AN EFFECTIVE SIEGEL THEOREM FOR POSITIVE INTEGRAL POINTS ON ACYCLIC CLUSTER VARIETIES

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Abstract. We determine the number of positive integral points on n-dimensional affine varieties associated to $n \times n$ generalized Cartan matrices. An application of this to the theory of cluster algebras and combinatorics is the resolution of the Fontaine–Plamondon conjecture, which says that there are exactly 4400 and 26952 positive integral friezes of type E_7 and E_8 respectively. An application of this to number theory is a simultaneous refinement and generalization of theorems of Mohanty, Mordell, and Schinzel to positive integers and higher dimensions by exhibiting examples of Diophantine equations $xyz = G(x, y)$ and $xyzw = G(x, y, z)$ of every degree greater than 3 with infinitely many positive integral solutions.

Contents

1. Introduction

Counting positive integral points. General finiteness problems about integral and rational points on affine varieties date back to at least 1901, when Poincaré [Poi01] asked whether the group $E(\mathbb{Q})$ of $\mathbb{Q}$-rational points on each elliptic curve E has only finitely many generators. Poincaré's problem was affirmatively resolved by Mordell [Mor23] in 1922, generalized by Weil's thesis [Wei29] in 1929 to number fields, and generalized by Faltings [Fal83, Fal84] in 1983 to curves of higher genus.

Theorem 1 (Mordell–Weil and Faltings). *Let K be a number field with ring of integers $\mathcal{O}_K$ and let X be a smooth projective curve of genus g defined over K.*
(a) If $g = 0$, then $\#X(K) = 0$ or ∞.
(b) If $g = 1$, then $X(K)$ is a finitely-generated abelian group.
(c) If $g \geq 2$, then $X(K)$ is finite.

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Long before Faltings's proof of Theorem 1(c), Siegel [Sie29][1] used Theorem 1(b) and the Thue–Siegel–Roth theorem from the theory of Diophantine approximations to prove an integral version of Theorem 1(c) in 1929.

THEOREM 2 (Siegel's theorem on integral points). *Let* K *be a number field with ring of integers* $\mathcal{O}_K$ *and let* X *be a smooth affine curve of genus* g *defined over* K*. If* $g \geq 1$*, then* $X(\mathcal{O}_K)$ *is finite.*

The theorems of Siegel and Faltings were first posed as open questions at the end of Mordell's 1922 paper [Mor23]. In this article, we consider Mordell's question over the positive integers $\mathbb{N}$ for a class of affine varieties of dimension ≥ 2.

The main number theoretic result in this paper is a positive refinement and higher-dimensional generalization of the Mordell–Schinzel program, which looks for infinitely many integer solutions to Diophantine equations of the form

$$xyz = G(x, y) \tag{1}$$

for a polynomial $G \in \mathbb{Z}[x, y]$. After proving that Equation 1 has infinitely many integer solutions when $G(x, y) = ax^3 + by^3 + c$, Mordell [Mor52] claimed (without proof) that Equation 1 also has infinitely many integer solutions for any $G \in \mathbb{Z}[x, y]$; quadratic counterexamples to this general statement were found by [Jac39, Bar53, Mil54]. Mordell's claim was recovered in several cases by [Sch15, Sch18, KL24], including when $G(x, y) = x^m + y + 1$ in [Sch15, Theorem 3]. We give a refinement of Schinzel's result by showing that there are infinitely many *positive* integer solutions for $G(x, y) = x^m + y + 1$ and also prove a new higher-dimensional generalization.

THEOREM 3 (A Diophantine application of Theorem 5). *Let* a*,* b*,* c*, and* d *be positive integers.*

(a) If $ab \geq 4$*, then there are infinitely many positive integer solutions* (x, y, z) *to the equation*

$$xyz = (x^a + 1)^b + y.$$

Furthermore if $ab = 1$*,* 2 *or* 3*, then the number of positive integer solutions is* 5*,* 6*, or* 9 *respectively.*

(b) If $abcd \geq 3$*, then there are infinitely many positive integer solutions* (x, y, z, w) *to the equation*

$$xyzw = (x^a + 1)^b y + (x^c + 1)^d z.$$

Remark 4. The $ab \leq 3$ part of Theorem 3(a) is already known. Mohanty [Moh77, Theorem 2] proved that $xyz = x^3 + y + 1$ has exactly 9 positive integer solutions. We give a new proof of Mohanty's theorem when treating the $ab = 3$ case.

The surface in Theorem 3(a) has arithmetic genus $g \geq 1$ precisely when $ab \geq 4$, which suggests a higher-dimensional analogue of Siegel's theorem; the proof of Theorem 3 is an application (in Section 5) of such a result for certain acyclic cluster varieties. Generally, a cluster variety is defined as the spectrum of a cluster algebra. In this article, we consider an n-dimensional affine variety X to be "of *cluster algebra type* C" if it is the zero locus of n polynomials associated to an $n \times n$ generalized

[1] For an English translation, see [Zan14].

Cartan matrix $C = (c_{i,j})$, which can be taken to be the following polynomials for each $i \in \{1, \dots, n\}$ (see Section 2 and Definition 2.3 in particular):

$$f_{C,i} := x_i y_i - \prod_{j=1}^{i-1} x_j^{-c_{j,i}} - \prod_{j=i+1}^{n} x_j^{-c_{j,i}} \in \mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_n].$$

These are Diophantine models of cluster varieties for (acyclic) cluster algebras $\mathcal{A}_C$ with trivial coefficients associated to C. In particular, the hypersurfaces of Theorem 3 are affine varieties of cluster algebra type with generalized Cartan matrices $\left(\begin{smallmatrix} 2 & -a \\ -b & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & -a & 0 \\ -b & 2 & -d \\ 0 & -c & 2 \end{smallmatrix}\right)$.

The main geometric result in this paper is a "positive Siegel theorem" for integral points on affine varieties of cluster algebra type $\mathcal{A}_C$ in terms of the invariant

$$t_C := \min_{\mathcal{I} \subset \{1,\dots,n\}} C_{\mathcal{I}} = \text{the smallest principal minor of } C,$$

where $C_{\mathcal{I}}$ is the deteminant of the submatrix of C obtained by removing the i-th column and i-th row of C for all $i \in \mathcal{I}$. Recall that a generalized Cartan matrix C is finite type if and only if $t_C \geq 1$.

THEOREM 5. *Let n be a positive integer and let X be an n-dimensional affine variety of cluster algebra type $\mathcal{A}_C$ associated to an $n \times n$ generalized Cartan matrix C.*

(a) If $t_C \leq 0$, then $\#X(\mathbb{N}) = \infty$.
(b) If $t_C \geq 1$, then $\#X(\mathbb{N})$ is finite and precisely given by the formulae in Table 1.

Remark 6. We give an effective version of Theorem 5(b) in Proposition 4.1, which specifies bounds on the heights of $X(\mathbb{N})$ that are sharp for all finite types except B_n and D_n and explicit for all finite types.

In Section 3, we prove Theorem 5 in two steps: the qualitative finiteness/infinitude and the quantitative enumeration. The key cluster algebra ingredients of the proof are:

- the classification of cluster algebras by Fomin–Zelevinsky [FZ03];
- the lower-bound model of Berenstein–Fomin–Zelevinsky [BFZ05];
- the finiteness of Dynkin friezes by Gunawan–Muller [GM22] and Muller [Mul23]; and
- the correspondence between positive integral friezes and positive integral points by de Saint Germain–Huang–Lu [dSGHL23] (Proposition 2.4).

The key arithmetic geometry ingredients of the proof are:

- the study of $(\mathbb{Z}/P_C\mathbb{Z})$-orbits on X, where P_C is a positive integer associated to each finite type C (as listed in Table 1); and
- a reduction procedure of taking intersections with certain affine hyperplanes (which is shown to be equivalent to removing degree-1 nodes from the corresponding Dynkin diagram by Proposition 2.6).

In Theorem 5(b), our counts of $\#X(\mathbb{N})$ for types A_n, B_n, C_n, D_n, E_6, F_4, and G_2 follow from direct applications of the de Saint Germain–Huang–Lu correspondence to the Dynkin frieze enumerations of [CC73a, MGOT12, FP16, BFG+21]. We also give a *uniform* proof for all types by reducing the enumeration of $X(\mathbb{N})$ to reasonable finite computational searches; we perform this computation explicitly for the E_7 and E_8 cases, which do not follow from combining previous studies. In Appendix A, we provide direct elementary proofs for finite types A_1, A_2, A_3, $B_2 = C_2$, and G_2 because even the small rank finite types are interesting from an arithmetic point of view: the corresponding affine varieties have infinitely many *integral* points but only finitely many *positive integral* points when $n \geq 2$.

Counting positive integral friezes. Using Theorem 5(b), we give an application of Diophantine geometry to combinatorics and the theory of cluster algebras. In particular, we look at the enumeration of friezes when $\mathcal{A}$ is of finite type Δ_n.

In 1971, Coxeter [Cox71] introduced frieze patterns for each positive integer $n \in \mathbb{N}$: arrays of numbers $x_{i,j}$ with $n+2$ rows and infinitely many columns of the shape

$$\begin{array}{ccccccccccc}
\cdots & & 1 & & 1 & & 1 & & 1 & & \cdots \\
& \cdots & & x_{1,1} & & x_{1,2} & & x_{1,3} & & \cdots & \\
\cdots & & x_{2,0} & & x_{2,1} & & x_{2,2} & & x_{2,3} & & \cdots \\
& \cdots & & x_{3,0} & & x_{3,1} & & x_{3,2} & & \cdots & \\
\cdots & & x_{4,-1} & & x_{4,0} & & x_{4,1} & & x_{4,2} & & \cdots \\
& \ddots & & \ddots & & \ddots & & \ddots & & \ddots & \\
\cdots & & x_{n,-2} & & x_{n,-1} & & x_{n,0} & & x_{n,1} & & \cdots \\
& \cdots & & 1 & & 1 & & 1 & & \cdots &
\end{array}$$

such that all diamonds

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array}$$

satisfy the unimodular relation $ad - bc = 1$. These generalize Gauss's formulae for the *pentagramma mirificum*, which correspond to the $n = 2$ case. Conway–Coxeter [CC73b, CC73a] proved that all frieze patterns are periodic and that there are only finitely many frieze patterns with positive integer entries for each $n \in \mathbb{N}$. They furthermore constructed a bijection with the number of triangulations of $(n+3)$-sided polygons to show that the number of friezes is the $(n+1)$-st Catalan number $\frac{1}{n+2}\binom{2n+2}{n+1}$.

Frieze patterns were generalized to all finite Dynkin types Δ_n by Caldero–Chapoton [CC06] and Assem–Reutenauer–Smith [ARS10], who constructed Dynkin friezes as special $\mathbb{Z}$-valued homomorphisms of cluster algebras and showed that there is a finite period P_{Δ_n} associated to each finite type. The Conway–Coxeter theorem corresponds to the enumeration of A_n-friezes and has been extended to all Dynkin types except E_7 and E_8 by [CC73a, MGOT12, FP16, BFG$^+$21] using triangulations of Euclidean n-gons, Poisson geometry, triangulations of once-punctured n-gons, and foldings of Dynkin diagrams.

