POINCARÉ DUALITY

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Abstract. This expository work aims to provide a self-contained treatment of the Poincaré duality theorem in algebraic topology expressing the symmetry between the homology and cohomology of closed orientable manifolds. In order to explain this fundamental result, we first define the orientability of manifolds in an algebraic topology setting. After covering the statement and proof of the main theorem, we provide a few examples and well-known applications.

1. Introduction

The Poincaré duality theorem is a fundamental theorem in algebraic topology that links the dual notions of homology groups and cohomology groups for manifolds. As a result, Poincaré duality provides a useful structure theorem for understanding the homology and cohomology of a large class of well-studied objects. This paper aims to provide an introduction to Poincaré duality and offer an exposition of a well-known proof of the theorem and some interesting consequences.

Remark 1.1. In this paper, the absence of a coefficient group from the homology or cohomology group notation means that it is implicitly $\mathbb{Z}$ unless a ring R has been specified.

1.1. **Orientation.**

Let us first define the orientability of a manifold in an algebraic topology context. Here we will not use the Jacobian determinants of atlas transition functions, tangent bundles, or volume forms as one may find in definitions of manifold orientation in other mathematical contexts. Instead, we will use the choice of generators of homology groups. Before we give the definition, however, we need to establish the following fact.

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Proposition 1.2. *Let* M *be a manifold of dimension* n*. Then for all* $x \in M$*, the local homology of* M *at* x *in dimension* k *is*

$$H_k(M, M - \{x\}) = \begin{cases} \mathbb{Z}, \textit{ if } k = n \\ 0, \textit{ if } k \neq n \end{cases}$$

Proof. Let U be a chart around x. Then

$$\begin{aligned} H_k(M, M - \{x\}) &\cong H_k(U, U - \{x\}) \\ &\cong H_k(\mathbb{R}^n, \mathbb{R}^n - \{x\}) && \text{by excision} \\ &\cong \widetilde{H}_{k-1}(\mathbb{R}^n - \{x\}) && \text{by long exact sequence} \\ &\cong \widetilde{H}_{k-1}(S^{n-1}) \cong \begin{cases} \mathbb{Z}, \text{ if } k = n \\ 0, \text{ if } k \neq n \end{cases} \end{aligned}$$

□

With this knowledge about the local homology of M at x, we may give the following localized definition of orientation.

Definition 1.3. A *local orientation* of an n-manifold M at $x \in M$ is a choice of a generator for $H_n(M, M - \{x\}) \cong \mathbb{Z}$. Denote $\widetilde{M} := \{\mu_x \mid x \in M \text{ and } \mu_x \text{ is a local orientation of } M \text{ at } x\}$.

To verify that this indeed satisfies the standard properties of orientability, we observe that for any rotation ρ and reflection τ of a chart around x that fix x, $\rho_*(\alpha) = \alpha)$ and $\tau_*(\alpha) = -\alpha$ for any generator α of $H_n(M, M - \{x\})$.

We can extend this local definition of orientation to a global definition by the following definition.

Definition 1.4. An *orientation* of an n-manifold M is a section $\phi : M \to \widetilde{M}$ of $\pi : \widetilde{M} \to M$ such that each $x \in M$ has a chart containing an open ball B about x such that all local orientations μ_y at points $y \in B$ are the images of the same generator $\mu_B \in H_n(M, M-B)$ under the maps $H_n(M, M - B) \to H_n(M, M - \{y\})$.

Remark 1.5. Note that these definitions of local orientation and global orientation can easily be extended to coefficient groups R that are commutative rings with unity. This generalized notion of orientability is denoted R-*orientation*.

Another definition that will be useful in the discussion of Poincaré duality is the notion of a fundamental class.

Definition 1.6. A *fundamental class* for M with coefficients in R is an element of $H_n(M;R)$ whose image in $H_n(M, M-\{x\}; R)$ is a generator for all $x \in M$.

1.2. **Cap product.** Finally, we define a useful bilinear map on homology and cohomology that allows one to adjoin a chain of degree p with a cochain of degree q to form a chain of degree $p-q$.

