A SIGNATURE FORMULA FOR MANIFOLDS WITH CORNERS OF CODIMENSION TWO

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Abstract. We present a signature formula for compact $4k$-manifolds with corners of codimension two which generalizes the formula of Atiyah, Patodi and Singer for manifolds with boundary. The formula expresses the signature as a sum of three terms, the usual Hirzebruch term given as the integral of an L-class, a second term consisting of the sum of the eta invariants of the induced signature operators on the boundary hypersurfaces with Atiyah-Patodi-Singer boundary condition (augmented by the natural Lagrangian subspace, in the corner null space, associated to the hypersurface) and a third 'corner' contribution which is the phase of the determinant of a matrix arising from the comparison of the Lagrangians from the different hypersurfaces meeting at the corners. To prove the formula, we pass to a complete metric, apply the Atiyah-Patodi-Singer formula for the manifold with the corners 'rounded' and then apply the results of our previous work [11] describing the limiting behaviour of the eta invariant under analytic surgery in terms of the b-eta invariants of the final manifold(s) with boundary and the eta invariant of a reduced, one-dimensional, problem. The corner term is closely related to the signature defect discovered by Wall [25] in his formula for nonadditivity of the signature. We also discuss some product formulæ for the b-eta invariant.

1. Introduction

In their celebrated paper [2], Atiyah, Patodi and Singer prove an index theorem for Dirac operators on compact manifolds with boundary, under the assumption that the metric is a product near the boundary and using a global boundary condition arising from the induced Dirac operator on the boundary. In their formula the index-defect, i.e. the difference between the analytic index and the interior term (which is the integral over the manifold of the form representing the appropriate absolute characteristic class), is expressed in terms of the the eta invariant of the boundary operator and the dimension of its null space. An important special case of this formula is an analytic expression for the signature of a compact $4k$ dimensional manifold with boundary. Their formula generalizes Hirzebruch's original signature theorem on compact, boundaryless manifolds, and was motivated by Hirzebruch's conjecture for the form of the signature-defect for Hirzebruch modular surfaces. This conjecture was proved later by Atiyah, Donnelly and Singer [1] and Müller [21], [20]. There have been a number of other generalizations of the APS theorem, notably those of Cheeger [7] and Stern [24], [23].

In a somewhat different direction, one of us [16] reinterpreted and systematized the proof of the original APS theorem using the calculus of b-pseudodifferential operators. One advantage of this method is the natural way the eta invariant appears

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in the course of the proof. This has led to families versions of this theorem [17], [18], generalizing those of Bismut and Cheeger [3]. This proof uses in a fundamental way the fact (used already by Atiyah, Patodi and Singer) that the global APS boundary condition is induced by restricting to sections which, if a cylindrical end is attached to each boundary of the manifold, extend as square-integrable solutions. This discussion also makes it seem more natural to interpret the APS index theorem as a result about complete 'b-metrics' on a manifold with boundary which slightly generalize the cylindrical end metrics. The same is true of the discussion below; it seems most natural to interpret our results in the realm of complete b-metrics on manifolds with corners and deduce the results for incomplete metrics as corollaries. Nevertheless we state our result first in the context of an incomplete metric of product type on a manifold with corners.

In this paper we generalize the signature formula of Atiyah, Patodi and Singer to the case of compact manifolds with corners up to codimension two. Topologically these are compact manifolds with boundary so we obtain a different formula for the same signature. This amounts to a decomposition of the eta invariant, which is exactly how we approach the problem using the results of [11]. As part of the definition of a manifold with corners we always assume that the boundary hypersurfaces are embedded. Thus, if X is a compact manifold with corners, near each boundary face it has a decomposition as the product of the boundary face and a neighbourhood of 0 in $[0,1)^k$, where k is the codimension of the boundary face. We will show that it is possible to choose consistent product decompositions near all boundary faces and a product-type metric on X which respects these decompositions. If $\eth$ is the Dirac operator associated to an Hermitian Clifford module with unitary Clifford connection for this metric then it can be expressed, near each boundary face, in terms of natural Dirac operators induced on the boundary faces; all are formally self-adjoint.

In the case of a compact manifold with boundary, Y, let Π^+ be the orthogonal projection onto the span of the eigenspaces, with positive eigenvalues, of the induced Dirac operator, $\eth_{\partial Y}$, on the boundary. Since ∂Y itself has no boundary this Dirac operator is self-adjoint with discrete spectrum. Let Π^0 be the orthogonal projection onto the null space of $\eth_{\partial Y}$. The null space of $\eth_Y$ acting on the Sobolev space $H^1(Y;E)$, where E is the Hermitian Clifford module, with Atiyah-Patodi-Singer boundary condition $\Pi^+(u\restriction\partial Y)=0$, is finite dimensional and the image of the map

$$\begin{aligned}\big\{u\in H^1(Y;E);\eth_Y u=0,\ \Pi^+(u\restriction\partial Y)=0\big\}\ni u\longmapsto\\ \Pi^0(u\restriction\partial Y)\in\big\{v\in L^2(\partial Y;E);\eth_{\partial Y}u=0\big\}\end{aligned}\tag{1}$$

is a Lagrangian subspace, which we denote Λ_Y, for the symplectic structure arising from the metric and the Clifford action of the normal variable to the boundary. Let $\Pi^{\mathrm{APS}}=\Pi^+ +\Pi^{\Lambda_Y}$ where Π^{Λ_Y} is the orthogonal projection, inside the null space of $\eth_{\partial Y}$, onto Λ_Y; we call this the augmented Atiyah-Patodi-Singer boundary operator (since in their work they use Π^+). Acting on the space

$$\big\{u\in H^1(Y;E);\Pi^{\mathrm{APS}}(u\restriction\partial Y)=0\big\}\tag{2}$$

the operator $\eth_Y$, which with this boundary condition we denote $\eth_{Y,\mathrm{APS}}$, is self-adjoint with discrete spectrum of finite multiplicity. As noted earlier, its nullspace consists of those spinors u which extend, if a cylindrical end is attached to each boundary, as L^2 solutions.

In case the Clifford module and connection are $\mathbb{Z}_2$-graded the index theorem of Atiyah, Patodi and Singer can be written

$$\operatorname{ind}(\eth_Y^+) = \int_Y \mathrm{AS} - \frac{1}{2}\eta(\eth_{\partial Y}). \tag{3}$$

Here AS is the index density constructed from the Clifford module and connection. The usual term involving the dimension of the null space of $\eth_{\partial Y}$ has been absorbed in the index, since this is defined with respect to the *augmented* Atiyah-Patodi-Singer projection. In the odd-dimensional case when the Clifford module is not assumed to be $\mathbb{Z}_2$-graded the eta invariant of $\eth_{Y,\mathrm{APS}}$, with augmented Atiyah-Patodi-Singer boundary condition, can be defined by the same prescription as in [2]; we shall denote it $\eta(\eth_{Y,\mathrm{APS}})$.

Theorem 1 (Signature Theorem)**.** *Let X be a compact $4k$-dimensional manifold with corners up to codimension two with a product-type metric specified. Let M_α, for $\alpha = 1, \dots, N$, be the boundary hypersurfaces and let $\eth_{\alpha,\mathrm{APS}}$ be the Dirac operator on M_α, induced by the signature operator on X, with the augmented Atiyah-Patodi-Singer boundary condition just discussed. Then the signature can be written*

$$\operatorname{sign}(X) = \int_X \mathcal{L}(p) - \frac{1}{2}\left(\sum_{\alpha=1}^{N} \eta(\eth_{\alpha,\mathrm{APS}}) + \frac{1}{i\pi}\operatorname{tr} P_\Lambda\right) \tag{4}$$

The first term is the analogue of Hirzebruch's formula in the boundaryless case. The eta invariants have been described briefly above. The third term on the right in (4) may be described as the 'corner correction term', though it is *not* solely determined by the corner; it depends on the collection of Lagrangian subspaces given by (1) for each boundary hypersurface of X. We proceed to describe the corner term more precisely.

Let M_α, $\alpha = 1, \dots, k$, be an ordering of the codimension-one boundary components of X. The intersections $H_{\alpha\beta} \equiv M_\alpha \cap M_\beta$ are the codimension-two boundaries, i.e. the corners. Note that these $H_{\alpha\beta}$ need not be connected. Let $\eth_{\alpha\beta}$ denote the Dirac operator on $H_{\alpha\beta}$ induced either by $\eth_\alpha$ on M_α or $\eth_\beta$ on M_β (they are the same up to sign). For each α define

$$V_\alpha = \bigoplus_\beta \operatorname{null}(\eth_{\alpha\beta}), \tag{5}$$

the sum of the null spaces of the Dirac operators on the boundary hypersurfaces of M_α. The Lagrangian subspace (1) for $Y = M_\alpha$ is denoted $\Lambda_\alpha \subset V_\alpha$. Let Π_α, $\Pi_\alpha^\perp$ be the orthogonal projection onto Λ_α, resp. $\Lambda_\alpha^\perp$ with respect to the natural L^2 metrics induced on each $\operatorname{null}(\eth_{\alpha\beta})$, and let $S_\alpha \equiv \Pi_\alpha - \Pi_\alpha^\perp = 2\Pi_\alpha - \mathrm{Id}$ be the orthogonal reflection across Λ_α. Define

$$V = \bigoplus_{\alpha<\beta} \operatorname{null}(\eth_{\alpha\beta}),$$

the direct sum of the null spaces of all the Dirac operators. It is important to note that the direct sum of the V_α over all α is isomorphic to $V \oplus V$, not V. Define S_L and $S_R : V \to V$ by

$$S_L(v_{\alpha\beta}) = S_\alpha v_{\alpha\beta}$$
$$S_R(v_{\alpha\beta}) = S_\beta v_{\alpha\beta}$$

on $v_{\alpha\beta} \in \operatorname{null}(\eth_{\alpha\beta})$, $\alpha < \beta$. These decompose according to the $\mathbb{Z}_2$ grading defined by Clifford multiplication by either of the normal variables to ∂M_α. Thus we may express

$$S_L = \begin{pmatrix} 0 & S_{L,-} \\ S_{L,+} & 0 \end{pmatrix}, \ S_R = \begin{pmatrix} 0 & S_{R,-} \\ S_{R,+} & 0 \end{pmatrix} \tag{6}$$

with respect to this grading. The matrix in (4) is then given by

$$P_\Lambda = \log' GA$$

where

$$\begin{gathered} A = \begin{pmatrix} \mathrm{Id} & -S_{L,-} \\ -S_{R,+} & \mathrm{Id} \end{pmatrix}, \\ G \text{ is the generalized inverse of } \begin{pmatrix} \mathrm{Id} & -S_{R,-} \\ -S_{L,+} & \mathrm{Id} \end{pmatrix}, \end{gathered} \tag{7}$$

and for a diagonalizable matrix (such as GA is shown to be in Lemma 2 below) $\log'(M) = \log(M + \operatorname{proj} \operatorname{null} M)$ is given by the standard branch of the logarithm.

