Solutions to 18.650 PS2 due Wednesday, Sept. 23, 2015

1. Before part (a) (5 points): We have

$$\frac{X(n-X)}{n} = \frac{106602 \cdot (315672 - 106602)}{315672} = \frac{106602 \cdot 209070}{315672} > 70602 > 15 > 8$$

(by far). Also $n = 315672 \geq 4.43 \cdot 10^4 = 44,300 > 25,500$, so the criteria given in the handout are satisfied so that endpoints of both the 95% and 99% confidence intervals for p by the plug-in and quadratic methods are within .0001 of each other, so we can use the plug-in intervals to sufficient accuracy.
(a) (5 points) We have $z_{.975} \doteq 1.96$, or from R $1.959964 \doteq 1.9600$ to the given number of digits, and $\hat{p} = 106602/315672 \doteq .337699$, $\hat{q} \doteq .662301$, so the endpoints are

$$\hat{p} \pm 1.96\sqrt{\hat{p}\hat{q}/n} \doteq .337699 \pm .001650 = [.3360, .3393],$$

giving only 4 places as for the quadratic interval, the 4th place could be different by 1.
(b) (5 points) The only difference is that now we have $z_{u(\alpha)} = z_{.995} = 2.575829$ according to R. The rounded values 2.5758 and 2.576 have both been mentioned, so any of these is OK. The endpoints are now $\hat{p} \pm 0.002168$, which gives $[0.3355, .3399]$.
(c) (5 points) No, 1/3 is in neither of the confidence intervals. The dice were not fair. Suggested physical explanation (not requested): numbers are marked by hollowed-out pips, so that the 5 and 6 faces are lighter and have some extra tendency to come up, while the opposite 2 and 1 faces are heavier and tend to fall on the bottom.

2. (a) (10 points) We have $X = 77$ and $n = 77 + 54 = 131$, so $\hat{p} = 77/131$. Certainly $n \geq 20$, and $\min(X, n - X) \geq 9$. So we use a quadratic interval. The quadratic equation to be solved is $(p - \hat{p})^2 = 1.96^2 p(1-p)/n$ and the solutions are $a(X) = a(X, n) = 0.5022$ and $b(X) = b(X, n) = .6684$. The interval does not contain 1/2 (although it comes close).
(b) (10 points) In this case $X = 75$ and $n = 75 + 94 = 169$. The quadratic equation is $(p-\hat{p})^2 = 1.96^2 p(1-p)/n$ and the solutions are $a(X) = .3710$ and $b(X) = .5191$, so this interval does contain 1/2. The intervals do overlap, in the short interval $[.5022, .5191]$.

3(a). (4 points) Since $0 \leq p \leq 1$ there is no point in having $a(X) < 0$. For a given n, if $p_0 > 0$ is extremely small (say $<< 1/n$), then the probability that

$X = 0$, namely $(1-p_0)^n$, is so close to 1 that $X \geq 1$ is virtually impossible. So with very high confidence we can choose $a(1) \geq p_0 > 0$.
(b) (16 points) Let $X = 2$, is

$$\frac{2}{n} - \frac{1.96\sqrt{2(n-2)}}{n^{3/2}} < 0? \quad \frac{4}{n^2} < \frac{1.96^2 \cdot 2(n-2)}{n^3}?$$

$$\frac{2}{1.96^2} < \frac{n-2}{n} = 1 - \frac{2}{n}, \quad \frac{2}{n} < 1 - \frac{2}{1.96^2} \doteq 0.4794, \quad n > \frac{2}{0.4794} \doteq 4.17?$$

Yes for $n \geq 5$.

For $X = 3$, a similar calculation leads to the question whether

$$\frac{3}{n} < 1 - \frac{3}{1.96^2} \doteq 0.2191? \quad n > 13.69?$$

Yes for $n \geq 14$.

For $X = 4$, $16/(4 \cdot 1.96^2) > 1$, and it follows that $a(4) > 0$ for all n, so no. Likewise for $X \geq 5$.

4. We have for the plug-in interval

$$b(5) = \frac{5}{20} + 2.5758\sqrt{\frac{1}{4} \cdot \frac{3}{4}}/\sqrt{20} = 0.4994037 < 0.5,$$

$$b(6) = \frac{6}{20} + 2.5758\sqrt{0.21}/\sqrt{20} = 0.563944 > 0.5.$$

By symmetry, $a(15) > 0.5$ and $a(14) < 0.5$. So 1/2 is in $[a(X), b(X)]$ if and only if $6 \leq X \leq 14$. The probability of this for $p = 1/2$ is $1 - 2B(5, 20, .5)$. From Rice's Table p. A5 $B(5, 20, .5) \doteq 0.021$, or more exactly from R it's pbinom(5,20,.5) = 0.02069473. The latter gives the coverage probability $\kappa(.5) \doteq 0.95861$, or Rice's number gives 0.958. Either way it's notably less than the target probability 0.99.

5. (a) (6 points) For $a(0) = 0 \leq p < a(1)$, the only interval $[a(k), b(k)]$ that p can be in is $[a(0), b(0)]$ since $a(1) \leq a(2) \leq \cdots \leq a(n)$ as assumed. So

$$\kappa(p) = \Pr_p(X = 0) = b(0, n, p) = (1-p)^n.$$

(b) (6 points) That's a decreasing function of p, so its infimum for $0 \leq p < a(1)$ comes when p increases up toward $a(1)$ and is $(1 - a(1))^n$.

(c) (8 points) $(1-.0009)^{11} \doteq 0.99014 > 0.99$, $(1-.0008)^{13} \doteq 0.98965 < 0.99$, $(1-.0007)^{15} \doteq 0.98955 < 0.99$, $(1-.0006)^{18} \doteq 0.989255 < 0.99$. This last is smallest. It's smaller than 0.99 by about 0.000745. So the $(1-p(1))^n$'s do come pretty close to the target 0.99.

Although the default is to give only as many significant digits in final answers as the smallest number in numbers used in obtaining it, this problem is different. In this case we get $1-a(1)$ to four significant digits even though $a(1)$ has only one. Recognizing that having only one significant digit in $a(1)$ might be a problem, the question is, by how much does using these $a(1)$'s cause the smallest coverage probability to be different from, of most concern less than, the target 0.99? To one significant digit, all the above would round to 1.0, or to two significant digits, 0.99, but those would be uninformative. Subtracting from 0.99 would cause a loss of two significant digits.