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An exposition of the Uhlmann (1966) inequality bounding hypergeometric by binomial tail probabilities

This exposition is adapted from one originally written for a seminar in 1980. Its latest reference is from 1968, except for a recently added one from 1999.

Let S be a finite set with N elements and A a subset of S with m elements. Let B be a subset of S with r elements, chosen at random with probability $1/\binom{N}{r}$ of choosing each of the $\binom{N}{r}$ subsets with r elements. The set A could also be chosen randomly with an arbitrary distribution as long as B is chosen independently of A and with a uniform distribution. Let $h(k, m, r, N)$ be the probability that $A \cap B$ contains exactly k elements. Then

$$h(k, m, r, N) = \binom{m}{k}\binom{N-m}{r-k}\Big/\binom{N}{r} = \frac{m!(N-m)!r!(N-r)!}{k!(m-k)!(r-k)!(N-m-r+k)!N!} \quad (1)$$

if $\max(0, r+m-N) \le k \le \min(m, r)$ (for an integer $j < 0$ let $j! := +\infty$ and $1/j! := 0$), and let $h(k, m, r, N) := 0$ otherwise. It follows directly from (1) that $h(k, m, r, N) \equiv h(k, r, m, N)$.

A random variable X will be said to have a hypergeometric (m, r, N) distribution if $\Pr(X = k) = h(k, m, r, N)$ for each k. Let $H(k, m, r, N) := \Pr(X \le k)$ be the probability that $A \cap B$ has at most k elements, so that

$$H(k, m, r, N) = \sum_{j=0}^{k} h(j, m, r, N).$$

Then h is called an individual hypergeometric probability and H a cumulative hypergeometric probability.

Recall the individual binomial probability defined by $b(k, n, p) := \binom{n}{k} p^k (1-p)^{n-k}$ for $0 \le p \le 1$ and $k = 0, 1, ..., n$, and the cumulative binomial probability defined by $B(k, n, p) := \sum_{j=0}^{k} b(j, n, p)$.

It is well known, and not hard to prove, that as $t \to \infty$ through positive integers, while j, m, r, and N stay fixed, we have $h(j, m, rt, Nt) \to b(j, m, r/N)$, so

$$H(j, m, rt, Nt) \to B(j, m, r/N). \quad (2)$$

The idea is that viewing $h(j, m, rt, Nt)$ as $h(j, rt, m, Nt)$, so that m elements are chosen at random from a large set of Nt elements of which rt have a given property, even if the m elements (where m is fixed and finite) were chosen with replacement (as in the binomial case), they would with probability converging to 1 all be different, as in the hypergeometric case.

The mean of a hypergeometric variable is easily seen to be given by

$$EX = \sum_{j=0}^{m} jh(j, m, r, N) = mr/N.$$

If $k < rm/N$, it's plausible that the hypergeometric tail probability $H(k, m, r, N)$ should be less than (or equal to) the corresponding binomial tail probability $B(k, m, r/N)$, and likewise for the upper tail probabilities. This is not quite correct if k is within 1 of rm/N, otherwise the hypergeometric and binomial tail probabilities would have to be equal, as they are not. But for k at least 1 away from the mean, the domination holds, as was apparently first proved by Uhlmann (1966). In fact, he showed that the convergence in (2) is monotone in t:

Theorem 1 (Uhlmann). *For any positive integers N, $m \leq N$, $r \leq N$ and k such that $k \leq (rm/N) - 1$, for any $t = 1, 2, \ldots,$*

$$H(k, m, rt, Nt) \leq H(k, m, r(t+1), N(t+1)). \tag{3}$$

Thus as $t \uparrow \infty$, $H(k, m, rt, Nt) \uparrow B(k, m, r/N)$, so

$$H(k, m, r, N) \leq B(k, m, r/N). \tag{4}$$

Proof. The proof will be as given by Uhlmann (1966), with a few minor changes and some details filled in. Actually, a somewhat stronger result will be proved, where r and N are not necessarily integers.

Since $H(k, m, r, N) = H(N - m - r + k, N - m, N - r, N)$, and $N - m - r + k \leq ((N - m)(N - r)/N) - 1$ if and only if $k \leq (mr/N) - 1$, we can assume that $m \leq N/2$.

