THE BAKER–CAMPBELL–HAUSDORFF FORMULA

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1. Objectives

- Prove the Baker–Campbell–Hausdorff formula
- Apply it to the Lie group–Lie algebra correspondence

Knapp [7] omits this topic, so we generally follow the treatments from Hochschild [6] and Hall [4]. Similar treatments are offered by Sepanski [10] and Serre [11].

2. The formula

Theorem 2.1 (Baker–Campbell–Hausdorff formula)**.** *If $\mathfrak{g}$ is a Lie algebra defined over a field k of characteristic 0 and $X, Y \in \mathfrak{g}$, then $\exp(X)\exp(Y) = \exp(Z)$ for some formal infinite sum Z of elements in $\mathfrak{g}$. In particular,*

$$Z = \log(\exp(X)\exp(Y)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{r_i+s_i>0} \frac{[X^{(r_1)}, Y^{(s_1)}, \ldots, X^{(r_n)}, Y^{(s_n)}]}{(\sum_{i=1}^{n}(r_i+s_i)) \prod_{i=1}^{n} r_i! s_i!},$$

where $[Z_1, \ldots, Z_m]$ denotes the iterated Lie bracket

$$[Z_1, [\ldots, [Z_{m-2}, [Z_{m-1}, Z_m]] \ldots],$$

and $Z^{(r)}$ denotes the iteration of r copies of Z, i.e $[Z_1^{(2)}, Z_2] = [Z_1, [Z_1, Z_2]]$.

Remark 2.2. Applications of the Baker–Campbell–Hausdorff formula are sensitive to convergence issues. As such, we generally need to work on neighborhoods or connected components. This fact is evident in the analytic proof given in Section 5

Note 2.3 (Historical). As noted in Bourbaki "Groupes et Algebres de Lie, Chapitres 2 et 3" [2, Note Historique V], the existence was first outlined by Schur (1890, [9]) and described "unconvincingly" by Campbell (1897, [3]), Poincaré (1899, [8]), and Baker (1902, [1]) in studying whether elements of connected groups were exponential. Hausdorff (1906, [5]) gave a much clearer demonstration of existence by studying more universal expressions in associative algebras. Finally, Dynkin (1947) found the explicit coefficients when considering Lie algebras. Consequently, the Baker–Campbell–Hausdorff formula is sometimes known as Dynkin's formula.

Remark 2.4. The Z given by the Baker–Campbell–Hausdorff formula may not actually converge to an element of $\mathfrak{g}$. For instance, if $k = \mathbb{R}$ and

$$X = \begin{pmatrix} 0 & -4 \\ 4 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

then there is no real matrix Z such that $\exp(Z) = \exp(X)\exp(Y)$.

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3. Applications

Two of the main applications of the Baker–Campbell–Hausdorff formula is its role in proving the Lie group–Lie algebra correspondence, which consists of Lie's third theorem (Theorem 3.1), the homomorphisms theorem (Theorem 3.3), and the subgroups/subalgebras theorem. While the may be proved with other methods that are not dependent on the Baker–Campbell–Hausdorff formula (e.g. the proof of the homomorphisms theorem using the Frobenius theorem from differential topology), the proofs of the first two parts of the correspondence are quite simple.

Theorem 3.1 (Lie's Third Theorem ([11, Chapter V, Section 8]))**.** *Every finite-dimensional real Lie algebra is isomorphic to the Lie algebra of a simply connected Lie group.*

Proof. Let $\mathfrak{g}$ be given. It suffices to find an analytic group whose Lie algebra is $\mathfrak{g}$ since taking the connected component H of e in G and then taking the simply covering group of G, we obtain the desired simply connected analytic group with Lie algebra $\mathfrak{g}$.

Let H be the Baker–Campbell–Hausdorff group chunk corresponding to $\mathfrak{g}$, and let $t : \mathfrak{g} \to E(V)$ be a faithful representation from Ado's theorem (Theorem 6.3). Then t induces a local homomorphism $f : H \to \mathrm{GL}(V)$. Since t is faithful, f is an immersion at e, i.e. H corresponds to a subgroup chunk of $\mathrm{GL}(V)$. But then, H is equivalent to an analytic group. □

Note 3.2. For reference and comparison, we give an alternate short proof of Lie's third theorem without the Baker–Campbell–Hausdorff formula in the appendix (Theorem 6.2).