If the signature complex is twisted by a flat Hermitian bundle G with flat connection the discussion above still holds for the twisted signature with only notational modifications. The corresponding result is

$$\operatorname{sign}(X, G) = \int_X \mathcal{L} \cdot \operatorname{Ch}(G) - \frac{1}{2} \left(\sum_{\alpha=1}^{N} \eta(\eth_{\alpha,\mathrm{APS}}) + \frac{1}{i\pi} \operatorname{tr} P_\Lambda \right). \tag{8}$$

where the terms arise in the same way from the corresponding twisted Dirac operators. Of course $\operatorname{Ch}(G)$ is simply the rank of G here. Since the nullspace of the signature operator twisted by a flat bundle depends only on the topology and the representation of the fundamental group of X, the deformation arguments in section 5 hold just as in the untwisted case. When G is no longer assumed to be flat, or indeed when more general compatible Dirac operators $\eth$ are considered, this nullspace is no longer stable under perturbations. In this case we obtain only an $\mathbb{R}/\mathbb{Z}$ result:

Theorem 2 (General case, $\mathbb{R}/\mathbb{Z}$)**.** *Let X be a manifold of dimension 2ℓ with corners up to codimension two. Let $\eth$ be the Dirac operator associated to a $\mathbb{Z}_2$-graded Hermitian Clifford module E with unitary Clifford connection over X. Using the notation of Theorem 1 for the induced Dirac operators with augmented Atiyah-Patodi-Singer boundary conditions on the boundary hypersurfaces of X and for the corner term, and letting AS denote the index density associated to $\eth$, we have*

$$\int_X \mathrm{AS} \equiv \frac{1}{2} \left(\sum_{\alpha=1}^{N} \eta(\eth_{\alpha,APS}) + \frac{1}{i\pi} \operatorname{tr} P_\Lambda \right) \quad \mod \mathbb{Z}.$$

The proof is the same as before, except now the small eigenvalues can no longer be controlled in the deformation process. In the particular case in which $\eth$ is the signature operator twisted by an Hermitian bundle G which is no longer assumed to be flat, the left hand side of (2) is simply $\int_X \mathcal{L}(p) \operatorname{Ch}(G)$.

The complete metrics on the interior of X that we consider are the product-type b-metrics. These will be defined precisely in the next section, but they stand in the same relationship to the incomplete product metrics we have been considering as complete cylindrical end metrics do to incomplete metrics on a manifold with boundary with a product decomposition near the boundary. In our opinion, the most natural class of metrics to consider are actually *exact* b-metrics (see [16]), which converge exponentially to these product metrics. The proof for these involves some extra technicalities but no major obstacles. They will not be discussed further in this paper.

In a recent paper, Müller [19] has proved an L^2-index formula for compatible Dirac operators on manifolds with corners of codimension two relative to product b-metrics. Although some of his results require no extra hypotheses, his index formula applies only when the induced Dirac operators on the corners are invertible. This case is not too different, analytically, from the L^2 version of the Atiyah-Patodi-Singer theorem on cylindrical-end manifolds. In particular, there are no contributions from the corners. Unfortunately, it rarely applies to the twisted signature operator since invertibility of the operators on the corners translates to the vanishing of twisted cohomology of each $H_{\alpha\beta}$. Nonetheless, he proves, without this nondegeneracy hypothesis, that the L^2 index is always finite (even though the operator is not always Fredholm), and in particular for the signature operator that this index equals the signature of X.

A product b-metric g on X induces cylindrical-end metrics on each of the codimension one boundaries M_α, as well as metrics on each corner. The signature operator $\eth_\alpha$ on M_α has a unique self-adjoint extension on $L^2(M_\alpha)$, but because of the noninvertibility of the corner operators $\eth_{\alpha\beta}$ it is never Fredholm; it has a band of finite-multiplicity continuous spectrum reaching to zero. Because of this, the spectrum of $\eth_X$ is quite complicated near zero, making the general L^2-index theorem rather more difficult to prove. We shall circumvent this by considering the Atiyah-Patodi-Singer formula for a family of manifolds with boundary, X_ϵ, which exhaust X, and studying the behaviour of all terms in the formula as ϵ tends to zero.

The formula for this case is quite similar to the one above:

Theorem 3 (Signature Theorem, complete metric)**.** *Let X be a compact $4k$-dimensional manifold with corners up to codimension two with a (product-type) exact b-metric specified. Let M_α, for $\alpha = 1, \dots, N$, be the boundary hypersurfaces and let $\eth_\alpha$ be the Dirac operator on M_α, induced by the signature operator on X. Then the signature can be written*

$$\operatorname{sign}(X) = \int_X \mathcal{L} - \frac{1}{2}\left(\sum_{\alpha=1}^{N} {}^b\eta(\eth_\alpha) + \frac{1}{i\pi}\operatorname{tr} P_\Lambda\right). \tag{9}$$

As in Theorem 2, there is a similar extension to general compatible Dirac operators provided one interprets the formula mod $\mathbb{Z}$.

The b-eta invariants here are defined by replacing the trace over M_α in the heat equation definition by a Hadamard-regularized trace, called the b-trace, as introduced in [16]. Although these are defined with respect to the complete metrics, it is shown in [20] that they coincide with the previous expressions $\eta(\eth_\alpha, \Pi^{\mathrm{APS}})$. Thus all three terms of the right hand side of (9) coincide with the corresponding terms in (4).

Many of the results presented here are prefigured in Bunke [5, 4]. He also noted the possibility of 'rounding the corners' of a manifold with corners in order to apply the index theorem for manifolds with boundary. However, he did not address the problem that the APS index theorem cannot be applied directly to the rounded manifold since the metric is no longer a product near the boundary after rounding. Indeed, in principle there will be an extra local boundary term depending on the second fundamental form [9]. In this paper we treat this limit carefully, hence justifying Bunke's informal arguments. We also discuss the case of several boundary hypersurfaces, obtain a formula for the 'corner correction term' in terms of finite dimensional determinants and analyse the case when X is a product of two manifolds with boundary, in which case the equality (9) is satisfied in a rather curious way.

In the next two sections we discuss the class of metrics more carefully and set notational conventions. After that, the precise rounding of the corners will be described, followed by a discussion of the scattering Lagrangians. The proof of Theorem 3 is then easily obtained. We next discuss Wall's formula (closely following [5], cf. also the earlier article of Rees [22]) for the nonadditivity of the signature. In the final section we examine closely the case where X is the product of two manifolds with boundary, and we prove product formulæ for the b-eta invariant. These formulæ were obtained by Müller [19] independently.

2. Metrics

For simplicity we restrict attention to the direct subject of this paper, manifolds with corners up to codimension two, although the discussion here easily extends to the general (higher codimension) case. First we show the existence of metrics which are of 'product-type' near the boundary, both in the sense of incomplete and complete metrics.

Let X be a compact manifold with corners up to codimension two with M_α, for $\alpha = 1, \dots, N$, the boundary hypersurfaces. Each M_α has a neighbourhood $O_\alpha \subset X$ with a product decomposition given by a diffeomorphism

$$F_\alpha : O_\alpha \longrightarrow [0, \epsilon_\alpha) \times M_\alpha \tag{10}$$

for some $\epsilon_\alpha > 0$. Here, if x is the variable in $[0, \epsilon_\alpha)$ then $F_\alpha^* x$ can be taken as the restriction to O_α of a defining function x_α for M_α. We shall show that it is possible to choose these decompositions consistently in their intersections in the sense that

$$\begin{aligned} F_\alpha^*(x_\beta \restriction M_\alpha) = x_\beta,\ F_\beta^*(x_\alpha \restriction M_\beta) = x_\alpha \text{ on } O_\alpha \cap O_\beta \\ F_{\alpha\beta} \equiv F_\alpha \circ (F_\beta \restriction M_\alpha) = F_\beta \circ (F_\alpha \restriction M_\beta) \text{ on } O_\alpha \cap O_\beta. \end{aligned} \tag{11}$$

Indeed the decomposition (10) is determined by a vector field V_α which is F_α-related to the coordinate vector field $\partial/\partial x$ in the first factor of the image. Clearly $V_\alpha x_\alpha = 1$ determines x_α near M_α and F_α is the inverse of $\exp(x_\alpha V_\alpha)$ on any surface on which x_α is constant. Since V_α is just a vector field which is strictly inward-pointing across M_α and tangent to the other boundary hypersurfaces it can be defined locally and globalized by summing over a partition of unity. To ensure (11) it is necessary and sufficient that the vector fields for different boundary hypersurfaces commute near the intersections of the hypersurfaces. Such vector fields can be constructed directly near the corners, using the collar neighbourhood theorm, and then extended to neighbourhoods of the boundary hypersurfaces. Thus a consistent product decomposition exists.