Definition 1.7. Let X be an arbitrary space and R be a coefficient ring. The R-bilinear *cap product* is defined to be $\frown: C_k(X;R) \times C^\ell(X;R) \to C_{k-\ell}(X;R)$ for $k \geq \ell$ with

$$\sigma \frown \varphi := \varphi(\sigma\,|_{[v_0,\ldots,v_\ell]})\sigma|_{[v_\ell,\ldots,v_k]}$$

for $\sigma : \Delta^k \to X$ and $\varphi \in C^\ell(X;R)$.

One may check that the cap product in chain and cochain groups induces an R-bilinear cap product map in the homology and cohomology groups $\frown: H_k(X;R) \times H^\ell(X;R) \to H_{k-\ell}(X;R)$ by a simple calculation exercise. Once one checks that $\partial(\sigma \frown \varphi) = \pm(\partial\sigma \frown \varphi)$ then it follows that the cap product of a cycle σ and a cocycle φ is itself a cycle and that cap products involving boundaries or coboundaries are themselves boundaries. Furthermore, one can check that the same formulas apply to relative homology and relative cohomology so that there is a relative form of the cap product.

The naturality property that the cap product satisfies is the following formula

$$f_*(\alpha) \frown \varphi = f_*\big(\alpha \frown f^*(\varphi)\big)$$

coming from the "commutative diagram"

$$\begin{array}{ccccc}
H_k(X) & \times & H^\ell(X) & \xrightarrow{\frown} & H_{k-\ell}(X) \\
\downarrow{\scriptstyle f_*} & & {\scriptstyle f^*}\uparrow & & \downarrow{\scriptstyle f_*} \\
H_k(Y) & \times & H^\ell(X) & \xrightarrow{\frown} & H_{k-\ell}(Y)
\end{array}$$

and its relative versions.

2. Poincaré duality

First, we plan to demonstrate the existence of the fundamental class for a closed R-orientable n-manifold. In order to prove this, we need the following technical lemma.

Lemma 2.1. *Let M be a manifold of dimension n with a compact sucompact subset $A \subset M$. Define $M_R := \bigcup_{x \in M} \{\alpha_x \in H_n(M \mid x)\}$ Then*

(1) *If $x \mapsto \alpha_x$ is a section of the covering space $M_R \to M$, then there is a unique class $\alpha_A \in H_n(M, M - A; R)$ whose image in $H_n(M, M - \{x\}; R)$ is α_x for all $x \in A$.*
(2) $H_i(M, M - A; R) = 0$ *for* $i > n$.

Proof. The approach here is to build cases from simplest examples to the most general. Let the coefficient ring R be implicit.

Case 1. Let $M = \mathbb{R}^n$ and A be compact and convex. Then $\mathbb{R}^n - K \cong \mathbb{R}^n - \{x\}$.

Case 2. Let $M = \mathbb{R}^n$ and $A = A_1 \cup A_2$ where the theorem holds for A_1, A_2, and $A_1 \cap A_2$. Since $(M - A_1) \cap (M - A_2) = M - A$ and $(M - A_1) \cup (M - A_2) = M - (A_1 \cap A_2)$, applying Mayer–Vietoris yields

$$0 \to H_n(M, M - A) \xrightarrow{\Phi} H_n(M, M - A_1) \oplus H_n(M, M - A_2)$$
$$\xrightarrow{\Psi} H_n(M, M - (A_1 \cap A_2)) \to \dots$$

where the first zero arises from the fact that $H_{n+1}(M, M - (A_1 \cap A_2)) = 0$ and so claim (2) is satisfied.

If $x \mapsto \alpha_x$ is a section, then we have unique classes $\alpha_{A_1} \in H_n(M, M - A_1), \alpha_{A_2} \in H_n(M, M - A_2)$, and $\alpha_{A_1 \cap A_2} \in H_n(M, M - (A_1 \cap A_2)$ with image α_x for all x in A, B, or $A \cap B$ respectively by virtue of the hypothesis of this case. Exactness of the sequence implies that $(\alpha_{A_1}, -\alpha_{A_2}) = \Phi(\alpha_A)$ for some $\alpha_A \in H_n(M, M - A)$. Furthermore, α_A must map to α_x since it maps to α_{A_1} and α_{A_2} and it is in fact unique by injectiveness of Φ.

Case 3. Let $M = \mathbb{R}^n$ and $A = \bigcup_{i=1}^k A_i$ with A_i convex and compact. This follows from induction on Cases 1 and 2.

