STAR PRODUCTS AND LOCAL LINE BUNDLES
ÉTOILE-PRODUITS ET FIBRÉS EN DROITE LOCAUX

RICHARD MELROSE

Dedicated to Louis Boutet de Monvel for the occasion of his sixtieth birthday

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Abstract. The notion of a local line bundle on a manifold, classified by 2-cohomology with real coefficients, is introduced. The twisting of pseudodifferential operators by such a line bundle leads to an algebroid with elliptic elements with real-valued index, given by a twisted variant of the Atiyah-Singer index formula. Using ideas of Boutet de Monvel and Guillemin the corresponding twisted Toeplitz algebroid on any compact symplectic manifold is shown to yield the star products of Lecomte and DeWilde ([3]) see also Fedosov's construction in [7]. This also shows that the trace on the star algebra is identified with the residue trace of Wodzicki ([18]) and Guillemin ([10]).

La notion de fibrés en droite locaux sur une variété différentiable est introduite. Ces derniers sont classifiés par la cohomologie réelle de dimension 2. Le twist d'opérateurs pseudodifférentiels par de tels fibrés en droite donne lieu à un algébroïde contenant des éléments elliptiques dont l'indice à valeur dans les réels est donné par une variante de la formule de l'indice d'Atiyah et Singer. En utilisant des idées de Boutet de Monvel et Guillemin, on montre que, sur toute variété symplectique compacte, il est possible d'obtenir le produit étoilé de Lecomte et DeWilde [3] (voir aussi la construction de Fedosov [7]) à partir de l'algébroïde associé au twist des opérateurs de Toeplitz. Cela établit du même coup que la trace définie sur cette algèbre étoilée peut être identifiée avec la trace résiduelle de Wodzicki [18] et Guillemin [10].

Contents

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Introduction

If M is a symplectic manifold, Lecomte and DeWilde ([3], see also Fedosov's construction, [7]) showed that M carries a star product. That is, the space of formal power series in a parameter, t, with coefficients being smooth functions on M, carries an associative product

$$\begin{aligned}(1)\quad & \star : \mathcal{C}^\infty(M)[[t]] \times \mathcal{C}^\infty(M)[[t]] \longrightarrow \mathcal{C}^\infty(M)[[t]], \\ & (\sum_{j\ge 0} a_j t^j) \star (\sum_{l\ge 0} b_l t^l) = (\sum_{k\ge 0} c_k t^k),\ c_k = \sum_{j+l\le k} B_{k,j,l}(a_j, b_l)\end{aligned}$$

where each $B_{k,j,l}$ is a bilinear differential operator and

$$B_{0,0,0}(\alpha,\beta) = \alpha\beta,\ B_{1,0,0}(\alpha,\beta) = \{\alpha,\beta\}. \tag{2}$$

Here the second term is the Poisson bracket on the symplectic manifold; note that the normalization of the non-zero coefficient is arbitrary, since it can be changed by scaling the formal variable t.

The star product is *pure* if $B_{k,j,l} = \tilde{B}_{k-j-l}$ only depends on the 'change of order'. This means that it can be written

$$a \star b = \left(\sum_l t^l \tilde{B}_l(x,y) \right) a(x)b(y)\big|_{x=y}. \tag{3}$$

Boutet de Monvel and Guillemin had an alternate approach to 'formal quantization' in this sense, but it was limited to the *pre-quantized* case, in which the symplectic form is assumed to be a non-vanishing multiple of an integral class and hence arises from the curvature of a line bundle. The corresponding circle bundle is a contact manifold and the quantization of M arises from the choice of an $\mathbb{S}$-invariant Toeplitz structure on this contact manifold; note that this gives much more than the star product. This construction is reviewed below and extended to the general case using the notion of a local line bundle. In a certain sense the construction here is intermediate between that of Boutet de Monvel and Guillemin and that of Fedosov (which for the sake of brevity is not discussed) but still gives more than the latter in so far as the star algebra is shown to be the quotient of an algebroid of Toeplitz-like kernels near the diagonal, where composition is only restricted by the closeness of the support to the diagonal. The quotient is by the corresponding algebroid of smoothing operators. This carries the usual trace functional and the unique (normalized) trace on the star algebra is shown to arise as a residue trace in this way, with the trace-defect formula used to prove the homotopy invariance of the index as in [12].

1. Toeplitz operators

Under the assumption that (M,ω) is a symplectic manifold with $[\omega] \in H^2(M,\mathbb{Z})$ an integral class, Guillemin, in [9], exploiting his earlier work with Boutet de Monvel ([2]) used the existence of a Toeplitz algebra on the circle bundle with curvature ω to construct a star product on M. This construction is first sketched and then extended to the non-integral case.

Let L be an Hermitian line bundle over M with unitary connection having curvature $\omega/2\pi$; this exists in virtue of the assumed integrality of ω. Let Z be the circle bundle of L. The connection on L induces a connection 1-form, $\eta \in \mathcal{C}^\infty(Z;\Lambda^1)$, on Z, fixed by the two conditions

(1) $\eta(\partial_\theta) = 1$ for the derivative of the circle action and
(2) For each $p \in Z$, η_p is normal to any local section of Z over M which is covariant constant, at p, as a section of L.

It follows that η is $\mathbb{S}$-invariant and $d\eta = \omega$ is a basic form. Since ω is assumed to be symplectic, η is a contact form on Z.

Any contact manifold, Y, carries a natural space of 'Heisenberg' pseudodifferential operators, see [1], [17] and [8] and [5]. If the contact manifold is compact these form an algebra, in general the properly supported elements form an algebra. Let us denote by $\Psi^0_{\mathrm{H}}(Y)$ the space of Heisenberg pseudodifferential operators of order 0 on Z. This also has two ideals, corresponding to the two orientations of the contact bundle, namely the upper and lower Hermite ideals $\mathcal{I}^0_{\mathrm{H},\pm}(Y) \subset \Psi^0_{\mathrm{H}}(Y)$. The intersection of these ideals is the space of (properly supported) smoothing operators.

Although working more from the point of view of complex Lagrangian distributions, Boutet de Monvel and Guillemin introduced the notion of a 'quantized contact structure' which is the choice of a generalized Szegő projector $P \in \mathcal{I}^0_{\mathrm{H},+}(Y)$. Assuming Y to be compact, $P^2 = P$, otherwise it is properly supported and such that $P^2 - P$ is a smoothing operator. The defining property of such a projector is that its symbol, in the sense of the Heisenberg algebra, should be the field of projections, one for each point of Y, onto the null space of the field of harmonic oscillators arising from the choice of a compatible almost complex structure. It is shown in [4] that the set of components of such projections is mapped onto $\mathbb{Z}$ by the relative index, once a base point is fixed.