If $k = m - 1$, we must have $r = N$ and then $H(k, m, r, N) = 0$. The cases $k = 0$ and $m - 1$ will be dealt with specially in Proposition 7 below. For the present we assume $1 \leq k \leq m - 2$, and thus

$$3 \leq m \leq N/2. \tag{5}$$

Remark. Also, Theorem 1 for $k = 0$ and $0 < r < N$ follows from the inequality

$$\frac{A(A-1)\cdots(A-j)}{B(B-1)\cdots(B-j)} \leq \frac{C(C-1)\cdots(C-j)}{D(D-1)\cdots(D-j)}$$

if $0 < A/B = C/D \le 1$, $0 < A \le C$, $j \le A$, and $j < B$; then the inequality is strict if $A < C$ and $A < B$. (Let $A := (N-r)t$, $B := Nt$, $C := (N-r)(t+1)$, and $D := N(t+1)$.)

For any real number c and nonnegative integer k, the binomial coefficient $\binom{c}{k}$ is defined by $\binom{c}{k} := c(c-1)\cdots(c-k+1)/k!$ for $k \ge 1$, and $\binom{c}{0} := 1$. For $0 \le p \le 1$ let

$$L(k,m,N,p) \;:=\; \sum_{j=0}^{k} \binom{Np}{j}\binom{N-Np}{m-j}\Big/\binom{N}{m}. \tag{6}$$

This equals $H(k,m,Np,N)$ if Np is an integer. Until further notice, k, m, and N will be fixed. Then let $L(p) := L(k,m,N,p)$. Here L is a polynomial in p of degree m.

We will use the following:

Proposition 2. *For any real number p and integers $0 \le m \le N$ we have*

$$\sum_{j=0}^{m} \binom{Np}{j}\binom{N-Np}{m-j} \;=\; \binom{N}{m}.$$

Proof. If $m = 0$ the equation becomes $1 = 1$, so let $0 < m \le N$. For $p = i/N$, $i = 0, 1, ..., N$, the equation holds as $\sum_{j=0}^{m} h(j,m,i,N) = 1$. The left side is a polynomial of degree m in p and the right side is constant in p. The equation has at least $N+1 > m$ roots, so it holds identically. □

In case $m \le N\min(p, 1-p)$, we get a kind of extended hypergeometric distribution which is a special case of what Wimmer and Altmann (1999) call the "Kemp family" of distributions with probability at x equal to $\binom{a}{x}\binom{b}{n-x}/\binom{a+b}{n}$ for integers $0 \le x \le n$ and under some further restrictions.

We have $L(i/N) = 1$ for $i = 0, 1, ..., k$ and $L(i/N) = 0$ for $i = N-m+k+1, ..., N$. Thus by Rolle's theorem, dL/dp has zeroes at $p_1, ..., p_{m-1}$ with

$$\begin{gathered} 0 < p_1 < \frac{1}{N} < p_2 < \frac{2}{N} < \cdots < \frac{k-1}{M} < p_k < \frac{k}{N} < \\ < \frac{N-m+k+1}{N} < p_{k+1} < \cdots < \frac{N-1}{N} < p_{m-1} < 1. \end{gathered}$$

Since dL/dp has degree $m-1$ in p, the p_j must be all of its roots. Hence we have:

Proposition 3. *$L(p)$ is strictly decreasing for $k/N \le p \le (N-m+k+1)/N$.*

Since the p_j are simple roots of dL/dp, they must be alternately local maxima and minima of L, where p_k is a local maximum and p_{k+1} is a local minimum, etc. Again by Rolle's theorem, d^2L/dp^2 has zeroes a_j with

$$p_j < a_j < p_{j+1}, \quad j = 1, ..., m-2, \tag{7}$$

and these are all the zeroes of d^2L/dp^2. Set $a_0 := 0$ and $a_{m-1} := 1$. Using Proposition 3, and since $a_{k-1} < p_k < a_k < p_{k+1} < a_{k+1}$, we have

$$\frac{d^2L}{dp^2} \quad \begin{cases} < 0, & a_{k-1} < p < a_k; \\ > 0, & a_k < p < a_{k+1}. \end{cases} \tag{8}$$

We next consider some differences of L:

Lemma 4. *For $0 \le p \le 1 - 1/N$, we have $L(p) - L\left(p + \frac{1}{N}\right) = \binom{Np}{k}\binom{N-Np-1}{m-k-1}/\binom{N}{m}$.*

