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PERIODICITY AND THE DETERMINANT BUNDLE

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ABSTRACT. The infinite matrix 'Schwartz' group $G^{-\infty}$ is a classifying group for odd K-theory and carries Chern classes in each odd dimension, generating the cohomology. These classes are closely related to the Fredholm determinant on $G^{-\infty}$. We show that while the higher (even, Schwartz) loop groups of $G^{-\infty}$, again classifying for odd K-theory, do *not* carry multiplicative determinants generating the first Chern class, 'dressed' extensions, corresponding to a star product, do carry such functions. We use these to discuss Bott periodicity for the determinant bundle and the eta invariant. In so doing we relate two distinct extensions of the eta invariant, to self-adjoint elliptic operators and to elliptic invertible suspended families and show that the corresponding τ invariant is a determinant in this sense.

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Introduction

The Fredholm determinant is a character for the group of invertible operators of the form $\operatorname{Id}+T$ with T of trace class on a Hilbert space. Transferred to invertible operators of the form $\operatorname{Id}+A$ with A smoothing on the compact fibres of a fibration it induces the determinant bundle of families of elliptic pseudodifferential operators. For suspended families of smoothing operators, depending in a Schwartz fashion on an even number of Euclidean parameters, we introduce an adiabatic determinant with similar topological properties and use it to prove periodicity properties for the determinant bundle. The corresponding suspended eta invariants are also discussed and in a subsequent paper will be used to describe cobordism of the determinant bundle in a pseudodifferential setting, extending the result of Dai and Freed [6] that the eta invariant in the interior defines a trivialization of the determinant bundle on the boundary.

The basic notion of determinant is that on finite rank matrices. If $\mathrm{M}(N,\mathbb{C})$ is the algebra of $N\times N$ complex matrices then the determinant is the entire (polynomial) multiplicative map

$$\det:\mathrm{M}(N,\mathbb{C})\longrightarrow\mathbb{C},\ \det(AB)=\det(A)\det(B)$$

which is determined by the condition on its derivative at the identity

$$\frac{d}{ds}\det(\operatorname{Id}+sA)\big|_{s=0}=\operatorname{Tr}(A),\ A\in\mathrm{M}(N,\mathbb{C}).$$

It has the fundamental property that $\det(A)\neq 0$ is equivalent to the invertibility of A, so

$$\mathrm{GL}(N,\mathbb{C})=\{A\in\mathrm{M}(N,\mathbb{C});\det(A)\neq 0\}=\det^{-1}(\mathbb{C}^*).$$

As is well-known, such a map into $\mathbb{C}^*$ determines, through the winding number, an integral 1-cohomology class:

$$\alpha(c)=\operatorname{wn}(\det:c\longrightarrow\mathbb{C}^*),\ \alpha\in\mathrm{H}^1(\mathrm{GL}(N,\mathbb{C});\mathbb{Z}).\tag{1}$$

Conversely for any path-connected space

$$H^1(X;\mathbb{Z}) \equiv \{\alpha : \pi_1(X) \longrightarrow \mathbb{Z};\ \alpha(c_1 \circ c_2) = \alpha(c_1) + \alpha(c_2)\}$$

so each integral 1-cohomology class may be represented by a continuous function $f : X \longrightarrow \mathbb{C}^*$ such that $\alpha(c)$ is the winding number of f restricted to a curve representing c. Even if X is a group and the class is invariant, it may not be possible to choose this function to be multiplicative.

Each integral 1-cohomology class on X may also be represented as the obstruction to the triviality of a principal $\mathbb{Z}$ bundle over X. Such a bundle, with total space P, always admits a 'connection' in the sense of a map $h : P \longrightarrow \mathbb{C}$ such that $h(np) = h(p) + n$ for the action of $n \in \mathbb{Z}$. Given appropriate smoothness, the function on X associated to the connection, $f = \exp(2\pi i h)$, fixes the obstruction 1-class as a deRham form

$$\alpha = \frac{1}{2\pi i} f^{-1} df = dh.$$

In particular the triviality of the $\mathbb{Z}$-bundle is equivalent to the existence of a continuous (normalized) logarithm for f, that is a function $l : X \longrightarrow \mathbb{C}$ such that $h - \phi^* l$ is locally constant, where $\phi : P \longrightarrow X$ is the bundle projection.

Returning to the basic case of the matrix algebra and $\mathrm{GL}(N,\mathbb{C})$, these spaces can be naturally included in the 'infinite matrix algebra' which we denote abstractly $\Psi^{-\infty}$. For the moment we identify

$$\Psi^{-\infty} = \{a : \mathbb{N}^2 \longrightarrow \mathbb{C};\ \sup_{i,j\in\mathbb{N}} (i+j)^k |a_{ij}| < \infty\ \forall\ k \in \mathbb{N}\}.$$

The algebra structure is just the extension of standard matrix multiplication

$$(ab)_{ij} = \sum_{l=1}^{\infty} a_{il} b_{lj}.$$

Now, although $\mathrm{M}(N,\mathbb{C}) \longrightarrow \Psi^{-\infty}$ is included as the subalgebra with $a_{ij} = 0$ for $i, j > N$, for the determinant this is not natural, in part because $\Psi^{-\infty}$ is non-unital. Namely, we consider instead the isomorphic space $\mathrm{Id} + \Psi^{-\infty}$ which may be identified with $\Psi^{-\infty}$ with the product

$$a \circ b = a + b + ab.$$

Then the inclusion

$$\mathrm{M}(N,\mathbb{C}) \ni a \longmapsto (\mathrm{Id} - \pi_N) + \pi_N a \pi_N$$

is multiplicative and the determinant is consistent for all N with the Fredholm determinant which is the entire multiplicative function

$$\det_{\mathrm{Fr}} : \mathrm{Id} + \Psi^{-\infty} \longrightarrow \mathbb{C}$$

satisfying the normalization

$$\frac{d}{ds} \det_{\mathrm{Fr}}(\mathrm{Id} + sa)\big|_{s=0} = \mathrm{Tr}(a) = \sum_{i=1}^{\infty} a_{ii}.$$

Again for $a \in \Psi^{-\infty}$ the condition $\det_{\mathrm{Fr}}(\mathrm{Id} + a) \neq 0$ is equivalent to the existence of an inverse $\mathrm{Id} + b$, $b \in \Psi^{-\infty}$ and this defines the topological group

$$G^{-\infty} = \{\mathrm{Id} + a; a \in \Psi^{-\infty},\ \det_{\mathrm{Fr}}(\mathrm{Id} + a) \neq 0\}$$

in which the $\mathrm{GL}(N,\mathbb{C})$ are included as subgroups. Since these determinants are consistent we generally drop the distinction between the finite and Fredholm determinants.

Now, $G^{-\infty}$ is a classfying group for odd K-theory,

$$K^1(X) = \Pi_0\{f : X \longrightarrow G^{-\infty}\}$$

where the maps can be taken to be either continuous or smooth. As such,

$$\Pi_l(G^{-\infty}) = \begin{cases} \{0\} & l \text{ even} \\ \mathbb{Z} & l \text{ odd.} \end{cases}$$

The odd Chern forms (see for example [13]),

$$\beta_{2k-1} = \frac{1}{(2\pi i)^k}\frac{(k-1)!}{(2k-1)!}\operatorname{Tr}[((\operatorname{Id}+a)^{-1}da)^{2k-1}],\ k\in\mathbb{N}, \tag{2}$$

give an explicit isomorphism

$$h_{2k-1} : \Pi_{2k-1}(G^{-\infty}) \ni [f] \longmapsto \int_{\mathbb{S}^{2k-1}} f^*\beta_{2k-1} \in \mathbb{Z} \tag{3}$$

where $[f] \in \Pi_{2k-1}(G^{-\infty})$ is represented by a smooth map $f : \mathbb{S}^{2k-1} \longrightarrow G^{-\infty}$. The cohomology classes $[\beta_{2k-1}] \in H^{2k-1}(G^{-\infty});\mathbb{C})$ generate $H^*(G^{-\infty};\mathbb{C})$ as an *exterior algebra* over $\mathbb{C}$

$$H^*(G^{-\infty};\mathbb{C}) = \Lambda_{\mathbb{C}}(\beta_1,\beta_3,\dots,\beta_{2k-1},\dots).$$

However, as noted by Bott and Seeley in [5], even though they give integers when integrated over the corresponding spherical homology class, the classes $[\beta_{2k-1}]$ are not all integral. When $k=1$, the isomorphism (3) shows that as in the case of the matrix groups, a loop along which the winding number of the determinant is 1 generates $\Pi_1(G^{-\infty})$ and $\mathrm{H}^1(G^{-\infty})$ is generated by the deRham class

$$\alpha = \frac{1}{2\pi i}\operatorname{Tr}((\operatorname{Id}+a)^{-1}da). \tag{4}$$

Bott periodicity corresponds to the fact that the (reduced) loop groups of $G^{-\infty}$ are also classifying spaces for odd or even K-theory. Consistent with the 'smooth' structure championed here, we consider loop groups of 'Schwartz' type. In fact we can first identify $\Psi^{-\infty}$ above as the expansion of an operator with respect to the eigenvectors of the harmonic oscillator on $\mathbb{R}^n$ to identify

$$\Psi^{-\infty} \longleftrightarrow \Psi^{-\infty}(\mathbb{R}^n) = \mathcal{S}(\mathbb{R}^{2n})$$

where the product on $\mathcal{S}(\mathbb{R}^{2n})$ is the operator product

$$(ab)(x,y) = \int_{\mathbb{R}^n} a(x,z)b(z,y)dz.$$

With this identification the loop groups become

$$G^{-\infty}_{\mathrm{sus}(p)}(\mathbb{R}^n) = \{f : \mathbb{R}^p \longrightarrow G^{-\infty}(\mathbb{R}^n); f = \operatorname{Id}+a,\ a\in\mathcal{S}(\mathbb{R}^{p+2n})\}.$$

Thus $G^{-\infty}_{\mathrm{sus}(p)}$ is a classifying group for K-theory of the parity opposite to that of p. In fact we may regard $G^{-\infty}_{\mathrm{sus}(p)}$ as classifying for the groups K^{-p-1} and Bott periodicity as giving the identification between these for all even and all odd orders.

The analogues of the forms (2) are given by

$$\beta^{(p)}_{2k-1-p}(f) = \int_{\mathbb{R}^p} f^*\beta_{2k-1}, \quad p \le 2k-1,\ k \in \mathbb{N}.$$

For $p = 1$, this gives the even forms

$$\beta^{(1)}_{2k} = \frac{1}{(2\pi i)^{k+1}} \frac{k!}{(2k)!} \int_{\mathbb{R}} \operatorname{Tr}\left[(a^{-1}da)^{2k} a^{-1}\frac{da}{d\tau}\right] d\tau, \quad k \in \mathbb{N}_0,$$

where τ is the suspension parameter (*cf.*[13]).

It is equally possible to use the eigenbasis of a Laplacian on the sections of a vector bundle over a compact Riemannian manifold without boundary (or of any self-adjoint elliptic pseudodifferential operator of positive order) to identify $\Psi^{-\infty}$ with $\Psi^{-\infty}(X;E)$, the space of smoothing operators. Then the loop groups are realized as

$$G^{-\infty}_{\mathrm{sus}(p)}(X;E) = \\ \{f : \mathbb{R}^p \longrightarrow G^{-\infty}(X;E); f = \operatorname{Id} + a,\ a \in \mathcal{S}(\mathbb{R}^p \times X \times X; \operatorname{Hom}(E))\}.$$

Here, the space of Schwartz sections is defined for any vector bundle which is the pull-back to $\mathbb{R}^p \times Z$ of a vector bundle over a compact manifold Z.

Now, the basic issue considered here is the existence of a determinant on the spaces $G^{-\infty}_{\mathrm{sus}(2k)}$. One can simply look for a smooth multiplicative function which generates the 1-dimensional homology through the winding number formula (1). In what is really the opposite side of the 'Miracle of the loop group' of Pressley and Segal [16] there is in fact no such function as soon as $k > 0$. As we show below, there is a multiplicative function closely related to the determinant but which has a global logarithm (if $k > 0$). However, as we also show below, there is a determinant function, the 'adiabatic determinant' in this sense provided the group $G^{-\infty}_{\mathrm{sus}(2k)}$ is 'dressed' by replacing it by an extension with respect to a star product, of which $G^{-\infty}_{\mathrm{sus}(2k)}$ is the principal term. This extension is homotopically trivial, i.e. still gives a classifying space for K-theory.

More precisely, consider the space $\Psi^k_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]]$ of formal power series

$$\sum_{\mu=0}^{\infty} a_\mu \varepsilon^\mu, \quad a_\mu \in \Psi^k_{\mathrm{sus}(2n)}(X;E)$$

(see (2.9) for the definition) equipped with the star-product

$$(A * B)(u) = \left(\sum_{\mu=0}^{\infty} a_\mu \varepsilon^\mu\right) * \left(\sum_{\nu=0}^{\infty} b_\nu \varepsilon^\nu\right) \\ = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \varepsilon^{\mu+\nu} \left(\sum_{p=0}^{\infty} \frac{i\varepsilon^p}{2^p p!}\omega(D_v, D_w)^p a_\mu(v) b_\nu(w)\right)\Bigg|_{v=w=u} \tag{5}$$

for $A, B \in \Psi^*_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]]$, where ω is the standard symplectic form on $\mathbb{R}^{2n}$. This gives a corresponding group

$$G^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]] = \{\operatorname{Id} + Q;\ Q \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]], \\ \exists\ P \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]],\ (\operatorname{Id} + Q) * (\operatorname{Id} + P) = \operatorname{Id} \in \Psi^0_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]]\}$$

with group law given by the star-product (5). Then $G^{-\infty}_{\mathrm{sus}(2n)}(X;E)$ is a retraction of $G^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]]$.

Our first main result is the following.

Theorem 1. *There is a multiplicative 'adiabatic' determinant function*

$$\det{}_{\mathrm{a}} : G^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]] \longrightarrow \mathbb{C}^*,$$
$$\det{}_{\mathrm{a}}(A * B) = \det{}_{\mathrm{a}}(A)\det{}_{\mathrm{a}}(B) \ \forall\ A, B \in G^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]],$$

which generates $\mathrm{H}^1(G^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]])$.

This is proven in §3 by considering a corresponding determinant for mixed isotropic operators and taking the adiabatic limit.