Using such a product decomposition near the boundary faces an incomplete metric of product type can be constructed. That is, there is a smooth nondegenerate metric which takes the form

$$dx_\alpha^2 + \tilde{F}_\alpha^* h_\alpha \text{ near } M_\alpha,\ dx_\alpha^2 + dx_\beta^2 + \tilde{F}_{\alpha\beta}^* h_{\alpha\beta} \text{ near } H_{\alpha\beta} \equiv M_\alpha \cap M_\beta \tag{12}$$

where $\tilde{F}_a$ and $\tilde{F}_{\alpha\beta}$ are the second components of the maps in (10), and h_α, $h_{\alpha\beta}$ are metrics on M_α, $H_{\alpha\beta}$, respectively. By (11), h_α is a product near the boundary. It is a metric of this type which is involved in Theorem 1.

As well as such incomplete product-type metrics we can consider metrics on the interior of X satisfying

$$\begin{aligned} g &= \frac{dx_\alpha^2}{x_\alpha^2} + \tilde{F}_\alpha^* h_\alpha \text{ on } O_\alpha \\ g &= \frac{dx_\alpha^2}{x_\alpha^2} + \frac{dx_\beta^2}{x_\beta^2} + \tilde{F}_{\alpha\beta}^* h_{\alpha\beta} \text{ on } O_\alpha \cap O_\beta \end{aligned} \tag{13}$$

with h_α the induced exact b-metric on M_α and $h_{\alpha\beta}$ the induced metric on $H_{\alpha\beta}$. This type of metric we call a b-metric of product type; it is complete on X°, as can be seen by the change of variables $t_\alpha = -\log x_\alpha$.

The geometry associated to b-metrics is discussed in some detail in [16], at least for manifolds with boundaries. The discussion may be readily extended to the case of corners, cf. [15]. In particular, the b-tangent and b-cotangent bundles, ${}^bTX$ and ${}^bT^*X$, and the b-form bundles, ${}^b\bigwedge^* X$ are all well defined smooth vector bundles over X. We suppose, from now on, that $n = 2\ell$ is even. Define, for each p, the map

$$\tau = i^{p(p-1)+\ell} * : {}^b\bigwedge^p X \longrightarrow {}^b\bigwedge^{n-p} X;$$

this is an involution, cf. [2], [9]. Then ${}^b\bigwedge^* X$ splits into the direct sum

$${}^b\bigwedge^* X = {}^b\bigwedge^+ X \oplus {}^b\bigwedge^- X$$

of eigenspaces for τ with eigenvalues ± 1.

The deRham differential and its adjoint induce b-differential operators of order one [16], and we define

$$\eth = d + \delta.$$

Since $\eth$ and τ anticommute, $\eth$ induces maps on the metric Sobolev spaces

$$\eth_\pm : {}^bH^1(X; {}^b\bigwedge^\pm X) \longrightarrow {}^bL^2(X; {}^b\bigwedge^\mp X).$$

When X has no boundary these operators are Fredholm and the index of $\eth_+$ is the signature of X. If X has boundary (but no corners), and g is an exact b-metric, then $\eth_+$ is never Fredholm; its continuous spectrum extends to zero because the induced signature operator on the boundary, $\eth_{\partial X}$, is never invertible. Again, its L^2-index is the signature [16], where for manifolds with boundary, or corners, the signature is defined as the number of positive eigenvalues minus the number of negative eigenvalues for the intersection form on the image of relative cohomology in absolute cohomology in the middle degree. By [19], the L^2-index still yields the signature, even when X has corners of codimension two.

3. Rounding the corners

Let X be a manifold with corners of codimension two, with a b-metric of product type. We shall define a particular family of manifolds with boundary, X_ϵ, with (incomplete) metrics g_ϵ, depending on a parameter ϵ, such that as $\epsilon \to 0$ the X_ϵ exhaust X.

Let x_α be boundary defining functions in terms of which the metric has a product decomposition as in (13). Since the defining functions can be scaled by positive constants and their values are irrelevant except near the boundary hypersurfaces they define we can assume that $O_\alpha = \{x_\alpha < 2\}$ for each α and that $x_\gamma > 2$ on $O_\alpha \cap O_\beta$ if $\gamma \neq \alpha, \beta$. Choose $\phi \in \mathcal{C}^\infty(\mathbb{R})$ with $\phi(t) = t$ in $t \leq \frac{1}{2}$, $\frac{1}{2} < \phi(t) < 1$ in $\frac{1}{2} < t < 1$ and $\phi(t) = 1$ in $t \geq 1$. Thus $\{\phi(x_\alpha) < 1\} = \{x_\alpha < 1\} \subset O_\alpha$. Now, consider

$$r = \prod_{\alpha=1}^{N} \phi(x_\alpha) \tag{14}$$

which is a 'total boundary defining function' in the sense that it is a product over boundary defining functions. We set

$$X_\epsilon = \left\{ r \geq \epsilon^2 \right\}, \ 0 < \epsilon \leq \epsilon_0 < 2^{-N}. \tag{15}$$

If $r(p) = \epsilon^2$ then, for at least one α, $\phi(x_\alpha) < \frac{1}{2}$, hence $x_\alpha(p) < \frac{1}{2}$, so that $p \in O_\alpha$. The properties of the x_α ensure that $p \in O_\beta$ for at most one other β.

Clearly X_ϵ is a manifold with boundary, with $\partial X_\epsilon = \{r = \epsilon^2\}$, and the X_ϵ exhaust X as ϵ tends to zero. ∂X_ϵ decomposes into three subsets:

$$\begin{aligned}\{r = \epsilon^2\} = \bigcup_{\alpha=1}^{N} \{x_\alpha = \epsilon^2, x_\beta \geq 1, \beta \neq \alpha\} \\ \cup \bigcup_{\alpha,\beta=1}^{N} \{x_\alpha = \epsilon^2/\phi(x_\beta), \frac{1}{2} \leq x_\beta \leq 1\} \\ \cup \bigcup_{\alpha,\beta=1}^{N} \{x_\alpha x_\beta = \epsilon^2, x_\alpha, x_\beta \leq \frac{1}{2}\}.\end{aligned} \tag{16}$$

Consider the behaviour of the metric near the boundary of X_ϵ. In the first region in (16) the metric is certainly a product near ∂X_ϵ, as follows from (13), using $\log x_\alpha$ as the normal variable near the boundary. Similarly, near the third region in (16) the metric in (13) can be written

$$(d \log x_\alpha)^2 + (d \log x_\beta)^2 + \tilde{F}^*_{\alpha\beta} h_{\alpha\beta} = \frac{1}{2}(d(\log x_\alpha x_\beta))^2 + \frac{1}{2}(d(\log \frac{x_\alpha}{x_\beta}))^2 + \tilde{F}^*_{\alpha\beta} h_{\alpha\beta} \tag{17}$$

which agains shows that the metric is of product type. In the intermediate region in (16) the metric is not of product type in the usual sense. However, each of the sets in this second region is contained in a compact subset of some $O_\alpha \cap O_\beta$. It follows that the metric decomposes in this region into the product of a metric on a subset of $\mathbb{R}^2$ and one on the corner $H_{\alpha\beta}$.

We next discuss the restriction of the metric g to ∂X_ϵ, which we denote h_ϵ. We shall show that this family of restricted metrics on this smooth compact manifold without boundary undergoes, as $\epsilon \to 0$, a surgery degeneration as treated in [11] and [14]. It is only necessary to discuss the degeneration of h_ϵ in $O_\alpha \cap O_\beta$ since away from there the induced metric is independent of ϵ. In the region where $x_\alpha, x_\beta \le \frac{1}{2}$, the metric is of the form (17). Setting $t = x_\alpha/x_\beta$ reduces the metric to $2dt^2/t^2 + \tilde{F}_{\alpha\beta}h_{\alpha\beta}$. Now, we change variables again on this hypersurface, to $s = (x_\alpha - x_\beta)/2$. Since $x_\beta = \epsilon^2/x_\alpha$ here, we see that $2s = x_\alpha - \epsilon^2/x_\alpha$. Solving for x_α yields $x_\alpha = \frac{1}{2}(2s + \sqrt{4s^2 + 4\epsilon^2})$. We have

$$t = x_\alpha^2/\epsilon^2 = (2s + \sqrt{4s^2 + 4\epsilon^2})^2/4\epsilon^2.$$

Hence

$$dt = 2\frac{(s + \sqrt{s^2 + \epsilon^2})^2}{\epsilon^2\sqrt{s^2 + \epsilon^2}}\, ds,$$

and so

$$\frac{dt}{t} = 2\frac{ds}{\sqrt{s^2 + \epsilon^2}}.$$

Reinserting the corner variables, we conclude that, in the third region (16),

$$g_\epsilon = 8\frac{ds^2}{s^2 + \epsilon^2} + h_{\alpha\beta} \tag{18}$$

for some s- and ϵ-independent metric $h_{\alpha\beta}$. This is precisely the form of a metric family undergoing analytic surgery degeneration, as defined and studied in [11] and [14].

We have omitted the discussion of the second region in (16). Since $x_\beta > \frac{1}{2}$ this attaches to the metric (18) in a region where s is bounded away from 0, uniformly as $\epsilon \downarrow 0$. The effect of this is to make the additional term $h_{\alpha\beta}$ in (18) depend parametrically on ϵ away from $s = 0$. Such a perturbation affects the arguments of [11] only in a trivial way.

4. The corner term

In this section we discuss the form of the corner correction term, for a general Dirac operator (associated to a $\mathbb{Z}_2$-graded Hermitian Clifford module with unitary Clifford connection) on an even-dimensional compact manifold with corners (even though, in this paper, we only prove the formula for the twisted signature operator).