Theorem 3.3 (Homomorphisms theorem ([4, Theorem 3.7]))**.** *If G is a simply connected Lie group with Lie algebra $\mathfrak{g}$, H is a Lie group with Lie algebra $\mathfrak{h}$, and $\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism, then there exists a unique Lie group homomorphism $\Phi : G \to H$ such that $\phi = d\Phi$ for all $X \in \mathfrak{g}$.*

Proof. First, we define Φ locally (i.e. in a neighborhood of the identity). Recall that exp has a local inverse that maps a neighborhood U of the identity 1_G into $\mathfrak{g}$. Pick U small enough so that $\log A$ and $\log B$ are small enough to apply the Baker–Campbell–Hausdorff formula for all $A, B \in U$. On such a neighborhood U, define $\Phi : U \to H$ via $\Phi := \exp \circ \phi \circ \log$.

To see that Φ is a local homomorphism, for any $A, B \in U$, write $X := \log(A)$ and $Y := \log(B)$. By the Baker–Campbell–Hausdorff formula on $\mathfrak{g}$,

$$\exp(X)\exp(Y) = \exp(Z),$$

where

$$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \dots.$$

Using that ϕ is a Lie algebra homomorphism (to get line 4) and the Baker–Campbell–Hausdorff formula on $\mathfrak{h}$ (to get line 5),

$$\begin{aligned}\Phi(AB) &= \Phi\big(\exp(X)\exp(Y)\big)\\ &= \Phi\big(\exp(Z)\big)\\ &= \exp\left(\phi(Z)\right)\\ &= \exp\left(\phi(X)+\phi(Y)+\phi\left(\frac{1}{2}[X,Y]\right)+\dots\right)\\ &= \exp\left(\phi(X)\right)\exp\left(\phi(Y)\right)\\ &= \Phi\big(\exp(X)\big)\Phi\big(\exp(Y)\big)\\ &= \Phi(A)\Phi(B).\end{aligned}$$

To define Φ globally, we use the fact that G is path-connected. For any $A \in G$, we obtain a path $A(t) \in G$ such that $A(0) = I$ and $A(1) = A$. We can then find a partition $0 = t_0 < t_1 < \ldots < t_m = 1$ such that for all s and t satisfying $t_i \leq s \leq t \leq t_{i+1}$, we have that $A(t)A(s)^{-1} \in U$. In particular, note that

$$A = \big(A(t_m)A(t_{m-1}^{-1})\big)\big(A(t_{m-1})A(t_{m-2})^{-1}\big)\ldots\big(A_{t_2}A_{t_1}^{-1}\big)A(t_1).$$

We can then define Φ globally via

$$\Phi(A) = \Phi\big(A(t_m)A(t_{m-1}^{-1}\big)\ldots\Phi\big(A_{t_2}A_{t_1}^{-1}\big)\Phi\big(A(t_1)\big).$$

The simple-connectedness of G guarantees that our definition of Φ is independent of our choice of partition and path. Furthermore,

$$\frac{d}{dt}\Phi\big(\exp(tX)\big)\bigg|_{t=0} = \frac{d}{dt}\exp\left(t\phi(X)\right)\bigg|_{t=0} = \phi(X).$$

□

In certain cases, the Baker–Campbell–Hausdorff formula converges to nice closed forms.

Corollary 3.4. *When* $[X,Y]=0$*, then* $Z = X+Y$

Corollary 3.5. *When* $[X,Y]=sY$ *for* $s\in k$ *nonzero, then* $Z = X + \frac{sY}{1-\exp(-s)}$ *in which case*

$$\exp(X)\exp(Y) = \exp\left(e^sY\right)\exp(X).$$

Corollary 3.6. *When* $[X,[X,Y]] = [Y,[X,Y]] = 0$*, as in the case of the Heisenberg group,*

$$\exp(X)\exp(Y) = \exp\left(X+Y+\frac{1}{2}[X,Y]\right)$$

Corollary 3.7 ([6, Chapter XVI, Section 4])**.** *The theory of unipotent algebraic groups over fields of characteristic* 0 *is reduced completely to that of nilpotent Lie algebras.*

Remark 3.8. The Baker–Campbell-Hausdorff formula has various uses in quantum mechanics, particularly quantum optics, where X and Y are Hilbert space operators generating the Heisenberg group. A typical example is the annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$. The degenerate formula also finds usage in quantum field theory.