Given a (locally trivial) fibration of compact manifolds

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B \end{tikzcd} \tag{6}$$

and a family of elliptic $2n$-suspended operators

$$D \in \Psi^k_{\mathrm{sus}(2n)}(M/B;E,F)$$

with vanishing numerical index, one can construct an associated determinant line bundle $\mathrm{Det}_{\mathrm{a}}(D) \to B$ as described in §3, the definition being in terms of (a slightly extended notion of) principal bundles; a related construction can be found in [16]. More generally, this construction can be extended to a fully elliptic family of product-suspended operators (see the appendix and §2 for the definition)

$$D \in \Psi^{k,k'}_{\mathrm{psus}(2n)}(M/B;E,F)$$

with vanishing numerical index. Our second result is to relate this determinant line bundle with Quillen's definition via Bott periodicity. Let $D_0 \in \Psi^1(M/B;E,F)$ be a family of elliptic operators with vanishing numerical index. Define, by recurrence for $n \in \mathbb{N}$, the fully elliptic product-suspended familes by

$$D_n(t_1,\dots,t_n,\tau_1,\dots,\tau_n) = \begin{pmatrix} it_n - \tau_n & D^*_{n-1} \\ D_{n-1} & it_n + \tau_n \end{pmatrix} \in \Psi^{1,1}_{\mathrm{psus}(2n)}(M/B; 2^{n-1}(E \oplus F)),$$

where $2^{n-1}(E \oplus F)$ is the direct sum of 2^{n-1} copies of $E \oplus F$. In §5 we prove

Theorem 2 (Periodicity of the determinant line bundle)**.** *For each* $n \in \mathbb{N}$*, there is an isomorphism* $\mathrm{Det}_{\mathrm{a}}(D_n) \cong \mathrm{Det}(D_0)$ *as line bundles over* B*.*

In §6, we investigate the counterpart of the eta invariant for the determinant of theorem 1. After extending the definition given in [11] to product-suspended operators, we relate this invariant (denoted here η_{sus}) to the extension of the original spectral definition of Atiyah, Patodi and Singer given by Wodzicki [20]. Namely consider

$$\eta_z(A) = \sum_j \operatorname{sgn}(a_j)|a_j|^{-z} \tag{7}$$

where the a_j are the eigenvalues of A in order of increasing $|a_j|$ repeated with multiplicity.

Theorem 3. *If $A \in \Psi^1(X;E)$ is an invertible self-adjoint elliptic pseudodifferential operator and $A(\tau) = A + i\tau \in \Psi^{1,1}_{\mathrm{psus}}(X;E)$ is the corresponding product-suspended family then*

$$\eta_{\mathrm{sus}}(A(\tau)) = \mathrm{reg}_{z=0}\, \eta_z(A) = \eta(A) \tag{8}$$

is the regularized value at $z = 0$ of the analytic extension of (7) *from its domain of convergence.*

The eta invariant for product-suspended operators is, as in the suspended case discussed in ([11]), a log-multiplicative functional

$$\eta_{\mathrm{sus}}(AB) = \eta_{\mathrm{sus}}(A) + \eta_{\mathrm{sus}}(B),\ A \in \Psi^{k,k'}_{\mathrm{psus}}(X;E),\ B \in \Psi^{l,l'}_{\mathrm{psus}}(X;E).$$

Finally, in §7, we show (see Theorem 4) that in the appropiate context, this eta invariant can be interpreted as the logarithm of the determinant of Theorem 1.

To discuss these results, substantial use is made of various classes of pseudodifferential operators, in particular product-type suspended operators and mixed isotropic operators. An overview of the various classes used in this paper is given in §2 and some of their properties are discussed in the appendix.

1. Determinant line bundle

Quillen in [17] introduced the determinant line bundle for a family of $\overline{\partial}$ operators. Shortly after, Bismut and Freed in [4] and [3] generalized the definition to Dirac operators. We will show here that this is induced by the Fredholm determinant, as a representation of the group $G^{-\infty}$. To do so we need to slightly generalize the standard notion of a principal bundle.

1.1. **Bundles of groups.**

Definition 1.1. Let G be a topological group (possibly infinite dimensional). Then a fibration $\mathcal{G} \to B$ over a compact manifold B with typical fibre G is called a **bundle of groups with model** G if its structure group is contained in $\mathrm{Aut}(G)$, the group of automorphisms of G.

The main example of interest here is the bundle of smoothing groups, with fibre $G^{-\infty}(Z_b)$ on the fibres of a fibration (6). In this case the group is smooth and the bundle inherits a smooth structure.

Definition 1.2. Let $\phi : \mathcal{G} \longrightarrow B$ be a bundle of groups with model G, then a (right) **principal $\mathcal{G}$-bundle** is a smooth fibration $\pi : \mathcal{P} \longrightarrow B$ with typical fibre G together with a continuous (or smooth) fibrewise group action

$$h : \mathcal{P}_b \times \mathcal{G}_b \ni (p,g) \longmapsto p \cdot g^{-1} \in \mathcal{P}_b$$

which is continous (or smooth) in all variables, locally trivial and free and transitive on the fibres. An isomorphism of principal $\mathcal{G}$-bundles is an isomorphism of the total spaces which intertwines the group actions.

The fibre actions combine to give a continous map from the fibre product

$$\mathcal{P} \times_B \mathcal{G} = \{(p,g) \in \mathcal{P} \times \mathcal{G}; \pi(p) = \phi(g)\} \longrightarrow \mathcal{P}.$$

Definition 1.2 is a generalization of the usual notion of a principal bundle for a group G in the sense that a principal G-bundle $\pi : \mathcal{P} \longrightarrow B$ is naturally a principal $\mathcal{G}$-bundle for the trivial bundle of groups $\mathcal{G} = G \times B \to B$. Any bundle of groups

$\mathcal{G} \to B$ is itself a principal $\mathcal{G}$-bundle and should be thought of as the trivial principal $\mathcal{G}$-bundle. Thus a principal $\mathcal{G}$-bundle $\mathcal{P} \to B$ is trivial, as a principal $\mathcal{G}$-bundle, if it is isomorphic as a principal $\mathcal{G}$-bundle to $\mathcal{G}$.

1.2. **Classifying principal bundles.**

Lemma 1.3. *If G has a topological classifying sequence* of groups

$$G \longrightarrow \mathrm{E}\,G \longrightarrow \mathrm{B}\,G \tag{1.1}$$

(so $\mathrm{E}\,G$ is weakly contractible) which is a Serre fibration, $\mathcal{G}$ is a bundle of groups modelled on G with structure group $H \subset \mathrm{Aut}(EG, G)$, the group of automorphisms of EG restricting to automorphisms of G, then, principal $\mathcal{G}$-bundles over compact bases are classified up to $\mathcal{G}$-isomorphism by homotoply classes of global sections of a bundle $\mathcal{G}(\mathrm{B}\,G)$ of groups with typical fibre $\mathrm{B}\,G$.

Proof. The assumption that the structure group of $\mathcal{G}$ is a subgroup of $\mathrm{Aut}(EG, G)$ allows the bundle of groups $\mathcal{G}$ to be extended to a bundle of groups with model $\mathrm{E}\,G$. Namely taking an open cover of X by sets over which $\mathcal{G}$ is trivial, the fibres may be extended to $\mathrm{E}\,G$, the transition maps then extend to the larger fibres and the cocycle condition continues to hold. Denote the resulting bundle of groups, $\mathcal{G}(\mathrm{E}\,G) \supset \mathcal{G}$, with typical fibre $\mathrm{E}\,G$. The quotient bundle

$$\mathcal{G}(\mathrm{B}\,G) = \mathcal{G}(\mathrm{E}\,G)/\mathcal{G}$$

is a bundle of groups with typical fibre $\mathrm{B}\,G$ and structure group $\mathrm{Aut}(\mathrm{E}\,G, G)$ acting on $\mathrm{B}\,G$.

Similarly, any (right) principal $\mathcal{G}$-bundle, $\mathcal{P}$, has an extension to a principal $\mathcal{G}(\mathrm{E}\,G)$-bundle, $\mathcal{P}(\mathrm{E}\,G)$,

$$\mathcal{P}(\mathrm{E}\,G)_x = \mathcal{P}_x \times \mathcal{G}(\mathrm{E}\,G)_x/\mathcal{G}_x,\ (p, e) \equiv (pg^{-1}, eg^{-1}).$$

Since the group $\mathrm{E}\,G$ is, by hypothesis, weakly contractible, and the base is compact, the extended bundle $\mathcal{P}(\mathrm{E}\,G)$ has a continuous global section. As in the case of a traditional principal bundle, the quotient of this section by the fibrewise action of $\mathcal{G}$ gives a section of $\mathcal{G}(\mathrm{B}\,G)$. Since all sections of a bundle with contractible fibre are homotopic, the section of $\mathcal{G}(\mathrm{B}\,G)$ is well-defined up to homotopy. Bundles isomorphic as principal $\mathcal{G}$ bundles give homotopic sections and the construction can be reversed as in the standard case. Namely, given a continuous section $u : B \longrightarrow \mathcal{G}(\mathrm{B}\,G)$ we may choose a 'good' open cover, $\{U_i\}$ of B, so that each of the open sets is contractible and $\mathcal{G}$ is trival over them. By assumption, the sequence (1.1) is a Serre fibration, and the fibre is weakly contractible, so it follows that u lifts to a global section $\tilde{u} : B \longrightarrow \mathcal{G}(\mathrm{E}\,G)$. The subbundle, given by the fibres $G \subset \mathrm{E}\,G$ in local trivializations, is well-defined and patches to a principal $\mathcal{G}$ bundle from which the given section can be recovered. ☐

1.3. **Associated bundles.** As in the usual case there is a notion of a vector bundle associated to a principal $\mathcal{G}$-bundle. Suppose given a fixed (real or complex) vector space V and a smooth bundle map $r : \mathcal{G} \times V \to B \times V$ which is a family of representations,

$$r_b : \mathcal{G}_b \times V \to V$$

of the $\mathcal{G}_b$. Then, from $\mathcal{P}$ and r, one can form the *associated vector bundle* $\mathcal{P} \times_r V$ with fibre

$$(\mathcal{P} \times_r V)_b = \mathcal{P}_b \times V/ \sim_b$$

where $\sim_b$ is the equivalence relation

$$(pg, r_b(g^{-1}, v)) \sim_b (p, v).$$

1.4. $\mathrm{Det}(\mathcal{P})$**.** Consider again the fibration of closed manifolds (6) and let

$$D \in \Psi^m(M/B; \mathbb{E}),\ D : \mathcal{C}^\infty(M; E^+) \to \mathcal{C}^\infty(M; E^-) \tag{1.2}$$

be a family of elliptic operators parametrized by the base B. Then

$$\begin{tikzcd} \mathcal{G}^{-\infty}(M/B; E^+) \arrow[d] \\ B \end{tikzcd}$$

with fibres

$$G^{-\infty}(Z_b; E_b^+) = \left\{ \mathrm{Id} + Q; Q \in \Psi^{-\infty}(Z_b; E^+(b)),\ \mathrm{Id} + Q_b \text{ is invertible} \right\}$$

is a bundle of groups, with model $G^{-\infty}$. To the family D we associate the bundle

$$\begin{tikzcd} G^{-\infty} \arrow[r, dash] & \mathcal{P}(D) \arrow[d] \\ & B \end{tikzcd} \tag{1.3}$$

of invertible perturbations of D by smoothing operators where the fibre at b is

$$\mathcal{P}_b(D) = \left\{ D_b + Q_b; Q_b \in \Psi^{-\infty}(Z_b; E^+, E^-),\ D_b + Q_b \text{ is invertible} \right\}.$$

The assumption that the numerical index vanishes implies that $\mathcal{P}_b(D)$ is non-empty. In fact, for each $b \in B$, the group $G^{-\infty}(Z_b; E^+(b))$ acts freely and transitively on the right on $\mathcal{P}_b(D)$ to give $\mathcal{P}(D)$ the structure of a principal $\mathcal{G}^{-\infty}(M/B; E^+)$-bundle. On the other hand, the Fredholm determinant gives a smooth map

$$\det : \mathcal{G}^{-\infty} \longrightarrow \mathbb{C}^* \cong \mathrm{GL}(1, \mathbb{C})$$

which restricts to each fibre to a representation.

Thus the construction above gives a line bundle associated to the principal bundle (1.3); for the moment we denote it $\mathrm{Det}(\mathcal{P})$.

1.5. **Quillen's definition.**

Proposition 1.4. *For an elliptic family of pseudodifferential operators of order $m > 0$ with vanishing numerical index, the determinant line bundle of Quillen,* $\mathrm{Det}(D)$, *is naturally isomorphic to the line bundle,* $\mathrm{Det}(\mathcal{P})$, *associated to the bundle* (1.3) *and the determinant as a representation of the structure group.*

Proof. First we recall Quillen's definition (following Bismut and Freed [3]). Since it extends readily we consider a pseudodifferential version rather than the original context of Dirac operators. So, for a fibration as in (6), let D be the smooth family of elliptic pseudodifferential operators of (1.2). We also set $E^\pm(b) = E\big|_{Z_b}$, $\mathcal{E}_b^\pm = \mathcal{C}^\infty(Z_b, E^\pm(b))$ and consider the infinite dimensional bundles $\mathcal{E}^\pm$ over B.

By assumption, D_b has vanishing numerical index. Choosing inner products on $E^\pm$ and a positive smooth density on the fibres of M allows the adjoint D^* of D to be defined. Then, for each $b \in B$, $D_b^* D_b : \mathcal{E}_b^+ \longrightarrow \mathcal{E}_b^+$ and $D_b D_b^* : \mathcal{E}_b^- \longrightarrow \mathcal{E}_b^-$ have a discrete spectrum with nonnegative eigenvalues. They have the same positive eigenvalues with D_b an isomorphism of the corresponding eigenspaces. Given $\lambda > 0$, the sets

$$\mathcal{U}_\lambda = \{b \in B; \lambda \text{ is not an eigenvalue of } D_b^* D_b\}$$

are open and $\mathcal{H}^+_{[0,\lambda)} \subset \mathcal{E}^+$ and $\mathcal{H}^-_{[0,\lambda)} \subset \mathcal{E}^-$, respectively spanned by the eigenfunctions of $D_b^* D_b$ and of $D_b D_b^*$ with eigenvalues less than λ, are bundles over $\mathcal{U}_\lambda$ of the same dimension, $k = k(\lambda)$. Now, $\mathcal{H}_{[0,\lambda)} = \mathcal{H}^+_{[0,\lambda)} \oplus \mathcal{H}^-_{[0,\lambda)}$ is a superbundle to which we associate the local determinant bundle

$$\operatorname{Det}(\mathcal{H}_{[0,\lambda)}) = (\wedge^k \mathcal{H}^+_{[0,\lambda)})^{-1} \otimes (\wedge^k \mathcal{H}^-_{[0,\lambda)}).$$

A linear map $P : \mathcal{H}^+_{[0,\lambda)} \to \mathcal{H}^-_{[0,\lambda)}$ induces a section

$$\det(P) = \wedge^m P : \wedge^m \mathcal{H}^+_{[0,\lambda)} \longrightarrow \wedge^m \mathcal{H}^-_{[0,\lambda)} \tag{1.4}$$

of $\operatorname{Det}(\mathcal{H}_{[0,\lambda)})$.