Having chosen such a projector, the associated space of Toeplitz operators consists of the compressions of pseudodifferential operators (or Heisenberg pseudodifferential operators) to the range of P, i.e. the operators

$$\Psi^0_{\mathrm{Tp}}(Y) = \left\{PAP; A \in \Psi^0(Y)\right\}. \tag{1.1}$$

If Y is compact this is again an algebra; otherwise the properly supported elements form an algebra if the smoothing operators are appended. In either case, the quotient by the corresponding algebra of smoothing operators is the 'Toeplitz full symbol algebra'

$$\Psi^0_{\mathrm{Tp}}(Y)/\Psi^{-\infty}_{\mathrm{Tp}}(Y) \simeq \mathcal{C}^\infty(Y)[[\rho]]. \tag{1.2}$$

Here the formal power series parameter is the inverse of the homogeneous, length, function on the contact line bundle over Y.

Returning to the case that the contact manifold, now Z, is a circle bundle with $\mathbb{S}$-invariant contact structure we may choose the projection P to be $\mathbb{S}$-invariant and then consider the subspace of $\mathbb{S}$-invariant Toeplitz operators

$$\Psi^0_{\mathrm{ITp}}(Z) = \left\{A \in \Psi^0_{\mathrm{Tp}}(Z) : T^*_\theta A = AT^*_\theta \ \forall\ \theta \in \mathbb{S}\right\}. \tag{1.3}$$

For this subalgebra

$$\Psi^0_{\mathrm{ITp}}(Z)/\Psi^{-\infty}_{\mathrm{ITp}}(Z) \simeq \mathcal{C}^\infty(M)[[\rho]]. \tag{1.4}$$

Theorem 1. *(Guillemin [9]) If (M, ω) is a compact integral symplectic manifold then $\Psi^0_{ITp}(Z)$, the $\mathbb{S}$-invariant part of the Toeplitz operators for a choice of $\mathbb{S}$-invariant generalized Szegő projector on the circle bundle of an Hermitian line bundle with curvature $\omega/2\pi$, is an algebra and this algebra structure induces a star product on M through* (1.4).

In case M is non-compact, but still with integral symplectic structure, the choice of a properly supported $\mathbb{S}$-invariant projection, up to smoothing, leads to the same result for the properly supported Toeplitz operators, with properly supported smoothing operators appended.

For the proof see [2] or [5].

2. Closed 1-forms

As a slight guide to the discussion of local line bundles below we first discuss the analogous 'geometric model' for 1-dimensional real cohomology. This result is not used anywhere below.

As is well-known, the closed 1-forms inducing integral 1-dimensional cohomology classes on a manifold M can be realized in terms of functions into the circle. Thus,

$$a \in \mathcal{C}^\infty(M; \mathbb{S}) \longleftrightarrow \frac{1}{2\pi i} a^{-1} da \in \mathcal{C}^\infty(M; \Lambda^1) \tag{2.1}$$

is an isomorphism onto the real closed integral 1-forms. One can get a closely related realization of the cohomology with real coefficients in terms of 'local circle functions' on M.

Definition 1. A *local circle function* on M is a smooth map defined on a neighbourhood of the diagonal in M^2, $A \in \mathcal{C}^\infty(W; \mathbb{S})$, $W \subset M^2$ open, $\mathrm{Diag} \subset W$, such that

$$A(x,y)A(y,z) = A(x,z) \ \forall \ (x,y,z) \in V, \tag{2.2}$$

where V is some neighbourhood of the triple diagonal in M^3.

In fact we will only consider germs of such functions at the diagonal, i.e. identify two such functions if they are equal in some neighbourhood of the diagonal. If $a \in \mathcal{C}^\infty(M; \mathbb{S})$ then $A(x,y) = a(x)a^{-1}(y)$ satisfies (2.2). Setting $x = y = z$ in (2.2) shows that $A\big|_{\mathrm{Diag}} = 1$. Similarly $A(x,y)A(y,x) = 1$ near the diagonal. If $U \subset M$ is a small open set so that $U \times U \subset W$ and $U \times U \times U \subset V$ then choosing $p \in U$ and setting $a_p(x) = A(x,p)$, $x \in U$, gives $a_p : U \longrightarrow \mathbb{S}$ and $A(x,y) = a_p(x)a_p^{-1}(y)$ on $U \times U$. Thus A does define such a map, a, locally. Changing the base point p to another $q \in U$ changes a_p to $a_q(x) = A(x,q) = A(x,p)A(p,q) = A(p,q)a_p(x)$, i.e. only by a multiplicative constant. It follows that the 1-form

$$\begin{aligned} \alpha &= \frac{1}{2\pi i} A^{-1} d_x A \text{ at } \mathrm{Diag} \simeq M \\ &= \frac{1}{2\pi i} a_p^{-1} da_p \in \mathcal{C}^\infty(M; \Lambda^1) \end{aligned} \tag{2.3}$$

is well-defined on U independently of the choice of $p \in U$ and hence is globally well-defined on M. From the local identification with $\frac{1}{2\pi 1} a_p^{-1} da_p$ it is clearly closed. Furthermore, the vanishing of α implies that a_p is locally constant and hence $A \equiv 1$ near the diagonal. Thus we have proved

Proposition 1. *The group of germs at the diagonal of local circle functions is isomorphic to the space of closed 1-forms.*

Similarly, if we consider real functions, $\beta \in \mathcal{C}^\infty(W, \mathbb{R})$, defined near the diagonal which satisfy the additivity condition

$$\beta(x,y) + \beta(y,z) = \beta(x,z) \text{ on } V \tag{2.4}$$

and identify the local circle functions A and $e^{2\pi i\beta}A$ then we arrive at a geometric realization of $H^1(M,\mathbb{R})$.

3. Local line bundles

The problem with extending Theorem 1 to the general case is, of course, that in the non-integral case there can be no line bundle over M with curvature the symplectic form. Nevertheless there is a 'virtual object' which plays at least part of the same role. This is closely related to, but rather simpler than, Fedosov's theorem on the classification of star products, up to isomorphism, in terms of $H^2(M;\mathbb{R})$.

We give a 'geometric realization' of $H^2(M;\mathbb{R})$ on any manifold, possibly with corners. The construction here of 'local line bundles' over M is related to ideas of Murray concerning bundle gerbes ([15]) and more particularly to the discussion of extensions of Azumaya bundles in [12].