The main combinatorial result in this article is the resolution of the conjecture of Fontaine–Plamondon [FP16, Conjecture 4.5], which states that there are 4400 and

26952 many positive integral friezes of Dynkin type E_7 and E_8 respectively. This completes the enumeration of all positive integral friezes of Dynkin type.

Theorem 7 (A frieze application of Theorem 5(b)). *For each finite Dynkin type Δ_n, the number of positive integral friezes of type Δ_n is precisely given by the formula in Table 1. In particular, there are exactly* 4400 *and* 26952 *positive integral friezes of Dynkin type* E_7 *and* E_8 *respectively.*

Remark 8. Theorem 7 follows from the enumerative part of Theorem 5(b) and a bijection between $X(\mathbb{N})$ and positive integral friezes given by de Saint Germain–Huang–Lu [dSGHL23] (Proposition 2.4). This gives new a uniform Diophantine proof of enumeration theorems in [CC73a, MGOT12, FP16, BFG$^+$21].

The study of friezes over $\mathbb{Z}$, $\mathbb{Z}[\sqrt{d}]$, and $\mathbb{Z}/N\mathbb{Z}$ has been fruitfully initiated in [Fon14, CH19, CHJ20, GS20, HJ20, MG21, BFT$^+$23, CHP24, SvSZ25]. One may generalize our Diophantine approach by studying $X_C(R)$ for these and other integral domains R, both with or without positivity conditions. We plan to give new counts of Dynkin friezes over finite fields in a forthcoming work. In another direction, friezes with general SL(k) unimodular conditions were introduced by Cordes–Roselle [CR72]; these were shown by Baur–Faber–Gratz–Serhiyenko–Todorov [BFG$^+$21] to arise as specializations of cluster algebras to the positive integers just like the classical SL(2)-friezes; so one might also use the corresponding variety to study general SL(k)-friezes.

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2. Affine varieties of cluster algebra type

2.1. **Positive integral friezes.** By the work of Fomin–Zelevinsky [FZ03], there is a classification of cluster algebras into types that mirrors the Killing–Cartan classification of simple Lie groups and the classification of Dynkin diagrams. Throughout this paper, we will restrict ourselves to cluster algebras $\mathcal{A} = \mathcal{A}_C$ associated to an $n \times n$ generalized Cartan matrix $C = (c_{i,j})$ with trivial coefficients and exchange matrix

$$B = \begin{pmatrix} 0 & -c_{1,2} & \cdots & -c_{1,n} \\ c_{2,1} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -c_{n-1,n} \\ c_{n,1} & \cdots & c_{n,n-1} & 0 \end{pmatrix}.$$

When $\mathcal{A}$ is of finite type, there is a Dynkin type Δ_n of rank n such that C is equivalent to the generalized Cartan matrix C_{Δ_n} of Δ_n up to a change of basis (see Table 1 for the list of finite Dynkin types Δ_n and a choice of C_{Δ_n}); we will simply use Δ_n to refer to the type of $\mathcal{A}$ in this case.

Following Caldero–Chapoton [CC06] and Assem–Reutenauer–Smith [ARS10], a positive integral frieze is a ring homomorphism from a cluster algebra $\mathcal{A}$ to $\mathbb{Z}$ such that the cluster variables are sent to the set $\mathbb{N}$ of positive integers. Positive integral friezes were used in the work of Assem–Reutenauer–Smith to recover results of [FZ02, FZ03, BFZ05, FZ07] and to prove new explicit formulae for cluster variables. In this generalization, the classical Coxeter friezes are the positive integral friezes for cluster algebras of type A_n.

By the work of Fomin–Zelevinsky [FZ03] (cf. [ARS10, Corollary 1], [MG15, Theorem 2.7]), if C is of finite type Δ_n then each C-frieze F (which we also call a Δ_n-frieze) is periodic under horizontal translation with some period P_{Δ_n}. In particular, each Δ_n-frieze can be uniquely represented by an $n \times P_{\Delta_n}$ grid of positive integers

$$\begin{matrix} F_{1,1} & F_{1,2} & \cdots & F_{1,P_{\Delta_n}} \\ F_{2,1} & F_{2,2} & \cdots & F_{2,P_{\Delta_n}} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n,1} & F_{n,2} & \cdots & F_{n,P_{\Delta_n}} \end{matrix}$$

such that it satisfies the mesh relations for all (i,j) in terms of the Cartan matrix $C = C_{\Delta_n} = (c_{i,j})$:

$$F_{i,j}F_{i,j+1} = 1 + \prod_{k=1}^{i-1} F_{k,j+1}^{-c_{k,i}} \prod_{k=i+1}^{n} F_{k,j}^{-c_{k,i}}.$$

Example 2.1 (A frieze of type E_8).

...	1		1		1		1		1		1	...
...		4		3		3		4		4		...
...	15		11		8		11		15		11	...
...	7	41	6	29	5	29	6	41	7	41	6	...
...	16		18		21		18		16		18	...
...		7		13		13		7		7		...
...	3		5		8		5		3		5	
...		2		3		3		2		2		...
...	1		1		1		1		1		1	...

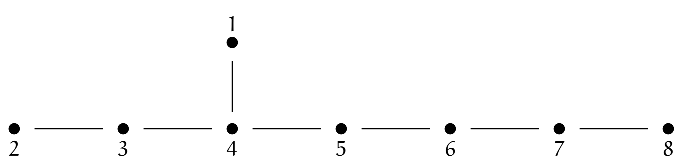
is a frieze of type E_8. In particular, all diamonds

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & f & \\ e & g & i \\ & h & \end{array}$$

satisfy the unimodular relations $ad - bc = 1$ and $ei - fgh = 1$. This E_8-frieze has exact period 4 and can be represented as the 8×4 grid:

6	5	6	7
4	3	3	4
11	8	11	15
29	29	41	41
21	18	16	18
13	7	7	13
5	3	5	8
2	2	3	3

Notice that the first entry in each column of the grid corresponds to a subdiagonal of the frieze and the rest of each column corresponds to the adjacent diagonal of the frieze. A different choice of Cartan matrix for E_8 would give rise to a different presentation of the frieze. Our choice of Cartan matrix given in Table 1 corresponds to the following labeling of the Dynkin diagram for E_8:



2.2. **Frieze polynomials and affine varieties.** Let C be an $n \times n$ generalized Cartan matrix with entries $c_{i,j}$. Consider the complex affine space $\mathbb{A}^{2n}_{\mathbb{C}}$ with coordinates $x = (x_1, \ldots, x_n; y_1, \ldots y_n)$. The following definition is based on the lower bound model of Berenstein–Fomin–Zelevinsky [BFZ05, Section 1.3] and the model of Geiss–Leclerc–Schröer [GLS13, Proposition 7.4] for the cluster algebra $\mathcal{A}_C$ with trivial coefficients associated to C.

Definition 2.2. For each integer $i \in \{1, \ldots, n\}$, the i-th *lower bound* C*-frieze polynomial* is

$$f_{C,i} := x_i y_i - \prod_{j=1}^{i-1} x_j^{-c_{j,i}} - \prod_{j=i+1}^{n} x_j^{-c_{j,i}} \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

and the i-th *Geiss–Leclerc–Schröer* C*-frieze polynomial* is

$$g_{C,i} := x_i z_i - \prod_{j=i+1}^{n} x_j^{-c_{j,i}} \prod_{j=1}^{i-1} z_j^{-c_{j,i}} - 1 \in \mathbb{Z}[x_1, \ldots, x_n, z_1, \ldots, z_n].$$

For the coordinate rings $R_C := \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]/(f_{C,1}, \ldots, f_{C,n})$, and $S_C := \mathbb{Z}[x_1, \ldots, x_n, z_1, \ldots, z_n]/(g_{C,1}, \ldots, g_{C,n})$ define the affine varieties $X_C := \mathrm{Spec}(R_C)$ and $Y_C := \mathrm{Spec}(S_C)$.

In general, a cluster variety is the spectrum of a cluster algebra over $\mathbb{C}$. Since the cluster algebra $\mathcal{A}_C$ associated to C is always acyclic, there are isomorphisms $\mathrm{Spec}(\mathcal{A}_C) \cong X_C \cong Y_C$ (see [dSGHL23, Propositions 6.2 and 6.7]) so we can equivalently use the lower bound and Geiss–Leclerc–Schröer models.

Definition 2.3. We say that an affine variety X is of *cluster algebra type* C if X is isomorphic to $\mathrm{Spec}(\mathcal{A}_C)$, X_C, or Y_C for some generalized Cartan matrix C.

The n-dimensional affine variety X_C can be given as the zero locus of the lower bound C-frieze polynomials:

$$X_C := V\big(\{f_{C,i}\}_{1\leq i\leq n}\big) = \left\{x \in \mathbb{A}^{2n}_{\mathbb{C}} \,\middle|\, f_{C,1}(x) = \ldots = f_{C,n}(x) = 0\right\} \subset \mathbb{A}^{2n}_{\mathbb{C}};$$

similarly Y_C is the zero locus of the ideal generated by the $g_{C,i}$. The affine variety X_C is smooth at its positive integral points but can have singularities; see [BFMS23], which exhibits singularities of $\mathrm{Spec}(\mathcal{A}_C)$ for the cluster algebra $\mathcal{A}_C$ of type Δ_n. Note that the variety X_C is different than the frieze variety $X(Q)$ associated to an acyclic quiver Q as defined by Lee–Li–Mills–Schiffler–Seceleanu [LLM+20]. Also note that the set of Δ_n-frieze polynomials is only unique up to the choice of Cartan matrix for Δ_n, but the number of positive integral points does not depend on the choice of Cartan matrix C by Proposition 2.4. See Table 1 for a list of frieze polynomials of each Dynkin type.

A key ingredient in our proofs is the following correspondence between friezes and affine varieties due to de Saint Germain–Huang–Lu [dSGHL23, Section 6.3] (see [dSG23, Example 3.4.2] for an illustration in type E_7).

Proposition 2.4 ([dSGHL23, Section 6.3]). *Let C be a generalized $n \times n$ Cartan matrix.*

(a) There is a one-to-one correspondence between the set of positive integral C-friezes and $X_C(\mathbb{N})$.
(b) There is a one-to-one correspondence between the set of positive integral C-friezes and $Y_C(\mathbb{N})$.

In the finite type case, a consequence of Proposition 2.4 is that the positive integrality of the first diagonal and n other entries of a Dynkin frieze F are sufficient to both uniquely determine all other entries and guarantee their positive integrality. A point $x \in X_{\Delta_n}(\mathbb{N})$ corresponds to the Δ_n-frieze with the first column given by the $x_i = F_{i,1}$ and with n other entries given by the y_i.