Proof. We can multiply both sides by $\binom{N}{m}$. Then for $k = 0$ the Lemma says

$$\binom{N-Np}{m} - \binom{N-Np-1}{m} = \binom{N-Np-1}{m-1},$$

which is true (Pascal's identity, with its algebraic rather than combinatorial proof). To induct from $k-1$ to k we have, again by Pascal's identity,

$$\binom{Np}{k-1}\binom{N-Np-1}{m-k} + \binom{Np}{k}\binom{N-Np}{m-k} - \binom{Np+1}{k}\binom{N-Np-1}{m-k}$$
$$= \binom{Np}{k}\binom{N-Np}{m-k} - \binom{Np}{k}\binom{N-Np-1}{m-k} = \binom{Np}{k}\binom{N-Np-1}{m-k-1},$$

which finishes the proof of the lemma. $\square$

For $1 \le k \le m-2$, Lemma 4 gives the second difference

$$L\left(p - \frac{1}{N}\right) - 2L(p) + L\left(p + \frac{1}{N}\right) =$$
$$\left\{\binom{Np-1}{k}\binom{N-Np}{m-k-1} - \binom{Np}{k}\binom{N-Np-1}{m-k-1}\right\}/\binom{N}{m}$$
$$= \binom{Np-1}{k-1}\binom{N-Np-1}{m-k-2}T/\binom{N}{m}$$

where

$$T := \frac{(Np-k)(N-Np) - Np(N-Np-1-[m-k-2])}{k(m-k-1)} = \frac{N((m-1)p-k)}{k(m-k-1)}.$$

Thus $T = 0$ when $p = c := k/(m-1)$. Let $\delta := 1/N$. So the following three points are on a straight line $\mathcal{L}$: $(c-\delta, L(c-\delta)$; $(c, L(c))$; and $(c+\delta, L(c+\delta))$. By the mean value theorem there are points where dL/dp equals the slope of $\mathcal{L}$ in each of the intervals $(c-\delta, c)$ and $(c, c+\delta)$. Thus $d^2L/dp^2 = 0$ somewhere in the interval $(c-\delta, c+\delta)$. Since $2m \le N$ from (5), and $1 \le k \le m-2$, we have

$$p_k \ < \ k/N \ \le c-\delta \ < \ c+\delta \ < \ 1-(m-k-1)/N \ < \ p_{k+1}.$$

So by (7), and since d^2L/dp^2 has a unique zero a_k in $[p_k, p_{k+1}]$, we have

$$c-\delta \ < \ a_k \ < \ c+\delta. \tag{9}$$

Lemma 5. *For $k/N \le p \le k/(m-1)$, the chord C_p through the points $(p-1/N, L(p-1/N))$ and $(p, L(p))$ is entirely below the graph $\mathcal{G}$ of L, i.e. for $0 < \lambda < 1$,*

$$L\left(\lambda\left(p-\frac{1}{N}\right)+(1-\lambda)p\right) \ > \ \lambda L\left(p-\frac{1}{N}\right)+(1-\lambda)L(p),$$

or $\lambda(L(p-(1/N))-L(p)) \ < \ L(p-(\lambda/N))-L(p)$.

For $k/(m-1)+(1/N) \le p \le 1-(m-k-2)/N$, the chord C_p lies above $\mathcal{G}$, so for $0 < \lambda < 1$,

$$\lambda(L(p-(1/N))-L(p)) \ > \ L(p-(\lambda/N))-L(p).$$

Proof. Whenever $0 \le u < v \le 1$ let $C_{uv} := C(u, v)$ be the chord (line segment) joining $(u, L(u))$ to $(v, L(v))$ and let s_{uv} be its slope. Then $s_{c-\delta,c} = s_{c,c+\delta} = s_{\mathcal{L}}$, the slope of $\mathcal{L}$.

In the first sentence of the lemma we assume $k/N \le p \le c$. Recall that in the interval $[p_k, p_{k+1}]$, $L'' = d^2L/dp^2$ has just one zero, namely a_k, and by (9), $c-\delta < a_k < c+\delta$. On $[a_{k-1}, a_k]$, L is strictly concave, hence above all its chords.

Claim 1. $C_{c-\delta,c}$ is below $\mathcal{G}$.