Recall from (1) in the introduction the Lagrangian subspaces $\Lambda_\alpha \subset \mathcal{H}^*(\partial\mathcal{M}_\alpha)$ associated to each codimension one boundary of X. The definition (1) applies only for the incomplete product metric on M_α, but it is clear that the Lagrangian Λ_α is independent of the length of the cylinder, hence could also be defined as the set of limiting values of solutions of $\eth_\alpha\phi = 0$ in

$$\mathcal{H}^*(\partial M_\alpha) = \bigoplus_{\beta \text{ s.t. } M_\beta \cap M_\alpha \neq \emptyset} \mathcal{H}^*(H_{\alpha\beta}).$$

with respect to a complete product b-metric on M_α. For this reason we call Λ_α the scattering Lagrangian associated to M_α.

The term in the signature formula coming from the corners depends only on these Lagrangians. This corner term is the eta invariant for a one-dimensional problem for ΓD_s, $D_s = -id/ds$ acting on functions valued in the vector space

$$V = \bigoplus_{\alpha<\beta} \mathcal{H}^*(H_{\alpha\beta}).$$

Here $\Gamma = \oplus_\alpha \gamma_\alpha$ where γ_α is the bundle map on V_α induced by Clifford multiplication by the normal variable at the boundary of M_α. The interval $[-1, 1]$ can be viewed formally as representing the links between M_α and M_β, with multiplicity the dimension of the cohomology of $H_{\alpha\beta}$. (In [11] and [14] this formal link is explained geometrically.) Define $\Lambda = \oplus_\alpha \Lambda_\alpha \subset V \oplus V$, a Lagrangian in $V \oplus V$ with respect to the symplectic form determined by Γ. We consider the operator ΓD_s acting on functions v with values in V with the boundary conditions

$$(v(-1), v(1)) \in \Lambda. \tag{19}$$

Lemma 1. *The operator ΓD_s acting on functions $\Phi : [-1, 1] \longrightarrow V$ is symmetric and essentially self-adjoint for this boundary condition.*

It is the eta invariant of this problem, which we denote $\eta(\Lambda)$, which appears in the index formula as the contribution from the corners. We emphasize that it depends only on the Lagrangians Λ_α. Below we will compute it explicitly in terms of these Lagrangians; it will be expressed as the logarithm of a quotient of determinants of two finite dimensional matrices constructed from the Λ_α.

Before doing this, however, we discuss briefly a slightly different way of viewing this corner term $\eta(\Lambda)$ which has the virtue of reflecting the geometry of X more closely. We first define a directed graph $\mathcal{G}$ as follows. Each vertex v_α of $\mathcal{G}$ will correspond to a codimension one boundary M_α of X. We associate an edge $e_{\alpha\beta}$ to each corner $H_{\alpha\beta}$ provided $M_\alpha \cap M_\beta$ is nonempty. (If $M_\alpha \cap M_\beta$ has several components, we let $\mathcal{G}$ have that many edges connecting these two vertices.) We identify each $e_{\alpha\beta}$ with the interval $[-1, 1]$ with orientation according to the (arbitrary, but fixed) ordering of the codimension one boundary faces; thus the end of $e_{\alpha\beta}$ at v_α will correspond to the end $s = -1$ of $[-1, 1]$ provided $\alpha < \beta$.

Over each edge $e_{\alpha\beta}$ we consider the trivial vector bundle $\mathcal{H}^*(H_{\alpha\beta})$. On sections $s_{\alpha\beta}$ of this vector bundle the Dirac operator $\gamma_\alpha D_s$ is defined. We define a smooth section of the graph to be a collection of smooth sections $\{s_{\alpha\beta}\}$ on each edge of the graph. The Dirac operator $\eth_{\mathcal{G}}$ on the graph is defined to be the operator $\gamma_\alpha D_s$ on each edge, with a boundary condition at each vertex. For vertex v_α we have a value $s_{\alpha\beta}(v_\alpha) \in \mathcal{H}^*(H_{\alpha\beta})$ for each edge $e_{\alpha\beta}$. The collection of these values lies in $\oplus_\beta \mathcal{H}^*(H_{\alpha\beta}) = V_\alpha$. The boundary condition for $\eth_{\mathcal{G}}$ is that this lie in the Lagrangian subspace Λ_α for each α. Then it is not hard to see that this operator is symmetric and that its eta invariant, $\eta(\eth_{\mathcal{G}})$, is identical to the eta invariant $\eta(\Lambda)$ defined above.

When there are only two codimension one boundary components, M_1 and M_2, and a single connected corner H, then $\mathcal{G}$ is simply the interval $[-1, 1]_s$, and the boundary conditions for ΓD_s are simply that $v(-1) \in \Lambda_1$ and $v(+1) \in \Lambda_2$. This was the situation considered in [11], and the associated eta invariant $\eta(\Lambda)$ was computed directly in terms of determinants of matrices associated to the Λ_α. We generalize this formula now.

To state the result, recall the notation introduced in the first section. The direct sum of the reflections $S_\alpha : V_\alpha \to V_\alpha$ across the Lagrangian Λ_α is $S : V \oplus V \to V \oplus V$. We define two operators S_L and S_R from V to V by

$$S_L\Big(\bigoplus_{\alpha<\beta} v_{\alpha\beta}\Big) = \bigoplus_{\alpha<\beta} S_\alpha(v_{\alpha\beta}) \tag{20}$$

$$S_R\Big(\bigoplus_{\alpha<\beta} v_{\alpha\beta}\Big) = \bigoplus_{\alpha<\beta} S_\beta(v_{\alpha\beta}). \tag{21}$$

S is the reflection across $\Lambda = \oplus_\alpha \Lambda_\alpha$, and interchanges the ± 1 eigenspaces of $\Gamma = \oplus \gamma_\alpha$ (γ_α is the Clifford action of the normal to the boundary on V_α). Since Γ is diagonal with respect to the decomposition of V into the nullspaces of the $\eth_{\alpha\beta}$, the matrices S_L and S_R also interchange the ± 1 eigenspaces of Γ, and therefore both matrices decompose into the off-diagonal block form (6). Define $\overline{S} = S_L + S_R$; $\overline{S}$ may also be expressed directly in terms of S via the addition map $T : V \oplus V \to V$ defined by $(\Upsilon', \Upsilon'') \mapsto \Upsilon' + \Upsilon''$:

$$\overline{S}(\Upsilon) = T \circ \big(S(\Upsilon, \Upsilon)\big). \tag{22}$$

Finally recall the definitions of the matrices G and A in (7). With this notation, we may state our result.

Lemma 2. *The matrix GA is diagonalizable, so P_Λ is defined, and the eta invariant $\eta(\Lambda)$ satisfies*

$$\eta(\Lambda) = \frac{1}{i\pi} \operatorname{tr} P_\Lambda. \tag{23}$$

Proof. We first observe that the spectrum of ΓD_s is π-periodic; this follows by direct inspection. The eta invariant of an operator with π-periodic spectrum is given by the sum

$$\sum \left(1 - \frac{2z_j}{\pi}\right)$$

over eigenvalues $z_j \in (0, \pi)$. We first show how to obtain any of these eigenvalues in terms of the matrices G and A, and after that show how to eliminate the zero eigenvalues.

If $z \in [0, \pi)$ is an eigenvalue, then the eigenfunction v can be written as a sum over all α and β (such that M_α meets M_β nontrivially) of functions of the form

$$e^{izs}\phi_{\alpha\beta} + e^{-izs}\psi_{\alpha\beta},$$

where $\phi_{\alpha\beta}$ and $\psi_{\alpha\beta}$ lie in the $+1$ and -1 eigenspaces of Γ, respectively. The boundary conditions require that at each M_β, the boundary value of the eigenfunction lies in Λ_β. Notice that with our conventions, the boundary $H_{\alpha\beta}$ in ∂M_β corresponds to $s = 1$ for $\alpha < \beta$, while for $\gamma > \beta$ the boundary $H_{\beta\gamma}$ in ∂M_β corresponds to $s = -1$. The boundary condition on V_β is

$$\Pi_\beta^\perp\big(\bigoplus_{\alpha<\beta} e^{iz}\phi_{\alpha\beta} \oplus \bigoplus_{\beta<\gamma} e^{-iz}\phi_{\beta\gamma} + \bigoplus_{\alpha<\beta} e^{-iz}\psi_{\alpha\beta} \oplus \bigoplus_{\beta<\gamma} e^{iz}\psi_{\beta\gamma}\big) = 0.$$

This can be rewritten

$$\Pi_\beta^\perp\big(\bigoplus_{\alpha<\beta} \phi_{\alpha\beta} \oplus \bigoplus_{\beta<\gamma} \psi_{\beta\gamma}\big) = -e^{-2iz}\Pi_\beta^\perp\big(\bigoplus_{\beta<\gamma} \phi_{\beta\gamma} \oplus \bigoplus_{\alpha<\beta} \psi_{\alpha\beta}\big) \text{ on } V_\beta. \tag{24}$$

Now apply Γ and use the identity $\Gamma\Pi_\beta^\perp = \Pi_\beta\Gamma$. This gives

$$\Pi_\beta\big(\bigoplus_{\alpha<\beta} \phi_{\alpha\beta} \oplus \bigoplus_{\beta<\gamma} -\psi_{\beta\gamma}\big) = -e^{-2iz}\Pi_\beta\big(\bigoplus_{\beta<\gamma} \phi_{\beta\gamma} \oplus \bigoplus_{\alpha<\beta} -\psi_{\alpha\beta}\big) \text{ on } V_\beta.$$