Since friezes of Dynkin type Δ_n are horizontally periodic with period P_{Δ_n}, there is an action of $\mathbb{Z}/P_{\Delta_n}\mathbb{Z}$ on $X_{\Delta_n}(\mathbb{N})$. For $x \in X_{\Delta_n}(\mathbb{N})$, let F denote the corresponding Δ_n-frieze. For each $\sigma_m \in \mathbb{Z}/P_{\Delta_n}\mathbb{Z}$, define $\sigma_m(x)$ to be the point of $X_{\Delta_n}(\mathbb{N})$ corresponding to the horizontal translation of F to the right by m columns. de Saint Germain has kindly pointed out to the author that the σ_1 action has the following names in different contexts:

- the Auslander–Reiten translation in finite-dimensional representation theory (cf. [LLM+20, Section 2.1]);
- the Fomin–Zelevinsky twist in Lie theory (cf. [dSG23, Section 5.6]);
- the Donaldson–Thomas transformation in Calabi–Yau theory (cf. [Wen21]);
- the maximal green sequence in combinatorics (cf. [Mil17]).

We note that σ_1 acts on $Y_C(\mathbb{N})$ by sending a positive integral point $(x_1, \dots, x_n, z_1, \dots, z_n)$ to another positive integral point $(z_1, \dots, z_n, z'_1, \dots, z'_n)$.

In this article, we will generally focus on the lower bound frieze polynomials and the affine variety X_C due to the lower polynomial degrees. For the remainder of this article, "frieze polynomial" will generally refer to the lower bound frieze polynomial.

Remark 2.5. An immediate corollary of the theorem of Conway–Coxeter [CC73a] and Proposition 2.4 is an expression for the n-th Catalan number as the number of positive integer solutions to a Diophantine system of equations:

$$\frac{1}{n+1}\binom{2n}{n} = X_{A_{n-1}}(\mathbb{N}) = \#\left\{ x \in \mathbb{N}^{2n-2} \;\middle|\; \begin{array}{c} x_1y_1=x_2+1, \\ x_2x_2=x_1+x_3, \\ x_3x_3=x_2+x_4, \\ \vdots \\ x_{n-2}y_{n-2}=x_{n-3}+x_{n-1} \\ x_{n-1}y_{n-1}=x_{n-2}+1 \end{array} \right\}.$$

2.3. **Reduction to smaller types.** We observe that for our affine varieties defined in Section 2.2, removing a node from the Dynkin diagram of type Δ_n corresponds to taking an intersection of X_{Δ_n} with the affine hyperplane to the removed node. For notational convenience, we will fix a Cartan matrix $C_{\Delta_n} = (c_{i,j})$ and label the nodes from 1 to n of the Dynkin diagram accordingly.

PROPOSITION 2.6. *Let Δ_n be a Dynkin type of rank n and let $\Delta_n^{(k)}$ be a Dynkin type obtained by removing the k-th node from the Dynkin diagram of Δ_n. If the degree of the k-th node in Δ_n is 1, then $X_{\Delta_n^{(k)}} \cong X_{\Delta_n} \cap \{y_k = 1\} \cong X_{\Delta_n} \cap \{x_k = 1\}$.*

Proof. Since the k-th node has degree 1, it is connected to a unique node k'. The k-th row and column of C_{Δ_n} are given by:

$$c_{i,k} = c_{k,i} = \begin{cases} 2 & \text{if } i = k, \\ -1 & \text{if } i = k', \\ 0 & \text{otherwise.} \end{cases}$$

This degree-1 condition therefore implies that the only two frieze polynomials involving x_k and y_k are:

$$f_{\Delta_n,k} = x_ky_k - x_{k'} - 1,$$
$$f_{\Delta_n,k'} = \begin{cases} x_{k'}y_{k'} - x_k - x_{k''}x_{k'''} & \text{if the } k'\text{-th node has in-degree } 3, \\ x_{k'}y_{k'} - x_k - x_{k''} & \text{otherwise,} \end{cases}$$

where k'' is possibly equal to k''' (such as when $\Delta_n = F_4$).

The Cartan matrix $C_{\Delta_n^{(k)}}$ is the submatrix of C_{Δ_n} obtained by deleting the k-th row and the k-th column. Therefore, the frieze polynomials of $\Delta_n^{(k)}$ are, up to a relabeling $f : \{1, \dots, n\} \setminus k \longrightarrow \{1, \dots, n-1\}$ of the nodes, identical to the frieze polynomials of

Δ_n that do not involve x_k and y_k. Denote $X_{\Delta_n} = \mathrm{Spec}(R_{\Delta_n})$ and $X_{\Delta_n^{(k)}} = \mathrm{Spec}(R_{\Delta_n^{(k)}})$, where R_{Δ_n} and $R_{\Delta_n^{(k)}}$ are the coordinate rings defined by the frieze polynomials. The closed subscheme $Y = V(y_k - 1) \subset X_{\Delta_n}$ defined by $y_k = 1$ is obtained by taking the quotient of R_{Δ_n} by the ideal $(y_k - 1)$. Similarly, let $Z = V(x_k - 1) \subset X_{\Delta_n}$ be the closed subscheme obtained by taking the quotient of R_{Δ_n} by the ideal $(x_k - 1)$.

In the quotient ring $R_{\Delta_n}/(y_k - 1)$, $f_{\Delta_n,k} = x_k - x_{k'} - 1$; substituting $x_k = x_{k'} + 1$ into the other polynomial involving x_k and y_k yields

$$f_{\Delta_n,k'}|_{x_k=x_{k'}+1,y_k=1} = \begin{cases} x_{k'}(y_{k'} - 1) - 1 - x_{k''}x_{k'''} & \text{if the } k'\text{-th node has in-degree 3,} \\ x_{k'}(y_{k'} - 1) - 1 - x_{k''} & \text{otherwise.} \end{cases}$$

Since $-1 - x_{k''}x_{k'''} \neq 0$ and $-1 - x_{k''} \neq 0$ for positive integer $x_{k''}$ and $x_{k'''}$, a solution to $f_{\Delta_n,k'}|_{x_k=x_{k'}+1} = 0$ requires that $y_{k'} \geq 2$. But taking a change of variables $y_{k'} - 1 = y_{f(k')}$ yields the frieze polynomial $f_{\Delta_n^{(k)},f(k')}$. Hence the corresponding morphism of coordinate rings

$$\begin{aligned} R_{\Delta_n}/(y_k - 1) &\xrightarrow{\phi^*} R_{\Delta_n^{(k)}} \\ x_i &\longmapsto x_{f(i)} && \text{for } i \neq k \\ y_i &\longmapsto y_{f(i)} && \text{for } i \neq k, k' \\ x_k &\longmapsto x_{f(k')} + 1. \\ y_k &\longmapsto 1. \\ y_{k'} &\longmapsto y_{f(k')} + 1. \end{aligned}$$

gives a morphism of schemes $\phi : Y \longrightarrow X_{\Delta_n^{(k)}}$ with an explicit bijection of positive integral points. The case with $Z = V(x_k - 1) \subset X_{\Delta_n}$ is nearly identical. □

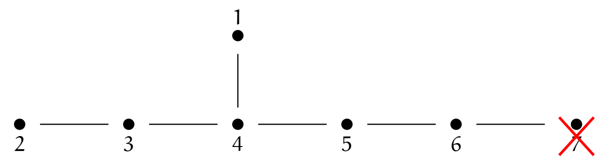
Observe that for every fixed alphabetic family of Dynkin diagrams, Δ_{n+1} is obtained from the Dynkin diagram for Δ_n by connecting a single new node to the first node of Δ_n; this corresponds to adding a row and column with a 2 and a -1 to the top and left of the Cartan matrix (in the orientation given in the Table 1). Then for any Dynkin type Δ_n, the specialization of the frieze equations for Δ_{n+1} to $x_1 = 1$ and $y_2 = x_2 + 1$ is equivalent to the frieze equations for Δ_n. In other words, $X_{\Delta_n} \cong X_{\Delta_{n+1}} \cap \{x_1 = 1\}$. Conversely, one can go from X_{Δ_n} to $X_{\Delta_{n+1}}$ by replacing an equation of the form

$$x_i y_i = x_j + 1$$

with two equations

$$\begin{aligned} x_i y_i &= x_j + x_k, \\ x_k y_k &= x_i + 1. \end{aligned}$$

EXAMPLE 2.7. Let $\Delta_n = E_7$. We can obtain the Dynkin diagram for E_6 by removing the node labeled "7" from the Dynkin diagram of E_7:



The E_7-frieze equations for the Cartan matrix in Table 1 are:

$$\begin{aligned} x_1y_1 &= x_4 + 1, \\ x_2y_2 &= x_3 + 1, \\ x_3y_3 &= x_2 + x_4, \\ x_4y_4 &= x_1x_3 + x_5, \\ x_5y_5 &= x_4 + x_6, \\ x_6y_6 &= x_5 + x_7, \\ x_7y_7 &= x_6 + 1. \end{aligned}$$

Observe that the $\mathbb{N}^{14}$-solutions of the specialization of the E_7-frieze equations to $y_7 = 1$:

$$\begin{aligned} x_1y_1 &= x_4 + 1, \\ x_2y_2 &= x_3 + 1, \\ x_3y_3 &= x_2 + x_4, \\ x_4y_4 &= x_1x_3 + x_5, \\ x_5y_5 &= x_4 + x_6, \\ x_6(y_6 - 1) &= x_5 + 1, \end{aligned}$$

are in bijection with the $\mathbb{N}^{12}$-solutions to the E_6-frieze equations in $\mathbb{Z}[x_1, \dots, x_6, y_1, \dots, y_6']$ (after observing that there are no positive integral solutions with $y_6 = 1$ and changing variables $y_6 - 1 \mapsto y_6'$). Similarly, the $\mathbb{N}^{14}$-solutions of the specialization of the E_7-frieze equations to $x_7 = 1$:

$$\begin{aligned} x_1y_1 &= x_4 + 1, \\ x_2y_2 &= x_3 + 1, \\ x_3y_3 &= x_2 + x_4, \\ x_4y_4 &= x_1x_3 + x_5, \\ x_5y_5 &= x_4 + x_6, \\ x_6y_6 &= x_5 + 1, \end{aligned}$$

are in bijection with the $\mathbb{N}^{12}$-solutions to the E_6-frieze equations in $\mathbb{Z}[x_1, \dots, x_6, y_1, \dots, y_6]$.

By Proposition 2.6, for every positive integral point of X_{Δ_n}, either all of its coordinates away from a double edge must be at least 2 or it must corresponds to a point on $X_{\Delta'_m}$ for some Dynkin type Δ'_m of rank $m < n$ after taking an intersection with a hyperplane. For E_7 and E_8, this recovers (given the enumerations of smaller ranks) the observation of [GM22, Remark 6.21] that it is sufficient to search for friezes whose entries are at least 2. In fact, their observation uses considerations of cluster algebras and generalized associahedra after removing vertices from Dynkin diagrams in [GM22, Appendix A].

We translate the observation of [GM22, Remark 6.21] into a statement about X_{Δ_n}, which will be useful in Section 3.2 when proving the E_8 enumeration theorem. Recall from Section 2.2 that there is an action of $\mathbb{Z}/P_{\Delta_n}\mathbb{Z}$ on $X_{\Delta_n}(\mathbb{N})$ defined via horizontal translations σ_m of the corresponding friezes by m columns.