Proof. If $C_{c-\delta,c+\delta}$ intersects $\mathcal{G}$ at $(s, L(s))$ for some $s \in (c-\delta, c+\delta)$ other than $s = c$, then $L' = s_{c-\delta,c}$ at three or more points in $(c-\delta, c+\delta)$, and $L'' = 0$ at two or more points in the interval, a contradiction. Also $L'(c) \ne s_{\mathcal{L}}$ for the same reason. So in $(c-\delta, c)$, L is entirely on one side of $\mathcal{L}$ and in $(c, c+\delta)$, on the other side. Since L'' is first negative, then positive, we must have $L'(c) < s_{\mathcal{L}}$ and $C_{c-\delta,c}$ is below $\mathcal{G}$, proving Claim 1.

Claim 2. The chords $C_{x,k/n}$ are below $\mathcal{G}$ for $(k-1)/N \le x < k/N$.

Proof. Since $L > 1$ on $((k-1)/N, k/N)$, the horizontal chord $C_{(k-1)/N,k/N}$ is below $\mathcal{G}$. For $x \ge a_{k-1}$, the claim holds by concavity. For $(k-1)/N \le x < a_{k-1}$, L' is positive and increasing, while $s_{x,k/N} < 0$, so $C_{x,k/N}$ lies below the part of $\mathcal{G}$ between x and a_{k-1}, hence also below the chord and graph from a_{k-1} to k/N by concavity, proving Claim 2.

Claim 3. For $(k-1)/N \le u < k/N \le v \le a_k$, C_{uv} is below $\mathcal{G}$.

Proof. We have $s_{u,k/N} > L'(k/N)$ by Claim 2, and $L'(k/N) > s_{k/N,v}$ by concavity, so Claim 3 follows.

Claim 4. Each chord $C_{c-\delta,q}$ for $c - \delta < q \le c$ is below $\mathcal{G}$.

Proof. For $q \le a_k$, the claim holds by concavity. If $a_k < q \le c$, then on $[a_k, c]$, L' is increasing but $L' < s_{\mathcal{L}}$. Thus the (perpendicular or vertical) distance to $\mathcal{L}$ is decreasing there, while on $C_{c-\delta,q}$ it is increasing. So, as in Claim 2, each chord $C_{c-\delta,q}$ is below the part of $\mathcal{G}$ between a_k and q, and thus below $\mathcal{G}$ on $(c-\delta, q)$, proving Claim 4.

Now to finish the proof of Lemma 5, consider any chord $C_p = C_{p-\delta,p}$ for $k/N \le p \le c$. If $p \le a_k$, then C_p is below $\mathcal{G}$ by Claim 3 if $p - \delta \le k/N$ and by concavity otherwise.

If $a_{k-1} \le p - \delta < a_k < p$, where $p - \delta \le c - \delta < a_k$ by (9) and Claim 3 dealt with $a_k \ge p$, then since $p \le c$,

$$s_{p-\delta,c-\delta} \;>\; L'(c-\delta) \;>\; s_{c-\delta,a_k} \;>\; s_{c-\delta,p}$$

where the first two inequalities hold by concavity and the third by Claim 4. Since for the same reasons, $C_{p-\delta,c-\delta}$ and $C_{c-\delta,p}$ are both below $\mathcal{G}$, $C_{p-\delta,p}$ is below $\mathcal{G}$ as desired.

The remaining case is

$$(k-1)/N \le p - \delta \;<\; a_{k-1} \;<\; k/N \;\le\; c - \delta \;<\; a_k \;<\; p, \tag{10}$$

because the first, third, fourth and fifth inequalities always hold, we just treated the case where the second fails, and Claim 3 treated the case that the sixth fails.

If (10) holds we have $s_{p-\delta,k/N} > s_{a_{k-1},k/N}$ by Claim 2, which is larger than $s_{k/N,c-\delta} > s_{c-\delta,a_k}$ by concavity, which is larger than $s_{c-\delta,p}$ by Claim 4. Since the chords from $p - \delta$ to k/N to $c - \delta$ to p are each below $\mathcal{G}$, and their slopes decrease from each to the next, $C_{p-\delta,p}$ is below $\mathcal{G}$. This proves the first sentence of the Lemma.