For $0 < \lambda < \mu$, $\mathcal{H}_{[0,\mu)} = \mathcal{H}_{[0,\lambda)} \oplus \mathcal{H}_{(\lambda,\mu)}$ over $\mathcal{U}_\lambda \cap \mathcal{U}_\mu$, where $\mathcal{H}_{(\lambda,\mu)} = \mathcal{H}^+_{(\lambda,\mu)} \oplus \mathcal{H}^-_{(\lambda,\mu)}$ and $\mathcal{H}^+_{(\lambda,\mu)}$ and $\mathcal{H}^-_{(\lambda,\mu)}$ are respectively the local vector bundles spanned by the eigenfunctions of $D_b^* D_b$ and $D_b D_b^*$ with associated eigenvalues between λ and μ. Thus, if $D_{(\lambda,\mu)}$ denotes the restriction of D to $\mathcal{H}^+_{(\lambda,\mu)}$, then (1.4) leads to transition maps

$$\phi_{\lambda,\mu} : \operatorname{Det}(\mathcal{H}_{[0,\lambda)}) \ni s \longmapsto s \otimes \det(D_{(\lambda,\mu)}) \in \operatorname{Det}(\mathcal{H}_{[0,\mu)}) \text{ over } \mathcal{U}_\lambda \cap \mathcal{U}_\mu.$$

The cocycle conditions holds over triple intersections and the resulting bundle, which is independent of choices made (up to natural isomorphism), is Quillen's determinant bundle, $\operatorname{Det}(D)$.

Let $Q_b \in \Psi^{-\infty}(Z_b; E^+, E^-)$, for $b \in \mathcal{U} \subset B$ open, be a smooth family of perturbations such that $D_b + Q_b$ is invertible; it therefore gives a section of $\mathcal{P}$ over $\mathcal{U}$. The associated bundle $\operatorname{Det}(\mathcal{P})$ is then also trivial over $\mathcal{U}$ with

$$\mathcal{U} \ni b \longrightarrow (D_b + Q_b, 1)$$

being a non-vanishing section. For $\lambda > 0$, let $P_{[0,\lambda)}$ be the projection onto $\mathcal{H}_{[0,\lambda)}$, and denote by $P^+_{[0,\lambda)}$ and $P^-_{[0,\lambda)}$ the projections onto $\mathcal{H}^+_{[0,\lambda)}$ and $\mathcal{H}^-_{[0,\lambda)}$ respectively. Then, on $\mathcal{U} \cap \mathcal{U}_\lambda$ for λ large enough, $P^-_{[0,\lambda)}(D_b + Q_b)P^+_{[0,\lambda)}$ is invertible, and one can associate to the section $D_b + Q_b$ of $\mathcal{P}$ the isomorphism

$$\begin{aligned} F_{\mathcal{U},\lambda} : \operatorname{Det}(\mathcal{P}) \ni [(D_b + Q_b, c)] \longmapsto& \\ &\det(P^-_{[0,\lambda)}(D_b + Q_b)P^+_{[0,\lambda)}) \det(A(Q_b,\lambda))c \in \operatorname{Det}(D), \end{aligned} \tag{1.5}$$

where $\det(P^-_{[0,\lambda)}(D_b + Q_b)P^+_{[0,\lambda)})$ is defined by (1.4),

$$A(Q_b, \lambda) = (D_b + P^-_{[0,\lambda)} Q_b P^+_{[0,\lambda)})^{-1}(D_b + Q_b) \in G_b^{-\infty}, \tag{1.6}$$

and $\det(A(Q_b,\lambda)) \in \mathbb{C}^*$ is the determinant defined on $G_b^{-\infty}$.

The map $\mathcal{F}_{\mathcal{U},\lambda}$ induces a global isomorphism of the two notions of determinant bundle since it is independent of choices. Indeed, it is compatible with the equivalence relation $\sim_b$ in the sense for each $g \in G^{-\infty}(Z_b; \mathcal{E}^+)$ such that both $P^-_{[0,\lambda)}(D_b + Q_b)P^+_{[0,\lambda)}$ and $P^-_{[0,\lambda)}(D_b + Q_b)gP^+_{[0,\lambda)}$ are invertible,

$$\mathcal{F}_{\mathcal{U},\lambda}((D + Q_b)g, \det(g^{-1})c) = \mathcal{F}_{\mathcal{U},\lambda}((D + Q_b), c).$$

It is also compatible with increase of λ to μ in that $\phi_{\lambda,\mu}\circ F_{\mathcal{U},\lambda} = F_{\mathcal{U},\mu}$ on $\mathcal{U}\cap\mathcal{U}_\lambda\cap\mathcal{U}_\mu$. This is readily checked

(1.7)
$$\begin{aligned}
\phi_{\lambda,\mu}\circ F_{\mathcal{U},\lambda}(D_b+Q_b,c) &= \phi_{\lambda,\mu}[\det(P^-_{[0,\lambda)}(D_b+Q_b)P^+_{[0,\lambda)})\det(A(Q_b,\lambda))c]\\
&= \det(A(Q_b,\lambda))\det(P^-_{[0,\lambda)}(D_b+Q_b)P^+_{[0,\lambda)})\otimes\det(D^+_{(\lambda,\mu)})c\\
&= \det(A(Q_b,\lambda))\det(P^-_{[0,\mu)}(D_b+P^-_{[0,\lambda)}Q_bP^+_{[0,\lambda)})P^+_{[0,\mu)})c\\
&= \det(A(Q_b,\lambda))\det(P^-_{[0,\mu)}(D_b+Q_b)P^+_{[0,\mu)})\ \times\\
&\det((D_b+P^-_{[0,\mu)}Q_bP^+_{[0,\mu)})^{-1}(D_b+P^-_{[0,\lambda)}Q_bP^+_{[0,\lambda)}))c\\
&= \det(A(Q_b,\mu))\det(P^-_{[0,\mu)}(D_b+Q_b)P^+_{[0,\mu)})c\\
&= F_{\mathcal{U},\mu}(D_b+Q_b,c)\ .
\end{aligned}$$

□

1.6. **Metric on** $\mathrm{Det}(\mathcal{P})$**.** The Quillen metric has a rather direct expression in terms of the definition of the determinant bundle as $\mathrm{Det}(\mathcal{P})$. Namely, if (D_b+Q_b) is a section of $\mathcal{P}$ over the open set $\mathcal{U}\subset B$, then

$$|(D_b+Q_b,1)|_Q = \exp(-\frac{1}{2}\zeta_b'(0)), \tag{1.8}$$

where ζ_b is the ζ-function associated to the self-adjoint positive elliptic operator $(D_b+Q_b)^*(D_b+Q_b)$ as constructed by Seeley [18]. When $A_b = \mathrm{Id}+R_b \in G^{-\infty}_b$ with $R_b : \mathcal{H}^+_{[0,\lambda)} \to \mathcal{H}^+_{[0,\lambda)}$ for some $\lambda>0$, Proposition 9.36 of [2], adapted to this context, shows that

$$|(D_b+Q_b)A_b|_Q = |\det(A_b)|\,|D_b+Q_b|_Q \tag{1.9}$$

but then by continuity the same formula follows in general. Moreover, in the form (1.8), Quillen's metric generalizes immediatly to the case of an arbitrary family of elliptic pseudodiffrential operators with vanishing numerical index.

1.7. **Primitivity.**

Lemma 1.5. *The deteminant bundle is 'primitive' in the sense that there is a natural isomorphism*

$$\mathrm{Det}(PQ)\simeq\mathrm{Det}(P)\otimes\mathrm{Det}(Q) \tag{1.10}$$

for any elliptic families $Q\in\Psi^m(M/B;E,F)$, $P\in\Psi^{m'}(M/B;F,G)$ *of vanishing numerical index.*

Proof. Let $\mathcal{P}$ and $\mathcal{Q}$ denote the principal bundles of invertible smoothing perturbations of P and Q. Let P_b+R_b and Q_b+S_b be local smooth sections over some open set U. Certainly $L_b = (P_b+R_b)(Q_b+S_b)$ is a local section of the principal bundle for PQ and $(L_b,1)$ as a local section of $\mathrm{Det}(PQ)$ may be identified with the product of the sections $(P_b+R_b,1)$ and $(Q_b,S_b,1)$ as a section of $\mathrm{Det}(P)\otimes\mathrm{Det}(Q)$. Changing the section of $\mathcal{P}$ to $(P_b+R_b)g_b$ modifies the section L_b to L_bg_b', $g_b' = (Q_b+S_b)^{-1}g_b(Q_b+S_b)$. Since

$$\det(g_b') = \det(g_b)$$

the identification is independent of choices of sections and hence is global and natural. □

Later, it will be convenient to restrict attention to first order elliptic operators. This is not a strong restriction since for $k \in \mathbb{Z}$, let $D \in \Psi^k(M/B; E, F)$ be a smooth family of elliptic pseudodifferential operators with vanishing numerical index. Let $\Delta_{M/B} \in \Psi^2(M/B; F)$ be an associated family of Laplacians, so that $\Delta_{M/B} + \mathrm{Id}$ is a family of invertible operators.

Corollary 1.6. *The family $D' = (\Delta_{M/B} + \mathrm{Id})^{-\frac{k-1}{2}} D \in \Psi^1(M/B; E, F)$ has determinant bundle isomorphic to the determinant bundle of D.*

2. Classes of pseudodifferential operators

Since several different types, and in particular combinations of types, of pseudodifferential operators are used here it seems appropriate to quickly review the essentials.

2.1. $\Psi^m(X; E, F)$. On a compact manifold without boundary the 'traditional' algebra (so consisting of 'classical' operators) may be defined in two steps using a quantization map. The smoothing operators acting between two bundles E and F may be identified as the space

$$\Psi^{-\infty}(X; E, F) = \mathcal{C}^\infty(X^2; \mathrm{Hom}(E, F) \otimes \Omega_R). \tag{2.1}$$

Here $\mathrm{Hom}(E, F)_{x,x'} = E_x \otimes F'_{x'}$ is the 'big' homomorphism bundle and $\Omega = \pi_R^* \Omega$ is the lift of the density bundle from the right factor under the projection $\pi_R : X^2 \longrightarrow X$. The space $\Psi^m(X; E, F)$ may be identified with the conormal sections, with respect to the diagonal, of the same bundle

$$\Psi^m(X; E, F) = I^m_{\mathrm{cl}}(X^2, \mathrm{Diag}; \mathrm{Hom}(E, F) \otimes \Omega_R). \tag{2.2}$$

More explicitly Weyl quantization, given by the inverse fibre Fourier transform from T^*X to TX,

$$\begin{aligned} q_g : \rho^{-m}\mathcal{C}^\infty(\overline{T^*X}; \pi^* \hom(E, F)) \ni a \longmapsto \\ (2\pi)^{-n} \int_{T^*X} \chi \exp(iv(x, y) \cdot \xi) a(m(x, y), \xi) d\xi dg \in \Psi^m(X; E, F) \end{aligned} \tag{2.3}$$

is surjective modulo $\Psi^{-\infty}(X; E, F)$. Here a Riemann metric, g, is chosen on X and used to determine a small geodesically convex neighbourhood U of the diagonal in X^2 which is identified as a neighbourhood U' of the zero section in TX by mapping $(x, y) \in U$ to $m(x, y)$, the mid-point of the geodesic joining them in X and to $v(x, y) \in T_{m(x,y)}X$, the tangent vector to the geodesic at that mid-point in terms of the length parameterization of the geodesic from y to x. The cut-off $\chi \in \mathcal{C}^\infty_c(U')$ is taken to be identically equal to 1 in a smaller neighbourhood of the diagonal. Connections on E and F are chosen and used to identify $\mathrm{Hom}(E, F)$ over U with the lift of $\hom(E, F)$ to U', $d\xi$ is the fibre density from g on T^*X and dg is the Riemannian density on the right. The symbol a is a classical symbol of order k on T^*X realized as $\rho^{-k}a'$ where $a' \in \mathcal{C}^\infty(\overline{T^*X})$ with $\overline{T^*X}$ the compact manifold with boundary arising from the radial compactification of the fibres of T^*X and $\rho_g = |\xi|_g^{-1}$ outside a compact set in T^*X is a boundary defining function for that compactification.

Then $q_g(a) \in \Psi^{-\infty}(X; E, F)$ if and only if $a \in \dot{\mathcal{C}}^\infty(\overline{T^*X})$ is a smooth function vanishing to all orders on the boundary of $\overline{T^*X}$, i.e. is a symbol of order $-\infty$. This leads to the short exact 'full symbol sequence'

$$\Psi^{-\infty}(X;E,F) \longrightarrow \Psi^{\infty}(X;E,F) \xrightarrow{\sigma_g} \mathcal{C}^\infty(S^*X;\hom(E,F))[[\rho,\rho^{-1}]] \tag{2.4}$$

with values in the Laurent series in ρ (i.e. formal power series in ρ with finite factors of ρ^{-1}). The leading part of this is the principal symbol

$$\Psi^{m-1}(X;E,F) \longrightarrow \Psi^m(X;E,F) \xrightarrow{\sigma_m} \mathcal{C}^\infty(S^*X;\hom(E,F)\otimes R_m) \tag{2.5}$$

where R_m is the trivial bundle with sections which are homogeneous of degree m over $T^*X\setminus 0$. Pseudodifferential operators act from $\mathcal{C}^\infty(X;E)$ to $\mathcal{C}^\infty(X;F)$ and composition gives a filtered product,

$$\Psi^m(X;F,G)\circ\Psi^{m'}(X;E,F)\subset\Psi^{m+m'}(X;E,G) \tag{2.6}$$

which induces a star product on the image spaces in (2.4),

$$a\star_g b = ab + \sum_{j=1}^{\infty} B_j(a,b) \tag{2.7}$$

where the B_j are smooth bilinear differential operators with polynomial coefficients on T^*X lowering total order, in terms of power series, by j. The leading part gives the multiplicativity of the principal symbol.

2.2. $\Psi^m_{\mathrm{sus}(p)}(X;E,F)$**.** There is a natural Fréchet topology on $\Psi^m(X;E,F)$, corresponding to the $\mathcal{C}^\infty$ topology on the symbol and the kernel away from the diagonal. Thus, smoothness of maps into this space is well-defined. The p-fold suspended operators are a subspace

$$\Psi^m_{\mathrm{sus}(p)}(X;E,F)\subset\mathcal{C}^\infty\left(\mathbb{R}^p;\Psi^m(X;E,F)\right) \tag{2.8}$$

in which the parameter-dependence is symbolic (and classical). In terms of the identification (2.2) this reduces to

$$\begin{multline}\Psi^m_{\mathrm{sus}(p)}(X;E,F)= \\ \mathcal{F}^{-1}_{\mathbb{R}^p}\left(I^M_{\mathrm{cl},\mathcal{S}}(X^2\times\mathbb{R}^p,\operatorname{Diag}\times\{0\};\operatorname{Hom}(E,F)\otimes\Omega_R)\right),\ M=m+\frac{p}{4}.\end{multline} \tag{2.9}$$

Here we consider conormal distributions on the non-compact space $X^2\times\mathbb{R}^p$ but with respect to the compact submanifold $\operatorname{Diag}\times\{0\}$; the suffix $\mathcal{S}$ denotes that they are to be Schwartz at infinity and then the inverse Fourier transform is taken in the Euclidean variables $\mathbb{R}^p$ giving the 'symbolic' parameters. The shift of m to M is purely notational. These kernels can also be expressed directly as in (2.3) with a replaced by

$$a\in\rho^{-m}\left(\overline{T^*X\times\mathbb{R}^p};\pi^*\hom(E,F)\right). \tag{2.10}$$

Composition, mapping and symbolic properties are completely analogous to the 'unsuspended' case. Note that we use the abbreviated notation for suffixes $\mathrm{sus}(1)=\mathrm{sus}$.