Lemma 2.8. *Let* $\mathcal{O}_{\mathbb{Z}/P_{\Delta_n}\mathbb{Z}}(x) := \left\{x, \sigma_1(x), \ldots, \sigma_{P_{\Delta_n}-1}(x)\right\}$ *denote the* $(\mathbb{Z}/P_{\Delta_n}\mathbb{Z})$*-orbit of a point* $x \in X_{\Delta_n}(\mathbb{N})$*.*

(a) If there are no points $x \in X_{E_7}(\mathbb{N})$ *such that* $\mathcal{O}_{\mathbb{Z}/10\mathbb{Z}}(x) \subset X_{E_7}(\mathbb{Z}_{\geq 2})$*, then* $\#X_{E_7}(\mathbb{N}) = 4400$*.*

(b) If there are exactly 4 *points* $x \in X_{E_8}(\mathbb{N})$ *such that* $\mathcal{O}_{\mathbb{Z}/16\mathbb{Z}}(x) \subset X_{E_8}(\mathbb{Z}_{\geq 2})$*, then* $\#X_{E_8}(\mathbb{N}) = 26952$*.*

Proof. A point $x = (x_1, \ldots, x_n; y_1, \ldots, y_n) \in X_{\Delta_n}(\mathbb{N})$ corresponds under Proposition 2.4 to a frieze given by $F_{i,j} = \sigma_{j-1}(x_i)$. Therefore, a point $x \in X_{\Delta_n}(\mathbb{N})$ whose entire $(\mathbb{Z}/P_{\Delta_n}\mathbb{Z})$-orbit has coordinates at least 2 necessarily corresponds to a Δ_n-frieze whose entries are all at least 2.

As listed in Table 1, $P_{E_7} = 10$; hence the assumption of (a) that there are no points $x \in X_{E_7}(\mathbb{N})$ such that $\mathcal{O}_{\mathbb{Z}/10\mathbb{Z}}(x) \subset X_{E_7}(\mathbb{Z}_{\geq 2})$ implies the absence of E_7-friezes whose entries are all at least 2. Then there are exactly 4400 friezes of type E_7 by [GM22, Remark 6.21] (or the aforementioned application of Proposition 2.6). The same argument with $P_{E_8} = 16$ and the assumption that there are exactly 4 friezes of type E_8 yields (b). □

2.4. **Positive integral values of rational functions.** By rearranging the vanishing of frieze polynomials for y_i, one obtains n equations $y_i = f_{C,i}$ involving rational functions $f_{C,i}$. Hence finding a positive integral point $(x_1, \ldots, x_n; y_1, \ldots, y_n) \in X_C(\mathbb{N})$ is equivalent to finding positive integral points $x = (x_1, \ldots, x_n) \in \mathbb{N}^n$ on which $f_{C,i}(x) \in \mathbb{N}$ simultaneously for all i.

Example 2.9. In the case $\Delta_n = E_7$ (cf. [dSG23, Example 3.4.2]), $X_{E_7}(\mathbb{N})$ is in bijection with the points $x = (x_1, \ldots, x_7) \in \mathbb{N}^7$ at which the following 7 rational functions simultaneously take positive integral values:

$$f_{E_7,1}(x) := \frac{x_4 + 1}{x_1}, \qquad f_{E_7,5}(x) := \frac{x_4 + x_6}{x_5},$$

$$f_{E_7,2}(x) := \frac{x_3 + 1}{x_2}, \qquad f_{E_7,6}(x) := \frac{x_5 + x_7}{x_6},$$

$$f_{E_7,3}(x) := \frac{x_2 + x_4}{x_3}, \qquad f_{E_7,7}(x) := \frac{x_6 + 1}{x_7}.$$

$$f_{E_7,4}(x) := \frac{x_1 x_3 + x_5}{x_4},$$

By Theorem 5(b), there are exactly 4400 7-tuples of positive integers x such that these rational functions $f_{E_7,i}$ take values in the positive integers.

Therefore, the determination of points in $X_C(\mathbb{N})$ can be viewed as a variant of another famous problem in number theory with *positivity*. Over the integers, the general question of determining when there are points $x \in \mathbb{Z}^n$ for two polynomials $p, q \in \mathbb{Z}[x_1, \dots, x_n]$ such that $p(x)$ divides $q(x)$ is a fascinating open problem and is closely related to other famous problems such as the dynamical Mordell–Lang conjecture. Some finiteness and Zariski density criteria for S-integral points are known in the 1- and 2-variable cases for of any number field due to Siegel [Sie29, Zan14] and Corvaja–Zannier [CZ04, CZ10] respectively. In the 1-variable case, Siegel's theorem (Theorem 2) is equivalent to the statement for a ring of S-integers R that: if $p(X), q(X) \in R[x]$ are coprime polynomials such that $p(X)$ has more than one complex root, then there are only finitely many $x \in R$ such that $p(x)$ divides $q(x)$ in R; the works of Corvaja–Zannier extend Siegel's theorem to the 2-variable setting. The general n-variable case is part of the Lang–Vojta conjecture and the broader conjectures of Vojta (see [HS00, Conjecture F.5.3.2, Conjecture F.5.3.6, Conjecture F.5.3.8], [CZ10, Section 1], [Zha24, Section 2]), which in particular predict the Zariski density of S-integral points of our affine varieties over number fields. A forthcoming work of Corvaja–Zannier [CZ25] provides interesting discussions about rational functions that take very few integral values, many of which are closely related to those obtained from frieze polynomials (also see Remark A.7).

3. A positive Siegel theorem

In this section, we prove the main theorem about integral points. In Section 3.1, we prove the qualitative parts of Theorem 5; in Section 3.2, we prove the enumerative parts of Theorem 5(b).

3.1. **Finiteness and infinitude.** First, we prove Theorem 5(a) and the finiteness in 5(b).

Let X be an affine variety of cluster algebra type C for a generalized $n \times n$ Cartan matrix C. By the classification of cluster algebras due to Fomin–Zelevinsky [FZ03], $\mathcal{A} = \mathcal{A}_C$ has only finitely many cluster variables if and only if C is the Cartan matrix of a finite-dimensional simple Lie algebra (i.e. C is of finite type). Hence if C is of infinite type, then there are infinitely many positive integral C-friezes. Gunawan–Muller [GM22, Theorem B] proved the converse: if C is of finite type then there are only finitely many positive integral C-friezes. Therefore we have shown that there are only finitely many positive integral C-friezes if and only if C is of finite type.

By Proposition 2.4, there are only finitely many positive integral points on an affine variety of cluster algebra type C if and only if C is of finite type. Since C is of finite type precisely when $t_C > 0$, we can conclude that $\#X(\mathbb{N}) < \infty$ if and only if $t_C > 0$.

3.2. **Enumeration.** We now prove the enumerative part of Theorem 5(b). For finite types Δ_n, the one-to-one correspondence of Proposition 2.4 implies that $\#X_{\Delta_n}(\mathbb{N})$ equals the number of positive integral friezes of type Δ_n. For each $\Delta_n \notin \{E_7, E_8\}$, the number of positive integral friezes of type Δ_n was enumerated in [CC73a, MGOT12, FP16, BFG+21]. We complete the enumeration by determining $\#X_{E_7}(\mathbb{N})$ and $\#X_{E_8}(\mathbb{N})$.

We directly prove that $\#X_{E_8}(\mathbb{N}) = 26952$ and then deduce that $\#X_{E_7}(\mathbb{N}) = 4400$ from the E_8 case using Proposition 2.6. Note that the method of proof for E_8 also works with slight modification for E_7.

Type E_8. Using the structure of the affine variety X_{E_8}, we give explicit bounds on its positive integral points to prove the E_8 case of Theorem 5(b).

The E_8-frieze equations are:

$$\begin{aligned}
x_1y_1 &= x_4 + 1\\
x_2y_2 &= x_3 + 1\\
x_3y_3 &= x_2 + x_4\\
x_4y_4 &= x_1x_3 + x_5\\
x_5y_5 &= x_4 + x_6\\
x_6y_6 &= x_5 + x_7\\
x_7y_7 &= x_6 + x_8\\
x_8y_8 &= x_7 + 1
\end{aligned}$$

First, we observe that all of the coordinates of a positive integral solution are essentially bounded from above by x_4.

Lemma 3.1. *Let* $(x_1, \dots, x_8; y_1, \dots y_8) \in X_{E_8}(\mathbb{N})$. *If* $x_4, y_3, y_5, y_6, y_7 \geq 2$, *then* $x_i, y_i \leq x_4 + 4$ *for all* $i \in \{1, \dots, 8\}$.

Proof. The bounds for x_1 and y_1 follow immediately from the equation $x_1y_1 = x_4+1$: we have that

- $x_1, y_1 \leq x_4 + 1$.

From the equation $x_2y_2 = x_3 + 1$, we have that $x_2 \leq x_3 + 1$. Substituting this into the third equation, we have that $x_3(y_3 - 1) \leq x_4 + 1$. But $y_3 > 1$, so:

- $x_3 \leq x_4 + 1$ and $y_3 \leq x_4 + 2$;
- $x_2, y_2 \leq x_4 + 2$.

From substituting $x_8 \leq x_7+1$ into the seventh equation, we have that $x_7(y_7-1) \leq x_6 + 1$. But $y_7 > 1$, so $x_7 \leq x_6 + 1$. Then applying $x_7 \leq x_6 + 1$ to the sixth equation, we have that $x_6(y_6 - 1) \leq x_5 + 1$. But $y_6 > 1$, so $x_6 \leq x_5 + 1$. Applying this to the fifth equation, we have that $x_5(y_5 - 1) \leq x_4 + 1$. But $y_5 > 1$, so $x_5 \leq x_4 + 1$. Hence:

- $x_5 \leq x_4 + 1$ and $y_5 \leq x_4 + 2$;
- $x_6 \leq x_4 + 2$ and $y_6 \leq x_4 + 3$;
- $x_7 \leq x_4 + 3$ and $y_7 \leq x_4 + 4$;
- $x_8, y_8 \leq x_4 + 4$.

Finally by the previous bounds, $y_4 = \frac{x_1x_3+x_5}{x_4} \leq \frac{x_4^2+3x_4+1}{x_4}$, which implies that $y_4 \leq x_4 + 3$. □

Remark 3.2. For computational purposes, the bounds in Lemma 3.1 can be refined further. When searching for friezes with entries all at least 2, i.e. $x_i, y_i \geq 2$ for all i, we have the following bounds by similar arguments:

- $x_1, y_1 \leq \frac{x_4+1}{2}$,
- $x_2, y_2 \leq \frac{x_4+2}{3}$,
- $x_3, y_3 \leq \frac{2x_4+1}{3}$,
- $y_4 \leq \frac{x_4}{3} + 1$,
- $x_5, y_5 \leq \frac{4x_4+1}{5}$,
- $x_6, y_6 \leq \frac{3x_4+2}{5}$,
- $x_7, y_7 \leq \frac{2x_4+3}{5}$,
- $x_8, y_8 \leq \frac{x_4+4}{5}$.

Muller [Mul23, Proposition 2.3 and Example 3.1] gives an upper bound on each entry of a frieze. Since x_4 equals $F_{4,1}$ on the corresponding frieze, this translates into upper bounds on x_4 of roughly $1.7 \cdot 10^{650}$ for a general E_8-frieze and roughly $4.4 \cdot 10^{266}$ for an E_8-frieze whose entries are all at least 2. These are unfortunately far too large for a computational search on X_{E_8}, but we can obtain an exponential improvement from a combination of our Diophantine model and a modification of the proof of [Mul23, Proposition 2.3] with the specific mesh relations for E_8.