4. Algebraic Approach

First, we present an algebraic proof of the Baker–Campbell–Hausdorff formula ([6, Chapter XVI Section 2]). To set this up, we define the exp and log functions in a general setting of associative algebras.

Let k be a field of characteristic 0. We can equip a k-algebra A that is graded by the non-negative integers with a topology by taking the powers of the ideal I of elements in A with constant term 0 as a fundamental system of neighborhoods of 0. Then A has a completion $\overline{A}$ defined to be the set of equivalence classes of Cauchy sequences of in A.

Definition 4.1. Suppose B is any complete topological k-algebra, with topology defined by the powers of an ideal $I \subset B$.

Define the exponential map $\exp : I \to 1 + I$ by setting

$$\exp(X) := \lim \left\{ \sum_{r=0}^{n} \frac{X^r}{r!} \right\}_{n \in \mathbb{N}}.$$

Define the logarithm map $\log : 1 + I \to I$ by setting

$$\log(1 - X) := \lim \left\{ \sum_{r=0}^{n} -\frac{X^r}{r} \right\}_{n \in \mathbb{N}}.$$

Remark 4.2. In this general setting, exp and log are continuous and mutually inverse. The proof is the same as the usual.

Observe that the Baker–Campbell–Hausdorff formula gives a series with terms consisting only of Lie brackets of X and Y. Let us fix $X, Y \in \mathfrak{g}$ so we can work over the free Lie algebra L generated by X and Y over k.

Definition 4.3. Let $\overline{\mathfrak{U}} := k[[X, Y]]$, the algebra of non-commuting formal power series over k in X and Y. Let $I \subset \overline{\mathfrak{U}}$ be the ideal of elements of $\overline{\mathfrak{U}}$ with constant term 0.

Define the coproduct map on $\overline{\mathfrak{U}}$

$$\Delta : \overline{\mathfrak{U}} \to \overline{\mathfrak{U}} \otimes \overline{\mathfrak{U}}$$

as the k-algebra homomorphism where $\Delta(X) := X \otimes 1 + 1 \otimes X$ and $\Delta(Y) := Y \otimes 1 + 1 \otimes Y$.

We say $z \in \overline{\mathfrak{U}}$ is grouplike if $\Delta(z) = z \otimes z$ and we say z is primitive if $\Delta(z) = z \otimes 1 + 1 \otimes z$.

Remark 4.4. The universal enveloping algebra $\mathfrak{U}$ of L is isomorphic to the free associative algebra $k\langle X, Y\rangle$ which has completion $\overline{\mathfrak{U}} = k[[X, Y]]$..

Let $X := \{z \in 1 + I \mid z \text{ is grouplike}\}$. We would like to show that $\exp|_{\overline{L}} : \overline{L} \to X$ is a bijection with inverse $\log|_{X}$. We know that the closure $\overline{L}$ of L is precisely the set of all primitive elements of $\overline{\mathfrak{U}}$ because L is precisely the set of primitive elements of $\mathfrak{U}$ (Theorem 6.1) and a continuity argument.

To see that $\exp(\overline{L}) \subset X$, let $z \in \overline{L}$. Then,

$$\Delta\big(\exp(z)\big) = \exp\big(\Delta(z)\big) = \exp(z \otimes 1 + 1 \otimes z) = \exp(z) \otimes \exp(z),$$

since Δ is a homomorphism and $\exp(u + v) = \exp(u)\exp(v)$ whenever $uv = vu$ by the usual Taylor series expansion (we define exp on the tensor product here in the sense of its formal power series).

To see that $\log(X) \subset \overline{L}$, let $x \in X$. Then,

$$\begin{aligned}
\Delta\big(\log(x)\big) &= \Delta\big(\log(1-(1-x))\big) \\
&= \log\big(1\otimes 1 - \Delta(1-x)\big) \\
&= \log\big(\Delta(x)\big) \\
&= \log(x\otimes x) \\
&= \log(x\otimes 1) + \log(1\otimes x) \\
&= \log(x)\otimes 1 + 1\otimes \log(x),
\end{aligned}$$

using the fact that $\log(uv) = \log(u) + \log(v)$ whenever $uv = vu$ (again, log on the tensor product is given in the sense of its formal power series).