The proof of the second sentence is symmetrical, specifically as follows. We have

$$1 - L(1-p) \;=\; 1 - \sum_{j=0}^{k} \binom{N(1-p)}{j}\binom{Np}{m-j} \Big/ \binom{N}{m}.$$

By Proposition 2 and a change of indices this equals

$$\sum_{j=k+1}^{m} \binom{N(1-p)}{j}\binom{Np}{m-j}\Big/\binom{N}{m} = \sum_{r=0}^{m-k-1} \binom{Np}{r}\binom{N(1-p)}{m-r}\Big/\binom{N}{m}$$
$$= L(m-k-1, m, N, p).$$

By the first sentence of the Lemma applied to $m-k-1$ in place of k, the chords C_p of this function of p are below its graph for $(m-k-1)/N \le p \le (m-k-1)/(m-1) = 1-k/(m-1)$. Thus for the function $p \mapsto L(1-p)$, the chords connecting $(p-\delta, L(1-p+\delta))$ to $(p, L(1-p))$ are above the graph of the function for the same interval of values of p. Via the substitution $x \leftrightarrow 1-x$ for the x coordinates while leaving the arguments of L and so the y coordinates unchanged, the chords will join points on the graph of L itself and be above it. Letting $q := 1-p+\delta$, it follows that the chord C_q connecting $(q-\delta, L(q-\delta))$ to $(q, L(q))$ is above the graph of L for $\delta + k/(m-1) \le q \le 1-(m-k-2)/N$, which gives the second sentence of the lemma and so completes its proof. □

Using Lemma 4 in Lemma 5 gives:

Lemma 6. *Let $0 < k < m-1 < N$ and $0 < \lambda < 1$. Then if $k/N \le p < k/(m-1)$, we have*

$$L\left(p - \frac{\lambda}{N}\right) - L(p) \ > \ \lambda\binom{Np-1}{k}\binom{N-Np}{m-k-1}\Big/\binom{N}{m}.$$

Or, if $k/(m-1) + (1/N) \le p < 1-(m-k-2)/N$, then we have

$$L\left(p - \frac{\lambda}{N}\right) - L(p) \ < \ \lambda\binom{Np-1}{k}\binom{N-Np}{m-k-1}\Big/\binom{N}{m}.$$

Now we consider zeroes of the polynomials

$$Q(p) \ := \ Q(k,m,N,p) \ := \ L(k,m,N,p) - L(k,m,N+1,p).$$

We have $Q(0) = Q(1) = 0$. If $k > 1$, then $L(k,m,N+1,j/(N+1)) = 1$ for $j = k, k-1, \dots, 1$, while $L(k,m,N,j/(N+1))$, by the facts mentioned in the paragraphs just before and after Proposition 3, is alternately > 1 and < 1 since $(j-1)/N < j/(N+1) < j/N$ for $j = 1, \dots, k$. So each of the $k-1$ intervals $((j-1)/(N+1), j/(N+1))$ for $j = 2, ..., k$ contains at least one zero of Q. Likewise, if $k < m-2$, so does each of the $m-k-2$ open intervals $(1-(j+1)/(N+1), 1-j/(N+1))$ for $j = 1, ..., m-k-2$. For $0 < k < m-1$ we have by Proposition 3 since $k/(N+1) < k/N < 1-(m-k-1)/N < (N+2-m+k)/(N+1)$,

$$L(k,m,N+1,k/N) \ < \ 1 \ = \ L(k,m,N,k/N), \tag{11}$$

and

$$L(k, m, N+1, 1-(m-k-1)/N) \ > \ 0 \ = \ L(k, m, N, 1-(m-k-1)/N). \qquad (12)$$

Thus the interval $(k/N, 1-(m-k-1)/N)$ contains another zero of Q. So we have mentioned altogether m different zeroes of Q if $2 \le k < m-2$. If $k = 0, 1, m-2$, or $m-1$, then we omit the parts of the above argument which do not apply and find that we have still counted m zeroes of Q. For $m \ge 3$ by (5), we see that Q is not identically 0: in case $0 < k < m-1$ for $p = k/N$ by (11); or by the case "$k > 1$" if $k = m-1$, for $p = 2/(N+1)$; or the case "$k < m-2$" if $k = 0$ with $p = 1 - 2/(N+1)$. So all the m roots have been counted.