Case 4. Let $M = \mathbb{R}^n$ and A be any compact subset. Let $\alpha \in H_i(\mathbb{R}^n, \mathbb{R}^n - A)$ be represented by a relative cycle z and let $C \subset \mathbb{R}^n - A$ be the union of the images of the singular simplices in ∂z. The set C is compact, so it has some positive distance δ from A. We can then cover A by finitely many closed balls of radius less than δ centered at points of A whose union U is thus disjoint from C. Then, the relative cycle z defines an $\alpha_U \in H_i(\mathbb{R}^n, \mathbb{R}^n - U)$ mapping to $\alpha \in H_i(\mathbb{R}^n, \mathbb{R}^n - A)$.

If $i > n$ then $H_i(\mathbb{R}^n, \mathbb{R}^n - U) = 0$ by Case 3 since U is a finite union of convex compact subsets, so $\alpha_U = 0$ and $\alpha = 0$. This gives that $H_i(\mathbb{R}^n, A) = 0$ and claim (1). If $i = n$ and $\alpha_x = 0$ in $H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\})$ for all $x \in A$ then this transmits to U where α_x is the image of α_U and so $\alpha_U = 0$ and $\alpha = 0$. This gives uniqueness. Existence follows from taking α_A to be the image of α_B for any of the balls B in U.

Case 5. Let M be any n-manifold and $A \subset U \subset M$ for U some open chart of M. By excision, $H_i(M, M - A) \cong H_i(U, U - A)$ and we know the latter case by Case 4.

Case 6. Let M be any n-manifold and $A \subset M$ any compact subset. Since A is a compact subset of a manifold, A can be written as $A = \bigcup_{i=1}^{k} A_i$ for compact sets A_i each contained in an open chart of M. Then the combination of Case 5 for each A_i and induction on Case 2 gives the full result.

□

Now, the existence of the fundamental class of closed connected R-orientable manifolds follows with a brief argument.

Theorem 2.2 (Existence of the Fundamental Class)**.** *Let M be a closed connected manifold of dimension n. Then*

(1) *If M is R-orientable, the map $H_n(M; R) \to H_n(M, M - \{x\}; R) \cong R$ is an isomorphism for all $x \in M$.*
(2) *If M is not R-orientable, the map $H_n(M; R) \to H_n(M, M - \{x\}; R) \cong R$ is injective with image $\{r \in R \mid 2r = 0\}$ for all $x \in M$.*
(3) $H_i(M; R) = 0$ *for* $i > n$.

Proof. Let $A = M$ in Lemma 2.1. Part (3) immediately follows from part (2) of Lemma 2.1. For the rest, let $\Gamma_R(M)$ be the R-module of sections of $M_R \to M$. The homomorphism $H_n(M;R) \to \Gamma_R(M)$ sending a class α to the section $x \mapsto \alpha_x$ is an isomorphism by part (1) of the lemma. Furthermore, each section is uniquely determined by its value at a single point in M since M is connected so $\Gamma_R(M) \cong M_R$. Now recall the canonical isomorphism $H_n(M, M-\{x\}; R) \cong H_n(M, M-\{x\}; R) \otimes R$ with each $r \in R$ determining a subcovering of M_R consisting of points $\pm\mu_x \otimes r \in H_n(M, M-\{x\}; R)$. We get the desired result by observing that M being not R-orientable means that there is no unit of order 2 in R. $\square$

Before we are ready to state Poincaré duality in full, we need another algebraic fact. However, we will state it here without proof because the argument is an involved series of steps involving excision, chains and cochains, and barycentric subdivision. Since it uses cohomology with compact support, we quickly give a definition below before proceeding onto the lemma statement.

Definition 2.3. Let $C^i_c(X;G)$ be the subgroup of $C^i(X;G)$ consisting of cochains for which there is a compact set of X outside of which they are zero. The *cohomology groups with compact supports* are the cohomology groups $H^i_c(X;G)$ of this subcomplex.