Consider the diagonal

$$\mathrm{Diag} = \{(z,z) \in M^2\} \tag{3.1}$$

which is naturally diffeomorphic to M under either the left or right projection from M^2 to M. Similarly consider the triple diagonal

$$\mathrm{Diag}_3 = \{(z,z,z) \in M^3\}. \tag{3.2}$$

There are three natural projections from M^3 to M^2 which we label π_F, π_S and π_C (for 'F'irst, 'S'econd and 'C'entral or 'C'omposite):

$$\begin{gathered}\pi_F : M^3 \ni (x,y,z) \longmapsto (y,z) \in M^2,\\ \pi_S : M^3 \ni (x,y,z) \longmapsto (x,y) \in M^2 \text{ and}\\ \pi_C : M^3 \ni (x,y,z) \longmapsto (x,z) \in M^2.\end{gathered} \tag{3.3}$$

Definition 2. A *local line bundle* over a manifold M is a (complex) line bundle $\mathcal{L}$ over a neighbourhood of $\mathrm{Diag} \subset M^2$ together with a smooth 'composition' isomorphism over a neighbourhood of $\mathrm{Diag}_3 \subset M^3$

$$H : \pi_{\mathrm{S}}^*\mathcal{L} \otimes \pi_{\mathrm{F}}^*\mathcal{L} \longrightarrow \pi_{\mathrm{C}}^*\mathcal{L} \tag{3.4}$$

with the associativity condition that for all (x,y,z,t) sufficiently close to the total diagonal in M^4, the same map

$$\mathcal{L}_{(x,y)} \otimes \mathcal{L}_{(y,z)} \otimes \mathcal{L}_{(z,t)} \longrightarrow \mathcal{L}_{(x,y)} \tag{3.5}$$

arises either by first applying H in the left two factors and then on the composite, or first in the right two factors and then in the composite, i.e. the following diagramme commutes:

$$\begin{tikzcd}[column sep=small] & \mathcal{L}_{(x,z)} \otimes \mathcal{L}_{(z,t)} \arrow[dr, "H_{(x,z,t)}"] & \\ \mathcal{L}_{(x,y)} \otimes \mathcal{L}_{(y,z)} \otimes \mathcal{L}_{(z,t)} \arrow[ur, "H_{(x,y,z)}\otimes \mathrm{Id}"] \arrow[dr, "\mathrm{Id}\otimes H_{(y,z,t)}"'] & & \mathcal{L}_{(x,t)}. \\ & \mathcal{L}_{(x,y)} \otimes \mathcal{L}_{(y,t)} \arrow[ur, "H_{(x,y,t)}"'] & \end{tikzcd} \tag{3.6}$$

We will really deal with germs at the diagonal of these objects.

Over the triple diagonal itself H necessarily gives an isomorphism

$$H_{\mathrm{Diag}} : \mathcal{L}_{\mathrm{Diag}} \otimes \mathcal{L}_{\mathrm{Diag}} \longrightarrow \mathcal{L}_{\mathrm{Diag}}. \tag{3.7}$$

If $e_z \neq 0$ is an element of $\mathcal{L}_{(z,z)}$ then H_{Diag} maps $e_z \otimes e_z$ to ce_z for some $0 \neq c \in \mathbb{C}$. Thus, corresponding to the two square-roots of c there are exactly two local sections of $\mathcal{L}_{\mathrm{Diag}}$ such that $H(c \otimes c) = c$. On the other hand $\mathcal{L}_{(x,x)}$ acts on the right on each $\mathcal{L}_{(t,x)}$ for t sufficiently close to x as the space of homomorphisms. The associativity condition (3.6) means that $\mathcal{L}_{(x,x)}$ is identified with the linear space of homomorphisms on $\mathcal{L}_{(y,x)}$ in a way consistent with its product. Thus either e or $-e$ must be locally the identity. Hence Id must exist as a global section so there is a canonical identification

$$\mathcal{L}_{\mathrm{Diag}} \longrightarrow M \times \mathbb{C} \tag{3.8}$$

consistent with the action of $\mathcal{L}_{\mathrm{Diag}}$ on the left or right on $\mathcal{L}$ through H.

Now, consider local trivializations of $\mathcal{L}$. Choosing a sufficiently fine open cover $\{U_i\}$ of M, the products $U_i \times U_i$ give an open cover of $\mathrm{Diag} \subset M^2$ and are contained in a given neighbourhood of the diagonal. Thus, for some such open cover, $\mathcal{L}$ is defined over each $U_i \times U_i$. Choose a point $p_i \in U_i$ and consider the bundles

$$L_{i,p_i} = \mathcal{L}\big|_{U_i \times \{p_i\}},\ R_{i,p_i} = \mathcal{L}\big|_{\{p_i\} \times U_i}. \tag{3.9}$$

The composition law H gives an identification

$$\mathcal{L} \overset{H}{\equiv} L_{i,p_i} \otimes R_{i,p_i} \text{ over } U_i \times U_i \tag{3.10}$$

and also an identification

$$R_{i,p_i} = L_{i,p_i}^{-1}, \tag{3.11}$$

since over the diagonal $\mathcal{L}$ has been canonically trivialized.

Lemma 1. *Any local line bundle over M has a multiplicative connection, i.e. a connection ∇ such that if u is a local section of $\mathcal{L}$ near (x, y) with $\nabla u = 0$ at (x, y) and v is a local section of $\mathcal{L}$ near (y, z) with $\nabla v = 0$ at (y, z) then $H(u, v)$ is locally constant at (x, y, z). Similarly $\mathcal{L}$ has a multiplicative unitary structure, so*

$$|H(u, v)| = |u||v| \tag{3.12}$$

and has a multiplicative Hermitian connection.

Proof. Using an open cover as described above, choose $p_i \in U_i$ for each i and a connection ∇_i on L_{i,p_i}. If we make a different choice, q_i, of point in U_i then the composition law H gives an identification of $L_{i,q_i} = \mathcal{L}\big|_{U_i \times \{q_i\}}$ with $L_{i,p_i} \otimes \mathcal{L}_{p_i,q_i}$. The second factor is a fixed complex line, so a connection on L_{i,p_i} induces, through H, a connection on L_{i,q_i}. Now, choose a partition of unity subordinate to the cover, $\rho_i \in \mathcal{C}^\infty_c(U_i)$ with $\sum\limits_i \rho_i = 1$. We shall modify the connection on L_{i,p_i} and replace it by

$$\nabla = \sum_j \rho_i \nabla_j \text{ on } L_{i,p_i} \text{ over } U_i. \tag{3.13}$$

Here we use the fact that over $U_{ij} = U_i \cap U_j$ the line bundles L_{i,p_i} and L_{j,p_j} are identified by H after tensoring with the fixed line $\mathcal{L}_{p_i,p_j}$, as discussed above, so ∇_j is well-defined on L_{i,p_i} over U_{ij} which contains the support of ρ_j in U_i. Directly from the definition, this new connection is consistent with the identification of L_{i,p_i} and L_{j,p_j} over U_{ij}.

Now the connection on $\mathcal{L}$ over $U_i \times U_i$ induced by taking the dual connection to ∇ on R_{i,p_i}, using the identification (3.11), and then the tensor product connection on $\mathcal{L}$ using (3.10) is independent of i. That is, it is a global connection on $\mathcal{L}$ and from its definition has the desired product property.