Let

$$J(k, m, N, p) := \begin{cases} 1, & 0 \le p \le k/N; \\ L(k, m, N, p), & k/N \le p \le (N-m+k+1)/N; \\ 0, & (N-m+k+1)/N \le p \le 1. \end{cases} \qquad (13)$$

Then $J(k, m, N, p) = H(k, m, Np, N)$ whenever Np is an integer and $0 \le p \le 1$. Let

$$D \ := \ D(k, m, N, p) \ := \ J(k, m, N, p) - J(k, m, N+1, p). \qquad (14)$$

By the Remark after (5), and by Proposition 2 for the second statement, we have the following for the special cases $k = 0$ or $m-1$:

Proposition 7. *If $k = 0 < m \le N$, then $D = 0$ if $p = 0$ or if $1-(m-1)/(N+1) \le p \le 1$; for $0 < p < 1-(m-1)/(N+1)$ we have $D < 0$.*

For $1 \le k = m-1 < N$, we have $D = 0$ for $0 \le p \le (m-1)/(N+1)$ and for $p = 1$; if $(m-1)/(N+1) < p < 1$ we have $D > 0$.

From the above treatment of zeroes of Q, we have:

Lemma 8. *If $0 < k < m-1 < N$ there is exactly one p_0 with $k/N < p_0 < (N-m+k+1)/N$ and $D(k, m, N, p_0) = 0$.*

If $m \ge 3$ is odd, $k = (m-1)/2$, and $m-1 < N$, then

$$L(k, m, N, 1/2) \ = \ \sum_{j=0}^{k} \binom{N/2}{j} \binom{N/2}{m-j} = \frac{1}{2}$$

where the second equation holds by interchanging j and $m-j$ and Proposition 2. Thus in this case, p_0 in Lemma 8 equals $1/2$. Hence Proposition 3, Lemma 8, (11) and (12) imply:

Proposition 9. *If m is odd, $m \geq 3$, and $k = (m-1)/2$, then $D = 0$ for $0 \leq p \leq (m-1)/(2N+2)$, also for $p = 1/2$ and for $1 - (m-1)/(2N+2) \leq p \leq 1$; for other $p < 1/2$ we have $D > 0$, and for other $p > 1/2$, we have $D < 0$.*

For $0 \leq k < m \leq N$ we have

$$\binom{N}{m}(L(k,m,N,p) - L(k,m,N+1,p))$$

$$\begin{aligned}
&= \textstyle\sum_{j=0}^{k} \binom{Np}{j}\binom{N-Np}{m-j} - \frac{N-m+1}{N+1}\sum_{j=0}^{k}\binom{Np+p}{j}\binom{N-Np-p+1}{m-j} \\
&= \textstyle\binom{N}{m}(L(k,m,N,p) - L(k,m,N,p+(p/N))) \qquad (15)\\
&\quad + \sum_{j=0}^{k}\binom{Np+p}{j}\binom{N-Np-p}{m-j-1}\frac{j-mp}{m-j}
\end{aligned}$$

because

$$N - Np - p - m + j + 1 - \frac{(N-m+1)(N-Np-p+1)}{N+1} = j - mp.$$

Next we have:

Lemma 10. *For $0 \leq k < m$ we have*

$$\sum_{j=0}^{k}\binom{Np+p}{j}\binom{N-Np-p}{m-j-1}\frac{j-mp}{m-j} = -p\binom{Np+p-1}{k}\binom{N-Np-p}{m-k-1}.$$

Proof. We use induction on k. For $k=0$ the result is clear. Assuming it holds for $k-1$, where $1 \leq k < m$, we have

$$\begin{aligned}
&\textstyle\sum_{j=0}^{k}\binom{Np+p}{j}\binom{N-Np-p}{m-j-1}\frac{j-mp}{m-j} \\
&= \textstyle -p\binom{Np+p-1}{k-1}\binom{N-Np-p}{m-k} + \binom{Np+p}{k}\binom{N-Np-p}{m-k-1}\frac{k-mp}{m-k} \\
&= \textstyle\binom{Np+p-1}{k-1}\binom{N-Np-p}{m-k-1}F
\end{aligned}$$

where

$$F := -\frac{p(N-Np-p-m+k+1)}{m-k} + \frac{(k-mp)(Np+p)}{k(m-k)},$$

so

$$\binom{Np+p-1}{k-1}\binom{N-Np-p}{m-k-1}F = -p\binom{Np+p-1}{k}\binom{N-Np-p}{m-k-1},$$

which proves the lemma. $\square$

Substituting Lemma 10 into (15) gives

$$\begin{aligned}\tbinom{N}{m}\left(L(k,m,N,p)-L(k,m,N+1,p)\right) \; &= \\ \tbinom{N}{m}\left(L(k,m,N,p)-L(k,m,N,p+(p/N))\right)-p\tbinom{Np+p-1}{k}\tbinom{N-Np-p}{m-k-1}&. \end{aligned} \tag{16}$$

