Fact. If X and Y are independent random variables, X is $N(\mu, \sigma^2)$ and Y is $N(\nu, \tau^2)$, then $X + Y$ is $N(\mu + \nu, \sigma^2 + \tau^2)$.

Note. For any two random variables X and Y with finite means (independent or not), $E(X + Y) = EX + EY$. And, for any two random variables X and Y with $E(X^2) < \infty$, $E(Y^2) < \infty$, and $\mathrm{Cov}(X, Y) = 0$, for example, if X and Y are independent, we have $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$. So, if $X + Y$ has a normal distribution, it must have the given mean and variance.

Proof. Clearly, $X - \mu$ has a $N(0, \sigma^2)$ distribution and likewise $Y - \nu$ has a $N(0, \tau^2)$ distribution. If we can show that $X + Y - \mu - \nu$ has a $N(0, \sigma^2 + \tau^2)$ distribution, it will follow that $X + Y$ has a $N(\mu + \nu, \sigma^2 + \tau^2)$ distribution. So we can assume that $\mu = \nu = 0$.

Recall that $\exp(u)$ means e^u. The convolution of the $N(0, \sigma^2)$ and $N(0, \tau^2)$ densities, omitting the constant factor $A = 1/(2\pi\sigma\tau)$, is

$$h(t) = \int_{-\infty}^{+\infty} \exp\left[-\frac{(t-y)^2}{2\sigma^2} - \frac{y^2}{2\tau^2}\right] dy.$$

We can bring a factor $\exp(-t^2/(2\sigma^2))$ not depending on y outside the integral. The remaining expression inside the integral, whose exponential is taken, if put over a common denominator, becomes $-((\sigma^2 + \tau^2)y^2 - 2t\tau^2 y)/(2\sigma^2\tau^2)$. Completing the square, then subtracting a term to compensate, this becomes

$$\frac{-(\sigma^2 + \tau^2)[(y - v)^2 - v^2]}{2\sigma^2\tau^2}$$

where $v = \tau^2 t/(\sigma^2 + \tau^2)$. Then

$$\exp\left(\frac{(\sigma^2 + \tau^2)v^2}{2\sigma^2\tau^2}\right) = \exp\left(\frac{\tau^2 t^2}{2\sigma^2(\sigma^2 + \tau^2)}\right)$$

and we can bring this factor outside the integral because it doesn't depend on y. Then, the value of the remaining integral doesn't depend on v and so doesn't depend on t; it's a constant B depending on σ and τ, specifically, $B = \sqrt{2\pi}\sigma\tau/\sqrt{\sigma^2 + \tau^2}$. The function of t we wind up with, leaving aside such constant multiples, is

$$\exp\left[-\frac{t^2}{2\sigma^2}\left\{1 - \frac{\tau^2}{\sigma^2 + \tau^2}\right\}\right] = \exp\left[-\frac{t^2}{2(\sigma^2 + \tau^2)}\right].$$

This is just the function of t we wanted. The constant multiplier, taking the product of those that were left aside, is

$$AB = \frac{1}{2\pi\sigma\tau}\frac{\sqrt{2\pi}\sigma\tau}{\sqrt{\sigma^2 + \tau^2}} = \frac{1}{\sqrt{2\pi(\sigma^2 + \tau^2)}},$$

which is also the correct normalizing constant (as it would have to be, since the convolution of two probability densities is a probability density). The proof is complete.