Fact 4.5. *The grouplike elements of $\overline{\mathfrak{U}}$ form a group under multiplication. In fact, they are a subgroup of the units of $\overline{\mathfrak{U}}$.*

Proof. Clearly, $XX \subset X$ and $x^{-1} := \sum_{n=1}^{\infty}(1-x)^n$ is the inverse of x in X since $\Delta(x^{-1}) = \Delta(x)^{-1} = (x\otimes x)^{-1} = x^{-1}\otimes x^{-1}$. □

Thus, log and exp are mutually inverse functions mapping $\overline{L}$ to X and X to $\overline{L}$ respectively. Since X and Y are primitive, we know that $\exp(X)$ and $\exp(Y)$ must be grouplike and hence $\exp(X)\exp(Y)$ as well. Then its logarithm $\log\big(\exp(X)\exp(Y)\big)$ must be primitive, i.e. in $\overline{L}$. Hence, we have existence of the desired Z.

In particular, we can expand log from its definition to obtain a precise formal sum

$$\begin{aligned}
\log\big(\exp(X)\exp(Y)\big) &= \log\Big(1 - \big(1-\exp(X)\exp(Y)\big)\Big) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\big(\exp(X)\exp(Y) - 1\big)^n.
\end{aligned}$$

Then, we can expand the exp terms:

$$\begin{aligned}
\big(\exp(X)\exp(Y)-1\big)^n &= \left(\left(\sum_{r=0}^{n}\frac{X^(r)}{r!}\right)\left(\sum_{s=0}^{n}\frac{Y^(s)}{s!}\right) - 1\right)^n \\
&= \sum_{r_i+s_i>0} \frac{[X^{(r_1)}, Y^{(s_1)}, \ldots, X^{(r_n)}, Y^{(s_n)}]}{\left(\sum_{i=1}^{n}(r_i+s_i)\right)\prod_{i=1}^{n} r_i! s_i!}.
\end{aligned}$$

5. Analytic Approach

We now present an analytic proof of the Baker–Campbell–Hausdorff formula ([4, Section 3.4] and [10, Section 5.2]). By Ado's theorem (Theorem 6.3), we can move to the setting in which $\mathfrak{g}$ is a matrix Lie algebra when $\mathfrak{g}$ is finite-dimensional over k of characteristic 0. Note that throughout, we often mean equality by equality of formal power series.

Define log and exp in terms of its Taylor series as usual. For $t \in [0,1]$, let

$$Z(t) := \log\big(\exp(tX)\exp(tY)\big),$$

and note that it is a smooth function in t. If X and Y are in a sufficiently small neighborhood of the identity, then $Z(t)$ is well-defined. Differentiating with respect to t yields

$$\frac{d}{dt}\exp\big(Z(t)\big) = X\exp\big(Z(t)\big) + \exp\big(Z(t)\big)Y.$$

Lemma 5.1. *If $X(t)$ is a smooth matrix-valued function, then*

$$\frac{d}{dt}\exp\big(X(t)\big)\bigg|_{t=0} = \exp\big(X(t)\big)\left(\frac{I-\exp(-\mathrm{ad}_{X(t)})}{\mathrm{ad}_{X(t)}}\right)\left(\frac{dX}{dt}\right)$$
$$= \left(\frac{\exp(\mathrm{ad}_{X(t)})-I}{\mathrm{ad}_{X(t)}}\right)\left(\frac{dX}{dt}\right)\exp\big(X(t)\big).$$

Hence, using Lemma 5.1, we have

$$\left(\frac{\exp(\mathrm{ad}_{Z(t)})-I}{\mathrm{ad}_{Z(t)}}\right)\left(\frac{dZ}{dt}\right)\exp\big(Z(t)\big) = X\exp\big(Z(t)\big)+\exp\big(Z(t)\big)Y.$$