The same approach allows one to define a multiplicative Hermitian structure by taking as Hermitian structure $\langle \cdot, \cdot \rangle_i$ on each L_{i,p_i}. This induces Hermitian structures on the inverses L_{i,p_i}^{-1} and hence on $\mathcal{L}$ over $U_i \times U_i$. Then if ρ_i is a partition of unity subordinate to the cover, the inner product on $\mathcal{L}$ over $U_i \times U_i$

$$\langle u, v \rangle = \sum_i (\rho_i \times \rho_i) \langle u, v \rangle_i \tag{3.14}$$

is consistent with the inner products over the other $U_j \times U_j$. Unitary metrics on each of the L_{i,p_i} then induce a connection on $\mathcal{L}$ which is both multiplicative and unitary, i.e. is consistent with the Hermitian structure. □

Proposition 2. *The left curvature of a product Hermitian connection, i.e. the restriction of the curvature at the diagonal to the left tangent space, is an arbitrary real closed* 2*-form on M lying in a fixed class in $H^2(M, \mathbb{R})$ determined by the local line bundle and product Hermitian structure. Two local line bundles are isomorphic in some neighbourhood of the diagonal under a unitary isomorphism intertwining the product structures if and only if the left curvatures define the same cohomology class; all cohomology classes arise in this way.*

Proof. Two product connections on a fixed local line bundle differ (in a small neighbourhood of the diagonal where they are both defined) by a 1-form $i\alpha$. The multiplicative condition on the connections implies that

$$\alpha_{(x,y)}(v, w) = \alpha_{(x,z)}(v, u) + \alpha_{(y,z)}(u, w) \tag{3.15}$$

for all points (x, z, y) in a small enough neighbourhood of the triple diagonal and all $v \in T_xX$, $u \in T_yX$ and $w \in T_zX$. In particular $\alpha_{(x,x)} = 0$ and $\alpha_{(x,y)}(u, w) + \alpha_{(y,x)}(w, v) = 0$ so the 1-form defined locally on M by

$$\beta_x(v) = \alpha_{(x,y)}(v, 0), \ v \in T_xX, \tag{3.16}$$

for y fixed close to x is actually globally well-defined and satisfies

$$\alpha = \pi_L^* \beta - \pi_R^* \beta. \tag{3.17}$$

Similarly the curvature of a product connection is locally the curvature of a connection on $L_{i,p_i} \boxtimes L_{i,p_i}^{-1}$ coming from a connection on L_{i,p_i}. It is therefore locally of the form $\pi_L^* \omega - \pi^* \omega$ for a closed 2-form on M. Again it follows that the 2-form ω is well-defined, as the restriction of the curvature to left tangent vectors at the diagonal.

Thus it remains to show that any real closed 2-form, ω, on M arises this way. Consider a good cover of M, so each of the open sets U_i and all of their non-trivial intersections are contractible. Then on each U_i there exists a smooth 1-form, $\alpha_i \in \mathcal{C}^\infty(U_i; \Lambda^1)$ such that

$$\omega = d\alpha_i \text{ on } U_i. \tag{3.18}$$

On non-trivial overlaps there exists a smooth function ϕ_{ij} such that

$$\alpha_i - \alpha_j = d\phi_{ij} \text{ on } U_i \cap U_j. \tag{3.19}$$

It follows that on non-trivial triple intersections

$$\phi_{ij} + \phi_{jk} + \phi_{ki} = \phi_{ijk} \text{ is constant on } U_i \cap U_j \cap U_k. \tag{3.20}$$

Now, consider L_i which is the trivial line bundle over U_i, with the connection $d + i\alpha_i$ which is unitary for the standard Hermitian structure. Over the open neighbourhood of the diagonal

$$U = \bigcup_i (U_i \times U_i) \tag{3.21}$$

use the product isomorphism $e^{i\phi_{ij}} \times e^{-i\phi_{ij}}$ to identify $L_i \boxtimes L_i^{-1}$ with $L_j \boxtimes L_j^{-1}$ over the intersection $(U_i \cap U_j) \times (U_i \cap U_j)$. The constancy of the ϕ_{ijk} means that these unitary isomorphisms satisfy the cocycle condition on triple overlaps, so this gives a well-defined bundle $\mathcal{L}$ over U. That this is a local line bundle follows immediately from its definition and the connections patch to give a global unitary connection with curvature ω. □

Thus the collection of Hermitian local line bundles modulo unitary multiplicative isomorphisms in some neighbourhood of the diagonal is identified with $H^2(M;\mathbb{R})$ and the collection of Hermitian local line bundles with unitary product connections modulo isomorphisms identifying the connections is identified with the space of real closed 2-forms on M. By extension from the standard case we call the cohomology class

$$\frac{\omega}{2\pi} \in H^2(M,\mathbb{R}) \tag{3.22}$$

the first Chern class of $\mathcal{L}$ and $\exp(\omega/2\pi) \in H^{\mathrm{even}}(M;\mathbb{R})$ its Chern character.

4. Atiyah-Singer index formula

The Atiyah-Singer formula expresses the analytic index, the difference of the dimension of the null space and the null space of the adjoint, for an elliptic pseudodifferential operator in terms of topological data determined by the principal symbol. Namely if $A \in \Psi^m(X; E, F)$ is a pseudodifferential operator acting between sections of the two (complex) vector bundles E and F then its principal symbol $\sigma_m(A) \in \mathcal{C}^\infty(S^*X; \hom(E,F) \otimes N_m)$ defines a homomorphism between the lifts of E and F to the cosphere bundle, up to a positive diagonal factor; ellipticity of A is by definition equivalent to invertibility of this homomorphism. This in turn fixes a compactly supported K-class on T^*X, or equivalently a K-class for the radial compactification $\overline{T^*X}$ of the cotangent bundle relative to its boundary, the cosphere bundle

$$[\sigma_m(A)] \in K(\overline{T^*X}, S^*X]. \tag{4.1}$$

The Atiyah-Singer formula is

$$\begin{aligned} \operatorname{ind}(A) &= \dim \operatorname{null}(A) - \dim \operatorname{null}(A^*) \\ &= \operatorname{Tr}(AB - \mathrm{Id}_F) - \operatorname{Tr}(BA - \mathrm{Id}_E) = \int_{T^*X} \operatorname{Ch}(\sigma_m(A)) \operatorname{Td}(X). \end{aligned} \tag{4.2}$$

Here $B \in \Psi^{-m}(X; F, E)$ is a parametrix for A, so is such that $AB - \mathrm{Id}_F$, $BA - \mathrm{Id}_E$ are smoothing operators, $\operatorname{Ch} : K(M, \partial M) \longrightarrow H^*(M, \partial M)$ is the Chern character and $\operatorname{Td}(X)$ is the Todd class of the cotangent bundle.

