18.650 PS2 due Wednesday, Sept. 23, 2015
in class (preferably), or by 4 P.M. at E18-412

It is shown in the handout on binomial confidence intervals, subsection 2.4, that for an observation of X successes in n independent trials with probability p of success on each, if n and $X(n-X)/n$ are large enough, the endpoints of the plug-in confidence interval for p agree with those of the quadratic interval to accuracies 10^{-m} for m depending on n.

As usual, each problem counts 20 points in this set.

1. *Weldon's dice data.* A British scientist, Walter Frank Raphael Weldon (1860-1906), was primarily a zoologist but also interested in statistics. The Wikipedia article on him (titled with his full name) has a section "Weldon's Dice" on his "dice data," which are given in W. Feller's famous textbook on probability. Wikipedia cites some longer biographical articles on Weldon.

Summarizing the data as given in Feller, there were n =315,672 throws of individual dice (actually 12 at a time) and a "success" was counted each time a die came up 5 or 6. There were X =106,602 successes (5's or 6's). To justify using the plug-in interval, check the conditions for $\alpha = 0.01$ for agreement with the quadratic interval to accuracy 10^{-4} given on pp. 11-12 of the handout binomialconfints.pdf.
(a) Find a 95% confidence interval and
(b) a 99% confidence interval for the probability p of success, for the dice used to generate the data.
(c) If the dice were "fair" the probability of success should have been 1/3. Is this value in either of the confidence intervals?

2. In a study of a kind of insects, at a dry site in one year, 77 males and 54 females were observed. Find an approximate 95% confidence interval for the probability of being male at that site in that year. Does it contain 1/2?
(b) In the same study and year, at a wet site, 75 males and 94 female insects were observed. Answer the question in part (a) for this site. Also, do the two confidence intervals overlap?

3. (a) If we observe a number $X \geq 1$ of successes, is it reasonable or not for the lower endpoint $a(X)$ of a confidence interval for p to be ≤ 0? Why?
(b) It's mentioned in the handout on binomial confidence intervals that for the 95% plug-in interval, if $X = 1$, then $a(X) < 0$ for all $n \geq 2$. Can $a(X) < 0$ happen for other values of X, and if so, for what values of X and for each such X, for what values of n?

4. Find the coverage probability of the 99% plug-in interval for $n = 20$ at $p = 1/2$. *Hint:* Find the upper endpoint $b(X)$ when $X = 5$ (and as a check, also for $X = 6$). By symmetry find the lower endpoints $a(X)$ when $X = 15$ or 14. Deduce that $\kappa(1/2) = 1 - 2B(5, 20, 0.5)$, where $B(5, 20, 0.5)$ can be found from Table 1 in Rice, or computed as pbinom(5,20,0.5) in R. Compare $\kappa(1/2)$ to the target value 0.99.

5. (a) For any way of choosing confidence intervals for p such that $a(0) = 0 < a(1) \leq b(0)$ with $a(1) \leq a(2) \leq \cdots \leq a(n)$, what is the coverage probability $\kappa(p)$ for $0 \leq p < a(1)$? *Hint:* $p < b(0)$ by the assumptions. Show that p is in the interval $[a(X), b(X)]$ if and only if $X = 0$. What is the probability that $X = 0$ for a given p?
(b) What is the infimum (greatest lower bound) of $\kappa(p)$ for p in the given range?
(c) For $11 \leq n \leq 19$, the lower endpoint $a(1)$ of the adjusted Clopper-Pearson 99% confidence interval for p if $X = 1$ is only given to one significant digit of accuracy in Table 2 at the end of the handout. One might suspect that the actual coverage probability is smallest at some $a(1)-$ for the largest n such that $a(1)$ has a given value to the given accuracy, specifically $a(1) = 0.0009$, $n = 11$; $a(1) = 0.0008$, $n = 13$; $a(1) = 0.0007$, $n = 15$; and $a(1) = 0.0006$, $n = 18$. What is the minimal coverage probability of part (b) in each of these cases, and for all four cases? Is it less than the target 0.99? By how much?