THE MORDELL CONJECTURE 100 YEARS LATER
OPEN PROBLEMS

TRANSCRIBED BY ROBIN ZHANG

1. Introduction

The Mordell conjecture was formulated by Louis J. Mordell in 1922–1923 [Mor23] and proved by Gerd Faltings in 1983 [Fal83]. More than a century after Mordell's landmark paper, related tractable unsolved problems were proposed by participants of a problem session on July 11, 2024 during the "The Mordell conjecture 100 years later" conference held at MIT July 8–12, 2024. Any errors in this transcribed list are due to the notetaker.

2. Problems

Problem 1 (Michael Stoll). *Fix a basepoint P on a curve $X/\mathbb{C}$ and consider the associated Abel–Jacobi map. The torsion packet of P is the set*

$$T(X,P) := \{Q \in X(\mathbb{C}) : Q - P \in \mathrm{Jac}(X)(\mathbb{C})_{\mathrm{tors}}\}.$$

For $g \geq 2$, define

$$T(g) := \max\{T(X,P) : X/\mathbb{C} \text{ is a curve of genus } g,\ P \in X(\mathbb{C})\},$$

the maximal size of a torsion packet on a genus g curve $X/\mathbb{C}$. It is known that $T(g) < \infty$ for all $g \geq 2$ and that $T(2) \geq 34$. Does $T(2) = 34$?

Problem 2 (Matt Broe). *How can one compute the L-function $L(E,T)$ of an elliptic curve E (or an abelian variety) over a global function field? One way is to compute the zeta function of $\mathscr{E}/\mathbb{F}_q$, going from $E \to \operatorname{Spec} k(B)$ to $\mathscr{E} \to B$ (where B is a curve over $\mathbb{F}_q$). What is the complexity in practice?*

Problem 3 (Chris Xu). *Find an efficient way to describe an isomorphism class of polarized abelian varieties defined over $\mathbb{Q}$.*

Remark on Problem 3 (Andrew Sutherland). This likely reduces to the problem of efficiently determining when two abelian varieties over $\mathbb{Q}$ are isomorphic.

Problem 4 (Héctor Pastén, endorsed by Jordan Ellenberg). *Let H denote the multiplicative height function. Can one prove that there is a constant K such that for all $\epsilon > 0$,*

$$\#\big\{c \in \mathbb{Q} : H(c) < T \text{ and } \#\mathrm{Preper}(z^2 + c) > K\big\} \ll_\epsilon T^\epsilon$$

as $T \to \infty$?

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PROBLEM 5 (Hyun Jong Kim). *For a curve X of genus ≥ 2 over a number field K, is there some notion of energy for $X(K)$?*

PROBLEM 6 (Bianca Viray). *For a curve X over a number field, is there an effective method to determine whether a closed point on X is AV-parametrized? Example: the appendix of Derickx–van Hoeij in* [BHK+24, Appendix].

PROBLEM 7 (Bjorn Poonen et al.). *Is there an algorithm that, given an abelian variety $A/\overline{\mathbb{Q}}$ and a subvariety X, determines the families of maximal cosets of abelian subvarieties contained in X?*

Remark on Problem 7 (Robin Zhang). This has recently been given by Baldi–Urbanik [BU24, Theorem 1.5] in a theoretical sense. This is a special case of the geometric Zilber–Pink conjecture for variations of mixed Hodge structures, which they prove effectively (the proof by Baldi–Klingler–Ullmo [BKU24, Theorem 3.1] is non-effective). Baldi–Urbanik gives an algorithm that outputs all atypical intersections for $\mathbb{Z}$-variations of mixed Hodge structures.

PROBLEM 8 (Alexander Borisov). *Is there an analogue of the Nyman–Beurling criterion for the Riemann hypothesis that uses theta functions?*

PROBLEM 9 (Barinder Banwait). *For every prime p, is the cardinality of the set*

$$\{X/\mathbb{Q} \text{ curve of genus } p : \mathrm{Jac}(X) \text{ has good reduction outside } p\}$$

a power of p? There are currently 2^9 such X known so far for $p = 2$, due to Robin Visser [Vis24].

PROBLEM 10 (Jordan Ellenberg). *For which infinite algebraic extensions K of $\mathbb{Q}$ is it true that every curve of genus ≥ 2 over K has only finitely many K-points?*

Remark on Problem 10 (Wei Zhang). It is essentially a theorem of Rohrlich [Roh84b, Roh84a] and Kato [Kat04] that for any curve whose Jacobian is split completely (i.e., isogenous to a product of elliptic curves over $\mathbb{Q}$) and any prime p, one has $\#X(\mathbb{Q}(\mu_{p^\infty})) < \infty$ but $\mathrm{rank}\,(\mathrm{Jac}(X)(\mathbb{Q}(\mu_{p^\infty}))) < \infty$.