Recall that there is an action of $\mathbb{Z}/16\mathbb{Z}$ on $X_{E_8}(\mathbb{N})$ induced by horizontal translation of the corresponding friezes. As in Lemma 2.8, let $\mathcal{O}_{\mathbb{Z}/16\mathbb{Z}}(x)$ denote the $(\mathbb{Z}/16\mathbb{Z})$-orbit of x and let $\mathbb{Z}_{\geq 2}$ denote the integers at least 2. Let $\pi_i : X_{E_8} \to \mathbb{C}$ denote the projection onto the i-th coordinate.

Lemma 3.3. *Let $x \in X_{E_8}(\mathbb{N})$ with $\mathcal{O}_{\mathbb{Z}/16\mathbb{Z}}(x) \subset X_{\mathbb{E}_8}(\mathbb{Z}_{\geq 2})$. Then there exists an $x' \in \mathcal{O}_{\mathbb{Z}/16\mathbb{Z}}(x)$ such that $\pi_4(x') < 16966221628$.*

Proof. Let $F = (F_{i,j})_{i,j} = (\pi_i(\sigma_j(x)))_{i,j}$ be the frieze corresponding to x. Since $\mathcal{O}_{\mathbb{Z}/16\mathbb{Z}}(x) \subset X_{\mathbb{E}_8}(\mathbb{Z}_{\geq 2})$, we have that $\pi_i(\sigma_j(x)) \in \mathbb{Z}_{\geq 2}$ for all i, j, i.e. all entries of the frieze are at least 2. Define $\nu := (\frac{1}{16}\log_2(\prod_{j=1}^{16} F_{i,j}))_i$ to be the vector of averages of 2-logarithms of the rows of the frieze F. As in [Mul23, Example 3.1], the i-th coordinate ν_i is bounded above by

$$\nu_i \leq \sum_{j=1}^{8} c_{i,j}^{-1} \log_2\left(2^{\sum_{k\neq j} c_{j,k}} + 1\right).$$

Observe that if all the entries of F are at least 2, then $F_{3,k}, F_{4,k}, F_{7,k} \geq 3$ for all k due to the third, fourth, and eighth E_8-frieze equations. The E_8 mesh relations are:

$$\begin{aligned}
F_{1,k}F_{1,k+1} &= 1 + F_{4,k+1} & F_{5,k}F_{5,k+1} &= 1 + F_{4,k}F_{6,k+1}\\
F_{2,k}F_{2,k+1} &= 1 + F_{3,k+1} & F_{6,k}F_{6,k+1} &= 1 + F_{5,k}F_{7,k+1}\\
F_{3,k}F_{3,k+1} &= 1 + F_{2,k}F_{4,k+1} & F_{7,k}F_{7,k+1} &= 1 + F_{6,k}F_{8,k+1}\\
F_{4,k}F_{4,k+1} &= 1 + F_{1,k}F_{3,k}F_{5,k+1} & F_{8,k}F_{8,k+1} &= 1 + F_{7,k}.
\end{aligned}$$

Adapting the steps of [Mul23, Section 2], we obtain even better bounds on each ν_i except ν_7 (since it does not involve the third, fourth, nor seventh rows). Consider the square of the product of all entries in each row and apply the E_8 mesh relations:

$$\begin{aligned}
\prod_{k=1}^{16} F_{1,k}^2 &= \prod_{k=1}^{16}(1+F_{4,k+1}) \le \prod_{k=1}^{16}(3^{-1}+1)F_{4,k+1}\\
\prod_{k=1}^{16} F_{2,k}^2 &= \prod_{k=1}^{16}(1+F_{3,k+1}) \le \prod_{k=1}^{16}(3^{-1}+1)F_{3,k+1}\\
\prod_{k=1}^{16} F_{3,k}^2 &= \prod_{k=1}^{16}(1+F_{2,k}F_{4,k+1}) \le \prod_{k=1}^{16}(6^{-1}+1)F_{2,k}F_{4,k+1}\\
\prod_{k=1}^{16} F_{4,k}^2 &= \prod_{k=1}^{16}(1+F_{1,k}F_{3,k}F_{5,k+1}) \le \prod_{k=1}^{16}(12^{-1}+1)F_{1,k}F_{3,k}F_{5,k+1}\\
\prod_{k=1}^{16} F_{5,k}^2 &= \prod_{k=1}^{16}(1+F_{4,k}F_{6,k+1}) \le \prod_{k=1}^{16}(6^{-1}+1)F_{4,k}F_{6,k+1}\\
\prod_{k=1}^{16} F_{6,k}^2 &= \prod_{k=1}^{16}(1+F_{5,k}F_{7,k+1}) \le \prod_{k=1}^{16}(6^{-1}+1)F_{5,k}F_{7,k+1}\\
\prod_{k=1}^{16} F_{8,k}^2 &= \prod_{k=1}^{16}(1+F_{7,k+1}) \le \prod_{k=1}^{16}(3^{-1}+1)F_{7,k+1}.
\end{aligned}$$

With

$$B_j := \begin{cases} 2 & \text{if } i = 7,\\ 3 & \text{if } i = 1,2,8,\\ 6 & \text{if } i = 3,5,6,\\ 12 & \text{if } i = 4,\end{cases}$$

one obtains the bounds

$$0 < (C_{E_8}\cdot v)_j \le \log_2\Big(B_j^{-1}+1\Big).$$

As noted in [Mul23, Proposition 2.3], the positivity of the entries in $C_{E_8}^{-1}$ gives an upper bound for v_i in terms of the bounds on $(C_{E_8}\cdot v)_i$:

$$v_i = \Big(C_{E_8}^{-1}(C_{E_8}v)\Big)_i \le \sum_{j=1}^{8} c_{i,j}^{-1}\log_2\Big(B_j^{-1}+1\Big).$$

In particular, $v_4 < 16966221628$ and

$$\prod_{j=1}^{16} F_{4,j} \le \prod_{j=1}^{8}\Big(B_j^{-1}+1\Big)^{16c_{i,j}^{-1}} < 16966221628^{16}.$$

Take the frieze F' in the $(\mathbb{Z}/16\mathbb{Z})$-orbit of F such that $F'_{4,1} \le F'_{4,j}$ for all $1 \le j \le 16$, and let x' denote the point in $X_{E_8}(\mathbb{Z}_{\ge 2})$ corresponding to F'. Since $\pi_4(x') = F'_{4,1}$ is the smallest factor in the product of 16 terms, we have that $\pi_4(x') < 16966221628$. $\square$

Proof of Theorem 5(b) in the E_8 *case.* By Lemma 2.8, it is sufficient to show that there are exactly 4 points $x \in X_{E_8}$ whose entire $(\mathbb{Z}/16\mathbb{Z})$-orbit $\mathcal{O}_{\mathbb{Z}/16\mathbb{Z}}(x)$ lies in $X_{E_8}(\mathbb{Z}_{\geq 2})$. By Proposition 3.3, for such an $x \in X_{E_8}$, there exists an element $x' \in \mathcal{O}_{\mathbb{Z}/16\mathbb{Z}}(x) \subset X_{E_8}(\mathbb{Z}_{\geq 2})$ such that $\pi_4(x') < 16966221628$. But Lemma 3.1 says that the other coordinates of x' are less than $\pi_4(x') + 4 < 16966221632$. Hence for every $x \in X_{E_8}$ such that $\mathcal{O}_{\mathbb{Z}/16\mathbb{Z}}(x) \subset X_{E_8}(\mathbb{Z}_{\geq 2})$, there exists an element

$$x' \in \mathcal{O}_{\mathbb{Z}/16\mathbb{Z}}(x) \cap X_{E_8}(\mathbb{N}) \cap \left\{ x \in \mathbb{A}^{16}_{\mathbb{C}} \,\middle|\, x_i, y_i < 16966221632 \text{ for all } i \right\}.$$

A computer search[2] exhaustively finds 26952 points in

$$X_{E_8}(\mathbb{N}) \cap \left\{ x \in \mathbb{A}^{16}_{\mathbb{C}} \,\middle|\, x_i, y_i < 16966221632 \text{ for all } i \right\},$$

the largest of which is

$$(1320, 165, 16994, 2820839, 134632, 6433, 461, 21; 2137, 103, 166, 8, 21, 21, 14, 22).$$

Searching among these 26952 points finds exactly 4 points whose entire orbit is in $X_{E_8}(\mathbb{Z}_{\geq 2})$:

- $(6, 4, 11, 29, 21, 13, 5, 2; 5, 3, 3, 3, 2, 2, 3, 3)$,
- $(5, 3, 8, 29, 18, 7, 3, 2; 6, 3, 4, 2, 2, 3, 3, 2)$,
- $(6, 3, 11, 41, 16, 7, 5, 3; 7, 4, 4, 2, 3, 3, 2, 2)$,
- $(7, 4, 15, 41, 18, 13, 8, 3; 6, 4, 3, 3, 3, 2, 2, 3)$.

□

Remark 3.4. The four points of $X_{E_8}(\mathbb{N})$ whose $(\mathbb{Z}/16\mathbb{Z})$-orbits lie in $X_{\mathbb{E}_8}(\mathbb{Z}_{\geq 2})$ actually lie in the same orbit of order 4. They correspond to the four E_8-friezes with all entries at least 2 given by horizontal translations of the frieze in Example 2.1. Note that this affirmatively resolves [BFG+21, Conjecture 5.14].

3.3. **Reduction from E_8 to smaller types.** Using the reduction principle of Proposition 2.6, the affine variety X_{Δ_n} for these Dynkin types can be obtained as the intersection of X_{E_8} with hyperplanes as follows:

$$X_{A_n} \cong X_{E_8} \cap \left(\bigcap_{i=1}^{8-n} \{x_i = 1\} \right) \qquad \text{for } n \leq 7,$$

$$X_{D_n} \cong X_{E_8} \cap \{x_2 = 1\} \cap \left(\bigcap_{i=8-n}^{7} \{x_i = 1\} \right) \qquad \text{for } n \leq 7,$$

$$X_{E_6} \cong X_{E_8} \cap \{x_8 = 1\} \cap \{x_7 = 1\}$$

$$X_{E_7} \cong X_{E_8} \cap \{x_8 = 1\}.$$

Since we have computationally found all 26952 points in $X_{E_8}(\mathbb{N})$, we can directly verify the other the ADE-type formulae for $n \leq 7$ by taking the corresponding subsets of the 26952 elements of $X_{E_8}(\mathbb{N})$. The other finite types of rank $n \leq 7$ follow from the ADE types by the observations of [FP16, Section 4] on foldings of Dynkin diagrams.

[2] This took a few dozen hours on optimized C++ programs on 64 NVIDIA V100 GPUs (via CUDA) and 1000 x86 CPU cores.