We wish to generalize this formula to include twisting by a local line bundle as discussed above. To do so, recall the definition of the space of pseudodifferential operators of order m. If $\mathrm{Diag} \subset X^2$ is the diagonal in the product then in terms of Schwartz' kernels

$$\Psi^m(X; E, F) = I^m(X^2, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \Omega_R). \tag{4.3}$$

Here $I^m(M, Y; G)$ is the space of conormal distributions of order m, introduced explicitly by Hörmander in [11] for any embedded submanifold Y of a manifold M and any vector bundle G over M. In the particular case (4.3), $\mathrm{Hom}(E, F) \equiv \pi_L^* F \otimes \pi_R^* E'$ is the 'big' homomorphism bundle over the product and $\Omega_R = \pi_R^* \Omega$ is the right-density bundle, allowing invariant integration; here $\pi_L, \pi_R : X^2 \longrightarrow X$ are the two projections. The symbol $\sigma_m(A)$ for $A \in \Psi^m(X; E, F)$ is then the leading asymptotic term in the Fourier transform, in directions transversal to the diagonal, of the kernel and is naturally identified with a section of the 'little' homomorphism bundle $\mathrm{hom}(E, F) = \mathrm{Hom}(E, F)|_{\mathrm{Diag}}$ lifted to the conormal bundle of Diag, which is to say the cotangent bundle of X.

As in [12] we can general this space of kernels by twisting with a vector bundle, even if this bundle is only defined in a neighbourhood of the diagonal. In this way we will obtain a space of 'kernels' with supports in a sufficiently small neighbourhood of $\mathrm{Diag} \subset X^2$. In [12] this was considered in the case of the homorphism bundle for a projective vector bundle and also for local line bundles which are Nth roots of line bundles over X. Here we can allow the more general case of a local line bundle as discussed above and define

$$\Psi^m_{\mathcal{L},\epsilon}(X; E, F) = I^m(B_\epsilon, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \mathcal{L} \otimes \Omega_R) \tag{4.4}$$

where B_ϵ is a sufficiently small neighbourhood of the diagonal over which the local line bundle $\mathcal{L}$ exists. For definiteness, and because it is related to Weyl quantization, we will take B_ϵ to be the points of X^2 distant less than ϵ from the diagonal with respect to a metric on X on the two factors of X^2.

In general the elements if $\Psi^m_{\mathcal{L},\epsilon}(X; E, F)$ do not compose freely as do those of $\Psi^m(X; E, F)$; rather it is necessary for the supports to be sufficiently close to the diagonal. Thus suppose E, F and G are three vector bundles over X. Composition

$$\Psi^m(X; F, G) \cdot \Psi^{m'}(X; E, F) \subset \Psi^{m+m'}(X; E, G) \tag{4.5}$$

in the standard case, reduces to a push-forward operation on the kernels. Namely

$$\begin{aligned} I^m(X^2, \mathrm{Diag}, \mathrm{Hom}(F, G) \otimes \Omega_R) \times I^{m'}(X^2, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \Omega_R) \\ \longrightarrow I^{m+m'}(X^2, \mathrm{Diag}, \mathrm{Hom}(E, G) \otimes \Omega_R). \end{aligned} \tag{4.6}$$

This push-forward result, and correspondingly the composition (4.5) can be localized on X^2 in each factor. Thus, localizing away from the diagonal gives a smooth term and this results in a smoothing operator. Localization near a point on the diagonal in either factor allows the vector bundles to be trivialized and then the result reduces to the scalar case for open sets in Euclidean space. There, or even globally, the elements of $I^m(X^2; \mathrm{Diag})$, which are the *classical* (so polyhomogeneous) conormal distributions may be approximated by smooth functions within the somewhat larger class of conormal distributions 'with bounds' (i.e. of type $1, 0$.)

In the smoothing case, i.e. for $m = m' = -\infty$, (4.6) becomes

$$\begin{aligned}(4.7)\quad & \mathcal{C}^\infty(X^2;\mathrm{Hom}(F,G)\otimes\Omega_R)\times\mathcal{C}^\infty(X^2;\mathrm{Hom}(E,F)\otimes\Omega_R)\\ &\longrightarrow \mathcal{C}^\infty(X^3;\pi_R^*\otimes\pi_M^*F'\otimes\pi_M^*F\otimes\pi_R^*E\otimes\pi_M^*\Omega\otimes\pi_R^*\Omega)\\ &\longleftrightarrow \mathcal{C}^\infty(X^2;\mathrm{Hom}(E,F)\otimes\Omega_R).\end{aligned}$$

Here the three projections $\pi_L, \pi_M, \pi_R : X^3 \longrightarrow X$ are used and for the second map the pairing between E and E' leads to a density in the middle factor which is integrated out. Thus (4.7) is a more explicit, and invariant, version of the composition formula

$$A\circ B(x,y) = \int_X A(x,z)B(z,y)dz. \tag{4.8}$$

In particular to extend (4.5) to the kernels in (4.4) it is only necessary to see that the smooth composition makes sense as in (4.7) since the singularities of the kernels behave exactly as in the standard case. In fact in the presence of a local line bundle $\mathcal{L}$, (4.7) is replaced by

$$\begin{aligned}(4.9)\quad & \mathcal{C}^\infty(B_\epsilon;\mathrm{Hom}(F,G)\otimes\mathcal{L}\otimes\Omega_R)\times\mathcal{C}^\infty(B_{\epsilon'};\mathrm{Hom}(E,F)\otimes\mathcal{L}\otimes\Omega_R)\\ &\longrightarrow \mathcal{C}^\infty(\pi_S^{-1}B_\epsilon\cap\pi_F^{-1}B_{\epsilon'};\pi_R^*G\otimes\pi_M^*F'\otimes\pi_M^*F\pi_S^*\mathcal{L}\otimes\pi_F^*\mathcal{L}\otimes\pi_R^*E\otimes\pi_M^*\Omega\otimes\pi_R^*\Omega)\\ &\longleftrightarrow \mathcal{C}^\infty(B_\eta;\mathrm{Hom}(E,F)\otimes\mathcal{L}\Omega_R).\end{aligned}$$

Here $\pi_S, \pi_F, \pi_C : X^3 \longrightarrow X^2$ are the projections from (3.3) and give, over a sufficiently small neighbourhood of the triple diagonal, an identification

$$H : \pi_S^*\mathcal{L}\otimes\pi_F^*\mathcal{L}\longrightarrow\pi_C^*\mathcal{L}. \tag{4.10}$$

Notice that if B_ϵ and B'_ϵ are sufficiently small neighbourhoods of the diagonals then $\pi_S^{-1}B_\epsilon\cap\pi_F^{-1}B_{\epsilon'}$ is indeed an arbitrarily small neigbourhood of the triple diagonal, projecting under π_C to a neighbourhood of the diagonal B_ϵ, small with $\epsilon+\epsilon'$.