If D is a first order elliptic differential operator acting on a bundle on X then $D+i\tau\in\Psi^1_{\mathrm{sus}}(X;E)$ is elliptic in this sense and invertible, with inverse in $\Psi^{-1}_{\mathrm{sus}}(X;E)$, if D is self-adjoint and invertible. However this is not the case for general (elliptic self-adjoint) $D\in\Psi^1(X;E)$; we therefore introduce larger spaces which will capture these operators and their inverses.

2.3. $\Psi^{m,m'}_{\mathrm{psus}(p)}(X;E,F)$. By definition in (2.9), before the inverse Fourier transform is taken, the singularities of the 'kernel' are constrained to $\mathrm{Diag}\times\{0\}\subset X^2\times\mathbb{R}^p$. For product-type (really partially-product-type corresponding to the fibration of $X\times\mathbb{R}^p$ with base $\mathbb{R}^p$) the singularities are allowed to fill out the larger submanifold

$$X^2\times\{0\}\supset\mathrm{Diag}\times\{0\}. \tag{2.11}$$

Of course they are not permitted to have arbitrary singularities but rather to be conormal with respect to these two, nested, submanifolds

$$\begin{multline}\Psi^{m,m'}_{\mathrm{psus}(p)}(X;E,F)=\\ \mathcal{F}^{-1}_{\mathbb{R}^p}\left(I^{M'M}_{\mathrm{cl},\mathcal{S}}(X^2\times\mathbb{R}^p,X^2\times\{0\},\mathrm{Diag}\times\{0\};\mathrm{Hom}(E,F)\otimes\Omega_R)\right),\\ M=m+\frac{p}{4},\ M'=m'+\frac{p}{4}-\frac{n}{2}.\end{multline} \tag{2.12}$$

The space of classical product-type pseudodifferential operators is discussed succinctly in an appendix below. Away from $\mathrm{Diag}\times\{0\}$ the elements of the space on the right are just classical conormal distributions at $\{0\}\times\mathbb{R}^p$, so if $\chi\in\mathcal{C}^\infty(X^2)$ vanishes near the diagonal (or even just to infinite order on it)

$$K\in\Psi^{m,m'}_{\mathrm{psus}(p)}(X;E,F)\Longrightarrow\chi K\in\rho^{-m'}\mathcal{C}^\infty(\overline{\mathbb{R}^p}\times X^2;\mathrm{Hom}(E,F)\otimes\Omega_R) \tag{2.13}$$

is just a classical symbol in the parameters depending smoothly on the variables in X^2. Conversely, if $\chi'\in\mathcal{C}^\infty(X^2)$ has support sufficiently near the diagonal then the kernel is given by a formula as in (2.3)

$$\begin{gathered}\chi'K=(2\pi)^{-n}\int_{T^*X}\chi\exp\left(iv(x,y)\cdot\xi\right)a(m(x,y),\xi,\tau)d\xi dg,\\ a\in(\rho'')^{-m}(\rho')^{-m'}\mathcal{C}^\infty(S;\pi^*\hom(E,F)),\ S=[\overline{T^*X\times\mathbb{R}^p},0_{T^*X}\times\partial\overline{\mathbb{R}^p}].\end{gathered} \tag{2.14}$$

Here the space on which the 'symbols' are smooth functions (apart from the weight factors) is the same compactification as in (2.10) but then blown up at the part of the boundary (i.e. infinity) corresponding to finite points in the cotangent bundle. Then ρ'' is a defining function for the 'old' part of the boundary and ρ' for the new part, produced by the blow-up. Conversely (2.14) and (2.13) together (for a partition of unity) define the space of kernels.

From the general properties of blow-up, if $\rho\in\mathcal{C}^\infty(\overline{T^*X\times\mathbb{R}^p})$ is a defining function for the boundary then $\rho=\rho'\rho''$ after blow-up. From this it follows easily that

$$\Psi^m_{\mathrm{sus}(p)}(X;E,F)\subset\Psi^{m,m}_{\mathrm{psus}(p)}(X;E,F). \tag{2.15}$$

Again these 'product suspended' operators act from $\mathcal{S}(X\times\mathbb{R}^p;E)$ to $\mathcal{S}(X\times\mathbb{R}^p;F)$ and have a doubly-filtered composition

$$\Psi^{m_1,m_1'}_{\mathrm{psus}(p)}(X;F,G)\circ\Psi^{m_2,m_2'}_{\mathrm{psus}(p)}(X;E,F)\subset\Psi^{m_1+m_2,m_1'+m_2'}_{\mathrm{psus}(p)}(X;E,G). \tag{2.16}$$

The symbol map remains, but now only corresponds to the part of the amplitude in (2.14) at $\rho''=0$

$$\Psi^{m-1,m'}_{\mathrm{psus}(p)}(X;E,F)\longrightarrow\Psi^{m,m'}_{\mathrm{psus}(p)}(X;E,F)\xrightarrow{\sigma_m}\mathcal{S}^{m,m'}_{\mathrm{psus}(p)}(X;E,F) \tag{2.17}$$

with

$$\mathcal{S}^{m,m'}_{\mathrm{psus}(d)}(X;E,F)=\mathcal{C}^\infty([S(T^*\times\mathbb{R}^p),0\times\mathbb{S}^{p-1}];\hom(E,F)\otimes R_{m,m'})$$

the space of smooth sections of a bundle over the sphere bundle corresponding to $T^*X \times \mathbb{R}^p$, blown up at the image of the zero section and with $R_{m,m'}$ a trivial bundle capturing the weight factors.

The other part of the amplitude corresponds to a more global 'symbol map' called here the 'base family'

(2.18)
$$\Psi^{m,m'-1}_{\mathrm{psus}(p)}(X;E,F) \longrightarrow \Psi^{m,m'}_{\mathrm{psus}(p)}(X;E,F) \xrightarrow{\beta_{m'}} \mathcal{C}^\infty(\mathbb{S}^{p-1};\Psi^m(X;E,F)\otimes R_{m'})$$

taking values in pseudodifferential operators on X depending smoothly on the parameters 'at infinity', i.e. in $\mathbb{S}^{p-1}$ with the appropriate homogeneity bundle (over $\mathbb{S}^{p-1}$).

These two symbol maps are separately surjective and jointly surjective onto pairs satisfying the natural compatibility condition

$$\sigma_m(\beta_{m'}(A)) = \sigma_m(A)\big|_\partial \tag{2.19}$$

that the symbol family, restricted to the boundary of the space on which it is defined, is the symbol family of the base family.

An operator in this product-suspended class is 'fully elliptic' if both its symbol and its base family are invertible. If it is also invertible then its inverse is in the corresponding space with negated orders. An elliptic suspended operator is automatically fully elliptic when considered as a product-suspended operators using (2.15).

2.4. $\Psi^m_{\mathrm{iso}(2n,\epsilon)}(\mathbb{R}^n)$**.** The suspension variables for these product-suspended operators are purely parameters. However, for the adiabatic limit constructions here, on which the paper relies heavily, we use products which are non-local in the parameters.

In the trivial case of $X = \{\mathrm{pt}\}$ we are dealing just with symbols above and the corresponding non-commutative product is the 'isotropic' algebra of operators on symbols on $\mathbb{R}^{2n}$, as operators on $\mathbb{R}^n$, for any n. This is variously known as the Weyl algebra or the Moyal product (although both often are taken to mean slightly different things). The isotropic pseudodifferential operators of order k act on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and, using Euclidean Weyl quantization, may be identified with with the spaces $\rho^{-k}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}})$. Thus, in terms of their distributional kernels on $\mathbb{R}^{2n}$, this space of operators is given by essentially the same formula as (2.3)

$$\begin{gathered} q_W : \rho^{-k}\mathcal{C}^\infty(\overline{\mathbb{R}^{2n}}) \ni b \longmapsto \\ q_W(b)(t,t') = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{i(t-t')\cdot\tau} b(\frac{t+t'}{2},\tau)d\tau \in \Psi^{-k}_{\mathrm{iso}}(\mathbb{R}^n). \end{gathered} \tag{2.20}$$

This map is discussed extensively in [10]. In this case q_W, with inverse σ_W, is an isomorphism onto the algebra and restricts to an isomorphism of the 'residual' algebra $\Psi^{-\infty}_{\mathrm{iso}}(\mathbb{R}^n) = q_W(\mathcal{S}(\mathbb{R}^{2n}))$. The corresponding star product is the Moyal product.

The full product on symbols on $\mathbb{R}^{2n}$ may be written explicitly as

$$a \circ_\omega b(\zeta) = \pi^{-2n}\int_{\mathbb{R}^{8n}} e^{i\xi\cdot\xi'+i\eta\cdot\eta'+2i\omega(\xi',\eta')}a(\zeta+\xi)b(\zeta+\eta)|\omega_\xi|^n|\omega_\eta|^n \tag{2.21}$$

where the integrals are not strictly convergent but are well defined as oscillatory integrals. Here ω is the standard symplectic form on $\mathbb{R}^{2n}$. By simply using linear changes of variables, it may be seen that this product and the more general ones in which ω is replaced by an arbitrary non-degenerate antisymmetric bilinear form on $\mathbb{R}^{2n}$ are all isomorphic. In fact the product depends smoothly on ω as an antisymmetric bilinear form, even as it becomes degenerate. When $\omega \equiv 0$ the product reduces to the pointwise, commutative, product of symbols. In fact it is not necessary to assume that the underlying Euclidean space is even dimensional for this to be true; of course in the odd-dimensional case the form cannot be non-degenerate and correspondingly there is always at least one 'commutative' variable.

The adiabatic limit here corresponds to replacing the standard symplectic form ω by $\epsilon\omega$ and allowing $\epsilon \downarrow 0$. As already noted, this gives a family of products on the classical symbol spaces which is smooth in ϵ and is the commutative product at $\epsilon = 0$. We denote the resulting smooth family of algebras by $\Psi^m_{\mathrm{iso}(2n,\epsilon)}(\mathbb{R}^n)$.

2.5. $\Psi^{m,m'}_{\mathrm{iso}(2n,\epsilon)}(X;E,F)$**.** Now, we may replace the parameterized product on the product-suspended algebra by 'quantizing it' as in (2.21), in addition to the composition in X itself. For the 'adiabatic' choice of $\epsilon\omega$ this induces a one parameter family of quantized products

(2.22)
$$[0,1]_\epsilon \times \Psi^{m_1,m_1'}_{\mathrm{psus}(2n)}(X;F,G) \times \Psi^{m_2,m_2'}_{\mathrm{psus}(2n)}(X;E,F) \longrightarrow \Psi^{m_1+m_2,m_1'+m_2'}_{\mathrm{psus}(2n)}(X;E,G).$$

The suspended operators still form a subalgebra. The Taylor series as $\epsilon \downarrow 0$ is given by

$$(A \circ_\epsilon B)(u) \sim \sum_{k=0}^{\infty} \frac{(i\epsilon)^k}{2^k k!} \omega(D_v, D_w) A(v) B(w)\Big|_{v=w=u}. \tag{2.23}$$

A more complete discussion of product suspended operators and the mixed isotropic product may be found in the appendix.

2.6. $\Psi^{m,m'}_{\mathrm{psus}(2n)}(X;E,F)[[\epsilon]]$**.** This is the space of formal power series in ϵ with coefficients in $\Psi^{m,m'}_{\mathrm{psus}(2n)}(X;E,F)$. The product (2.23) projects to induce a product

(2.24)
$$\Psi^{m_1,m_1'}_{\mathrm{psus}(2n)}(X;F,G)[[\epsilon]] \times \Psi^{m_2,m_2'}_{\mathrm{psus}(2n)}(X;E,F)[[\epsilon]] \longrightarrow \Psi^{m_1+m_2,m_1'+m_2'}_{\mathrm{psus}(2n)}(X;E,G)[[\epsilon]]$$

which is consistent with the action on formal power series

$$\Psi^{m,m'}_{\mathrm{psus}(2n)}(X;E,F)[[\epsilon]] \ni A : \mathcal{C}^\infty(X;E)[[\epsilon]] \longrightarrow \mathcal{C}^\infty(X;F)[[\epsilon]].$$

3. Adiabatic determinant

Let $E \longrightarrow X$ a complex vector bundle over a compact manifold X. Consider the infinite dimensional group

$$G^{-\infty}_{\mathrm{sus}(2n)}(X;E) = \{\mathrm{Id} + Q;\ Q \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(X;E),\ \mathrm{Id} + Q \text{ is invertible}\}$$

of invertible 2n-suspended smoothing perturbations of the identity. A naive notion of determinant would be given by using the 1-form

$$d \log d(A) = \mathrm{Tr}_{\mathrm{sus}(2n)}(A^{-1} dA)$$

where

$$\mathrm{Tr}_{\mathrm{sus}(2n)}(B) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} \mathrm{Tr}(B(t,\tau)) dt d\tau$$

is the regularized trace for suspended operators as defined in [11]. The putative determinant is then given by

$$d(A) = \exp\left(\int_0^1 \operatorname{Tr}_{\mathrm{sus}(2n)}(\gamma^{-1}\frac{d\gamma}{ds})ds\right) \tag{3.1}$$

where $\gamma : [0,1] \to G^{-\infty}_{\mathrm{sus}(2n)}(X;E)$ is any smooth path such that $\gamma(0) = \mathrm{Id}$ and $\gamma(1) = A$. Although $d(A)$ is multiplicative, it is topologically trivial, in the sense that for any smooth loop $\gamma : \mathbb{S}^1 \to G^{-\infty}_{\mathrm{sus}(2n)}(X;E)$, one has

$$\int_{\mathbb{S}^1} \operatorname{Tr}_{\mathrm{sus}(2n)}(\gamma^{-1}\frac{d\gamma}{ds})ds = \frac{1}{(2\pi)^{2n}}\int_{\mathbb{R}^{2n}}\left(\int_{\mathbb{S}^1}\operatorname{Tr}(\gamma^{-1}\frac{d\gamma}{ds})ds\right)dt\,d\tau = 0. \tag{3.2}$$

So this is not a topological analogue of the usual determinant.