Lemma 2.4. *If M is the union of two open sets U and V, then there is a diagram of Mayer–Vietoris sequences, commutative up to sign:*

$$\begin{tikzcd}
\cdots \arrow[r] & H^k_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & H^k_c(U)\oplus H^k_c(V) \arrow[r] \arrow[d, "D_U \oplus -D_V"] & H^k_c(M) \arrow[r] \arrow[d, "D_M"] & H^{k+1}_c(U\cap V) \arrow[r] \arrow[d, "D_{U\cap V}"] & \cdots \\
\cdots \arrow[r] & H_{n-k}(U\cap V) \arrow[r] & H_{n-k}(U)\oplus H_{n-k}(V) \arrow[r] & H_{n-k}(M) \arrow[r] & H_{n-k-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Now, we may finally state the theorem of Poincaré duality.

Theorem 2.5 (Poincaré Duality)**.** *If M is a closed R-orientable manifold of dimension n with fundamental class $[M] \in H_n(M;R)$, then the map $D_M : H^k_c(M;R) \to H_{n-k}(M;R)$ given by $D(\alpha) = [M] \frown \alpha$ is an isomorphism for all k.*

Proof. As in the proof of Lemma 2.1, this proof will build from specific cases to general cases.

Case 1. Let $M = \operatorname{int}\Delta^n$. The map D_M can be identified with the map $D'_M : H^k(\Delta^n, \partial\Delta^n) \to H_{n-k}(\Delta^n)$ given by

$$D'_M(\alpha) := [\Delta^n] \frown \alpha,$$

where $[\Delta^n]$ is defined by the identity map of Δ^n. The homology and cohomology groups are 0 when $k \neq n$, so we only need to consider that case. With $k = n$, the generator of $H^n(\Delta^n, \partial\Delta^n) \cong \hom(H_n(\Delta^n, \partial\Delta^n), \mathbb{Z})$ is represented by a cocycle ϕ that is 1 on Δ^n. Thus,

$$[\Delta^n] \frown \phi = \phi\big([\Delta^n]\,|_{[v_0,\dots,v_n]}\big)[\Delta^n]\,|_{[v_n]} = [\Delta^n]\,|_{[v_n]},$$

the right-hand side of which is the last vertex of Δ^n and a generator of $H_0(\Delta^n)$. By taking a generator to a generator, D'_M is an isomorphism.

Case 2. Let $M = \mathbb{R}^n$. This case follows from Case 1 by homotopic equivalence.

Case 3. Let $M = U \cup V$ where $U, V \subset M$ open where the theorem holds for U, V, and $U \cap V$. This follows immediately from Lemma 2.4 and the five lemma.

Case 4. Let $M = \bigcup_\alpha U_i$ where $U_1 \subset U_2 \subset \cdots$ is a sequence of open sets and the theorem holds for each U_i. By excision, $H^k_c(U_i)$ is the limit $\varinjlim H^k(M, M - A)$ over $A \subset U_i$ compact so we have natural maps $H^k_c(U_i) \to H^k_c(U_{i+1})$ since $U_i \subset U_{i+1}$. This allows us to form $\varinjlim H^k_c(U_i) \cong H^k_c(M)$. We also have that $H_{n-k} \cong \varinjlim H_{n-k}(U_i)$ for all i and G since $M = \bigcup_i U_i$ and each compact set in M is contained in one of the U_i. Thus, D_M is the limit of the isomorphisms D_{U_i} and is therefore itself an isomorphism.

Case 5. Let M be an open subset of $\mathbb{R}^n$. For any convex open subset V of M, the theorem holds for V by Case 1 and the fact V is homeomorphic to $\mathbb{R}^n$. If V, W are both convex open subsets of $\mathbb{R}^n$ then so is $V \cap W$ and so the theorem holds for $V \cup W$ by Case 3. By induction, it also holds for any $\bigcup_{i=1}^k V_i$. Now, we may write $M = \bigcup_{i=1}^\infty V_i$ with V_i rational radius balls centered around rational points, i.e. a countable union of convex open subsets. Letting $W_j = \bigcup_{i=1}^j V_i$, we have a sequence of open subsets in M that satisfy the theorem. Thus, we obtain the desired result for M by Case 4.

Case 6. Let M be any general such manifold. The collection of open sets $U \subset M$ for which the duality maps D_U are isomorphisms is partially ordered by inclusion. The union of every totally ordered subcollection is in the collection by Case 4 (which did not use any fact of countable indexing). By Zorn's Lemma, we have the existence of a maximal open set U for which the theorem holds.