Thus, when the product (4.9) is localized near a point on the diagonal in each factor, and these points can always be taken to be the same, $\mathcal{L}$ reduces to $\pi_L^*L\otimes\pi_R^*L^{-1}$ and it becomes (4.7) with E, F and G all replaced by $E\otimes L$, $F\otimes L$ and $G\otimes L$. Thus indeed we arrive at the restricted, but associative, product

$$\Psi^m_{\mathcal{L},\epsilon}(X;F,G)\cdot\Psi^{m'}_{\mathcal{L},\epsilon'}(X;E,F)\subset\Psi^{m+m'}_{\mathcal{L},\eta}(X;E,G) \tag{4.11}$$

for $\epsilon+\epsilon'$ small compared to η. This product can still be written as in (4.8) but with the associative product H giving the pairing on $\mathcal{L}$. Notice that the symbol is well-defined, as it is in the standard case, and since it is just a section of the restriction to the diagonal of the bundle it leads to a short exact sequence

$$\Psi^{m-1}_{\mathcal{L},\epsilon}(X;E,F)\longrightarrow\Psi^m_{\mathcal{L},\epsilon}(X;E,F)\xrightarrow{\sigma_m}\mathcal{C}^\infty(S^*X;\mathrm{hom}(E,F)\otimes N_m) \tag{4.12}$$

in which the twisting local bundle $\mathcal{L}$ does not appear in the symbol.

Theorem 2. *If $A\in\Psi^m_{\mathcal{L},\epsilon}(X;E,F)$ is elliptic, in the sense that $\sigma_m(A)$ is invertible, and $\epsilon>0$ is sufficiently small then there is a parametrix $B\in\Psi^{-m}_{\mathcal{L},\epsilon'}(X;F,E)$, for any $\epsilon'>0$ sufficiently small, such that $AB-\mathrm{Id}$ and $BA-\mathrm{Id}$ are smoothing operators and then the index*

$$\mathrm{ind}(A) = \mathrm{Tr}[A,B]\in\mathbb{R} \tag{4.13}$$

is independent of the choice of B, is log-multiplicative for elliptic operators

$$\operatorname{ind}(AA') = \operatorname{ind}(A) + \operatorname{ind}(A') \tag{4.14}$$

is homotopy invariant under elliptic deformations, is additive under direct sums and so defines an additive map

$$\operatorname{ind}_{\mathcal{L}} : K(\overline{T^*X}; S^*X) \longrightarrow \mathbb{R},\ \operatorname{ind}(A) = \operatorname{ind}_{\mathcal{L}}([E, F, \sigma_m(A)]), \tag{4.15}$$

which is given by a variant of the Atiyah-Singer formula

$$\operatorname{ind}_{\mathcal{L}}(a) = \int_{T^*X} \operatorname{Ch}(a) \operatorname{Td}(X) \exp(\omega/2\pi) \tag{4.16}$$

where $\omega/2\pi \in H^2(X;\mathbb{R})$ is the first Chern class of the local line bundle $\mathcal{L}$.

Proof. The proof of this result is essentially the same as that of the twisted index theorem in [12]; we recall the steps.

First we recall that the symbol calculus for the twisted pseudodifferential operators has the same formal properties as in the untwisted case – it is the smoothing part which is restricted by support conditions. Thus the standard proofs of the existence of a parametrix, B, for an elliptic element carry over unchanged. Furthermore the set of parametrices is affine over the smoothing operators (with restricted support).

First we consider the special case that $E = F$. The smoothing operators are only constrained away from the diagonal and the trace functional is well-defined as usual as the integral of the Schwartz kernel over the diagonal

$$\operatorname{Tr}(A) = \int_X A\big|_{\mathrm{Diag}}. \tag{4.17}$$

When the composition of smoothing operators is defined,

$$\operatorname{Tr}([A, B]) = 0 \tag{4.18}$$

and this identity extends, by continuity as indicated above, to the case where one of the operators is a pseudodifferential operator.

Now, from this it follows that the definition, (4.13), of $\operatorname{ind}(A)$ is independent of the choice of parametrix B since if B, B' are two parametrices then $B_t = (1-t)B + tB'$ is a family of parametrices and

$$\frac{d}{dt}\operatorname{Tr}([A, B_t]) = \operatorname{Tr}([A, B' - B]) = 0 \tag{4.19}$$

since $B' - B$ is smoothing.

It is vital to establish the homotopy invariance of this index. To do so we use the trace-defect formula from [13]; it is very closely related to the proof of the Atiyah-Patodi-Singer index theorem in [14]. First we may define the residue trace on $\Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;E)$ following the idea of Guillemin [10]. Namely, the residue trace can be defined as the reside at $z = 0$ of the meromorphic function

$$\operatorname{Tr}_{\mathrm{R}}(A) = \lim_{z\to 0} zF(z),\ F(z) = \operatorname{Tr}(AQ(z)) \tag{4.20}$$

provided $Q(z)$ is a family of pseudodifferential operators of complex order z which is everywhere elliptic and satisfies $Q(0) = \operatorname{Id}$. Even in the twisted case we can find such a family. One approach, indicated in [12], is to take a generalized Laplacian, $L \in \Psi^2_{\mathcal{L},\epsilon}(X;E)$, construct the singularity of its formal heat kernel and then take the Laplace transform. The construction of the singularity, at $\{t = 0\} \times \operatorname{Diag}$ of the heat kernel e^{-tL} is known to be completely local and symbolic (see for example [14] where this is done in a more general case) and hence can be carried out in the twisted case, up to a smoothing error term and with support in any preassigned neighbourhood of the diagonal. The virtue of this construction is that it gives an entire family of operators $Q(z)$ of complex order z such that

$$\begin{gathered} Q(0) = \operatorname{Id} \text{ and } Q(z)Q(-z) = \operatorname{Id} + R(z), \\ \mathbb{C} \ni z \longrightarrow \Psi^{-\infty}_{\mathcal{L},\epsilon}(X;E) \text{ entire with } R(0) = 0,\ R'(0) = 0. \end{gathered} \tag{4.21}$$

The vanishing of $R'(0)$ follows by direct differentiation of the defining identity.