3.1. **Isotropic determinant.** To obtain a determinant which generates the 1-dimensional cohomology, we instead use the isotropic quantization of §B. At the cost of slightly deforming the composition law on $G^{-\infty}_{\mathrm{sus}(2n)}(X;E)$, this determinant will be multiplicative as well.

Notice first that because of the canonical identification

$$\Psi^{-\infty,-\infty}_{\mathrm{psus}(2n)}(X;E) = \Psi^{-\infty}_{\mathrm{sus}(2n)}(X;E)$$

there is no distinction between $G^{-\infty}(X;E)$ and the group

$$G^{-\infty,-\infty}_{\mathrm{psus}(2n)}(X;E) = \{\mathrm{Id}+Q;\ Q \in \Psi^{-\infty,-\infty}_{\mathrm{psus}(2n)}(X;E),\ \mathrm{Id}+Q \text{ is invertible,}\},$$

so in this context, we can interchangeably think in terms of suspended or product-suspended operators. For $\epsilon \in [0,1]$, we use the $\circ_\epsilon$-product of Theorem 5 to define the group

$$\begin{aligned} G^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(X;E) = \{\mathrm{Id}+Q;\ Q \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(X;E),\ \exists\, P \in \Psi^{0,0}_{\mathrm{psus}(2n)}(X;E),& \\ P\circ_\epsilon(\mathrm{Id}+Q) = (\mathrm{Id}+Q)\circ_\epsilon P = \mathrm{Id}\}.& \end{aligned} \tag{3.3}$$

For $\epsilon = 0$, we have the canonical group isomorphism

$$G^{-\infty}_{\mathrm{iso}(2n,0)}(X;E) = G^{-\infty}_{\mathrm{sus}(2n)}(X;E).$$

On the other hand, for $\epsilon > 0$, the group $G^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(X;E)$ is isomorphic to $G^{-\infty}$ so that it is possible to transfer the Fredholm determinant to it.

Proposition 3.1. *For $\epsilon > 0$, there is a natural multiplicative determinant*

$$\det\nolimits_\epsilon(A) : G^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(X;E) \to \mathbb{C}^*$$

defined for $A \in G^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(X;E)$ by

$$\det\nolimits_\epsilon(A) = \exp\left(\int_0^1 \operatorname{Tr}_\epsilon(\gamma^{-1}\circ_\epsilon\frac{d\gamma}{ds})ds\right)$$

where $\gamma : [0,1] \to G^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(X;E)$ is any smooth path with $\gamma(0) = \mathrm{Id}$ and $\gamma(1) = A$ so

$$d\log\det\nolimits_\epsilon(A) = \operatorname{Tr}_\epsilon(A^{-1}\circ_\epsilon dA).$$

Proof. To show that $\det_\epsilon$ is well-defined and multiplicative, it suffices to show that it reduces to the Fredholm determinant under a suitable identification of $G^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(X;E)$ with $G^{-\infty}$. From Appendix C it follows that $G^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(X;E)$ acts on $\mathcal{S}(X\times\mathbb{R}^n;E)$. Fix a Riemannian metric g on X and a Hermitian metric h on E. Let $\Delta\in\Psi^2(X;E)$ be the corresponding Laplace operator. Then consider the mixed isotropic operator

$$\Box_\epsilon = \Delta + \epsilon\sum_{i=1}^n\left(-\frac{\partial^2}{\partial t_i^2}+t_i^2\right)\in\Psi^2_{\mathrm{iso}(2n,\epsilon)}(X;E)$$

where $\sum_{i=1}^n\left(-\frac{\partial^2}{\partial t_i^2}+t_i^2\right)$ is the harmonic oscillator on $\mathbb{R}^n$. As an operator acting on $\mathcal{S}(X\times\mathbb{R}^n;E)$, $\Box_\epsilon$ has a positive discrete spectrum. Let $\{\lambda_k\}_{k\in\mathbb{N}}$ be the eigenvalues, in non-decreasing order, with corresponding eigensections

$$\Box_\epsilon f_k=\lambda_k f_k,\quad f_k\in\mathcal{S}(X\times\mathbb{R}^n;E)$$

such that $\{f_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of $\mathrm{L}^2(X\times\mathbb{R}^n;E)$. This gives an algebra isomorphism

$$\mathcal{F}_\epsilon:\Psi^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(X;E)\ni A\longmapsto\langle f_i,Af_j\rangle_{\mathrm{L}^2}\in\Psi^{-\infty}.$$

and a corresponding group isomorphism $\mathcal{F}_\epsilon:G^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(X;E)\to G^{-\infty}$. Under these isomorphisms, one has

$$\mathrm{Tr}_\epsilon(A)=\mathrm{Tr}(\mathcal{F}_\epsilon(A))$$

and consequently

$$\det{}_\epsilon(\mathrm{Id}+A)=\det{}_{\mathrm{Fr}}(\mathcal{F}_\epsilon(\mathrm{Id}+A)).\tag{3.4}$$

□

3.2. **Asymptotics of** $\det_\epsilon$**.** Now, for any $\delta>0$ we can consider the group of sections,

$$\begin{aligned}G^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E)=\{A\in\mathcal{C}^\infty([0,\delta];\mathrm{Id}+\Psi^{-\infty}(\mathbb{R}^{2n}\times X));\\ A(\epsilon)\in G^{-\infty}_{\mathrm{iso},\epsilon}(\mathbb{R}^{2n}\times X;E)\ \forall\ \epsilon\in[0,\delta]\}.\end{aligned}\tag{3.5}$$

Proposition 3.2. *The determinant with respect to the $\circ_\epsilon$ product defines*

$$\widetilde{\det}:G^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E)\longrightarrow\mathcal{C}^\infty((0,\delta])\tag{3.6}$$

which takes the form

$$\widetilde{\det}(A)(\epsilon)=\exp\left(\sum_{k=0}^{n-1}\epsilon^{k-n}a_k(A)\right)F_\epsilon(A)\ \forall\ A\in G^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E)\tag{3.7}$$

where

$$F:G^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E)\ni A\longmapsto F_\epsilon(A)\in\mathcal{C}^\infty([0,\delta])\tag{3.8}$$

and $a_k:G^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E)\longrightarrow\mathbb{C}$ are $\mathcal{C}^\infty$ functions and the a_k only depend on the Taylor series of A.

Proof. Since the group is open (for each $\epsilon\in[0,1]$ and also for the whole group) the tangent space at any point is simply $\mathcal{C}^\infty([0,\delta];\Psi^{-\infty}(\mathbb{R}^{2n}\times X;E))$. With the usual identifications for a Lie group the form $A^{-1}\circ_\epsilon dA$ therefore takes values in

$\mathcal{C}^\infty([0,\delta];\Psi^{-\infty}(\mathbb{R}^{2n}\times X;E))$. On the other hand, the trace functional is *not* smooth down to $\epsilon=0$. In fact it is rescaled by a factor of ϵ^{-n}. Thus,

$$d\log\det_\epsilon(A)=\operatorname{Tr}_\epsilon(A^{-1}\circ_\epsilon dA)\sim\sum_{k=0}^{\infty}\alpha_k\epsilon^{k-n} \tag{3.9}$$

is ϵ^{-n} times a smooth function.

For any smooth map

$$f:\mathbb{S}^1\to G^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E),$$

the integral $\int_{\mathbb{S}^1}f^*d\log\det_\epsilon(A)$ also has an asymptotic expansion

$$\int_{\mathbb{S}^1}f^*d\log\det_\epsilon(A)\sim\sum_{k=0}^{\infty}c_k\epsilon^{k-n}\ \ ,c_k=\int_{\mathbb{S}^1}\alpha_k\in\mathbb{C}. \tag{3.10}$$

On the other hand, by (3.4), this is a winding number so cannot depend on ϵ. Hence

$$c_k=\int_{\mathbb{S}^1}\alpha_k=0\text{ for }k\neq n. \tag{3.11}$$

So, for $k\neq n$, α_k is exact and then (3.7) follows directly by integration along any path $\gamma:[0,1]\to G^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E)$ with $\gamma(0)=\mathrm{Id}$ and $\gamma(1)=A$. The range space is path-connected, so

$$a_k(A)=\int_0^1\gamma^*\alpha_k,\quad k<n$$

is independent of the path and well-defined. □

3.3. **Star product.**

The restriction map at $\epsilon=0$

$$R:G^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E)\longrightarrow G^{-\infty}_{\mathrm{sus}(2n)}(X;E) \tag{3.12}$$

is surjective. From this it follows that if we let $\dot{G}^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E)$ be the subgroup of those elements which are equal to the identity to infinite order at $\epsilon=0$ then the quotient

$$\begin{multline}G^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E)/\dot{G}^{-\infty}_{\mathrm{iso}}([0,\delta]\times\mathbb{R}^{2n}\times X;E)=\\ G^{-\infty}_{\mathrm{sus}(2n)}(X;E)+\ \epsilon\Psi^{-\infty}(\mathbb{R}^{2n}\times X;E)[[\epsilon]]\end{multline} \tag{3.13}$$

is the obvious formal power series group, namely with invertible leading term and arbitrary smoothing lower order terms. The composition law is the one induced by the $\circ_\epsilon$-product. Since the higher order terms in ϵ amount to an affine extension of the leading part, this formal power series group is also a classifying group for odd K-theory.

Definition 3.3. We denote by $\Psi^{k,k'}_{\mathrm{psus}(2n)}(X;E)[[\varepsilon]]$, $k,k'\in\mathbb{R}\cup\{-\infty\}$, the space of formal series

$$\sum_{\mu=0}^{\infty}a_\mu\varepsilon^\mu$$

with coefficients $a_\mu\in\Psi^{k,k'}_{\mathrm{psus}(2n)}(X;E)$, where ε is a formal parameter.

For $A \in \Psi^{k,k'}_{\mathrm{psus}(2n)}(X;F,G)[[\varepsilon]]$ and $B \in \Psi^{l,l'}_{\mathrm{psus}(2n)}(X;E,F)[[\varepsilon]]$ the $*$-product $A * B \in \Psi^{k+l,k'+l'}_{\mathrm{psus}(2n)}(X;E,G)[[\varepsilon]]$ is

$$\begin{aligned} A * B(u) &= (\sum_{\mu=0}^{\infty} a_\mu \varepsilon^\mu) * (\sum_{\nu=0}^{\infty} b_\nu \varepsilon^\nu) \\ &= \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \varepsilon^{\mu+\nu} \left(\sum_{p=0}^{\infty} \frac{i^p \varepsilon^p}{2^p p!} \omega(D_v, D_w)^p A(v)B(w)\bigg|_{v=w=u} \right) \end{aligned}$$

where $u, v, w \in \mathbb{R}^{2n}$.

Since this is based on the asymptotic expansion (C.3) of Appendix B, its associativity follows immediately from the associativity of the $\circ_\epsilon$-product.

3.4. **Adiabatic determinant.** This product is consistent with that of the quotient group in (3.13), so

Lemma 3.4. *The quotient group* $G^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\epsilon]]$ *of* (3.13) *is canonically isomorphic to*

$$\begin{aligned} G^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]] = \{(\mathrm{Id} + Q); Q \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]],\ \exists\, P \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]] \\ \textit{such that } (\mathrm{Id} + Q) * (\mathrm{Id} + P) = \mathrm{Id} \in \Psi^{0}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]]\}. \end{aligned}$$

We can now prove Theorem 1 stated in the Introduction.

Theorem 1. *The functional*

$$F_0(A) : G^{-\infty}_{\mathrm{iso}}([0,\delta] \times \mathbb{R}^{2n} \times X; E) \to \mathbb{C}^*$$

induces a multiplicative determinant $\det_{\mathrm{a}}$ *on the formal power series group*

$$G^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]]$$

in the sense discussed above, i.e. it is a smooth multiplicative function which generates H^1 *.*

Proof. From (3.9),

$$F_0(A) = \exp\left(\int_\gamma \alpha_n\right), \tag{3.14}$$

where $\gamma : [0,1] \to G^{-\infty}_{\mathrm{sus}(2n)}(X;E)[[\varepsilon]]$ is any smooth path with $\gamma(0) = \mathrm{Id}$ and $\gamma(1) = A$. In the expansion (3.9), the only non-trivial cohomological contribution comes from α_n. Since $\det_\epsilon$ corresponds to the Fredholm determinant under the identification of $G^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(X;E)$ with $G^{-\infty}$ the integral of α_n along a generator of the fundamental group is $\pm 2\pi i$. Thus, the determinant induced by $F_0(A)$ has the desired topological behavior.

For the multiplicativity, from (3.9),

$$\begin{aligned} \mathrm{Tr}((AB)^{-1} d(AB)) &= \mathrm{Tr}(B^{-1}A^{-1}dAB + B^{-1}A^{-1}AdB) \\ &= \mathrm{Tr}(A^{-1}dA) + \mathrm{Tr}(B^{-1}dB)\ , \end{aligned} \tag{3.15}$$

where the $*$-product is used to compose elements and define the inverses. From the ε-expansion of (3.15),

$$\alpha_n(A * B) = \alpha_n(A) + \alpha_n(B). \tag{3.16}$$

As a consequence, the determinant defined in (3.1) is multiplicative. □

This determinant can be used to define the determinant line bundle of a fully elliptic family $D \in \Psi^{k,k}_{\mathrm{psus}(2n)}(M/B;E,F)[[\varepsilon]]$ of fibrewise product $2n$-suspended pseudodifferential operators on a fibration (6). Full ellipticity here corresponds to ellipticity of the leading term D_0 and its invertibility for large values of the parameters. Assume in addition that for each $b \in B$, $D_b \in \Psi^{k,k}_{\mathrm{psus}(2n)}(Z_b;E_b,F_b)[[\varepsilon]]$ can be perturbed by $Q_b \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(Z_b;E_b,F_b)[[\varepsilon]]$ to be invertible, where invertibility is equivalent to invertibility of the leading term. Then over the manifold B, consider the bundle of invertible smoothing perturbations with fibres

$$
\begin{aligned}
(3.17)\quad \mathcal{P}_b(D) = \{D_b + Q_b; Q_b \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(Z_b;E_b,F_b)[[\varepsilon]],\\
\exists\, P \in \Psi^{-k,-k}_{\mathrm{psus}(2n)}(Z_b;F_b,E_b)[[\varepsilon]],\ P*(D_b+Q_b) = (D_b+Q_b)*P = \mathrm{Id}\}.
\end{aligned}
$$

Let

$$
(3.18)\qquad \begin{array}{ccc}
G^{-\infty}_{\mathrm{sus}(2n)} & \longrightarrow & G^{-\infty}_{\mathrm{sus}(2n)}(M/B;E)\\
 & & \big\downarrow \phi\\
 & & B
\end{array}
$$

be the bundle of groups with fibre at $b \in B$

$$
\begin{aligned}
(3.19)\quad G^{-\infty}_{\mathrm{sus}(2n)}(Z_b;E_b)[[\varepsilon]] = \{\mathrm{Id} + Q; Q \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(Z_b;E_b)[[\varepsilon]],\\
\exists\, P \in \Psi^{0,0}_{\mathrm{psus}(2n)}(Z_b;E_b)[[\varepsilon]] P*(\mathrm{Id}+Q) = (\mathrm{Id}+Q)*P = \mathrm{Id}\}.
\end{aligned}
$$

Then $\mathcal{P}(D)$ is a principal $G^{-\infty}_{\mathrm{sus}(2n)}(M/B;E)[[\varepsilon]]$-bundle in the sense of Definition 1.2.