Suppose $U \neq M$. Choose a point $x \in M - U$ and an open neighborhood V of x homeomorphic to $\mathbb{R}^n$. The theorem then holds for V and $U \cap V$ by Cases 2 and 4. By Case 3 and the fact that the theorem holds for U, it then holds for $U \cup V$ which contradicts the maximality of U.

$\square$

3. Applications

The main theorem itself is interesting due to the symmetry it reveals underlying homology and cohomology. Here, we present various corollaries and examples where the theorem becomes useful on the level of examples. Although this section is limited to examples, the theorem itself has been generalized several times and adapted to other contexts (e.g. Poincaré–Lefschetz duality and Verdier duality).

There is an immediate corollary that follows from the application of Poincaré duality to knowledge about the top and bottom homology groups in Proposition 1.2 and knowledge of the first cohomology group as the group of homomorphisms from an Abelian group to $\mathbb{Z}$.

Corollary 3.1. *The top homology and top cohomology groups of* M *are* $\mathbb{Z}$. *The second highest homology group of* M *is free Abelian.*

Furthermore, Poincaré duality contains the fact that homology and cohomology groups are zero for orientable closed n-manifolds in degrees larger than n since homology and cohomology groups are defined to zero for negative degrees.

Some interesting applications of the theorem arise almost immediately from observations using Euler characteristics. These applications are well-rooted in history given the fact that Henri Poincaré's original formulation of the theorem was in terms of Betti numbers (homology theory and cohomology theory were invented decades after the original formulation of Poincaré duality).

Corollary 3.2. *A closed manifold of odd dimension has Euler characteristic 0.*

Proof. Suppose that M is any closed orientable n-manifold. The Poincaré duality implies that $H_i(M) \cong H^{n-i}(M)$ so naturally their ranks are equal. Furthermore, the universal coefficient theorem implies that rank $H^{n-i}(M) =$ rank $H_{n-i}(M)$. So if n is odd, the terms of $\sum_i (-1)^i \text{rank } H_i(M)$ cancels in pairs and so the Euler characteristic of M is zero.

If M is not orientable, then applying the same argument on the 2-sheeted cover $\widetilde{M}$, which is a closed orientable manifold itself, yields that $\chi(\widetilde{M}) = 0$. Thus, $\chi(M) = 0$ using the relation $\chi(\widetilde{M}) = 2\chi(M)$. $\square$

Corollary 3.3. $\mathbb{RP}^{2n}$, $\mathbb{CP}^{2n}$, *and* $\mathbb{HP}^{2n}$ *are not boundaries of any compact manifold.*

Proof. Suppose that X is either $\mathbb{RP}^{2n}, \mathbb{CP}^{2n}$, or $\mathbb{HP}^{2n}$ and is a boundary of some compact manifold M. Let M_1, M_2 be two copies of M. Gluing them along their boundaries to remove a copy of our chosen projective space X yields that

$$\begin{aligned}\chi(2M) &= \chi(M_1 \cup_X M_2) \\ &= \chi(M_1) + \chi(M_2) - \chi(X) \\ &= 2\chi(M) - \chi(X).\end{aligned}$$

Notice that the Euler characteristic of any of the above projective spaces are positive and odd. Furthermore, since they are of even dimension, any manifold of which they form the boundary must be odd-dimensional. If M is odd-dimensional then so must be $2M$. Thus, we see that $\chi(X) = 0$ but this contradicts the fact that the Euler characteristic of these even-dimensional projective spaces are positive. $\square$

It is worth mentioning that Poincaré duality has immediate applications to the study of Lie groups: one may see that even-dimensional spheres and even-dimensional real projective spaces cannot be Lie groups because compact connected Lie groups are compact connected orientable manifolds with Euler characteristic zero. Complex and quaternionic projective spaces also cannot be Lie groups for the same reason.

However, the larger applications of Poincaré duality lie in the computation and understanding of cohomology rings of manifolds. Several results regarding the torsion subgroups and algebraic structureof cohomology rings of manifolds arise directly from Poincaré duality. Another well-known application lies in the homotopy classification of Lens spaces and results on intersection products. Thus, any further study and appreciation of Poincaré duality necessitates further reading in these topics.

References

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