Solutions to 18.650 PS1 due Wednesday, Sept. 16, 2015

1. (a) (10 points) $\overline{X}$ has distribution $N(0, 1/36)$, so $6\overline{X}$ has $N(0, 1)$. So for any $c > 0$, $P(|\overline{X}| < c) = P(|Z| < 6c)$ where Z has $N(0, 1)$. To make this $= 0.6$ we want to make $P(|Z| > 6c) = 0.4$ and so by symmetry of $N(0, 1)$ around 0, we want $P(Z > 6c) = 0.2$. So for the standard normal distribution function Φ we want $\Phi(6c) = 0.8$. The closest approximation given in Rice's Table 2 of the "Cumulative Normal Distribution" is $\Phi(0.84) = 0.7795$. In R one could (but this was not suggested in the problem) get the more precise quantile qnorm(.8) = 0.8416. Dividing by 6 would give $c = 0.1403$. Taking 0.84/6 gives 0.14.

So: any answer for c equaling 0.140 to the given number of places can be accepted as orrect.

(b) (10 points, 5 each part): To solve $P(|t(5)| < t_0) = 0.9$ for t_0, from Rice's Table 4 for the t distribution, we want by symmetry of t distributions around 0 that $P(t(5) > t_0) = 0.05$ and so we want the 0.95 quantile of $t(5)$ which from Rice's Table 4 is 2.015. The other part has the identical answer 2.015. (If you got from R, or otherwise, the more precise qt(.95,5) = 2.01505, 5 points extra credit.)

2. (a) (5 points) The sample mean $\overline{X} = 6.676407$.

(b) (5 points) The sample variance $s_X^2 = 3.34239$e-05 (R's notation) meaning $3.34239 \cdot 10^{-5}$. The sample standard deviation is $s_X = 0.005781342$.

(c) (10 points) We need the 0.975 quantile of the t distribution with $7 = 8-1$ degrees of freedom, which is $tq := t_{.975}(7) = 2.365$ according to Rice's Table 4. (R gives qt(.975,7) = 2.364624.) The endpoints of the interval are $\overline{X} \pm h$ where $h = tq \cdot s_X/\sqrt{8} \doteq .0048341$. The endpoints are then $\overline{X} - h = 6.6716$, $\overline{X} + h = 6.6812$, giving as many significant digits (5) as in the data, or better rounded to $[6.672, 6.681]$, as only 4 digits are given in the t quantile.

3. The following answers are based on quantiles from Rice's tables. Using quantiles found from R, which was not required, would give more precise answers.

(a) (7 points) We have $SUM := \sum_{j=1}^{8}(X_j - \overline{X})^2 = 2.3397 \cdot 10^{-4}$, which is $7s_X^2$. For a $\chi^2(7)$ distribution, from Rice's Table 3, the .975 quantile $\chi^2_{.975}(7) \doteq 16.01$. So the lower endpoint of the 95% confidence interval for σ^2 is $SUM/16.01 \doteq 1.461 \cdot 10^{-5}$, rounding to 4 significant digits as in the quantile. The .025 quantile is 1.69 according to the table, and so the upper endpoint is $SUM/1.69 \doteq 1.38 \cdot 10^{-4}$, rounding to 3 significant digits as in

the quantile. For σ, taking square roots of the endpoints (before rounding), we get $[0.0038228, 0.011766]$ or $[0.00382, 0.01177]$.
(b) (3 points) Only the Schwarz et al. 1998 given error estimate of .0094 is within the confidence interval. All the others, including the CODATA number, are less than the lower endpoint 0.00382. This is related to the fact, mentioned in an email, that the experimenters are giving lower bounds for errors they can account for.
(c) (10 points) The sample variance of the two estimates is 1.757813e-06. Since $2 - 1 = 1$ that equals the new SUM. Divided by $\chi^2_{.975}(1) = 5.02$ from Rice Table 3 gives the lower endpoint $a(X) \doteq 3.50162\text{e-}07$ whose square root is $5.92 \cdot 10^{-4}$, rounded to 3 digits because the quantile has just 3. For the upper endpoint, $\chi^2_{.025}(1) = .00098$ from Rice's table, so we get $SUM/0.00098 \doteq 1.79369 \cdot 10^{-3}$. So the 95% confidence interval for σ^2 is $[3.50 \cdot 10^{-7}, 1.8 \cdot 10^{-3}]$. Taking the square root to get the upper endpoint of the interval for σ gives $\doteq 0.042$, rounding to 2 digits which is all the quantile has. So the confidence interval for σ is $[.000592, .042]$.