The proof that the function $F(z)$ in (4.20) is meromorphic is again the same as in the standard case, as in the (corrected version of) the original argument of Seeley ([16]) and $F(z)$ has at most a simple pole at $z = 0$. Furthermore the residue at z, defining $\operatorname{Tr}_{\mathrm{R}}(A)$ is independent of the choice of $Q(z)$ since if $Q'(z)$ is another such family then $Q'(z) - Q(z) = zE(z)$ where $E(z)$ is also an entire family of operators of complex order z. Since the residue reduces to the same local computation as in the untwisted case, it is given by the same formula, namely the integral over the cosphere bundle of the trace of the term of homogeneity $-n$ in the full symbol expansion. In particular

$$\operatorname{Tr}_{\mathrm{R}}(\operatorname{Id}) = 0. \tag{4.22}$$

The regularized trace of A,

$$\overline{\operatorname{Tr}}_Q(A) = \lim_{z\to 0}(\operatorname{Tr}(AQ(z)) - \operatorname{Tr}_{\mathrm{R}}(A)/z) \tag{4.23}$$

does depend on the choice of the family $Q(z)$. However once a choice is made, it gives a functional extending the trace. In the special case that we are considering where $A \in \Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;E)$ 'acts on a fixed bundle' this allows the index to be rewritten

$$\operatorname{ind}(A) = \overline{\operatorname{Tr}}_Q([A,B]). \tag{4.24}$$

The family Q also defines a derivation on the algebroid $\Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;E)$. Namely

$$D_Q T = \frac{d}{dz} Q(z)TQ(-z)\big|_{z=0}. \tag{4.25}$$

Notice that the family on the right is an entire family of fixed order, one less than that of T (since $Q(z)$ is principally diagonal) so

$$D_Q : \Psi^m_{\mathcal{L},\epsilon}(X;E) \longrightarrow \Psi^m_{\mathcal{L},\epsilon+\delta}(X;E) \tag{4.26}$$

if ϵ and δ are small enough. This is a derivation in the sense that

$$D_Q(T_1T_2) = (D_QT_1)T_2 + T_1(D_QT_2) \tag{4.27}$$

provided all supports are small enough. In fact D_Q is independent of the choice of Q up to an interior derivation. Notice that for any $T \in \Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;E)$

$$\operatorname{Tr}_{\mathrm{R}}(D_QT) = 0 \tag{4.28}$$

since this is the residue at $\tau = 0$ of

$$\begin{aligned}\operatorname{Tr}(D_Q TQ(\tau) = \frac{d}{dz}\operatorname{Tr}\left(Q(z)TQ(-z)Q(\tau)\right)\big|_{z=0}\\ = \frac{d}{dz}\operatorname{Tr}\left(TQ(-z)Q(\tau)Q(z)\right)\big|_{z=0}\\ = \frac{d}{dz}\operatorname{Tr}\left(T(Q(\tau)+E(z,\tau)\right)\big|_{z=0}\end{aligned}$$

where $W(z,\tau)$ is entire in both variables with values in the smoothing operators. Thus there is no singularity at $\tau = 0$.

The trace-defect formula now follows directly. If $T, S \in \Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;E)$ with $\epsilon > 0$ (and $\delta > 0$ from Q) small enough then

$$\overline{\operatorname{Tr}}_Q([T,S]) = \operatorname{Tr}_{\mathrm{R}}(SD_QT). \tag{4.29}$$

Indeed using the trace identity when $\operatorname{Re} z << 0$,

$$\begin{aligned}\operatorname{Tr}\left([T,S]Q(z)\right) &= \operatorname{Tr}\left(SQ(z)T - STQ(z)\right)\\ &= \operatorname{Tr}\left(S(Q(z)TQ(-z)-T)Q(z)\right) - \operatorname{Tr}\left(SQ(z)TR(-z)\right).\end{aligned}\tag{4.30}$$

The last term here is the trace of an entire family of smoothing operators, vanishing at $z = 0$. Furthermore, $Q(z)TQ(-z) - T$ is an entire family of operators of fixed order, vanishing at $z = 0$ so can be written $zD_QT + O(z^2)$ and we arrive at (4.29).

Now the residue trace vanishes on operators of low order so is a trace on the symbolic quotient

$$\Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;E)/\Psi^{-\infty}_{\mathcal{L},\epsilon}(X;E) \tag{4.31}$$

in which an elliptic operator is, by definition, invertible. Thus the index of an invertible element a in (4.31) is

$$\operatorname{ind}(a) = \operatorname{Tr}([A,B]) = \operatorname{Tr}_{\mathrm{R}}(a^{-1}D_Qa) \tag{4.32}$$

which is also independent of the choice of Q. In this case the homotopy invariance follows directly, since if a_t is an elliptic family depending smoothly on a parameter t then

$$\begin{aligned}\frac{d}{dt}\operatorname{ind}(a_t) &= \frac{d}{dt}\operatorname{Tr}_{\mathrm{R}}(a_t^{-1}D_Qa_t)\\ &= \operatorname{Tr}_{\mathrm{R}}\left(a_t^{-1}D_Q\dot a_t - a_t^{-1}\dot a_t a_t^{-1}D_Qa_t\right) = \operatorname{Tr}_{\mathrm{R}}\left(D_Q(\dot a_t a_t^{-1})\right) = 0\end{aligned}\tag{4.33}$$

where $\dot a_t$ denotes the t-derivative and (4.28) has been used.

This proof only covers directly the case of elliptic operators on a fixed bundle E. However, for $A \in \Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;E,F)$, elliptic between two different bundles, only relatively minor modifications are required. Namely we need to choose entire families as above, $Q_E(z)$ and $Q_F(z)$ for the two bundles. The definition of the index is modified to

$$\begin{aligned}\operatorname{ind}(A) &= \operatorname{Tr}(AB - \operatorname{Id}_F) - \operatorname{Tr}(BA - \operatorname{Id}_E)\\ &= \overline{\operatorname{Tr}}_F(AB) - \overline{\operatorname{Tr}}_E(BA) - \overline{\operatorname{Tr}}_F(\operatorname{Id}) + \overline{\operatorname{Tr}}_E(\operatorname{Id})\end{aligned}\tag{4.34}$$

where $\overline{\operatorname{Tr}}_E$ and $\overline{\operatorname{Tr}}_F$ are the regularized traces defined by Q_E and Q_F on twisted operators on E and F. Independence of choice follows as before.

To see homotopy invariance we define the operator

$$D_Q : \Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;E,F) \longrightarrow \Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;E,F),\ D_Q A = \frac{d}{dz}\Big|_{z=0} Q_F(z) A Q_E(-z) \tag{4.35}$$

for any bundles with a fixed choices of the regularizing families. From the vanishing of $\frac{d}{dz}Q_E(z)Q_E(-z)$ at $z=0$ it follows that these are again derivations in a module sense. From (4.34) the index is the regularized value at $z=0$ of

$$\begin{aligned}(4.36)\quad & \mathrm{Tr}(BQ_F(z)A) - \overline{\mathrm{Tr}}(BAQ_E(z)) - \overline{\mathrm{Tr}}_F(\mathrm{Id}) + \overline{\mathrm{Tr}}_E(\mathrm{Id}) \\ &= \mathrm{Tr}\left(B(Q_F(z)AQ(-z) - A)Q_E(z)\right) - \mathrm{Tr}\left(B(Q_F(z)R_F(-z)\right) - \overline{\mathrm{Tr}}_F(\mathrm{Id}) + \overline{\mathrm{Tr}}_E(\mathrm{Id}).\end{aligned}$$

The second term vanishes at $z=0$ and evaluating the first gives

$$\mathrm{ind}(A) = \mathrm{Tr}_{\mathrm{R}}(a^{-1}D_Q a) - \overline{\mathrm{Tr}}_F(\mathrm{Id}) + \overline{\mathrm{Tr}}_E(\mathrm{Id}) \tag{4.37}$$

where we have replaced A by its image in the quotient by the smoothing operators. From this the homotopy invariance follows as before.