Remark on Wei Zhang's remark on Problem 10 (Bjorn Poonen). See [ES93] for many examples of curves over $\mathbb{Q}$ whose Jacobian is $\mathbb{Q}$-isogenous to a product of elliptic curves over $\mathbb{Q}$.

Remark on Wei Zhang's remark on Problem 10 (Wei Zhang). Rohrlich and Kato theorems also apply when the Jacobian variety is more generally a product of GL(2)-type Abelian varieties over $\mathbb{Q}$. Examples: all modular curves over $\mathbb{Q}$ and Fermat curves over $\mathbb{Q}$, as long as $g \geq 2$.

Remark on Problem 10 (Robin Zhang). Mazur [Maz72] conjectured that the Mordell–Weil theorem remains true over the cyclotomic extension $K(\mu_{p^\infty})$ of any number field K. Parshin [ZP89, §6, Subsection 2] (cf. Zarhin [Zar10, Remark 1.3]) made the analogous conjecture for Faltings' theorem: $X(K(\mu_{p^\infty}))$ is finite for any number field K and any absolutely irreducible smooth projective curve X of genus ≥ 2 defined over $K(\mu_{p^\infty})$. Ribet [KL81, Appendix] proved that for number field K

and every abelian variety A defined over K, the torsion subgroup of $A(K(\mu_{p^\infty}))$ is finite.

Remark on Robin Zhang's remark on Problem 10 (Bjorn Poonen). Mazur's conjecture implies Parshin's conjecture, by the Mordell conjecture applied to X over a number field over which generators of the Mordell–Weil group of the Jacobian over $K(\mu_{p^\infty})$ are defined.

PROBLEM 11 (Jordan Ellenberg). *Does there exist an infinite algebraic extension $K/\mathbb{Q}$ and a curve X/K of genus ≥ 2 such that $\operatorname{Jac}(X)(K)$ has infinite rank and $X(K)$ is nonempty but finite?*

Remark on Problem 11 (Alexander Betts and Noam D. Elkies). The answer is yes. Sketch: Start with a curve X of gonality > 4 with a rational point over $\mathbb{Q}$, so that X has only finitely many quadratic points over any number field, and let K be the direct limit of a tower of suitable quadratic extensions.

PROBLEM 12 (Elyes Boughattas). *Does K being a Hilbertian field of characteristic 0 imply that $X(K)$ is finite for all curves X of genus ≥ 2 over K that are not definable over a non-Hilbertian subfield of K?*

Remark on Problem 12 (Boaz Moerman). Here is a counterexample: the Fried–Jarden book [FJ86] has an example of a pseudo-algebraically closed field that is Hilbertian of characteristic 0.

PROBLEM 13 (Mulun Yin). *Let E be an elliptic curve over $\mathbb{Q}$. Construct points in $E(\mathbb{Q})$ of infinite order. When $\operatorname{rank}_{\mathrm{an}}(E) = 1$, one can use the Heegner point construction.*

PROBLEM 14 (Robin Visser). *Given a genus 3 hyperelliptic curve $X/\mathbb{Q}$, is there a criterion or algorithm to determine whether X has potential good reduction at 2?*

PROBLEM 15 (Andrew Sutherland). *Let $A/\mathbb{Q}$ be an abelian variety of dimension g whose adelic Galois representation*

$$\rho\colon \operatorname{Gal}_{\mathbb{Q}} \to \operatorname{Gsp}_{2g}(\widehat{\mathbb{Z}})$$

has open image; for $g = 1$ this means A is a non-CM elliptic curve, and for $g = 2, 6$ or odd, it means $\operatorname{End}(A_{\overline{\mathbb{Q}}}) = \mathbb{Z}$ [Ser00].

Give an explicit function $B(x)$, as small as possible, for which

$$\left[\operatorname{GSp}_{2g}(\widehat{\mathbb{Z}}) : \rho(\operatorname{Gal}_{\mathbb{Q}})\right] < B(\operatorname{ht}(A)),$$

for all such A. For $g = 1$, the bound $B(x) = \exp(1.9\times 10^{10}) \max(1, x)^{12395}$ mentioned by Viada, due to Lombardo [Lom15], works. But Lombardo suggests that one can do better, and in fact $B(x) = 1536$ probably works [Zyw15].

One may also wish to consider bounds $B(x)$ that depend on GRH.

Remark on Problem 15 (Lorenzo Furio). For $g = 1$, we may use $B(x) = 10^{22}(x + 30)^5$.

PROBLEM 15′ (Jordan Ellenberg). *Does there exist a $B(x)$ as in Problem 15 if one replaces A by a curve X of genus ≥ 2, $\mathrm{GSp}_{2g}(\widehat{\mathbb{Z}})$ by the outer automorphism group $\mathrm{Out}(\pi_1^{\text{ét}}(X_{\overline{\mathbb{Q}},x}))$, and "open image" by another adjective?*

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