Proof of Theorem 5(b) in the E_7 *case.* In particular, we directly verify that there are exactly 4400 points in $X_{E_8}(\mathbb{N})$ whose eighth coordinate is 1. □

Remark 3.5. We can also directly verify that none of the points in $X_{E_7}(\mathbb{N})$ have $(\mathbb{Z}/10\mathbb{Z})$-orbits in $X_{E_7}(\mathbb{Z}_{\geq 2})$, affirmatively resolving [BFG+21, Conjecture 5.11].

4. Effective bounds on positive integral points in finite type

In this section, we establish general effective bounds on $X_{\Delta_n}(\mathbb{N})$ in terms of the following $\mathbb{Z}$-lattice in a hypercube:

$$S(N_1, N_2) := \left\{ (x_1, \ldots, x_n; y_1, \ldots, y_n) \in \mathbb{Z}^{2n} \,\middle|\, N_1 \leq x_i, y_i \leq N_2 \text{ for all } i \right\}.$$

This gives a further refinement of Theorem 5(b) by bounding the height of positive integral points; these bounds are sharp for A_n, C_n, and the exceptional Dynkin types.

Proposition 4.1. *Let* Δ_n *be a finite Dynkin type. For the following Dynkin types, the positive integral points* $X_{\Delta_n}(\mathbb{N})$ *are contained in* $S(1, N_{\Delta_n})$*, where*

- $N_{A_n} = F_{n+2}$;
- $N_{C_n} = F_{2n+1}$;
- $N_{E_6} = 307$;
- $N_{E_7} = 135503$;
- $N_{E_8} = 2820839$;
- $N_{F_4} = 307$;
- $N_{G_2} = 14$;

and F_n *denotes the* n*-th Fibonacci number.*

For the remaining two Dynkin types, the positive integral points $X_{\Delta_n}(\mathbb{N})$ *are contained in* $S(1, N_{\Delta_n}^{P_{\Delta_n}})$*, where*

- $N_{B_n} = 2^{\frac{(n+1)(n-2)}{2}}$, $P_{B_n} = n + 1$;
- $N_{D_n} = 2^{\frac{n^2}{2}}$, $P_{D_n} = n$;

Furthermore, every Δ_n*-frieze is the horizontal translation of the* Δ_n*-frieze corresponding to a point in* $X_{\Delta_n}(\mathbb{N}) \cap S(1, N_{\Delta_n})$.

As before, let $C_{\Delta_n} = (c_{i,j})$ be a Cartan matrix for Dynkin type Δ_n. Denote its inverse by $C_{\Delta_n}^{-1} = (c_{i,j}^{-1})$. Define the following associated values:

$$b_{i,\Delta_n} := \prod_{j=1}^{n} 2^{c_{i,j}^{-1}}$$

$$c_{i,\Delta_n} := \prod_{j=1}^{n} \left(1 + 2^{\sum_{k \neq i} c_{j,k}}\right)^{c_{i,j}^{-1}}.$$

Muller [Mul23, Proposition 2.3 and Example 3.1] gives bounds on frieze entries in terms of these values.

Lemma 4.2. [Mul23, Proposition 2.3 and Example 3.1] *Let* Δ_n *be a finite Dynkin type of rank* n *and let* F *be a positive integral* Δ_n*-frieze of period* P_{Δ_n}*. Then there is the following upper bound on the product of entries in its* i*-th row:* $\prod_{j=1}^{P_{\Delta_n}} F_{i,j} \leq b_{i,\Delta_n}^{P_{\Delta_n}}$*. Furthermore, if all entries in* F *are at least* 2*, then* $\prod_{j=1}^{P_{\Delta_n}} F_{i,j} \leq c_{i,\Delta_n}^{P_{\Delta_n}}$.

Remark 4.3. While friezes can be defined for more general Dynkin types, the proof of the following result crucially uses the result of Lusztig–Tits [LT92] that the inverse of a Cartan matrix C_{Δ_n} is positive if and only if Δ_n is finite type. We also note a misprint in [Mul23, Example 3.1]: the bound $(\frac{151875}{16384})^{16} \approx 2^{51}$ on the eighth row of an E_8-frieze should be $c_{8,E_8}^{16} = (\frac{177347025604248046875}{144115188075855872})^{16} \approx 2^{164}$.

An immediate consequence of Lemma 4.2 and Proposition 2.4 is that if the frieze F corresponds to the point $(x_1, \dots, x_n; y_1, \dots, y_n) \in X_{\Delta_n}(\mathbb{N})$, then $x_i \leq b_{i,\Delta_n}^{P_{\Delta_n}}$. Furthermore, if all entries in F are at least 2, then $x_i \leq c_{i,\Delta_n}^{P_{\Delta_n}}$.

The Diophantine model of friezes allows us to find a minimal element in each $(\mathbb{Z}/P_{\Delta_n}\mathbb{Z})$-orbit on which we can reduce existing bounds by a power of $P_{\Delta_n}^{-1}$.

Lemma 4.4. *Let Δ_n be a finite Dynkin type, let F be a positive integral Δ_n-frieze, and fix an $i \in \{1, \dots, n\}$. There is a Δ_n-frieze F' corresponding to $(x_1, \dots, x_n; y_1, \dots, y_n) \in X_{\Delta_n}(\mathbb{N})$ such that F is a horizontal translation of F' and $x_i \leq b_{i,\Delta_n}$. Furthermore, if all entries in F are at least 2, then $x_i \leq c_{i,\Delta_n}$.*

Proof. Each Δ_n-frieze F is a horizontal translation of a frieze F' in its $(\mathbb{Z}/P_{\Delta_n}\mathbb{Z})$-orbit such that $F'_{i,1} \leq F'_{i,j}$ for all $1 \leq j \leq P_{\Delta_n}$. By Lemma 4.2, there is a bound on $\prod_{j=1}^{P_{\Delta_n}} F'_{i,j} \leq b_{i,\Delta_n}^{P_{\Delta_n}}$. But $x_i = F'_{i,1}$ is the smallest factor in the product of P_{Δ_n}-many terms, hence $x_i \leq b_{i,\Delta_n}$. If the entries of F are all at least 2, then the entries of the translation F' are also all at least 2. Hence the same argument gives the bound in terms of c_{i,Δ_n} instead of b_{i,Δ_n}. □

We conclude this section by proving Proposition 4.1 by determining the largest row and compute the bounds b_{i,Δ_n} and c_{i,Δ_n} for each finite Dynkin type.

Proof of Proposition 4.1. We explicitly compute the largest $\log_2(b_i) = \sum_{j=1}^{n} c_{i,j}^{-1}$ for each Dynkin type. The Cartan matrix that we use for each type is listed in Table 1.

For A_n, the inverse of the Cartan matrix is given by: $c_{i,j}^{-1} = \min(i,j) - \frac{ij}{n+1}$. When n is even, simple manipulations show that $\log_2(b_i)$ attains its maximum at $i = \frac{n}{2}$ and $\log_2(b_{\frac{n}{2}}) = \frac{n(n+2)}{8}$, the n-th triangular number. When n is odd, $\log_2(b_i)$ attains its maximum at $i = \frac{n+1}{2}$ and $\log_2(b_{\frac{n}{2}}) = \frac{(n+1)^2}{8}$. This gives a bound of $N_{A_n} = 2^{\frac{(n+1)^2}{8}}$, but in fact a sharp bound of the $(n+2)$-nd Fibonacci number F_{n+2} was already achieved by [CdSG24, Theorem 1(1)].

For B_n, the inverse of the Cartan matrix is given by:

$$c_{i,j}^{-1} = \begin{cases} \frac{n-j+1}{2} \text{ if } i = 1, \\ n + 1 - \max(i,j) \text{ if } i > 1. \end{cases}$$

The entries of $C_{B_n}^{-1}$ are clearly the largest in row 2, so $\log_2(b_2) = n - 1 + \sum_{j=2}^{n}(n - j + 1) = \frac{n^2+n-2}{2}$.

For C_n, the Cartan matrix is the transpose of C_{B_n} so its inverse is given by:

$$c_{i,j}^{-1} = \begin{cases} \frac{n-i+1}{2} \text{ if } j = 1, \\ n + 1 - \max(i,j) \text{ if } j > 1. \end{cases}$$

The entries of $C_{B_n}^{-1}$ are clearly the largest in row 1, so $\log_2(b_2) = \frac{n}{2} + sum_{j=2}^{n}(n - j + 1) = \frac{n^2}{2}$. This gives a bound of $N_{C_n} = 2^{\frac{n^2}{2}}$, but in fact a sharp bound of the $(2n+1)$-st Fibonacci number F_{2n+1} was already achieved by [CdSG24, Theorem 1(2)].

For D_n, the inverse of the Cartan matrix is given by:

$$c_{i,j}^{-1} = \begin{cases} \frac{n}{4} \text{ if } i = j = 1, \\ \frac{n-2}{4} \text{ if } i = 1, j = 2 \text{ or } i = 2, j = 1 \\ \frac{n-j+1}{2} \text{ if } i \leq 2, j \geq 3 \\ \frac{n-i+1}{2} \text{ if } i \geq 3, j \leq 2 \\ n + 1 - \max(i,j) \text{ if } i \geq 3, j \geq 3. \end{cases}$$

We can directly calculate that $\log_2(b_1) = \log_2(b_2) = \frac{n(n-1)}{4}$ while $\log_2(b_i) = \frac{n^2-n-i^2+3i-2}{2}$ for $i \geq 3$. The $i \geq 3$ expression is decreasing in i, so we only need to compare $\log_2(b_1) = \log_2(b_2)$ with $\log_2(b_3) = \frac{n^2-n-2}{2} = \frac{(n+1)(n-2)}{2}$. Clearly, $\frac{(n+1)(n-2)}{2} > \frac{n(n-1)}{4}$ when $n \geq 3$.

For the exceptional types E_6, E_7, E_8, F_4, and G_2, we can use a computer algebra system to compute the sums of the rows of the inverses of each Cartan matrix:

- $\lfloor b_{4,E_6} \rfloor = 2097152$,
- $\lfloor b_{4,E_7} \rfloor = 281474976710656$,
- $\lfloor b_{4,E_8} \rfloor = 435561429658801233233119497512663310663 68$,
- $\lfloor b_{3,F_4} \rfloor = 32768$,
- $\lfloor b_{2,G_2} \rfloor = 8$.

For the exceptional types of rank ≤ 6, a short search finds all of the points in $X_{\Delta_n}(\mathbb{N})$ and the actual maximima given in the theorem statement. The explicit searches in Section 3.2 give the maxima for E_7 and E_8. $\square$

5. Positive Mordell–Schinzel in infinite type

It is a simple exercise to show that there are exactly 5 positive integer solutions to the Diophantine equation

$$S_1 : xyz = x + y + 1.$$

Similarly, there are exactly 6 positive integer solutions to the Diophantine equation

$$S_2 : xyz = x^2 + y + 1.$$

Mohanty [Moh77, Theorem 2] proved that there are exactly 9 positive integer solutions to the Diophantine equation

$$S_3 : xyz = x^3 + y + 1;$$

we give another proof of this fact in Proposition A.6. The purpose of this section is to show how the theory of friezes and cluster algebras of *infinite* type can establish the infinitude of positive integral solutions to Diophantine equations, using the example of $S_m : xyz = x^m + y + 1$ with degree $m \geq 4$.