Adding and subtracting these two equations gives

$$\begin{aligned} \bigoplus_{\alpha<\beta} \phi_{\alpha\beta} \oplus -S_\beta \bigoplus_{\beta<\gamma} \psi_{\beta\gamma} &= -e^{-2iz}\big(\bigoplus_{\beta<\gamma} \phi_{\beta\gamma} \oplus -S_\beta \bigoplus_{\alpha<\beta} \psi_{\alpha\beta}\big), \\ -S_\beta \bigoplus_{\alpha<\beta} \phi_{\alpha\beta} \oplus \bigoplus_{\beta<\gamma} \psi_{\beta\gamma} &= -e^{-2iz}\big(-S_\beta \bigoplus_{\beta<\gamma} \phi_{\beta\gamma} \oplus \bigoplus_{\alpha<\beta} \psi_{\alpha\beta}\big). \end{aligned} \tag{25}$$

on V_β. Next, sum over β. To write this in a convenient form let Φ denote the vector consisting of all the $\phi_{\alpha\beta}$ and similarly for Ψ. Then in the notation of (6), (25) becomes

$$\begin{pmatrix} \mathrm{Id} & -S_{L,-} \\ -S_{R,+} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = -e^{-2iz} \begin{pmatrix} \mathrm{Id} & -S_{R,-} \\ -S_{L,+} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix}. \tag{26}$$

For notational convenience, denote the matrices appearing on the left and right hand sides of (26) by A and B, respectively, and let Υ denote the column vector with entries Φ and Ψ. Also, let G be the generalized inverse of B which vanishes on null(B) and inverts B on the orthogonal complement of null(B). Multiplying both sides of (26) by G shows that if Υ corresponds to the eigenvalue $z \in [0, \pi)$ of ΓD_s as above, then

$$(GA)\Upsilon = -e^{-2iz}\Pi^{\perp}_{\mathrm{null}(B)}\Upsilon.$$

To continue, we must understand the relationship between the null space of B, and the eigenvalue $z = 0$ and its corresponding eigenvector Υ. First note that regardless of the value of z, if $B\Upsilon = 0$, then by (26) $A\Upsilon = 0$ as well.

We now show the equivalence of the following four conditions:

(i) $z = 0$
(ii) $(A + B)\Upsilon = 0$
(iii) $S(\Upsilon, \Upsilon) = (\Upsilon, \Upsilon)$
(iv) $A\Upsilon = B\Upsilon = 0$.

The equivalence of (i) and (iii) is clear. For, any element $\Upsilon = (\Upsilon_{\alpha\beta})$ of the null space of ΓD_s is, by definition, constant along the edges, and at each vertex v_β the sum $\sum_{\alpha<\beta} \Upsilon_{\alpha\beta} + \sum_{\beta<\gamma} \Upsilon_{\beta\gamma}$ is an element of Λ_β. But this corresponds exactly to the condition (iii). The implication (iv) $\Rightarrow$ (ii) is also obvious. Also, if $z = 0$, so (i) is satisfied, then immediately from (26) we get (ii). It remains only to demonstrate the implications (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv).

For the first of these, write the condition $(A + B)\Upsilon = 0$ as

$$\begin{pmatrix} 2 \cdot \mathrm{Id} & -\overline{S}_- \\ -\overline{S}_+ & 2 \cdot \mathrm{Id} \end{pmatrix} \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{27}$$

or equivalently, $\overline{S}\Upsilon = 2\Upsilon$. We shall show that this implies that (Υ, Υ) is in the intersection of Λ and the diagonal of $V \oplus V$, which is the same as (iii). To see this, let $S(\Upsilon, \Upsilon) = (\Sigma', \Sigma'')$; we must show that $\Sigma' = \Sigma'' = \Upsilon$. Since S is orthogonal, $|(\Sigma', \Sigma'')|^2 = |(\Upsilon, \Upsilon)|^2$; also, by the definition of $\overline{S}$, $\Sigma' + \Sigma'' = 2\Upsilon$. Subtract the square of the norm of this second equation from twice the first equation to get

$$|\Sigma'|^2 - 2\Sigma' \cdot \Sigma'' + |\Sigma''|^2 = |\Sigma' - \Sigma''|^2 = 0.$$

This proves the claim.

To prove the final implication, regard S as the direct sum $S_L \oplus S_R : V \oplus V \to V \oplus V$ and write S in block diagonal form with respect to this splitting, which in turn splits into Γ-eigenspaces

$$\begin{pmatrix} 0 & S_{L,-} & 0 & 0 \\ S_{L,+} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{R,-} \\ 0 & 0 & S_{R,+} & 0 \end{pmatrix}.$$

Applying this to (Υ, Υ) shows that $A\Upsilon = B\Upsilon = 0$, as desired.

Combining this chain of equivalences with (26), we conclude that

$$(GA)\Upsilon = \begin{cases} 0, \text{ if } z = 0 \\ -e^{-2iz}\Upsilon, \text{ if } z \neq 0. \end{cases} \tag{28}$$

It remains only to show that Υ runs over a complete set of eigenvalues of GA. The dimension of GA is $K \equiv \dim V$. The number of eigenvectors Υ is the number of eigenvalues, counted with multiplicity, in the interval $[0,\pi)$ of the first order system ΓD_s, which acts on functions valued in V, a vector space of dimension K. By Weyl's law the number of eigenvalues in the range $[0,\lambda]$ is $\frac{K\lambda}{\pi} + O(1)$ as $\lambda \to \infty$. Since the spectrum of ΓD_s is π-periodic, ΓD_s must have exactly K eigenvalues in the interval $[0,\pi)$. Thus all eigenvalues of GA are accounted for. In particular, GA is diagonalizable.

In conclusion, we have shown that the eigenvalues z of ΓD_s in $(0,\pi)$ correspond bijectively, with multiplicities, to the eigenvalues $-e^{-2iz}$ of GA. Summing over all such z, and writing $-e^{-2iz} = e^{i\pi(1-2z/\pi)}$, we see that the sum (23) is precisely $\frac{1}{i\pi}\operatorname{tr}\log' GA = \frac{1}{i\pi}\operatorname{tr} P_\Lambda$. □

In the special case where there is only one corner $H = H_{12}$, there is a simpler expression for this one-dimensional eta invariant. The Lagrangian Λ_1, representing scattering data on M_1, gives the boundary condition at $s = -1$ while Λ_2 gives the boundary condition at $s = 1$. As before, S_α are the orthogonal reflections across the Λ_α. Now define

$$\mu(\Lambda_1, \Lambda_2) = \frac{1}{i\pi}\operatorname{tr}\log' \Gamma(I - S_1 S_2),$$

A simple computation shows that this formula is equivalent to (23).

It was shown directly in [11] that

$$\mu(\Lambda_{M_1}, \Lambda_{M_2}) = \eta(\Gamma D_s, \Lambda).$$

This quantity first appeared in the work of Lesch and Wojciechowski [12] for arbitrary Lagrangian boundary values, and was employed by Bunke [5] in his gluing formula for the eta invariant, cf. also the work of Wojciechowski [26] and Dai-Freed [8].

5. Proof of Theorems

In this section we complete the proof of the signature theorem, Theorem 3, for a b-metric of product type, and then deduce Theorem 1 as a (trivial) corollary.

As noted in the introduction we start from the Atiyah-Patodi-Singer index formula for the submanifolds X_ϵ constructed in §3. There is a small difficulty, due to the fact that the metric is not quite of product type in a neighbourhood of the boundary, so the results of [2] do not apply immediately. The index theorem was extended to general metrics by Gilkey, [9]. Rather than use his general result we can proceed directly in the present special circumstances to arrive at

Lemma 3. *For a b-metric of product type on X and $\epsilon > 0$ sufficiently small the Atiyah-Patodi-Singer signature formula on X_ϵ becomes*

$$\operatorname{sign}(X) = \int_X \mathcal{L} - \frac{1}{2}\eta(\eth_\epsilon) \tag{29}$$

where $\mathcal{L}$ is the Hirzebruch L-polynomial of the metric on X and $\eth_\epsilon$ is the induced Dirac operator on ∂X_ϵ.

Proof. Let τ be the signed distance from ∂X_ϵ. Flowing by the gradient of τ recovers the product structure near the boundary wherever the metric is a product. In the second region of (16) the τ direction does not split metrically; however, there the metric is the product of the corner metric, $h_{\alpha\beta}$, and a metric on a two-dimensional factor parametrized by τ and one tangential variable s. Let us call a metric which splits off either a one- or two-dimensional factor a metric of 'generalized product type'. Consider the augmented region

$$X'_\epsilon = X_\epsilon \cup ([-\delta, 0)_\tau \times \partial X_\epsilon)\,. \tag{30}$$

Choosing $\delta > 0$ very small the original metric is still a generalized product in the collar. The metric g'_ϵ on X'_ϵ can be changed by a homotopy which is constant on X_ϵ through metrics of generalized product type on the collar to a metric which is strictly of product type near $\tau = -\delta = \partial X'_\epsilon$. Indeed this can be done in such a way that the final induced metric on $\partial X'_\epsilon$ equals h_ϵ, the metric on ∂X_ϵ.