Definition 3.5. The *adiabatic determinant line bundle* associated to the family D of product $2n$-suspended elliptic pseudodifferential operators is

$$
\mathrm{Det}_{\mathrm{a}}(D) = \mathcal{P}(D) \times_{\det_{\mathrm{a}}} \mathbb{C}
$$

induced by the adiabatic determinant as representation of the bundle of groups (3.18).

4. Periodicity of the numerical index

In the next section, we establish a relation between the determinant line bundles of a family of standard elliptic pseudodifferential operators and the determinant line bundle just defined for families of $2n$-suspended operators. Here we consider the corresponding question for the numerical index.

4.1. **Product-suspended index.** A product-suspended operator

$$
P \in \Psi^{m,m'}_{\mathrm{psus}(k)}(Z;E,F)
$$

is **fully elliptic** if both its symbol in the usual sense and its base family are invertible. Here the base family, elliptic because of the invertibility of the symbol, is parameterized by $\mathbb{S}^{k-1}$. As a family of operators over $\mathbb{R}^k$, P has a families index.

Since by assumption the family is invertible at, and hence near, infinity the family defines an index class in compactly-supported K-theory

$$\operatorname{ind}(P) \in K^0(\mathbb{R}^k) = \begin{cases} \mathbb{Z} & k \text{ even} \\ \{0\} & k \text{ odd.} \end{cases} \tag{4.1}$$

Thus by choosing a generator (i.e. Bott element) in $K^0(\mathbb{R}^{2n})$ a product $2n$-suspended family has a numerical index which we will denote $\operatorname{ind}_{\text{sus}(n)}$ (since it only arises for even numbers of parameters). The families index of Atiyah and Singer does not apply directly to this setting although it *does* apply if the operator is in the 'suspended' subspace (and so in particular $m' = m$.) Using the properties of the suspended eta invariant we will show in §9 that the suspended index can be expressed in terms of the 'adiabatic' η invariant discussed below. Namely, suppose a linear decomposition $\mathbb{R}^{2n} = \mathbb{R} \times \mathbb{R}^{2n-1}$ is chosen in which the variables are written τ and ξ. Then, for some $R \in \mathbb{R}$, $P(\tau, \xi)$ is invertible for $|\tau| \geq R$ for all $\xi \in \mathbb{R}^{2n-1}$. Furthermore, by standard index arguments we may find a family of smoothing operators, A, of compact support in (τ, ξ) such that $P' = P + A$ is invertible for all $\tau \leq R$. Then

$$\operatorname{ind}_{\text{sus}(n)}(P) = -\frac{1}{2}\left(\eta_{\text{a(n-1)}}(P\big|_{\tau=R}) - \eta_{\text{a(n-1)}}(P'\big|_{\tau=R})\right). \tag{4.2}$$

4.2. **Periodicity.** Here we show that there is a 'Bott map' from ordinary pseudo-differential operators into product-type suspended operators which maps the usual index to the suspended index (although most of the proof is postponed until later). Thus if $D \in \Psi^1(Z; E, F)$ is an elliptic operator then

$$\mathbb{R}^2 \ni (t, \tau) \longmapsto \hat{D}(t, \tau) = \begin{pmatrix} it - \tau & D^* \\ D & it + \tau \end{pmatrix}, \ \hat{D} \in \Psi^{1,1}_{\text{psus}(2)}(Z; E \oplus F) \tag{4.3}$$

is an associated twice-suspended fully elliptic operator. In [12], such a family is realized explicitly as the indicial family of a product-suspended cusp operator. The ellipticity of $\hat{D}$ follows from the fact that

$$\hat{D}^*\hat{D} = \begin{pmatrix} D^*D + t^2 + \tau^2 & 0 \\ 0 & DD^* + t^2 + \tau^2 \end{pmatrix} \in \Psi^{2,2}_{\text{psus}(2)}(Z; E \oplus F) \tag{4.4}$$

is an elliptic family which is invertible for $t^2 + \tau^2 > 0$.

Definition 4.1. Given an elliptic operator $D \in \Psi^1(Z; E, F)$, we define by recurrence on $n \in \mathbb{N}_0$, elliptic product-suspended operators $D_n \in \Psi^{1,1}_{\text{psus}(2n)}(Z; 2^{n-1}(E \oplus F)$ by

$$D_n(t_1, \ldots, t_n, \tau_1, \ldots, \tau_n) = \begin{pmatrix} it_n - \tau_n & D^*_{n-1} \\ D_{n-1} & it_n + \tau_n \end{pmatrix}$$

with $D_0 = D$.

Lemma 4.2. *If D is elliptic then D_n is a totally elliptic product $2n$-suspended operator for all n and* $\operatorname{ind}_{\text{sus}(n)}(D_n) = \operatorname{ind}(D)$.

Proof. Both the ordinary index and the n-suspended index (on fully elliptic $2n$-suspended operators) are homotopy invariant. Since the map $D \longmapsto D_n$ maps invertible operators to invertible operators it follows that $\operatorname{ind}(D) = 0$ implies $\operatorname{ind}_{\text{sus}(n)}(D_n) = 0$. Indeed, $\operatorname{ind}(D) = 0$ means there exists a smoothing operator

$Q \in \Psi^{-\infty}(Z; E, F)$ such that $D + Q$ is invertible. Then $(D + sQ)_n$ is a homotopy of fully elliptic $2n$-suspended operators which is invertible for $s = 1$ so $\operatorname{ind}_{\mathrm{sus}(n)}(D_n) = 0$.

The actual equality of the index is proved below in §9, using (4.2). $\square$

5. Periodicity of the determinant line bundle

5.1. **Adiabatic determinant bundle.** Returning to the setting of a fibration with compact fibres, $\phi : M \to B$, as in (6), let $D \in \Psi^1(M/B; E, F)$ be a family of elliptic pseudodifferential operators with vanishing numerical index. From Lemma 4.2 (the part that is already proved), the suspended index of the fully elliptic family $D_n \in \Psi^{1,1}_{\mathrm{psus}(2n)}(M/B; E, F)$, given by Definition 4.1, also vanishes. Thus the fibres

$$\begin{aligned}(5.1)\quad \mathcal{P}_{\mathrm{psus}(2n)}(D_n)_b = \{D_{n,b} + Q_b; Q_b \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(Z_b; 2^{n-1}(E_b \oplus F_b))[[\varepsilon]],\\ \exists\ (\hat{D}_{n,b} + Q_b)^{-1} \in \Psi^{-1}_{\mathrm{psus}(2n)}(Z_b; 2^{n-1}(E_b \oplus F_b))[[\varepsilon]]\}\end{aligned}$$

are non-empty and combine to give a principal-$G^{-\infty}_{\mathrm{sus}(2n)}(M/B; 2^{n-1}(E \oplus F))[[\varepsilon]]$-bundle as in (3.17). Since we have defined an adiabatic determinant on these groups we have an associated determinant bundle

$$(5.2)\qquad \operatorname{Det}_{\mathrm{sus}(2n)}(D) = \operatorname{Det}_{\mathrm{a}}(D_n) = \mathcal{P}_{\mathrm{psus}\,2n}(D) \times_{\det_{\mathrm{a}}} \mathbb{C}.$$

5.2. **Isotropic determinant bundle.** One can make a different, but similar, construction using the isotropic quantization of D_n.

Definition 5.1. For $\epsilon > 0$, let ${}^{\epsilon}\hat{D}_n \in \Psi^{1,1}_{\mathrm{iso}(2n,\epsilon)}(M/B; 2^{n-1}(E \oplus F))$ be the isotropic quantization of D_n as in Appendic C, so giving an operator on $\mathcal{S}(\mathbb{R}^n \times X; 2^{n-1}(E_b \oplus F_b))$.

As discussed earlier for families of standard elliptic operators, there are two equivalent definitions of the determinant line bundle for ${}^{\epsilon}\hat{D}_n$. Namely, Quillen's spectral definition or as an associated bundle to the principal bundle of invertible perturbations. In the latter case, the principal $G^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(M/B; 2^{n-1}(E \oplus F))[[\varepsilon]]$-bundle has fibre

$$\begin{aligned}(5.3)\quad \mathcal{P}_{\mathrm{iso}(2n,\epsilon)}({}^{\epsilon}\hat{D}_{en})_b = \{{}^{\epsilon}\hat{D}_{2n,b} + Q_b; Q_b \in \Psi^{-\infty}_{\mathrm{iso}(2n,\epsilon)}(Z_b; 2^{n-1}(E_b \oplus F_b)),\\ \exists\ ({}^{\epsilon}\hat{D}_{2n,b} + Q_b)^{-1} \in \Psi^{-1,-1}_{\mathrm{iso}(2n,\epsilon)}(Z_b; 2^{n-1}(E_b \oplus F_b))\}.\end{aligned}$$

Note that this fibre is non-empty as soon as the original family D has vanishing numerical index. Indeed, we know that D_n then has vanishing suspended index and hence has an invertible perturbation by a smoothing operator (in the suspended sense). The isotropic product is smooth down to $\epsilon = 0$, where it reduces to the suspended product (pointwise in the parameters). Thus such a perturbation is invertible with respect to the isotropic product for small $\epsilon > 0$. Since these products are all isomorphic for $\epsilon > 0$, it follows that perturbations as required in (5.3) do exist.

Proposition 5.2. *Let $D \in \Psi^1(M/B; E, F)$ be an elliptic family with vanishing numerical index, then for each $n \in \mathbb{N}_0$ and $\epsilon > 0$, the determinant line bundle* $\operatorname{Det}({}^{\epsilon}\hat{D}_n)$ *is naturally isomorphic to the determinant line bundle* $\operatorname{Det}(D)$.

Proof. The proof is by induction on $n \in \mathbb{N}_0$ starting with the trivial case $n = 0$. We proceed to show that $\operatorname{Det}({}^\epsilon\hat{D}_{n+1}) \cong \operatorname{Det}({}^\epsilon\hat{D}_n)$. In Quillen's definition of the determinant line bundle, only the eigenfunctions of the low eigenvalues are involved and the strategy is to identify the eigensections of the low eigenvalues of ${}^\epsilon\hat{D}_n$ with those of ${}^\epsilon\hat{D}_{n+1}$.

The isotropic quantization of the polynomial $\tau_n^2 + t_n^2$, is the harmonic oscillator, H_n^ϵ, so

$$
{}^\epsilon\hat{D}^*_{n+1,b}\,{}^\epsilon\hat{D}_{n+1,b} = \begin{pmatrix} {}^\epsilon\hat{D}^*_{n,b}\,{}^\epsilon\hat{D}_{n,b} + H^\epsilon_{n+1} - \epsilon & 0 \\ 0 & {}^\epsilon\hat{D}_{n,b}\,{}^\epsilon\hat{D}^*_{n,b} + H^\epsilon_{n+1} + \epsilon \end{pmatrix}. \tag{5.4}
$$

The eigenvalues of H^ϵ_{n+1} are positive, with the smallest being simple. The eigensections of ${}^\epsilon\hat{D}^*_{n+1,b}\,{}^\epsilon\hat{D}_{n+1,b}$ and ${}^\epsilon\hat{D}_{n+1,b}\,{}^\epsilon\hat{D}^*_{n+1,b}$ with small eigenvalues are of the form

$$
\Phi^+(f_b) = \begin{pmatrix} \varphi_{n+1} \otimes f \\ 0 \end{pmatrix}, \quad \Phi^-(f_b) = \begin{pmatrix} 0 \\ \varphi_{n+1} \otimes {}^\epsilon\hat{D}_{n,b} f \end{pmatrix} \tag{5.5}
$$

where f is an eigenfunction of ${}^\epsilon\hat{D}^*_{n,b}\,{}^\epsilon\hat{D}_{n,b}$ with eigenvalue less than 2ϵ. Note also that on such an eigenfunction, $\hat{D}_{n+1,b}$ acts as

$$
\begin{pmatrix} iC^*_{n+1} & {}^\epsilon\hat{D}^*_{n,b} \\ {}^\epsilon\hat{D}_{n,b} & iC_{n+1} \end{pmatrix} \Phi^+(f_b) = \Phi^-(f_b) \tag{5.6}
$$

since $C^*_{n+1}\varphi_{n+1} = 0$. For $0 < \lambda < 2\epsilon$, consider the open set

$$
\mathcal{U}_\lambda = \{b \in B; \lambda \text{ is not an eigenvalue of } D^*_b D_b\}. \tag{5.7}
$$

Let $\mathcal{H}^{+,k}_{[0,\lambda)}$ denote the vector bundle over $\mathcal{U}_\lambda$ spanned by the eigenfunctions of ${}^\epsilon\hat{D}^*_{k,b}\,{}^\epsilon\hat{D}_{k,b}$ with eigenvalues less than λ. Let $\mathcal{H}^{-,k}_{[0,\lambda)}$ denote the vector bundle over $\mathcal{U}_\lambda$ spanned by the eigenfunctions of ${}^\epsilon\hat{D}_{k,b}\,{}^\epsilon\hat{D}^*_{k,b}$ with eigenvalues less than λ. Then there are natural identifications

$$
F^{\pm,n}_{\mathcal{U},\lambda} : \mathcal{H}^{\pm,n}_{[0,\lambda)} \ni f_b \longmapsto \Phi^\pm(f_b) \in \mathcal{H}^{\pm,n+1}_{[0,\lambda)}. \tag{5.8}
$$

Thus, directly from Quillen's definition of the determinant bundle ${}^\epsilon\hat{D}_{n+1,b}$ and ${}^\epsilon\hat{D}_{n,b}$ have isomorphic determinant line bundles. □

5.3. Adiabatic limit of $\operatorname{Det}({}^\epsilon\hat{D}_n)$.

Proposition 5.3. *If $P \in \Psi^m_{\mathrm{psus}(2n)}(M/B; E, F)$ is a family of fully elliptic operators with vanishing numerical index then the bundle over $B \times [0,1]$ with fibre*

$$
\begin{aligned}
\mathcal{P}_{b,\epsilon} = \big\{ & Q \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(Z_b, E_b); \\
& \exists\ (P+Q)^{-1} \in \Psi^{-m}_{\mathrm{psus}(2n)}(Z_b; F_b, E_b) \text{ for the } \epsilon \text{ isotropic product.} \big\}
\end{aligned} \tag{5.9}
$$

is a principal $\mathcal{G}$-bundle for the bundle of groups with fibre

$$
\begin{aligned}
\mathcal{G}^{-\infty}_{\mathrm{sus}(2n),\epsilon}(Z_b, E_b) = \big\{ & \operatorname{Id} + A,\ A \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(Z_b, E_b); \\
& \exists\ (\operatorname{Id} + A)^{-1} = \operatorname{Id} + B,\ B \in \Psi^{-\infty}_{\mathrm{sus}(2n)}(Z_b, E_b) \text{ for the } \epsilon \text{ isotropic product} \big\}
\end{aligned} \tag{5.10}
$$

and the associated determinant bundle defined over $\epsilon > 0$ extends smoothly down to $\epsilon = 0$ and at $\epsilon = 0$ is induced by the adiabatic determinant.