In this section, we show that there is an injection $X(\mathbb{N}) \hookrightarrow S_m(\mathbb{N})$ from affine varieties X of cluster algebra type C (which happens to be a bijection when C is of finite type) and use Theorem 5 to prove Theorem 3. Rank-2 types are classified by generalized Cartan matrices of the form $\left(\begin{smallmatrix} 2 & -a \\ -b & 2 \end{smallmatrix}\right)$ for positive integers a and b; they are of infinite type precisely when $ab \geq 4$; the surface S_m corresponds to the special case $(a, b) = (m, 1)$. The threefold analogue arises from the rank-3 infinite types classified by generalized Cartan matrices of the form $\left(\begin{smallmatrix} 2 & -a & 0 \\ -b & 2 & -d \\ 0 & -c & 2 \end{smallmatrix}\right)$ for positive integers a, b, c, and d such that $abcd \geq 3$.

Before moving on to the proof of Theorem 3, we remark that it can be viewed in a *positive* refinement to the program of Mordell–Schinzel (as well as Siegel, Corvaja–Zannier, and Vojta [Sie29, Zan14, CZ04, CZ10]; see Section 2.4 and Remark A.7). Jacobsthal [Jac39], Barnes [Bar53], Mills [Mil54], and Schinzel [Sch15, Sch18] found counterexamples to Mordell's claim that Equation 1 has infinitely many integer solutions when $3 \leq \deg_x(G) \cdot \deg_y(G) \leq 4$, but several cases have been recovered by Schinzel [Sch15, Sch18] and Kollár–Li [KL24, Theorem 3]. In particular, Schinzel [Sch15, Theorem 3] established that if $m \geq 4$ and a, b, c are nonzero integers, then there are infinitely many integer solutions (x, y, z) to the Diophantine equation

$$xyz = ax^m + by + c$$

with $\gcd(y, c) = 1$. Theorem 3(a) gives a new proof of the $a = b = c = 1$ case. The Mordell–Schinzel conjecture, as formulated by Kollár–Li [KL24, Conjecture 2], predicts that there are infinitely many integral solutions to the Equation 1 when $\deg_x(G) \geq 3$ and $\deg_y(G) \geq 3$. It is unclear if these Diophantine equations should also have infinitely many *positive* integral solutions, but perhaps cluster algebras of infinite type should inform the formulation of a *positive* Mordell–Schinzel conjecture.

5.1. **The general rank 2 case.** We prove Theorem 3(a). Consider the generalized 2×2 Cartan matrix

$$C = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}.$$

C is symmetrizable for all positive integers a and b. The C-frieze equations are:

$$x_1 y_1 = x_2^a + 1,$$
$$x_2 y_2 = x_1^b + 1.$$

Applying the substitution $x_1 = \frac{x_2^a + 1}{y_1}$ to the second equation and relabeling

$$x_2 \longmapsto x, \qquad y_1 \longmapsto y, \qquad y_2 \longmapsto z.$$

yields the equation:

$$xy^b z = (x^a + 1)^b + y^b. \tag{5.1}$$

If $(a, b) \notin \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)\}$, then C is of infinite type and $t_C \leq 0$. The assumption that $ab \geq 4$ guarantees that C is of infinite type, so there are infinitely many positive integral solutions to the C-frieze equations by Theorem 5(a). This gives infinitely many positive integral solutions to Equation 5.1 and in particular to the equation

$$xyz = (x^a + 1)^b + y.$$

If $abcd \le 3$, then C is of finite type and $t_C \ge 1$. There are finitely many positive integral solutions to the C-frieze equations with precise counts given by Propositions A.2, A.5, and A.6.

5.2. **A rank 3 case.** We prove Theorem 3(b). Consider the generalized 3×3 Cartan matrix

$$C = \begin{pmatrix} 2 & -a & 0 \\ -b & 2 & -d \\ 0 & -c & 2 \end{pmatrix}.$$

C is symmetrizable for all positive integers a, b, c, and d. The C-frieze equations are:

$$\begin{aligned} x_1 y_1 &= x_2^a + 1, \\ x_2 y_2 &= x_1^b + x_3^d \\ x_3 y_3 &= x_2^c + 1. \end{aligned}$$

Applying the substitutions $x_1 = \frac{x_2^a+1}{y_1}$ and $x_3 = \frac{x_2^c+1}{y_3}$ to the second equation yields:

$$x_2 y_2 y_1^b y_3^d = (x_2^a + 1)^b y_3^d + (x_2^c + 1)^d y_1^b.$$

Relabeling

$$x_2 \longmapsto x, \qquad y_1 \longmapsto z, \qquad y_2 \longmapsto w \qquad y_3 \longmapsto y.$$

yields the equation:

$$x y^d z^b w = (x^a + 1)^b y^d + (x^c + 1)^d z^b.$$

Notice that if $abcd \ge 3$, then C is of infinite type and $t_C \le 0$. There are infinitely many positive integral solutions to the C-frieze equations by Theorem 5(a). This gives infinitely many positive integral solutions to Equation 5.1 and in particular to the equation

$$xyzw = (x^a + 1)^b y + (x^c + 1)^d z.$$

Appendix A. Elementary proofs for small n

Enumeration theorems for all Dynkin types for small enough n can be proved in the same manner as Theorem 5(b) using the bounds provided by Proposition 4.1. In this appendix, we instead illustrate that elementary arithmetic methods, relying solely on simple divisibility relations from the Diophantine equations defining X_{Δ_n}, can directly prove the enumeration theorems for various Δ_n of small n,

A.1. **Affine varieties of cluster type A_n for $n < 4$.** The classical frieze enumeration theorem of Conway–Coxeter [CC73a] shows that the number of positive integral friezes of type A_n is precisely the $(n+1)$-st Catalan number by constructing a bijection between the A_n friezes and the triangulations of an n-gon. Proposition 2.4, the number of positive integral points on X_{A_n} is precisely the $(n+1)$-st Catalan number.

We give another proof for small n by directly counting the points of $X_{A_n}(\mathbb{N})$.

Proposition A.1. *The affine curve X_{A_1} has exactly 2 positive integral points.*

Proof. The A_1-frieze equation is:

$$x_1 y_1 = 2.$$

Since x_1 and x_2 are positive integers, the only possibilities are $(1,2)$ and $(2,1)$. □

PROPOSITION A.2. *The affine surface* X_{A_2} *has exactly* 5 *positive integral points:*

- $(1,1,2,2)$,
- $(1,2,3,1)$,
- $(2,1,1,3)$,
- $(2,3,2,1)$,
- $(3,2,1,2)$.

Proof. The A_2-frieze equations are:

$$x_1 y_1 = x_2 + 1,$$
$$x_2 y_2 = x_1 + 1.$$

Observe that $X_{A_2}(\mathbb{N})$ is in bijection with the set of positive integral points on the smooth affine cubic surface,

$$xyz = x + y + 1,$$

under the change of variables

$$x_1 \longmapsto x, \qquad x_2 \longmapsto \frac{x_1+1}{y_2}, \qquad y_1 \longmapsto z, \qquad y_2 \longmapsto y.$$

Observe that x_2 is congruent to $-1 \pmod{x_1}$ from the initial A_2-frieze equations and that $y = y_2$ is congruent to $-1 \pmod{x_1}$ from the affine cubic equations. Writing $x_2 = mx_1 - 1$ and $y_2 = nx_1 - 1$, we have that $(mx_1-1)(nx_1-1) = x_1+1$ so $mnx_1^2 - (m+n+1)x_1 = 0$. Since $x_1 \neq 0$, we have $x_1 = \frac{m+n+1}{mn}$. This is a positive integer only if $m, n \in \{1,2,3\}$. These correspond to the 5 positive integral solutions $(x_1, x_2; y_1, y_2) \in X_{A_2}(\mathbb{N})$ listed. □

Remark A.3. The affine cubic $xyz = x+y+1$ has infinite families of integer points $(t,-1,-1)$, $(-1,t,-1)$, and $(t,-t-1,0)$. So $\Delta_n = A_2$ is the first Dynkin type with $\#X_{\Delta_n}(\mathbb{Z}^{2n}) = \infty$ but $\#X_{\Delta_n}(\mathbb{N}) < \infty$.

PROPOSITION A.4. *The affine three-fold* $X_{A_3} \cong X_{D_3}$ *has exactly* 14 *positive integral points:*

- $(1,1,1,2,2,2)$
- $(1,1,2,2,3,1)$,
- $(1,2,1,3,1,3)$,
- $(1,2,3,3,2,1)$,
- $(1,3,2,4,1,2)$,
- $(2,1,1,1,3,2)$,
- $(2,1,2,1,4,1)$,
- $(2,3,1,2,1,4)$,
- $(2,3,4,2,2,1)$,
- $(2,5,3,3,1,2)$,
- $(3,2,1,1,2,3)$,
- $(3,2,3,1,3,1)$,
- $(3,5,2,2,1,3)$,
- $(4,3,2,1,2,2)$.

Proof. The A_3-frieze equations (equivalently the D_3-frieze equations) are:

$$x_1 y_1 = x_2 + 1,$$
$$x_2 y_2 = x_1 + x_3,$$
$$x_3 y_3 = x_2 + 1.$$

Observe that a multiple of x_2 is the sum of x_1 and x_3, which are two divisors of x_2+1. In particular, $x_2 y_2 = x_1 + x_3 \leq 2x_2 + 2$, so $y_2 \leq 4$. We completely determine the positive integral solutions with case-work.

Case $y_2 = 1$. In this case, $x_2 = x_1 + x_3$. Then either x_1 or x_3 must be $\frac{x_2+1}{2}$ or $\frac{x_2+1}{3}$, since the other divisors of $x_2 + 1$ are too small (or large) to sum to x_2. If either x_1 or x_3 equals $\frac{x_2+1}{2}$, then the other must be $\frac{x_2-1}{2}$; these are both divisors of $x_2 + 1$ only if $x_2 = 2$, 3, or 5. These correspond to the 4 positive integer solutions:

- $(1, 3, 2, 4, 1, 2)$,
- $(2, 3, 1, 2, 1, 4)$,
- $(2, 5, 3, 3, 1, 2)$,
- $(3, 5, 2, 2, 1, 3)$.

The other case is if x_1 or x_3 is $\frac{x_2+1}{3}$, i.e. $x_2 = 2$. This corresponds to the single positive integer solutions:

- $(1, 2, 1, 3, 1, 3)$.

Case $y_2 = 2$. In this case, $2x_2 = x_1 + x_3$. Then either x_1 or x_3 must be $x_2 + 1$ or x_2 since the other divisors of $x_2 + 1$ are too small to sum to $2x_2$. If $x_2 + 1$ is either x_1 or x_3, then the other must be $x_2 - 1$. But $x_2 - 1$ is a positive integral divisor of $x_2 + 1$ only if $x_2 = 2$ or 3. These correspond to the 4 positive integer solutions:

- $(1, 2, 3, 3, 2, 1)$,
- $(2, 3, 4, 2, 2, 1)$,
- $(3, 2, 1, 1, 2, 3)$,
- $(4, 3, 2, 1, 2, 2)$.