If X and Y are small enough, then $Z(t)$ will also be small so that the leftmost term of the LHS is close to the identity and invertible (since exp is a local diffeomorphism near $Z(t)$). Then

$$\text{(1)} \qquad \left(\frac{dZ}{dt}\right) = \left(\frac{\mathrm{ad}_{Z(t)}}{\exp(\mathrm{ad}_{Z(t)})-I}\right)\Big(X+\exp\big(Z(t)\big)Y\exp\big(Z(t)\big)^{-1}\Big).$$

Let $\mathrm{Ad}_M : \mathfrak{g}\to\mathfrak{g}$ denote the usual matrix adjoint $\mathrm{Ad}_M(X) = MXM^{-1}$. It is well-known that $\exp(\mathrm{ad}_X) = \mathrm{Ad}_{\exp(X)}$. Since $\exp\big(Z(t)\big) = \exp(tX)\exp(tY)$, we have

$$\mathrm{Ad}_{\exp\left(Z(t)\right)} = \mathrm{Ad}_{\exp(tX)}\mathrm{Ad}_{\exp(tY)},$$

so

$$\exp\big(\mathrm{ad}_{Z(t)}\big) = \exp(\mathrm{ad}_X)\exp(t\mathrm{ad}_Y).$$

Going back to Equation 1, we now have

$$\left(\frac{dZ}{dt}\right) = \left(\frac{\mathrm{ad}_{Z(t)}}{\exp(\mathrm{ad}_{Z(t)})-I}\right)\Big(X+\mathrm{Ad}_{Z(t)}Y\Big)$$
$$= \left(\frac{\mathrm{ad}_{Z(t)}}{\exp(\mathrm{ad}_{Z(t)})-I}\right)\Big(X+\exp\big(\mathrm{ad}_{Z(t)}\big)Y\Big)$$
$$= \left(\frac{\mathrm{ad}_{Z(t)}}{\exp(\mathrm{ad}_{Z(t)})-I}\right)\Big(X+\exp\big(t\mathrm{ad}_X\big)\exp\big(t\mathrm{ad}_Y\big)Y\Big)$$
$$= \left(\frac{\mathrm{ad}_{Z(t)}}{\exp(\mathrm{ad}_{Z(t)})-I}\right)\Big(X+\exp\big(t\mathrm{ad}_X\big)Y\Big).$$

Observe that we have the relation

$$\mathrm{ad}_{Z(t)} = \log\Big(I+\big(\exp(\mathrm{ad}_{Z(t)})-I\big)\Big) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\big(\exp(\mathrm{ad}_{Z(t)})-I\big)^n.$$

Then,

$$
\begin{aligned}
\left(\frac{dZ}{dt}\right) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \big(\exp(t\mathrm{ad}_X) \exp(t\mathrm{ad}_Y) - I\big)^{n-1} \Big(X + \exp\big(t\mathrm{ad}_X\big)Y\Big) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\sum_{\substack{r\geq 0 \\ s \geq 0 \\ (r,s)\neq(0,0)}} \frac{t^{r+s}}{r!s!} (\mathrm{ad}_X)^r (\mathrm{ad}_Y)^s \right)^{n-1} \left(X + \left(\sum_{i=0}^{\infty} \frac{t^i}{i!} (\mathrm{ad}_X)^i \right) Y \right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\sum_{r_i+s_i>0} \frac{t^{\sum_{i=1}^{n-1}(r_i+s_i)} [X^{(r_1)}, Y^{(s_1)}, \ldots, X^{(r_{n-1})}, Y^{(s_{n-1})}, X]}{\prod_{i=1}^{n-1} r_i! s_i!} \right. \\
&\quad \left. + \sum_{r_i+s_i>0} \frac{t^{r_n + \sum_{i=1}^{n-1}(r_i+s_i)} [X^{(r_1)}, Y^{(s_1)}, \ldots, X^{(r_{n-1})}, Y^{(s_{n-1})}, X^{(r_n)}, Y]}{r_n! \prod_{i=1}^{n-1} r_i! s_i!} \right)
\end{aligned}
$$

Noting that $Z(0) = 0$, we can calculate from the above expression that

$$
Z(1) = \int_0^1 \frac{dZ}{dt} dt = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{r_i+s_i>0} \frac{[X^{(r_1)}, Y^{(s_1)}, \ldots, X^{(r_n)}, Y^{(s_n)}]}{(\sum_{i=1}^n (r_i+s_i)) \prod_{i=1}^n r_i! s_i!}.
$$