Proof. This is just the smoothness of the 'rescaled' determinant (i.e. with the singular terms removed) down to $\epsilon = 0$. □

We will now complete the proof of Theorem 2 in the Introduction which we slightly restate as

Theorem 2 (Periodicity of the determinant line bundle)**.** *Let $D \in \Psi^1(M/B; E, F)$ be an elliptic family with vanishing numerical index, then for $n \in \mathbb{N}$ and $\epsilon > 0$,*

$$\mathrm{Det_a}(D_n) \cong \mathrm{Det}({}^{\epsilon}\hat{D}_n) \cong \mathrm{Det}(D).$$

Proof. The existence of the second isomorphism follows from Proposition 5.2. The first follows from Proposition 5.3. □

6. Eta invariant

In [11] a form of the eta invariant was discussed for elliptic and invertible once-suspended families of pseudodifferential operators. Applied to the spectral family (on the imaginary axis) of a self-adjoint invertible Dirac operator this new definition was shown to reduce to the original definition, of Atiyah, Patodi and Singer in [1] of the eta invariant of a single operator. Here, the definition in [11] is shown to extend to (fully) elliptic, invertible, product-suspended families. In §9 it is further extended to such product-suspended families in any odd number of variables. The extension to single-parameter product-suspended operators allows us to apply the definition to $A + i\tau$, $\tau \in \mathbb{R}$, for $A \in \Psi^1(X; E)$ an invertible elliptic selfadjoint pseudodifferential operator and check that this reduces to the spectral definition, now as given by Wodzicki ([20]). Again the extended (and below also the 'adiabatic') eta invariant gives a log-multiplicative function for invertible families

$$\eta(AB) = \eta(A) + \eta(B) \tag{6.1}$$

and this allows us to show quite directly that the associated τ invariant is a determinant in the sense discussed above.

6.1. Product-suspended eta.

If $B \in \Psi^{m,m'}_{\mathrm{psus}}(X; E)$ is a product-suspended family it satisfies

$$\frac{\partial^N}{\partial \tau^N} B(\tau) \in \Psi^{m-N, m'-N}_{\mathrm{psus}}(X; E) \ \forall\ N \in \mathbb{N}_0. \tag{6.2}$$

This implies that for N large, say $N > \dim X + m$, the differentiated family takes values in operators of trace class on L^2.

Proposition 6.1. *For any m, $m' \in \mathbb{Z}$, if $N \in \mathbb{N}$ is chosen sufficiently large then,*

$$B \in \Psi^{m,m'}_{\mathrm{psus}}(X; E) \Longrightarrow \mathrm{Tr}_E \left(\frac{\partial^N}{\partial \tau^N} B(\tau) \right) \in \mathcal{C}^\infty(\mathbb{R}^p) \tag{6.3}$$

has a complete asymptotic expansion (possibly with logarithms) as $\tau \to \pm\infty$ and the coefficient of T^0 in the expansion as $T \to \infty$

$$\begin{gathered} \overline{\mathrm{Tr}}(B) = \operatorname*{LIM}_{T \to \infty} F_{B,N}(T), \\ F_{B,N}(T) = \int_{-T}^{T} \int_0^{t_1} \dots \int_0^{t_N} \mathrm{Tr}_E \left(\frac{\partial^N}{\partial s^N} B(s) \right) ds\, dt_N \dots dt_1 \end{gathered} \tag{6.4}$$

is independent of the choice of N and defines a trace functional

$$\overline{\mathrm{Tr}} : \Psi^{\mathbb{Z},\mathbb{Z}}_{\mathrm{psus}}(X;E) \longrightarrow \mathbb{C},\ \overline{\mathrm{Tr}}([A,B]) = 0\ \forall\ A,B \in \Psi^{\mathbb{Z},\mathbb{Z}}_{\mathrm{psus}}(X;E) \tag{6.5}$$

which reduces to

$$\overline{\mathrm{Tr}}(B) = \int_{\mathbb{R}} \mathrm{Tr}_E\,(B(\tau))\,d\tau\ \forall\ A \in \Psi^{-\infty,-\infty}_{\mathrm{psus}}(X;E). \tag{6.6}$$

Proof. As already noted, $\partial_s^N B(s)$ is a continuous family of trace class operators as soon as $N > \dim X + m$. Then (6.3) is a continuous function and further differentiation again gives a continuous family of trace class operators so the trace is smooth.

To see that this function has a complete asymptotic expansion we appeal to the discussion of the structure of the kernels of such product-suspended families in Appendix B. It suffices to consider the trace of a general element $B \in \Psi^{-n-1,0}_{\mathrm{psus}}(X;E)$. Since the kernels form a module over $\mathcal{C}^\infty(X^2)$ we can localize in the base variable (not directly in the suspended variable since that has global properties). Localizing near a point away from the diagonal gives a classical symbol in the suspending variable with values in the smoothing operators. Since the trace is the integral over the diagonal this makes no contribution to (6.3). Thus it suffices to suppose that B is supported in the product of a coordinate neighbourhood with itself over which the bundle E is trivial. Locally (see (2.3)) the kernel is given by Weyl quantization of a product-type symbol so the trace becomes the integral of the sum of the diagonal terms and hence we need only consider

$$\frac{1}{(2\pi)^n} \int a(x,\xi,\tau)dxd\xi \tag{6.7}$$

where a is compactly supported in the base variables x. Now by definition, a is a smooth function, with compact support, on the product $\mathbb{R}^n \times [\overline{\mathbb{R}\times\mathbb{R}^n}; \partial(\overline{\mathbb{R}}\times\{0\})]$. Thus we can further localize the support of a on this blown up space. There are three essentially different regions, corresponding to the part of the boundary which arises from the radial compactfication, the part arising from the blow up and the corner.

The first of these regions corresponds to a true suspended family, as considered in [11]. In this region the variable $|\xi|$ dominates, and $|\tau| \le C|\xi|$ on the support so the integral takes the form

$$\begin{aligned}\frac{1}{(2\pi)^n} \int_0^1 \int \phi(r\tau) r^{n+1} f(x,\omega,r,r\tau) r^{-n-1} dxd\omega dr \\ = \int_0^1 \int s\phi(R) f(x,\omega,Rs,R)dxd\omega dR,\ s = 1/\tau.\end{aligned} \tag{6.8}$$

Here, ϕ has compact support and f (with the factor of r^{n+1} representing the order $-n-1$) is smooth. The result is smooth in $s = 1/\tau$, which corresponds to a complete asymptotic expansion with only non-negative terms.

The second region corresponds to boundedness of the variable ξ with the function being a classical symbol (by assumption of order at most 0) in τ so integration simply gives a symbol

$$\frac{1}{(2\pi)^n} \int a(x,\xi,\tau)dxd\xi. \tag{6.9}$$

The third region is the most problematic. Here the two boundary faces of the compactification are defined by $r = 1/|\xi|$ and $|\xi|/\tau$ and with polar variables $\omega = \xi/|\xi|$. Thus the integral takes the form

$$\int r^{n+1} f(x,\omega,r,s/r) r^{-n-1} dx d\omega dr \in \mathcal{C}^\infty([0,1)_s) + (\log s)\mathcal{C}^\infty([0,1)_s), \tag{6.10}$$

where f is smooth and with compact support near 0 in the last two variables. This a simple example of the general theorem on pushforward under b-fibrations in [15], or the 'singular asymptotics lemma' of Brüning and Seeley (see also [9]) and is in fact a type of integral long studied as an orbit integral. In any case the indicated regularity follows and this proves the existence of a complete asymptotic expansion, possibly with single logarithmic terms.

It follows that the integral in (6.4) also has a complete asymptotic expansion as $T \to \infty$; where in principle there can be factors of $(\log T)^2$ after such integration. Thus the coefficient of T^0 does exist, and defines $\overline{\mathrm{Tr}}(B)$. Now if N is increased by one in the definition, the additional integral gives the same formula (6.4) except that a constant of integration may be added by the first integral. After N additional integrals, this adds a polynomial, so the result is changed by the integral over $[-T, T]$ of a polynomial. This is an odd polynomial, so has no constant term in its expansion at infinity. Thus the definition of $\mathrm{Tr}(B)$ is in fact independent of the choice of N.

The trace identity follows directly from (6.4), since if $B = [B_1, B_2]$, then any derivative is a sum of commutators between operators with order summing to less than $-n$ and the trace of such a term vanishes. Thus applied to a commutator (6.4) itself vanishes. □

Using this trace functional on product-suspended operators we extend the domain of the eta invariant.

Proposition 6.2. *The eta invariant for any fully elliptic, invertible element $A \in \Psi^{m,m}_{\mathrm{psus}}(X; E)$ defined using the regularized trace*

$$\eta(A) = \frac{1}{\pi i}\overline{\mathrm{Tr}}(A^{-1}\dot{A}),\ \dot{A} = \frac{\partial A}{\partial \tau} \tag{6.11}$$

is a log-multiplicative functional, in the sense of (6.1).

Proof. Certainly (6.11) defines a continuous functional on elliptic and invertible product-suspended families. The log-multiplicativity, (6.1), follows directly since if B is another invertible product-suspended family then

$$(AB)^{-1}\frac{\partial(AB)}{\partial \tau} = B^{-1}A^{-1}\dot{A}B + B^{-1}\dot{B} \tag{6.12}$$

and the trace identity shows that $\overline{\mathrm{Tr}}(B^{-1}A^{-1}\dot{A}B) = \eta(A)$. □

6.2. $\eta(A + i\tau) = \eta(A)$**.** To relate this functional on product-suspended invertible operators to the more familiar eta invariant for self-adjoint elliptic pseudodifferential operators we rewrite the definition in a form closer to traditional zeta regularization, starting with the regularized trace.

Consider the meromorphic family t_+^{-z} of tempered distributions with support in $[0, \infty)$. This family has poles only at the positive integers, with residues being derivatives of the delta function at the origin. For $\operatorname{Re} z$ sufficiently positive and non-integral, t_+^{-z} can be paired with the function $F_{B,N}(t)$ in (6.4), since this is smooth and of finite growth at infinity. This pairing gives a meromorphic function in $\operatorname{Re} z > C$, with poles only at the natural numbers since the poles of t_+^{-z} are associated with the behaviour at 0, where $F_{B,N}$ is smooth. In fact this pairing

$$g(z) = \langle T_+^{-z-1}, F_{B,N}(T)\rangle \tag{6.13}$$

extends to be meromorphic in the whole complex plane. Indeed, dividing the pairing into two using a cut-off $\psi \in \mathcal{C}_c^\infty([0,\infty))$ which is identically equal to 1 near 0,

$$g(z) = \langle T_+^{-z-1}, \psi(T)F_{B,N}(T)\rangle + \langle T_+^{-z-1}, (1-\psi(T))F_{B,N}(T)\rangle \tag{6.14}$$

the first term is meromorphic with poles only at $z \in \mathbb{N}$ and the poles of the second term arise from the terms in the asymptotic expansion of $F_{B,N}(T)$. Notice that there is no pole at $z = 0$ for the first term since the residue of T_+^{-z-1} at $z = 0$ is a multiple of the delta function and $F_{B,N}(0) = 0$. The pole at $z = 0$ for the second term arises exactly from the coefficient of T^0 in the asymptotic expansion so we see that

$$\overline{\operatorname{Tr}}(B) = \operatorname{res}_{z=0} g(z). \tag{6.15}$$

Any terms $a_k(\log T)^k$ for $k \in \mathbb{N}$, in the expansion do not contribute to the residue since they integrate to regular functions at $z = 0$ plus multiples of z^{-k}.

Proposition 6.3. *For $B \in \Psi_{\text{psus}}^{m,m'}(X;E)$ and any $N > m - \dim X - 1$, the regularized trace is the residue at $z = 0$ of the meromorphic continuation from $\operatorname{Re} z >> 0$, $z \notin \mathbb{Z}$, of*

$$\frac{(-1)^{N+1}}{(N-z)\dots(1-z)(-z)}\left\langle\left(\frac{(t+i0)^{N-z}}{1+e^{-\pi i z}} + \frac{(t-i0)^{N-z}}{1+e^{\pi i z}}\right), \operatorname{Tr}_E(\partial_t^N B(t)\right\rangle. \tag{6.16}$$

Proof. Consider the identity

$$t_+^{-z-1} = \frac{1}{(N-z)\dots(1-z)(-z)}\frac{d^{N+1}}{dt^{N+1}}t_+^{-z+N}. \tag{6.17}$$

After inserting this into (6.14), integration by parts is justified (since (6.17) holds in the sense of distributions on the whole real line, supported in $[0,\infty)$), and shows that

$$g(z) = \frac{1}{(N-z)\dots(1-z)(-z)}\langle t_+^{-z+N}, (-1)^{N+1}\operatorname{Tr}_E(\frac{\partial^N}{\partial t^N}B)(t) - \operatorname{Tr}_E(\frac{\partial^N}{\partial t^N}B)(-t)\rangle \tag{6.18}$$

where the pairings are defined, and holomorphic, for $\operatorname{Re} z$ large and z non-integral. Using the identity

$$t_+^z = \frac{(t+i0)^z}{1-e^{2\pi i z}} + \frac{(t-i0)^z}{1-e^{-2\pi i z}},$$

(6.18) becomes

$$g(z) = \frac{(-1)^{N+1}}{(N-z)\dots(1-z)(-z)}\langle D(t,z), \operatorname{Tr}_E(\frac{\partial^N}{\partial t^N}B)(t)\rangle,$$

$$\begin{aligned} D(t,z) = {} & \frac{(t+i0)^{N-z}}{1-e^{2\pi i(N-z)}} + \frac{(t-i0)^{N-z}}{1-e^{-2\pi i(N-z)}} \\ & + (-1)^N\frac{(-t+i0)^{N-z}}{1-e^{2\pi i(N-z)}} + (-1)^N\frac{(-t-i0)^{N-z}}{1-e^{-2\pi i(N-z)}}. \end{aligned} \tag{6.19}$$

Now, $(-t-i0)^{-z} = e^{\pi iz}(t+i0)^{-z}$ so

$$(6.20)\quad D(t,z) = \left(\frac{1}{1-e^{-2\pi iz}} + \frac{e^{\pi iz}}{1-e^{2\pi iz}}\right)(t+i0)^{N-z} + \left(\frac{1}{1-e^{2\pi iz}} + \frac{e^{-\pi iz}}{1-e^{-2\pi iz}}\right)(t-i0)^{N-z}$$

which reduces (6.19) to (6.16). □

This allows us to prove a result of which Theorem 3 in the introduction is an immediate corollary.

