcal{F}_R)_i) \text{ why can we take } j = i? \\
&= \pi_0\mathcal{F}_R(k \oplus k[i]) \\
&= \pi_0 \mathrm{Hom}_{Alg/k}(R, k \oplus k[i]) \\
&\simeq \pi_0 \mathrm{Hom}_{sR-mod}(\Omega_{R/\mathbb{Z}}, k[i]) \\
&\simeq D^i_{\mathbb{Z}}(R,k)
\end{aligned}$$

Lurie's Derived schlessinger

Theorem (Lurie's Derived Schlessinger Criterion)

Let

$$\mathcal{F} : \mathrm{sArt}_k \to \mathrm{sSets}$$

be a formally cohesive functor. Then $\mathcal{F}$ is **pro-representable** *if and only if the ith homology groups $H_i \mathfrak{t}\mathcal{F}$ of the tangent complex $\mathfrak{t}\mathcal{F}$ (a chain complex made up of k-vector spaces) vanish for $i > 0$:*

$$\text{for all } i > 0 \quad H_i(\mathfrak{t}\mathcal{F}) = 0$$

Technical addendum: If the k-vector spaces $H_i(\mathfrak{t}\mathcal{F})$ for all $i \in \mathbb{Z}$ have countable dimension, then the pro-representing object may be chosen to have countable indexing category.

Sketch of proof of Lurie's Derived Schlessinger, following [GV]

- Easy direction: First assume $\mathcal{F}$ is pro-representable, say by $R : J^{\mathrm{op}} \to \mathrm{Art}_k$, we need to show tangent complex $\mathfrak{t}(\mathrm{colim}_{\,j}\, \mathrm{Hom}(R_j, -))$ is co-connective.
- The tangent complex of any representable functor $\mathrm{Hom}(R_j, -)$ is co-connective.
- Tangent complex operation

$$\mathfrak{t} : \mathrm{Fun}(\mathrm{Art}_k, \mathrm{sSets}) \to \mathrm{Spectra}$$

  from functors to spectra takes filtered homotopy colimits of functors to filtered homotopy colimits of spectra, so in particular

$$\mathfrak{t}(\mathrm{colim}_j\ \mathrm{Hom}(R_j, -)) = \mathrm{colim}_\alpha\ \mathfrak{t}(\mathrm{Hom}(R_j, -))$$

- filtered colimits of co-connective spectra remains co-connective
- so in particular tangent complex of any pro-representable functor is co-connective.

harder direction

- Conversely suppose $\mathcal{F} : \mathsf{Art}_k \to \mathsf{sSets}$ is such that $\pi_i \mathfrak{t}\mathcal{F} = 0$ for $i > 0$, i.e. $\pi_i \mathcal{F}(k \oplus k[n]) = 0$ for $i > 0$. GV/Lurie build a projective system $R : J^{\mathsf{op}} \to \mathsf{Art}_k$ of simplicial rings and a natural weak equivalence (an actual morphism, not just a zig-zag)
$$\mathsf{hocolim}_j \, \mathsf{Hom}(R_j, -) \to \mathcal{F}$$
in $\mathrm{Fun}(\mathsf{Art}_k, \mathsf{sSets})$.
- Without loss of generality, we may assume that $\mathcal{F}\, \mathsf{Art}_k \to \mathsf{sSets}$ is a simplicially enriched functor and takes Kan values.
- The construction is by a (generally transfinite) recursive recipe, providing an "improvement" to any pair $(\mathcal{R}, \iota)$ consisting of a cofibrant $\mathcal{R} \in Art_k$ and a zero-simplex $\iota \in \mathcal{F}(\mathcal{R})$.
- Simplicial Yoneda says ι gives a natural transformation $\iota : \mathsf{Hom}(\mathcal{R}, -) \to \mathcal{F}(-)$
- To be continued...