6. Appendix

Theorem 6.1 ([6, Chapter XVI, Theorem 2.1])**.** *If $\mathfrak{g}$ is a Lie algebra over a field k of characteristic 0, then the space of primitive elements of the universal enveloping algebra $\mathfrak{U}$ of $\mathfrak{g}$ is $\mathfrak{g}$.*

Proof. Choose a totally ordered k-basis (x_α) for $\mathfrak{g}$. By the Poincaré–Birkhoff–Witt theorem, we have a k-basis of $\mathfrak{U}$ given by ordered monomials

$$\{x_{\alpha_1}^{e_1} \cdots x_{\alpha_n}^{e_n}\} \cup \{1_k\},$$

where $e_i \in \mathbb{N}$ and $x_{\alpha_i} < x_{\alpha_{i+1}}$.

We have

$$\Delta(x_{\alpha_1}^{e_1} \cdots x_{\alpha_n}^{e_n}) = (x_{\alpha_1}^{e_1} \cdots x_{\alpha_n}^{e_n}) \otimes 1 + 1 \otimes (x_{\alpha_1}^{e_1} \cdots x_{\alpha_n}^{e_n}) + \Sigma,$$

where Σ is the sum of the terms in $(\mathfrak{g}\mathfrak{U}) \otimes (\mathfrak{U}\mathfrak{g})$ resulting from the expansion of $\Delta(x_{\alpha_1})^{e_1} \cdots \Delta(x_{\alpha_n})^{e_n}$. These terms are

$$C_{e_1,f_1} \cdots C_{e_n,f_n} x_{\alpha_1}^{e_1-f_1} \cdots x_{\alpha_n}^{e_n-f_n},$$

where the C_{e_i,f_i} are the binomial coefficients and the summation runs over all n-tuples $(f_1, \ldots, f_n)$ such that $0 \leq f_i \leq e_i$ and $0 < \sum_{i=1}^n f_i < \sum_{i=1}^n e_i$. Since k is of characteristic 0, all of these binomial coefficients are nonzero and hence, if u is a linear combination of the ordered monomials and $\Delta(u) = u \otimes 1 + 1 \otimes u$, then u must be a linear combination of the x_α's (due to the linear independence of the ordered monomials). Thus, every primitive element of $\mathfrak{U}$ belongs to $\mathfrak{g}$. $\square$

Theorem 6.2 (Lie's third theorem ([7, Theorem B.7]))**.** *Every finite-dimensional real Lie algebra is isomorphic to the Lie algebra of a simply connected Lie group.*

Proof. (Without using the Baker–Campbell–Hausdorff formula)

Let $\mathfrak{g}$ be a finite-dimensional real Lie algebra. We can write $\mathfrak{g} = \mathfrak{g}_1 \oplus_\pi \mathfrak{g}_2$ with $\mathfrak{g}_1$ semisimple and $\mathfrak{g}_2$ solvable. Recall that every finite-dimensional solvable real Lie algebra is the Lie algebra of a simply connected Lie group [7, Corollary 1.103]. Let R be such a Lie group for $\mathfrak{g}_2$.

The group $\operatorname{Int}\mathfrak{g}_1$ is a Lie group with Lie algebra $\operatorname{ad}\mathfrak{g}_1$ isomorphic to $\mathfrak{g}_1$ since the center of $\mathfrak{g}_1$ is 0. Let S be the universal covering group of $\operatorname{Int}\mathfrak{g}_1$. Then there is a unique action τ of S on R by automorphisms $\overline{\tau}$ such that $d\overline{\tau} = \pi$ and $G = S \times_\tau R$ is a simply connected Lie group with Lie algebra isomorphic to $\mathfrak{g} = \mathfrak{g}_1 \oplus_\pi \mathfrak{g}_2$. □

Theorem 6.3 (Ado's theorem ([4, Theorem 2.40]))**.** *Every finite-dimensional Lie algebra* $\mathfrak{g}$ *over a field* k *of characteristic* 0 *can be viewed as a matrix Lie algebra under the commutator bracket.*

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