Next we can prove a basic monotonicity fact:

Proposition 11. *Suppose* $0 < k < m-1 < N$*. Then for* $D = D(k,m,N,p)$ *defined by (14),* $D = 0$ *for* $0 \leq p \leq k/(N+1)$ *and for* $1-(m-k-1)/(N+1) \leq p \leq 1$*;* $D > 0$ *for* $k/(N+1) < p \leq kN/((m-1)(N+1))$*; and* $D < 0$ *for*

$$\frac{k}{m-1}+\frac{m-k-1}{(m-1)(N+1)} \leq p < 1-\frac{m-k-1}{N+1}.$$

Proof. For $0 \leq p \leq k/N$ or $1-(m-k-1)/N \leq p \leq 1$, the result follows from (13), i.e. the definition of J, and Proposition 3. For the remaining p we can replace J by L. Using Lemma 6 with p replaced by $p+p/N$ and $\lambda = p$ gives

$$\binom{N}{m}(L(k,m,N,p)-L(k,m,N,p+p/N) \; > \; p\binom{Np+p-1}{k}\binom{N-Np-p}{m-k-1}$$

for $k/N \leq p+p/N \leq k/(m-1)$, or equivalently for $k/(N+1) \leq p \leq kN/((m-1)(N+1))$. Substituting this in (16) gives the desired result.

Likewise for

$$\frac{k}{m-1}+\frac{1}{N} \leq p+\frac{p}{N} < 1-\frac{m-k-2}{N},$$

or equivalently for

$$\frac{k}{m-1}+\frac{m-k-1}{(m-1)(N+1)} \leq p < 1-\frac{m-k-1}{N+1},$$

we get from the second half of Lemma 6 that

$$\binom{N}{m}(L(k,m,N,p)-L(k,m,N,p+p/N) \; < \; p\binom{Np+p-1}{k}\binom{N-Np-p}{m-k-1}.$$

Thus Proposition 11 is proved. □

To check that Proposition 11 implies Theorem 1, note that if $k \leq mp-1$, where $p := r/N$, then

$$p \;\geq\; \frac{k+1}{m} \;\geq\; \frac{k}{m-1}+\frac{m-k-1}{(m-1)(N+1)} \;\geq\; \frac{k}{m-1}+\frac{m-k-1}{(m-1)(Nt+1)}$$

since $(k+1)(m-1)(N+1) \geq km(N+1)+(m-k-1)m$, using $0 \leq (m-k-1)(N-m+1)$. Thus by Proposition 11 and its proof,

$$\begin{aligned} 0 \ > \ D(k,m,Nt,P) \ &= \ J(k,m,Nt,p) - J(k,m,Nt+1,p) \\ &= \ L(k,m,Nt,p) - L(k,m,Nt+1,p). \end{aligned}$$

The same holds if we replace Nt by $Nt+i-1$ for $i = 1, ..., N$, so by induction we get

$$H(k,m,rt,Nt) = L(k,m,Nt,p) < \cdots < L(k,m,Nt+N,p) = H(k,m,r(t+1),N(t+1)),$$

proving (3) and thus Theorem 1. $\square$

NOTES

The above proof is mainly as in Uhlmann (1966) with some details, especially the proof of Lemma 6 above, filled in. This exposition was originally prepared for a seminar in October 1980. Small revisions and corrections have been made in 2007.

Beside Theorem 1 and other facts developed in its proof, Uhlmann's 1966 paper contains two other theorems:

(a) the fact that both for hypergeometric and for binomial distributions, the median and the mean differ by at most 1. Using Theorem 1 (and its symmetrical form for upper tails) it suffices to prove this for binomial distributions. For this Uhlmann (1966) uses results from Uhlmann (1963). Jogdeo and Samuels (1968) gave an independent proof for binomial distributions.

(b) Finally, Uhlmann (1966) gives an incorrect theorem (and proof) conflicting with a result of Anderson and Samuels (1965).

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