Of the given error numbers given after $\pm$ in the two studies, the largest is 0.00014, which is less than the lower endpoint 0.000592. In this case the CODATA value .00080 is in the interval.

4. (a) (7 points) Following the hint, the solutions of $G'(t) = \gamma G(t)$ are of the form $G(t) = Ce^{\gamma t}$ for constants C. For a fixed time t_0 we will then have $G(t) = G(t_0) \exp(\gamma(t - t_0))$. In this case $t - t_0$ is no more than 117 years. We have $e^x = 1 + x$ to a very good approximation in case x is very small. The estimates for γ are no more than 10^{-11} per year and some of them are smaller. In 117 years G should have changed by a fraction no more than about 10^{-9}. The average of the estimates in 1895–97 minus the 2010 CODATA value is about $6.6577 - 6.674 \doteq -0.0163$ which divided by 6.67 gives a fraction about .0025. This is much larger than 10^{-9}, so *no*, it is not plausible that the average change in estimates between the 1890's and recent times resulted from actual changes in G.

(It seems that astrophysicists' measurements of possible changes in G are much more accurate than laboratory physicists' measurements of G itself.)
(b) (4 points) So the difference -0.016 given in part (a) seems to have resulted from bias in the older measurements.
(c) (3 points) Brayn (1897) declared the smallest error, ± 0.002.
(d) (4 points) Eötvös (1896) had the largest and so most realistic error estimate, in fact 0.013 is of the order of magnitude of the actual error (bias)

-0.016 as we now see it.
(e) (2 points) There is no mathematically "correct" answer to this question. Any reasonable argument can be given the 2 points. No calculation is expected. In my opinion: Brayn came later in time than the other two. Very possibly he was aware of their findings. This may have contributed to his giving an unrealistically small error estimate of $\pm .002$, which is wide enough to include the other two 1890's estimates, but far from including 1998–2010 estimates.

Eötvös invented an apparatus, improving on earlier "torsion balance" instruments, which others later called the Eötvös pendulum. Einstein cited Eötvös in regard to the equality of gravitational and inertial mass. Budapest University in Hungary was renamed for Eötvös in 1950. C. V. Boys, a then-leading English physicist, used his own, apparently different, form of torsion balance. In a brief Web search, nothing more was found about Brayn than his numbers in the G data base.

In over a century since 1900, the technology for estimating G has improved substantially.

5. (i) (a) (3 points) RFO - reject for other reasons; although normality is not rejected by the test, with p-value $> .05$, we know that the data were generated with $U[0,1]$ distribution, not a normal distribution.
(b) (3 points) RBT, Reject by test, the p-value is less than .05 (also, we know that the data were generated as standard exponential).
(c) (3 points) AP, accept provisionally.
(ii) (a′) (5 points) A $U[0,1]$ distribution is symmetric around its mean $\mu = 1/2$, so its skewness is 0. By a calculation neglecting the symmetry, and so not required, if X has this distribution, then

$$E\left(\left(X - \frac{1}{2}\right)^3\right) = E(X^3) - 3E(X^2)/2 + 3EX/4 - 1/8$$

$$= \frac{1}{4} - \frac{3}{2} \cdot \frac{1}{3} + \frac{3}{8} - 1/8 = (2 - 4 + 3 - 1)/8 = 0.$$

(b′) (6 points) For a standard exponential distribution, the mean is 1, and the variance is $E((X-1)^2) = 2 - 2 + 1 = 1$ also. We have

$$E((X-1)^3) = E(X^3) - 3E(X^2) + 3EX - 1 = 6 - 6 + 3 - 1 = 2$$

so the skewness is 2. A $N(0,1)$ distribution is symmetric around its mean 0, so the skewness for it is 0.

In regard to skewness, the $U[0,1]$ distribution is more like a normal distribution than is the standard exponential distribution. (This gives an indication why the exponential distribution is easier to distinguish from the normal with a relatively small n in the Shapiro–Wilk test.)