Now, we can apply the signature formula of [2] to X'_ϵ:

$$\operatorname{sign}(X) = \operatorname{sign}(X'_\epsilon) = \operatorname{sign}(X_\epsilon) = \int_{X'_\epsilon} \mathcal{L}' - \frac{1}{2}\eta(\eth_\epsilon). \tag{31}$$

Here $\eth_\epsilon$ is the induced Dirac operator on $\partial X'_\epsilon$, which of course is the same as the induced Dirac operator on ∂X_ϵ, and $\mathcal{L}'$ is the Hirzebruch class for the metric on X'_ϵ. Whenever the metric is locally a product of a metric on two factors, one with dimension less than 4, the volume part of the L-class vanishes pointwise. In particular this applies to the collar of X'_ϵ so the integral in (31) reduces to one over X_ϵ. The same argument applies to the original metric, so the integral over X reduces to that over X_ϵ and (29) follows. □

This index formula shows that $\eta(\eth_\epsilon)$ is independent of ϵ. Thus to prove the formula (9) we only need show that $\eta(\eth_\epsilon)$ reduces to the sum of the second and third terms in (9). Observe that the family of boundary operators is well-behaved in the sense that no eigenvalues cross zero; this is clear since the dimension of the null space is cohomological, hence independent of ϵ. (Note that it is only here where we use directly that $\eth_\epsilon$ is the signature operator; the same arguments lead to mod $\mathbb{Z}$ formulæ for general compatible Dirac operators on X.) Finally, the signature theorem is a direct consequence of the result in [11] on the degeneration of the eta invariant. We recall that result (in the more general context of Dirac operators for Hermitian Clifford modules)

Theorem 4. *Let g_ϵ be a family of metrics on some odd-dimensional manifold M undergoing analytic surgery degeneration. Let $\eth_\epsilon$ be a compatible Dirac operator associated to this family. Let M_α denote the family of manifolds with exact b-metrics obtained in the limit, and let Λ be the Lagrangian subspace determined by the scattering data on the M_α. Then for ϵ sufficiently small,*

$$\eta(\eth_\epsilon) = \sum_\alpha {}^b\eta(\eth_{M_\alpha}) + \eta(\Lambda) + r(ilg\,\epsilon) + \eta_{fd}(\epsilon),$$

where r is a smooth function of $ilg\,\epsilon \equiv 1/\log(1/\epsilon)$ vanishing at 0 and $\eta_{fd}(\epsilon)$ is the signature of $\eth_\epsilon$ acting on the sum of eigenspaces corresponding to very small eigenvalues, i.e. those which decay faster than any power of $ilg\,\epsilon$.

We remark that in [11] the additional term at the corners was only obtained in the special case where there are two M_α and a single intersection H. However, the extension of that proof to this more general case requires only notational changes. We note that there are several other proofs of this 'gluing formula,' cf. [5], [26], [8].

It is shown in [6] or [10] that for cohomology twisted by a flat bundle all of the very small eigenvalues are identically zero (it follows from the Mayer-Vietoris sequence for cohomology). Thus $\eta_{\mathrm{fd}}(\epsilon)$ is identically zero and so the limit of $\eta(\eth_\epsilon)$ as $\epsilon \downarrow 0$ is

$$\sum_\alpha {}^b\eta(\eth_{M_\alpha}) + \eta(\Lambda).$$

Inserting this into (29), and using the results of the last section, the proof of Theorem 3 and of (8) (for flat bundles) is complete. The $\mathbb{R}/\mathbb{Z}$ extension of Theorem 3, mentioned after the statement, follows once it is noted that the only dependence of the index density AS in (2) on the metric (on X) is through the Pontrjagin forms; these are unaffected by the deformation of the metric on the two-dimensional factor.

Theorem 1 is an immediate consequence of Theorem 3. We need simply compare the incomplete metric to a corresponding product b-metric. Then each term in (4) is equal to the corresponding term in (9). The signature is a topological invariant, so is unchanged by a change of metric. The integral of the L class is unchanged because the integrand vanishes pointwise in any region where the metric is a product of factors whose dimensions are not both divisible by four. The b-eta invariants are equal to the eta invariants as defined with respect to augmented APS boundary conditions; this has been shown by Müller [20]. Theorem 1 follows. This reasoning, along with the preceding comments on the index density, proves the general case Theorem 2 as well.

6. Wall's nonadditivity of the signature

A direct corollary of Theorem 1 is Wall's formula [25] for the nonadditivity of signatures for manifolds with boundary (and corners). As is well known, if X is a compact manifold without boundary, and it is written as a union $X = X_+ \cup_{M_0} X_-$, where $M_0 = X_+ \cap X_-$ is a closed hypersurface, then $\sigma(X) = \sigma(X_+) + \sigma(X_-)$. Wall considers the situation where X has boundary, and the dividing hypersurface M_0 intersects ∂X and divides it into two pieces M_+ and M_-. Now X disconnects into two pieces, X_+ and X_-, each of which are manifolds with corners of codimension two. Let $H = \partial X \cap M_0$. Assume that X is endowed with a metric which is a product near ∂X, M_0 and H. Associate to each M_α, $\alpha = -, 0, +$, the scattering Lagrangian Λ_α in $\mathcal{H}^*(H)$. Then Wall's formula is

$$\sigma(X) = \sigma(X_+) + \sigma(X_-) + \tau(W, \Lambda_-, \Lambda_0, \Lambda_+),$$

where $W = \mathcal{H}^*(H)$. The signature defect $\tau(W, \Lambda_-, \Lambda_0, \Lambda_+)$ was shown by Wall to be the Maslov index of the three Lagrangians in the symplectic vector space W (the symplectic structure being defined by composition of the cup product with Poincaré duality and taking the degree zero part). Wall, however, states his formula dually, in terms of homology; there $\Lambda_\alpha \subset \mathcal{H}_{2k-1}(H)$ is the nullspace of the map $\mathcal{H}_{2k-1}(H) \to \mathcal{H}_{2k-1}(M_\alpha)$.

Bunke [5] discusses this formula in relationship to gluing formulæ for the eta invariant. His intent is to use this formula to get a 'synthetic' derivation of this gluing formula (he also proves the gluing formula analytically). As noted in the introduction, to perform his calculation he has to use the signature formula for

$X_\pm$ which is only proved here. Nonetheless, he recalls from [13] the important observation that while τ is not a coboundary in the complex of measurable cochains invariant with respect to the symplectic group on the Lagrangian Grassmanian $\mathcal{L}$, it is one in the complex of K-invariant measurable cochains (where K is the unitary group on W associated to the complex structure induced by the Hodge star operator). An explicit K-invariant 1-cocycle μ with $d\mu = \tau$ is defined by

$$\mu(\Lambda_1, \Lambda_2) = \int_K \tau(\Lambda_1, \Lambda_2, k\Lambda)\, dk,$$

where dk is Haar measure on K and Λ is an arbitrary third Lagrangian. When Λ_+ and Λ_- are the scattering Lagrangians associated to the two components of $M = M_+ \cup M_-$, he shows in [5] that $\mu(\Lambda_+, \Lambda_-)$ is the extra term in the gluing formula for the eta invariant.

For completeness we present Bunke's derivation of the signature defect from the signature formula. The standard APS theorem for X says that

$$\sigma(X) = \int_X \mathcal{L}(p) - \frac{1}{2}\eta(\eth_{\partial X}),$$

while for $X_\pm$ Theorem 1 states that

$$\sigma(X_+) = \int_{X_+} \mathcal{L}(p) - \frac{1}{2}\left(\eta(\eth_{M_+}, \Lambda_+) + \eta(\eth_{M_0}, \Lambda_0) + \eta(\Lambda_0, \Lambda_+)\right),$$

and similarly

$$\sigma(X_-) = \int_{X_-} \mathcal{L}(p) - \frac{1}{2}\left(\eta(\eth_{M_-}, \Lambda_-) + \eta(\eth_{M_0}, \Lambda_0) + \eta(\Lambda_-, \Lambda_0)\right).$$

Here $\eta(\Lambda_-, \Lambda_0)$ and $\eta(\Lambda_0, \Lambda_+)$ are the eta invariants for the one dimensional problem discussed above, with boundary conditions $v_-(-1) \in \Lambda_-$, $v_-(1) \in \Lambda_0$, and $v_+(-1) \in \Lambda_0$, $v_+(1) \in \Lambda_+$. The sign for the second occurrence of $\eta(\eth_{M_0}, \Lambda_-)$ is positive because in the second occurrence M_0 is taken with the other orientation, which is equivalent to changing the sign of the operator and hence the eta invariant.

Subtracting the sum of the second and third formulæ from the first yields

$$\sigma(X) - \sigma(X_+) - \sigma(X_-) =$$
$$\frac{1}{2}\left(\eta(\eth_{M_+}, \Lambda_+) + \eta(\eth_{M_-}, \Lambda_-) - \eta(\eth_{\partial X}) - \eta(\Lambda_-, \Lambda_0) - \eta(\Lambda_0, \Lambda_+)\right).$$

Using the gluing formula for the eta invariant for the decomposition $M = M_- \cup M_+$ from [11] or [4] we get

$$\eta(\eth_{M_+}, \Lambda_+) + \eta(\eth_{M_-}, \Lambda_-) + \eta(\Lambda_+, \Lambda_-) = \eta(\eth_{\partial X}).$$

Inserting this above we find that

$$\sigma(X) - \sigma(X_+) - \sigma(X_-) = -\frac{1}{2}\left(\eta(\Lambda_-, \Lambda_0) + \eta(\Lambda_0, \Lambda_+) + \eta(\Lambda_+, \Lambda_-)\right).$$

Finally, using Bunke's identification of $\eta(\Lambda_\alpha, \Lambda_\beta)$ with $\mu(\Lambda_\alpha, \Lambda_\beta)$ and since $d\mu = \tau$, we arrive at Wall's formula for nonadditivity of the signature. (N.B. our formula is somewhat different from Bunke's because of different conventions.)

7. A model case and product formulæ for the b-eta invariant

We shall now examine the simplest model situation for a manifold with corners of codimension two, namely a product of two manifolds with boundary. We examine the signature formula here and reinterpret its terms in this context, thus giving a check on our formula. In the course of this, we need to find a formula for the b-eta invariant of a product of two manifolds. We examine this last question for more general Dirac-type operators.

To start, then, let X_1 and X_2 be two manifolds with boundary, with product (or exact) b-metrics g_1 and g_2. For notational simplicity, we only investigate the untwisted signature formula, and thus we assume that $\dim X_i = 4k_i$. Also, let $Y_i = \partial X_i$.