Not only does the formula (4.37) lead to the homotopy invariance of the index, but it also shows the multiplicativity. Given three bundles E, F, G with $A_1 \in \Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;E,F)$ and $A_2 \in \Psi^{\mathbb{Z}}_{\mathcal{L},\epsilon}(X;F,G)$ elliptic with full symbolic images a_1, a_2 we see that

$$\begin{aligned}(4.38)\quad & \mathrm{ind}(A_2A_1) = \mathrm{Tr}_{\mathrm{R}}((a_2a_1)^{-1}D_Q(a_2a_1)) - \overline{\mathrm{Tr}}_G(\mathrm{Id}) + \overline{\mathrm{Tr}}_E(\mathrm{Id}) \\ &= \mathrm{Tr}_{\mathrm{R}}((a_2^{-1}D_Qa_2)\,\mathrm{Tr}_{\mathrm{R}}((a_1^{-1}D_Qa_1) - \overline{\mathrm{Tr}}_G(\mathrm{Id}) + \overline{\mathrm{Tr}}_E(\mathrm{Id}) = \mathrm{ind}(A_2) + \mathrm{ind}(A_1).\end{aligned}$$

The index of the direct sum of two operators is trivially the sum of the indexes, so from (4.38) and the homotopy invariance we conclude that the index actually defines a group homomorphism from K-theory as in the untwisted case

$$K^0(\overline{T^*X}, S^*X) \longrightarrow \mathbb{R}. \tag{4.39}$$

Here, any class in the K-theory of $\overline{T^*X}$, the radial compactification of the cotangent bundle, relative to its bounding sphere bundle, is represented by an elliptic symbol $a \in \hom(E,F)$ over S^*X, for bundles E and F over X and the discussion above shows that the index is the same for two representatives of the same class.

Now, from (4.39) we deduce that the map is actually vanishes on torsion elements of K-theory, i.e. is well defined on $K^0(\overline{T^*X}, S^*X) \otimes \mathbb{R}$. Thus, to prove the desired formula (4.16) it suffices to check it on a set of elements of which span the K-theory, over $\mathbb{R}$ (or $\mathbb{Q}$). If X is even-dimensional the original observation of Atiyah and Singer is that the bundle-twisted signature operators are enough to do this. This argument applies directly here and the arguments of [12] again apply to show that the local index theorem for Dirac operators gives the formula in that case, and hence proves it in general in the even-dimensional case. For the odd-dimensional case it is enough to suspend with a circle to pass to the even-dimensional case. $\square$

5. Star products

Notice that in the construction of the star product in Theorem 1 only the $\mathbb{S}$-invariant part of the Heisenberg, and Toeplitz, algebra is used. There is a close connection between the notion of a local line bundle and the $\mathbb{S}$-invariance; this is enough to allow the invariant part of the algebroid to be constructed directly.

Proposition 3. *Let L be an Hermitian line bundle over a manifold M with S the circle bundle of L then there is a canonical isomorphism between distributions on $S\times S$ which are invariant under the conjugation $\mathbb{S}$-action and distributions on the circle bundle, Q, of $\pi_L^*L\otimes\pi_R^*L^{-1}=\operatorname{Hom}(L)$.*

Proof. The map from the total product to the exterior tensor product

$$\begin{aligned}(5.1)\quad S\times S\ni(p,\tau;p',\tau')&\longrightarrow(p,p',\tau\otimes(\tau')^{-1})\in Q\\&=\{(p,p',\sigma);(p,p')\in M\times M,\ \sigma\in L_p\otimes L_{p'}^{-1},\ |\sigma|=1\}\end{aligned}$$

is a circle bundle with fibre action of $\mathbb{S}$ given by the conjugation action on $\pi_L^*L\times\pi_R^*L$. Thus the invariant distributions on $S\times S$ are precisely the pull-backs of distributions on Q. □

Still in the integral case, under this identification, the $\mathbb{S}$-invariant Heisenberg operators are identified with the space of parabolic conormal distributions on the 'diagonal'

$$\Psi^m_{\mathrm{IH}}(S)=I^{m'}(Q;D;\lambda),\ D=\{(p,p,s)\in Q;p=p',\ s=1\} \tag{5.2}$$

with λ being the contact form on Q.

So, in the general case of a possibly non-integral symplectic form we simply define

$$\Psi^m_{\mathrm{IH},\epsilon}(S)=I_{\mathrm{c}}^{m+\frac12}(Q_\epsilon,D,\lambda;). \tag{5.3}$$

Proposition 4. *The kernels* (5.3) *form an algebroid with composition restricted only by supports:*

$$\Psi^m_{IH,\epsilon}(S)\circ\Psi^{m'}_{IH,\epsilon'}(S)\subset\Psi^{m+m'}_{IH,\epsilon+\epsilon'}(S) \tag{5.4}$$

for all sufficiently small ϵ, $\epsilon'>0$.

Proof. This is a local result once the composition formula is written down invariantly and therefore follows from the standard theory of Heisenberg operators. □

Theorem 3. *For any symplectic manifold there exists $P_\epsilon\in\Psi^0_{IH,\epsilon}(S)$ with $P_\epsilon^2=P_\epsilon$ modulo smoothing and $\sigma(P)$ the field of projections for a positive almost complex structure on M and the associated invariant Toeplitz algebroid*

$$\Psi^m_{ITp,\epsilon}(M)=P_{\epsilon/3}\Psi^m_{IH,\epsilon/3}(S)P_{\epsilon/3} \tag{5.5}$$

induces a star product on the quotient

$$\mathcal{C}^\infty(M)[[\rho]]=\Psi^0_{ITp,\epsilon}(M)/\Psi^{-\infty}_{ITp,\epsilon}(M). \tag{5.6}$$

Again, the global setup having been defined, this is in essence a local result and hence follows as in the integral case.

Note that only 'pure' star products arise directly this way, those classified by $H^2(M,\mathbb{R})$. As in Fedosov's original construction, one can pass to the general star product by twisting asymptotic sums of pure star products.

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Department of Mathematics, Massachusetts Institute of Technology
E-mail address: `rbm@math.mit.edu`