The other case is if x_2 divides $x_2 + 1$, i.e. $x_2 = 1$. This corresponds to the single positive integer solutions:

- $(1, 1, 1, 2, 2, 2)$.

Case $y_2 = 3$. Since $3x_2 = x_1 + x_3 \leq 2x_2 + 2$, we necessarily have that $x_2 \leq 2$. In particular, (x_1, x_3) must be $(1, 2)$, $(2, 1)$, or $(3, 3)$. This corresponds to the 3 positive integer solutions:

- $(1, 1, 2, 2, 3, 1)$,
- $(2, 1, 1, 1, 3, 2)$,
- $(3, 2, 3, 1, 3, 1)$.

Case $y_2 = 4$. Since $4x_2 = x_1 + x_3 \leq 2x_2 + 2$, we necessarily have that $x_2 = 1$, $x_1 = 2$, $x_3 = 2$. This corresponds to the single positive integer solution:

- $(2, 1, 2, 1, 4, 1)$.

Altogether we have the 14 positive integral solutions $(x_1, x_2, x_3; y_1, y_2, y_3) \in X_{A_3}(\mathbb{N})$ listed. □

A.2. **Friezes of type B_2.** Fontaine–Plamondon [FP16, Theorems 4.2 and 4.3] showed that there are precisely 6 positive integral friezes of type $B_2 = C_2$. By Proposition 2.4, there are precisely 6 positive integral points on X_{B_2}.

We give another proof by directly counting the points of $X_{B_2}(\mathbb{N})$.

PROPOSITION A.5. *The affine surface X_{B_2} has exactly 6 positive integral points:*

- $(1,2,1,2)$,
- $(1,3,2,1)$,
- $(2,1,1,5)$,
- $(2,3,5,1)$,
- $(3,1,2,5)$,
- $(3,2,5,2)$.

Proof. Observe that $B_2 = C_2$. We will work with the C_2-frieze equations for the sake of similarity with G_2. The C_2-frieze equations are:

$$x_1 y_1 = x_2 + 1,$$
$$x_2 y_2 = x_1^2 + 1.$$

Observe that $X_{C_2}(\mathbb{N})$ is in bijection with the positive integral points on the affine cubic surface,

$$xyz = x^2 + y + 1,$$

under the change of variables

$$x_1 \longmapsto x, \qquad x_2 \longmapsto \frac{x^2+1}{y}, \qquad y_1 \longmapsto z, \qquad y_2 \longmapsto y.$$

Observe that x_2 is congruent to $-1 \pmod{x_1}$ from the B_2-frieze equations and that y_2 is also congruent to $-1 \pmod{x_1}$ from the affine cubic equation. Writing $x_2 = mx_1 - 1$ and $y_2 = nx_1 - 1$, we have that $(mx_1 - 1)(nx_1 - 1) = x_1^2 + 1$ so $(1 - mn)x_1^2 + (m+n)x_1 = 0$. Since $x_1 \neq 0$, we have reduced the problem to solving $x_1 = \frac{m+n}{mn-1}$ over the positive integers. This is only possible for $m, n \in \{1, 2, 3\}$. These correspond to the 6 positive integral solutions $(x_1, x_2; y_1, y_2) \in X_{C_2}(\mathbb{N})$ that are listed. □

A.3. **Friezes of type G_2.** Fontaine–Plamondon [FP16, Theorem 4.4] showed that there are precisely 9 positive integral friezes of type G_2 by using the enumeration of D_4-friezes by Morier-Genoud–Ovsienko–Tabachnikov [MGOT12] and the observation that G_2-friezes lift to certain endomorphism-invariant D_4-friezes. By Proposition 2.4, there are precisely 9 positive integral points on X_{G_2}.

We give another proof by directly counting the points of $X_{G_2}(\mathbb{N})$.

Proposition A.6. *The affine surface X_{G_2} has exactly 9 positive integral points:*

- $(1,1,2,2)$,
- $(1,2,3,1)$,
- $(2,1,2,9)$,
- $(2,3,2,3)$,
- $(2,9,5,1)$,
- $(3,2,1,14)$,
- $(3,14,5,2)$,
- $(5,9,2,14)$,
- $(5,14,3,9)$.

Proof. The G_2-frieze equations are:

$$x_1 y_1 = x_2 + 1,$$
$$x_2 y_2 = x_1^3 + 1.$$

Observe that $X_{G_2}(\mathbb{N})$ is in bijection with the positive integer points on the affine cubic surface,

$$xyz = x^3 + y + 1,$$

under the change of variables

$$x_1 \longmapsto x, \qquad x_2 \longmapsto \frac{x^3+1}{y}, \qquad y_1 \longmapsto z, \qquad y_2 \longmapsto y.$$

Mohanty [Moh77, Theorem 2] proved that this affine cubic surface has exactly 9 positive points; we now give a shorter proof that it only has 9 positive integral points. Observe that x_2 is congruent to $-1 \pmod{x_1}$ from the G_2-frieze equations and that $y = y_2$ is also congruent to $-1 \pmod{x_1}$ from the affine cubic equation. Writing $x_2 = mx_1 - 1$ and $y_2 = nx_1 - 1$, we have that $(mx_1 - 1)(nx_1 - 1) = x_1^3 + 1$ so $x_1^3 - mnx_1^2 + (m+n)x_1 = 0$. Since $x_1 \neq 0$, we have reduced the problem to the quadratic equation

$$x_1^2 - mnx_1 + (m+n) = 0.$$

If the quadratic polynomial has a positive integral root, then it has positive integral roots a and b such that $a + b = mn$ and $ab = m + n$. This is only possible for $a, b, m, n \in \{1, 2, 3, 5\}$. These correspond to the 9 positive integral solutions $(x_1, x_2; y_1, y_2) \in X_{G_2}(\mathbb{N})$ that are listed. $\square$

Remark A.7. From a naïve arithmetic geometry perspective, it is not obvious *a priori* that the surface $xyz = x^3 + y + 1$ should have only finitely many positive integral points. Like in the A_2 case, there are infinite families of integer points $(t, -t-1, -t+1)$, $(-1, t, -1)$, and $(t, -t^3 - 1, 0)$ on this affine cubic. Furthermore, every choice of positive integer z specializes the surface to an affine cubic curve of geometric genus 0 with two complex points at infinity; Siegel's theorem does not suggest that each curve should have only finitely many integral points. But in fact, a descent argument can demonstrate that all integral points of the surface are obtained from the three curves listed above and the automorphism $T : (x, y, z) \mapsto (q, y, w)$ where $q = \frac{y+1}{x} = yz - x^2$ and $w = \frac{y+xz+1}{x^2}$. A forthcoming work of Corvaja–Zannier [CZ25] provides a more in-depth discussion on the integral points of this surface and other related examples with interesting topological considerations.

TABLE 1

For each finite Dynkin type Δ_n, we list the Cartan matrix used throughout the paper, the corresponding system of equations for X_{Δ_n} given by Definition 2.2, the frieze periods, and the number of positive integer solutions (equivalently, the number of positive integral friezes). Let $d(m)$ be the number of divisors of an integer m.

Finite type Δ_n	Cartan matrix C_{Δ_n}	Frieze equations for X_{Δ_n}	Period P_{Δ_n}	Positive integral point count $\#X_{\Delta_n}(\mathbb{N})$
A_n	$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & \ddots & & \\ & \ddots & \ddots & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$	$\begin{aligned} x_1y_1 &= x_2+1 \\ x_2y_2 &= x_1+x_3 \\ &\vdots \\ x_{n-1}y_{n-1} &= x_{n-2}+x_n \\ x_ny_n &= x_{n-1}+1 \end{aligned}$	$n+3$	$\frac{1}{n+2}\binom{2n+2}{n+1}$
B_n	$\begin{pmatrix} 2 & -1 & & & \\ -2 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$	$\begin{aligned} x_1y_1 &= x_2^2+1 \\ x_2y_2 &= x_1+x_3 \\ &\vdots \\ x_{n-1}y_{n-1} &= x_{n-2}+x_n \\ x_ny_n &= x_{n-1}+1 \end{aligned}$	$n+1$	$\sum_{m=1}^{\lfloor\sqrt{n+1}\rfloor}\binom{2n-m^2+1}{n}$
C_n	$\begin{pmatrix} 2 & -2 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$	$\begin{aligned} x_1y_1 &= x_2+1 \\ x_2y_2 &= x_1^2+x_3 \\ &\vdots \\ x_{n-1}y_{n-1} &= x_{n-2}+x_n \\ x_ny_n &= x_{n-1}+1 \end{aligned}$	$n+1$	$\binom{2n}{n}$
D_n	$\begin{pmatrix} 2 & & -1 & & & \\ & 2 & -1 & & & \\ -1 & -1 & 2 & -1 & & \\ & & -1 & \ddots & \ddots & \\ & & & \ddots & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$	$\begin{aligned} x_1y_1 &= x_3+1 \\ x_2y_2 &= x_3+1 \\ x_3y_3 &= x_1x_2+x_4 \\ x_4y_4 &= x_3+x_5 \\ &\vdots \\ x_{n-1}y_{n-1} &= x_{n-2}+x_n \\ x_ny_n &= x_{n-1}+1 \end{aligned}$	n	$\sum_{m=1}^{n} d(m)\binom{2n-m-1}{n-m}$
E_6	$\begin{pmatrix} 2 & & & -1 & & \\ & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ -1 & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix}$	$\begin{aligned} x_1y_1 &= x_4+1 \\ x_2y_2 &= x_3+1 \\ x_3y_3 &= x_2+x_4 \\ x_4y_4 &= x_1x_3+x_5 \\ x_5y_5 &= x_4+x_6 \\ x_6y_6 &= x_5+1 \end{aligned}$	14	868
E_7	$\begin{pmatrix} 2 & & & -1 & & & \\ & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ -1 & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{pmatrix}$	$\begin{aligned} x_1y_1 &= x_4+1 \\ x_2y_2 &= x_3+1 \\ x_3y_3 &= x_2+x_4 \\ x_4y_4 &= x_1x_3+x_5 \\ x_5y_5 &= x_4+x_6 \\ x_6y_6 &= x_5+x_7 \\ x_7y_7 &= x_6+1 \end{aligned}$	10	4400
E_8	$\begin{pmatrix} 2 & & & -1 & & & & \\ & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ -1 & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}$	$\begin{aligned} x_1y_1 &= x_4+1 \\ x_2y_2 &= x_3+1 \\ x_3y_3 &= x_2+x_4 \\ x_4y_4 &= x_1x_3+x_5 \\ x_5y_5 &= x_4+x_6 \\ x_6y_6 &= x_5+x_7 \\ x_7y_7 &= x_6+x_8 \\ x_8y_8 &= x_7+1 \end{aligned}$	16	26952
F_4	$\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -2 & 2 & -1 \\ & & -1 & 2 \end{pmatrix}$	$\begin{aligned} x_1y_1 &= x_2+1 \\ x_2y_2 &= x_1+x_3 \\ x_3y_3 &= x_2^2+x_4 \\ x_4y_4 &= x_3+1 \end{aligned}$	7	112
G_2	$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$	$\begin{aligned} x_1y_1 &= x_2+1 \\ x_2y_2 &= x_1^3+1 \end{aligned}$	4	9

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