Since the signature of $X_1 \times X_2$ is the product of the signatures of X_1 and X_2, and since the L class is also multiplicative, we may simply multiply the Atiyah-Patodi-Singer formulæ for the two manifolds to obtain

$$\sigma(X_1 \times X_2) = L_{12} - \frac{1}{2}\eta(Y_1)L_2 - \frac{1}{2}L_1\eta(Y_2) + \frac{1}{4}\eta(Y_1)\eta(Y_2),$$

where we have used the notation $L_i = \int_{X_i} L$ and $L_{12} = \int_{X_1\times X_2} L$. We could also replace L_i by $\sigma(X_i) + \eta(Y_i)$ to obtain the equivalent formula

$$\sigma(X_1 \times X_2) = L_{12} - \frac{1}{2}\eta(Y_1)\sigma(X_2) - \frac{1}{2}\sigma(X_1)\eta(Y_2) - \frac{1}{4}\eta(Y_1)\eta(Y_2).$$

Now we compare the terms in this formula to those which appear in Theorem 3. The term on the left and the first term on the right appear both here and in that formula. Thus we need only understand the relationship of the other terms on the right to the b-eta invariants of $X_1 \times Y_2$ and $Y_1 \times X_2$, and the term involving the scattering Lagrangians.

We first make a digression to discuss general compatible Dirac operators on product manifolds and the multiplicative behaviour of the b-eta invariant. Suppose now that X is a manifold of dimension $2k$ and Y has dimension $2\ell - 1$. Suppose X and Y have spin structures; their associated Dirac operators, $\eth_X$ and $\eth_Y$, act on their respective spin bundles S_X and S_Y. Note that $\eth_X$ is odd on the $\mathbb{Z}_2$-graded spin bundle $S_X = S_X^+ \oplus S_X^-$, while S_Y is ungraded. Now, even though $X \times Y$ is odd dimensional, its spin bundle has a $\mathbb{Z}_2$ grading inherited from a reduction of the principal bundle $\mathrm{Spin}\,(X \times Y)$ to $\mathrm{Spin}\,(X) \times \mathrm{Spin}\,(Y)$. This grading is just $S_{X\times Y} = S_X^+ \otimes S_Y \oplus S_X^- \otimes S_Y$. Correspondingly, the Dirac operator $\eth_{X\times Y}$ may be regarded as a matrix

$$\eth_{X\times Y} = \begin{pmatrix} I\otimes\eth_Y & \eth_X^*\otimes I \\ \eth_X\otimes I & -I\otimes\eth_Y \end{pmatrix} : \begin{matrix} \mathcal{C}^\infty(S_X^+\otimes S_Y) \\ \oplus \\ \mathcal{C}^\infty(S_X^-\otimes S_Y) \end{matrix} \longrightarrow \begin{matrix} \mathcal{C}^\infty(S_X^+\otimes S_Y) \\ \oplus \\ \mathcal{C}^\infty(S_X^-\otimes S_Y) \end{matrix}$$

This may be checked in several different ways. For example, since it is a local formula, we may assume that Y has no boundary and that $Y = \partial Z$, where Z is spin and has a product metric near the boundary. Then the Dirac operator on Z, $\eth_Z : \mathcal{C}^\infty(S_Z^+) \to \mathcal{C}^\infty(S_Z^-)$ may be written near the boundary as $\gamma(\partial_x + \eth_Y)$, where x is a defining function for Y in Z and γ, Clifford multiplication by $\frac{\partial}{\partial x}$, is an isomorphism between S_Z^+ and S_Z^-. The Dirac operator on $X \times Z$ may be written similarly as $\gamma(\partial_x + \eth_{X\times Y})$, but it may also be expressed in terms of the $\mathbb{Z}_2$ grading $S_{X\times Z} = S_{X\times Z}^+ \oplus S_{X\times Z}^-$ where $S_{X\times Z}^+ = S_X^+ \otimes S_Z^+ \oplus S_X^- \otimes S_Z^-$ and

$S^-_{X\times Z} = S^+_X \otimes S^-_Z \oplus S^-_X \otimes S^+_Z$. Writing $\eth_{X\times Z}$ as a matrix, corresponding to this splitting, and comparing with the other form, gives the desired expression.

One consequence of (7) is that if both X and Y are compact without boundary, and if λ is a nonzero eigenvalue of $\eth_X$ and μ is an eigenvalue of $\eth_Y$, then $\pm\sqrt{\lambda^2+\mu^2}$ are eigenvalues for $\eth_{X\times Y}$. In particular, many of the nonzero eigenvalues occur in pairs of opposite sign; the ones that do not can only arise when $\lambda = 0$. This leads to the following well-known result.

Lemma 4. *Let X^{2k} and $Y^{2\ell-1}$ be compact and without boundary, and let $\eth_X$ and $\eth_Y$ be (generalized) Dirac operators, as above. If $\eth_{X\times Y}$ is the induced generalized Dirac operator on $X\times Y$, then its eta invariant, $\eta(\eth_{X\times Y})$ is given by ind$(\eth_X)\eta(\eth_Y)$.*

One quick proof of this, at least when $Y = \partial Z$ (which we may as well assume by replacing Y by a finite number of disjoint copies of it), is to multiply the Atiyah-Singer formula for $\eth_X$ with the Atiyah-Patodi-Singer formula for $\eth_Z$ to get

$$\operatorname{ind}(\eth_X)\operatorname{ind}(\eth_Z) = \int_X \mathrm{AS}(\eth_X)\int_Z \mathrm{AS}(\eth_Z) - \frac{1}{2}(\operatorname{ind}(\eth_X)\eta(\eth_Y)).$$

It is easy to check that the index of $\eth_{X\times Z}$ is the product of the indices of $\eth_X$ and $\eth_Z$, and that the Atiyah-Singer integrands are also multiplicative in this case. On the other hand, the Atiyah-Patodi-Singer formula for $X\times Z$ labels the index defect as $\eta(\eth_{X\times Y})$. Comparing these two formulæ gives the result.

We shall present another proof which will generalize to when X or Y has boundary. Namely, we use the heat-kernel representation

$$\eta(\eth_{X\times Y}) = \frac{1}{\sqrt{\pi}}\int_{X\times Y} t^{-\frac{1}{2}}\operatorname{Tr}(\eth_{X\times Y}e^{-t\eth^2_{X\times Y}})\,dt.$$

using (7) and the fact that $I\times\eth_Y$ and $\eth^{\pm}_X\times I$ commute with one another, we reduce this immediately to

$$\frac{1}{\sqrt{\pi}}\int_{X\times Y} t^{-\frac{1}{2}}\operatorname{Str}(e^{-t\eth^2_X})\operatorname{Tr}(\eth_Y e^{-t\eth^2_Y})\,dt.$$

By the McKean-Singer identity, the supertrace $\operatorname{Str}(e^{-t\eth^2_X})$ is independent of t, and is equal to the index of $\eth_X$. This can be pulled out of the integral, and what is left is simply the eta density for $\eth_Y$. Performing the integration yields the result once again.

We now consider the first of two generalizations.

Proposition 1. *Suppose that X is a compact even-dimensional manifold without boundary, and that Y is an odd-dimensional compact manifold with boundary, with an exact b-metric. Suppose that $\eth_X$, $\eth_Y$ and $\eth_{X\times Y}$ are generalized Dirac operators, as above. Then*

$${}^b\eta(\eth_{X\times Y}) = \operatorname{ind}(\eth_X)\,{}^b\eta(\eth_Y).$$

The proof of this is again immediate from the formalism above. In fact, the b-regularized integral

$$\frac{1}{\sqrt{\pi}}\int_0^\infty {}^b\!\int_{X\times Y} t^{-\frac{1}{2}}\operatorname{tr}\left(\eth_{X\times Y}e^{-t\eth^2_{X\times Y}}\right)dV_{X\times Y}\,dt$$

reduces by the same considerations to

$$\mathrm{Str}\,(e^{-t\eth_X^2})\int_0^\infty \frac{1}{\sqrt{\pi}}t^{-\frac{1}{2}}\,{}^b\mathrm{Tr}\left(\eth_Y e^{-t\eth_Y^2}\right)\,dt.$$

As before, the term in front is simply the index of $\eth_X$, and the integral defines the b-eta invariant of $\eth_Y$.

The most interesting case is the one that pertains to our signature formula, namely when X has a boundary and Y does not.

Proposition 2. *Let X be a compact $2k$-dimensional manifold with boundary, with exact b-metric and suppose Y is a compact $2\ell-1$-dimensional manifold without boundary. Let $\eth_X$ and $\eth_Y$ be compatible generalized Dirac operators, and denote by $\eth_{\partial X}$ the induced compatible Dirac operator on ∂X. Then*

$$\begin{aligned}{}^b\eta(\eth_{X\times Y}) &= \mathrm{ind}\,(\eth_X)\eta(\eth_Y)\\ &-\int_{t=0}^\infty\int_{\tau=0}^t \frac{1}{\sqrt{\pi}}\tau^{-\frac{1}{2}}\,Tr\left(\eth_Y e^{-\tau\eth_Y^2}\right)\frac{1}{\sqrt{\pi}}t^{-\frac{1}{2}}\,Tr\left(\eth_{\partial X}e^{-t\eth_{\partial X}^2}\right)\,d\tau\,dt.\end{aligned}$$

Proof. We shall let $e_{\partial X}(t)$ and $e_Y(t)$ denote the eta-densities,

$$e(t)=(\pi t)^{-\frac{1}{2}}\mathrm{Tr}\,(\eth\exp\,(-t\eth^2)),$$

with $\eth=\eth_{\partial X}$ or $\eth_Y$. Similarly ${}^be_{X\times Y}(t)$ will denote the b-eta density for $\eth_{X\times Y}$ (where the Trace has been replaced by a b-Trace). Now we return to the original defining formula and use the same reductions as before to get

$$\int_0^\infty {}^be_{X\times Y}(t)\,dt=\int_0^\infty {}^b\mathrm{Str}\left(e^{-t\eth_X^2}\right)\,e_Y(t)\,dt.$$

Unlike before, though, neither of the factors in the integrand is constant in t. Instead, we integrate by parts, passing to the antiderivative $E_Y(t)=\int_0^t e_Y(\tau)\,d\tau$ and differentiating the b-supertrace. The all-important fact here is that this last derivative is precisely the eta-density $e_{\partial X}(t)$,

$$\frac{d}{dt}{}^b\mathrm{Str}\left(e^{-t\eth_X^2}\right)=e_{\partial X}(t).$$

This is proved in [16]. Now we perform the integration by parts to get

$$\begin{aligned}&{}^b\mathrm{Str}\left(e^{-t\eth_X^2}\right)\,E_Y(t)\Big|_{t=0}^\infty-\int_0^\infty e_{\partial X}(t)E_Y(t)\,dt\\ &=\mathrm{ind}\,(\eth_X)^b\eta(\eth_Y)-\int_{t=0}^\infty\int_{\tau=0}^t e_{\partial X}(t)e_Y(\tau)\,d\tau\,dt,\end{aligned}$$

as desired. □

Applying this to the signature complex yields

$${}^b\eta(X\times Y)=\sigma(X)\eta(Y)-\int_0^\infty\int_0^t e_Y(\tau)e_{\partial X}(t)\,d\tau\,dt.$$

We will establish below that the corner correction terms vanish for the product case, so using this proposition the defect terms reduce to

$$\begin{aligned}&{}^b\eta(\eth_{X_1\times Y_2})+{}^b\eta(\eth_{Y_1\times X_2})\\ &=\sigma(X_1)^b\eta(\eth_{Y_2})+\sigma(X_2)^b\eta(\eth_{Y_1})+\int_0^\infty\int_0^t\{e_{Y_1}(\tau)e_{Y_2}(t)+e_{Y_1}(t)e_{Y_2}(\tau)\}\,d\tau\,dt.\end{aligned}$$

Now finally observe that if we interchange t and τ in the second summand of this integral, then the two integrands together sum to

$$\int_0^\infty \int_0^\infty e_{Y_1}(\tau)e_{Y_2}(t)\, d\tau\, dt = \eta(\eth_{Y_1})\eta(\eth_{Y_2}).$$

We finally study the scattering term at the corner. We do this for general compatible Dirac operators. As before, we assume for simplicity that Y_1 and Y_2 are connected, so that $H = Y_1 \times Y_2$. Of course $H = \partial(X_1 \times Y_2) = \partial(Y_1 \times X_2)$. We need to identify the Lagrangians Λ_{12} and Λ_{21}, which are the asymptotic limits of solutions of the Dirac equation $\eth\phi = 0$ on $X_1 \times Y_2$ and $Y_1 \times X_2$, respectively. For convenience, let $V_i = \mathrm{null}\,(\eth_{Y_i})$ and $V = V_1 \otimes V_2 = \mathrm{null}\,(\eth_{Y_1\times Y_2})$. Also let $\Lambda_i \subset V_i$ denote the scattering Lagrangian for $\eth_{X_i}$.

First we prove an elementary result.

Lemma 5. *Let X^{2k} be a manifold with boundary carrying an exact b-metric and $Y^{2\ell-1}$ be compact without boundary. Then the scattering Lagrangian $\Lambda_{\partial X\times Y} \subset$ null$(\eth_{\partial X\times Y})$ may be identified with $\Lambda_{\partial X} \otimes$ null$(\eth_Y)$.*

Proof. Using the representation (7) for $\eth_{X\times Y}$ above, we see that we can generate all solutions of $\eth_{X\times Y}\psi = 0$ as column vectors with components $\alpha\otimes\phi_+$ and $\alpha\otimes\phi_-$, where $\eth_Y\alpha = 0$ and $\eth_X\phi_\pm = 0$. □

Using this lemma, we see that the set of limiting values of solutions from $X_1 \times Y_2$ is $\Lambda_1 \otimes V_2$, and the limiting values from $Y_1 \times X_2$ is $V_1 \otimes \Lambda_2$. This means that the contribution from the corner is the eta invariant of the operator γD_s on the interval $[-1, 1]$ acting on functions with values in V_{12}, with boundary conditions $\psi(-1) \in V_1 \otimes \Lambda_2$, $\psi(+1) \in \Lambda_1 \otimes V_2$. Now

$$V_1 \otimes \Lambda_2 = (\Lambda_1 \otimes \Lambda_2) \oplus (\Lambda_1^\perp \otimes \Lambda_2), \qquad \Lambda_1 \otimes V_2 = (\Lambda_1 \otimes \Lambda_2) \oplus (\Lambda_1 \otimes \Lambda_2^\perp).$$

Also, γD_s acting on functions with values in $\Lambda_1\otimes\Lambda_2$ is spectrally symmetric. Hence, the corner term reduces to the eta invariant for γD_s acting on functions with values in $V_1 \otimes V_2$ with boundary condition $\psi(-1) \in \Lambda_1^\perp \otimes \Lambda_2$, $\psi(+1) \in \Lambda_1 \otimes \Lambda_2^\perp$. It is not hard to see that this operator is spectrally symmetric as well. In fact, if $\phi(s)$ is a solution of $\gamma D_s\phi = z\phi$ and $\phi(-1) \in \Lambda_1^\perp\otimes\Lambda_2$, $\phi(+1) \in \Lambda_1\otimes\Lambda_2^\perp$, then $\tilde\phi(s) \equiv \gamma\phi(-s)$ is an eigenfunction of γD_s with eigenvalue $-z$ and satisfies the boundary conditions (since γ interchanges Λ_j and $\Lambda_j^\perp$). Hence $\tilde\phi$ is an eigenvalue of the same boundary problem with eigenvalue $-z$. Hence $\eta(\Lambda_{12}, \Lambda_{21})$ vanishes. Hence in the expression (9) for $X_1 \times X_2$, the sum of b-eta invariants accounts for the last *three* terms of (7) and the corner correction term vanishes. Thus we have verified the formula (9) when X is a product of $4k$-dimensional manifolds with boundary.

References

[1] M.F. Atiyah, H. Donnelly, and I.M. Singer, *Eta invariants, signature defects of cusps and values of L-functions*, Ann. of Math. **18** (1983), 131–177.

[2] M.F. Atiyah, V.K. Patodi, and I.M. Singer, *Spectral asymmetry and Riemannian geometry, I*, Math. Proc. Camb. Phil. Soc **77** (1975), 43–69.

[3] J.-M. Bismut and J. Cheeger, *Families index for manifolds with boundary, superconnections and cones*, Invent. Math. **89** (1990), 91–151.

[4] U. Bunke, *η-invariants for manifolds with boundary*, Humboldt Universität SFB preprint no. 52 (1993).

[5] ———, *On the gluing formula for the η invariant*, J. Diff. Geom. **41** (1995), no. 2, 397–448.

[6] S.E. Cappell, R. Lee, and E.Y. Miller, *Self-adjoint operators and manifold decomposition part i: Low eigenmodes and stretching*, preprint, 1992? could be published now.

[7] J. Cheeger, *Spectral geometry of singular Riemannian spaces*, J. Diff. Geom. **18** (1983), 175–221.

[8] X. Dai and D. Freed, *η-invariants and determinant lines*, J. Math. Phys. **35** (1994), 5155–5194.

[9] P. Gilkey, *The boundary integrand in the formula for the signature and the Euler characteristic of a Riemannian manifold with boundary*, Advances in Math. **15** (1975), 344–360.

[10] A. Hassell, *Analytic surgery and analytic torsion*, preprint, 1995.

[11] A. Hassell, R. Mazzeo, and R. Melrose, *Analytic surgery and the accumulation of eigenvalues*, To appear, Comm. Anal. Geom.

[12] M. Lesch and K.P. Wojciechowski, *On the η invariant of generalized Atiyah-Singer-Patodi boundary value problems*, To appear, Illinois Math. J.

[13] G. Lion and M. Vergne, *The Weil representation, Maslov index and theta series*, Birkhäuser, Boston, Basel, 1980.

[14] R. Mazzeo and R. Melrose, *Analytic surgery and the eta invariant*, Geom. and Func. Anal. **5** (1995), no. 1, 14–75.

[15] R.B. Melrose, *Pseudodifferential operators, corners and singular limits*, Proc. Int. Congress of Mathematicians, Kyoto, Springer Verlag, 1990, pp. 217–234.

[16] ———, *The Atiyah-Patodi-Singer index theorem*, A K Peters, Wellesley, MA, 1993.

[17] R.B. Melrose and P. Piazza, *Families of Dirac operators, boundaries and the b-calculus*, Preprint, 1993.

[18] ———, *An index theorem for families of Dirac operators on odd-dimensional manifolds with boundary*, Preprint, 1994.

[19] W. Müller, *On the index of Dirac operators on manifolds with corners of codimension two, I*, Max-Planck-Institut Preprint 95-23.

[20] ———, *Signature defects of cusps of Hilbert modular varieties*, J. Diff. Geom. **20** (1984), 55–119.

[21] ———, *Manifolds with cusps of rank one, spectral theory and L^2-index theorem*, Lecture Notes in Math., vol. 1244, Springer Verlag, Berlin, Heidelberg, New York, 1987.

[22] H.D. Rees, *The η-invariant and Wall non-additivity*, Math. Ann. **267** (1984), 449–452.

[23] M. Stern, *L^2-index theorems on locally symmetric spaces*, Invent. Math. **96** (1989), 231–282.

[24] ———, *Eta invariants and hermitian locally symmetric spaces*, J. Diff. Geom. **31** (1990), 771–789.

[25] C.T.C. Wall, *Non-additivity of the signature*, Inventiones Math. **7** (1969), 269–274.

[26] K. Wojciechowski, *The additivity of the η-invariant: the case of a noninvertible tangential operator*, To appear, Comm. Math. Phys.

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