Theorem 3. *If $A \in \Psi^1(X;E)$ is a self-adjoint elliptic and invertible pseudodifferential operator then $\eta(A+i\tau)$, defined through (6.11) reduces to the (regularized) value at $z=0$ of the analytic continuation from $\operatorname{Re} z >> 0$ of*

$$(6.21)\qquad \sum_j \operatorname{sgn}(a_j)|a_j|^{-z},$$

where the a_j are the eigenvalues of A, in order of increasing $|a_j|$ repeated with multiplicities.

Proof. With $A(\tau) = A + i\tau$ the eta invariant defined by (6.11) reduces to

$$(6.22)\qquad \eta(A+i\tau) = \frac{1}{\pi}\overline{\operatorname{Tr}}\left((A+i\tau)^{-1}\right) = \frac{1}{\pi}\operatorname{res}_{z=0} h(z)$$

where $h(z)$ is the function (6.16) with $B(t) = (A+it)^{-1}$. Computing the Nth derivative

$$(6.23)\qquad \frac{\partial^N}{\partial\tau^N}\left((A+i\tau)^{-1}\right) = i(-1)^{N+1}N!(\tau-iA)^{-N-1}.$$

The trace is therefore given, for any $N > n$, by

$$(6.24)\qquad \operatorname{tr}_E\left(\frac{\partial^N}{\partial\tau^N}(A+i\tau)^{-1}\right) = i(-1)^{N+1}N!\sum_j(\tau-ia_j)^{-N-1}.$$

This converges uniformly with its derivatives so can be inserted in the pairing (6.16) and the order exchanged. Thus

$$(6.25)\quad h(z) = \frac{a_N(z)i(-1)^{N+1}N!}{1+e^{-\pi iz}}\sum_j \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}+i\epsilon}\tau^{N-z}(\tau-ia_j)^{-N-1}d\tau + \frac{a_N(z)i(-1)^{N+1}N!}{1+e^{\pi iz}}\sum_j \lim_{\epsilon\downarrow 0}\int_{\mathbb{R}-i\epsilon}\tau^{N-z}(\tau-ia_j)^{-N-1}d\tau,$$

where

$$a_N(z) = \frac{(-1)^{N+1}}{(N-z)\dots(1-z)(-z)}.$$

Each of these contour integrals is actually independent of $\epsilon > 0$ for ϵ smaller than the minimal $|a_j|$. By residue computation, in the first sum by moving the contour to infinity in the upper half plane and in the second by moving the contour into the lower half plane

$$(6.26)\quad \int_{\mathbb{R}\pm i\epsilon}\tau^{N-z}(\tau-ia_j)^{-N-1}d\tau = \begin{cases} \pm 2\pi i\frac{(N-z)\cdots(1-z)}{N!}e^{\mp\pi iz/2}|a_j|^{-z} & \pm a_j > 0 \\ 0 & \pm a_j < 0 \end{cases}$$

Inserting this into (6.25) shows that $\eta(A+i\tau)$ is the residue at $z=0$ of

$$\frac{1}{z\cos(\pi z/2)}\sum_j \operatorname{sgn}(a_j)|a_j|^{-z}. \tag{6.27}$$

By definition, the usual eta invariant, $\eta(A)$, is the value at $z=0$ of the continuation of the series in (6.27). This series is the analytic continuation of the trace of an entire family of classical elliptic operators of complex order $-z$ (namely $A^{-z}(\Pi_+-\Pi_-)$ where $\Pi_\pm$ are the projections onto the span of positive and negative eigenvalues) which can have only a simple pole at $z=0$. In fact, here, it is known that there is no singularity, i.e. the residue vanishes. Even without invoking this we conclude the desired equality, since the explicit meromorphic factor in (6.27) is odd in z, so a pole in the continuation of the series would not affect the residue. $\square$

7. Universal η and τ, invariants

That the differential of the eta invariant of a family of self-adjoint Dirac operators is a multiple of the first (odd) Chern class of the index, in odd cohomology of the base, of the family is well-known. In the case of the suspended eta invariant discussed in [11] and above, we show that the η invariant is, in appropriate circumstances, the logarithm of a determinant, which is to say a multiplicative function giving the first odd Chern class. Initially we show this in the context of classifiying spaces for K-theory, then in the geometric context of $(2n+1)$-fold suspended odd elliptic families.

Consider again the algebra of once-suspended isotropic pseudodifferential operators of order 0 on $\mathbb{R}^n$, with values in smoothing operators on a compact manifold X. This can be identified with the smooth functions on $\overline{\mathbb{R}^{2n+1}}\times X^2$ and the subspace

$$\mathcal{I}_+=\left\{A\in\mathcal{C}^\infty(\overline{\mathbb{R}^{2n+1}}\times X^2); A\cong 0 \text{ in } \{t\le 0\}\cap\mathbb{S}^{2n}\times X^2\right\}, \tag{7.1}$$

is a subalgebra. Here, t is the suspending parameter and equality is in the sense of Taylor series at infinity on the compactified Euclidean space. Thus the subalgebra is just the sum of the smoothing ideal (identified with the functions vanishing to infinite order everywhere at the boundary) and the subalgebra of functions vanishing in $t<0$. In fact $\mathcal{I}_+$ is also an ideal. We consider the corresponding group

$$\mathcal{G}_+=\{B=\operatorname{Id}+A,\ A\in\mathcal{I}_+,\ B^{-1}=\operatorname{Id}+B',\ B'\in\mathcal{I}_+\}. \tag{7.2}$$

Now we may use the suspending variable t to identify the upper half-sphere $\{t>0\}\cap\mathbb{S}^{2n}$ of the boundary of $\overline{\mathbb{R}^{2n+1}}$ with $\mathbb{R}^{2n}$,

$$\{t>0\}\cap\mathbb{S}^{2n}\ni[(t,x,\xi)]\longmapsto(X,\Xi)=(x/t,\xi/t)\in\mathbb{R}^{2n}. \tag{7.3}$$

The inverse image under pull-back of $\mathcal{S}(\mathbb{R}^{2n})$ is then naturally identified with $\{a\in\mathcal{C}^\infty(\mathbb{S}^{2n}); a=0 \text{ in } t<0\}$ where $\mathbb{S}^{2n}$ is the boundary of the radial compactification of $\mathbb{R}^{2n+1}$. This allows the space of formal power series $\mathcal{S}(\mathbb{R}^{2n})[[t]]$ to be identified with the formal power series at the boundary of the subspace of $\mathcal{C}^\infty(\overline{\mathbb{R}^{2n+1}})$ consisting of the functions vanishing in $t<0$.

The same identifications carry over to the case of functions valued in the smoothing operators and so gives a short exact sequence of algebras

$$\Psi^{-\infty}_{\text{sus}}(\mathbb{R}^n\times X)\longrightarrow\mathcal{I}_+\longrightarrow\mathcal{S}(\mathbb{R}^{2n}\times X^2)[[t]] \tag{7.4}$$

Lemma 7.1. *In* (7.4)*, the product induced on the quotient is the standard $\star$ product (valued in smoothing operators on X) on $\mathbb{R}^{2n}$ (i.e. the 'Moyal product').*

Proof. Let $A, B \in \mathcal{I}_+$ be the symbols of two operators $\hat{A}, \hat{B} \in \Psi^0_{\mathrm{psus}(1)}(\mathbb{R}^n \times X)$. Then the asymptotic expansion at infinity of the symbol of $\hat{A}\hat{B}$ is given by the standard $\star$ product

$$\sigma(\hat{A}\hat{B}) \sim \sum_{k=0}^{\infty} \frac{1}{k!(2i)^k} (D_x D_\eta - D_y D_\xi)^k A(t,x,\xi) B(t,y,\eta) \Big|_{x=y,\eta=\xi} . \tag{7.5}$$

Under the map (7.3), the asymptotic expansion (7.5) becomes an asymptotic expansion at $\{t > 0\} \cap \mathbb{S}^{2n} \subset \overline{\mathbb{R}^{2n+1}}$

$$\sigma(\hat{A}\hat{B}) \sim \sum_{k=0}^{\infty} \frac{1}{k!(2i)^k} \frac{1}{t^{2k}} (D_X D_\Lambda - D_Y D_\Xi)^k A(t,tX,t\Xi) B(t,tY,t\Lambda) \Big|_{X=Y,\Lambda=\Xi} . \tag{7.6}$$

Thus, if

$$A(t,tX,t\Xi) \sim \sum_{k}^{\infty} \frac{1}{t^k} a_k(X,\Xi), \;\; B(t,tX,t\Xi) \sim \sum_{k}^{\infty} \frac{1}{t^k} b_k(X,\Xi) \tag{7.7}$$

are the asymptotic expansions of A and B at $\{t > 0\} \cap \mathbb{S}^{2n} \subset \overline{\mathbb{R}^{2n+1}}$, then

$$\sigma(\hat{A}\hat{B}) \sim \sum_{k,l,m \geq 0} \frac{1}{k!(2i)^k} \frac{1}{t^{2k+l+m}} \; (D_X D_\Lambda - D_Y D_\Xi)^k a_l(X,\Xi) b_m(Y,\Lambda) \big|_{X=Y,\Lambda=\Xi} \tag{7.8}$$

is the asymptotic expansion of $\sigma(\hat{A}\hat{B})$ at $\{t > 0\} \cap \mathbb{S}^{2n} \subset \overline{\mathbb{R}^{2n+1}}$. But the right hand side is precisely the standard $\star$ product on $\mathcal{S}(\mathbb{R}^{2n} \times X^2)[[\varepsilon]]$ with $\varepsilon = \frac{1}{t^2}$. $\square$

Corresponding to this exact sequence of algebras is the exact sequence of groups consisting of the invertible perturbations of the identity

$$G^{-\infty}_{\mathrm{sus}}(\mathbb{R}^n \times X) \longrightarrow \mathcal{G}_+ \longrightarrow G^{-\infty}(\mathbb{R}^n \times X)[[t]]. \tag{7.9}$$

Theorem 4. *In this 'delooping' sequence, the first group is classifying for even K-theory, the central group is (weakly) contractible and the quotient is (therefore) a classifying group for odd K-theory; the eta invariant, defined as in* (6.11)*,*

$$\eta : \mathcal{G}_+ \longrightarrow \mathbb{C} \tag{7.10}$$

restricts to twice the index on the normal subgroup and $e^{i\pi\eta} = \det_{\mathrm{a}}$ is the adiabatic determinant on $G^{-\infty}(\mathbb{R}^n \times X)[[t]]$.

Proof. As a first step in the proof we consider the behaviour of the regularized trace.

Lemma 7.2. *The regularized trace $\overline{\mathrm{Tr}}$ on the central algebra in* (7.4) *restricts to the integrated trace on the smoothing subalgebra and*

$$\overline{\mathrm{Tr}}(\frac{\partial b}{\partial t}) = \int_{\mathbb{R}^{2n}} b_{2n} dX d\Xi \tag{7.11}$$

for any $b \in \mathcal{I}_+$, where b_k is the term of order k in the formal power series of the image in (7.4)*.*

Proof. When the parameter t is fixed, an element $b \in \mathcal{I}_+$ is actually a smoothing operator, since the asymptotic behavior on the surface where t is constant is determined by the equatorial sphere $t = 0$ at infinity. Thus the definition, from (6.4), of $\overline{\mathrm{Tr}}(b)$ for any element $b \in \mathcal{I}_+$ may be modified by dropping all N integrals, i.e. we may take $N = 0$. Indeed, taking $N > 0$ and then integrating results in the case $N = 0$, plus a polynomial which, as noted earlier, does not affect the result. Carrying out the last integral by the fundamental theorem of calculus,

$$\overline{\mathrm{Tr}}(\dot{b}) = \operatorname*{LIM}_{T\to\infty}\Big(\int_{\mathbb{R}^{2n}} b(T,x,\xi)dxd\xi - \int_{\mathbb{R}^{2n}} b(-T,x,\xi)dxd\xi\Big) \tag{7.12}$$

where LIM stands for the constant term in the asymptotic expansion. The second term in (7.12) corresponds to $t < 0$ where b is rapidly decreasing so does not contribute to the asymptotic expansion. Now, making the scaling change of variable in (7.3), transforms (7.12) to

$$\overline{\mathrm{Tr}}(\dot{b}) = \operatorname*{LIM}_{T\to\infty} T^{-2n}\int_{\mathbb{R}^{2n}} \tilde{b}(T,X,\Xi)dXd\Xi \tag{7.13}$$

where $\tilde{b}$ is the transformed function. Thus (7.13) picks out the term of homogeneity $2n$ (in T) in the formal expansion of $\tilde{b}$. This gives exactly (7.11). □

Now, by definition, the eta invariant is $\frac{1}{\pi i}\overline{\mathrm{Tr}}(a^{-1}\dot{a})$. It follows directly that restricted to the smoothing subgroup this lies in $2\mathbb{Z}$. Thus $D = e^{i\pi\eta}$ does indeed descend to the quotient group in (7.9). This group is connected, so to check that it reduces to the 'adiabatic' determinant defined earlier we only need check the variation formula, both being 1 on the identity. Along a curve $a(s)$,

$$\frac{d}{ds}\eta(a(s)) = \frac{1}{\pi i}\overline{\mathrm{Tr}}(a^{-1}\frac{d\dot{a}}{ds} - a^{-1}\frac{da}{ds}a^{-1}\dot{a}) = \overline{\mathrm{Tr}}\left(\frac{d}{dt}(a^{-1}\frac{da}{ds})\right). \tag{7.14}$$

Thus the identity (7.11) shows that

$$\frac{d}{ds}\eta(a(s)) = \mathrm{Tr}\left[(\tilde{a}(s)\frac{d\tilde{a}}{ds})_{2n}\right] \tag{7.15}$$

where $\tilde{a}$ is the image of a in the third group in (7.9). The identity term in a does not affect the argument since it is annihilated by d/ds.

Since the right hand side of (7.15) is the variation formula for the logarithm of the adiabatic determinant this proves the theorem. □

8. Geometric η and τ invariants

Returning to the 'geometric setting' of a fibration (6) with compact fibres, consider a totally elliptic family $A \in \Psi^{m,m'}_{\mathrm{psus}}(M/B;E,F)$. Although we allow for operators between different bundles here, (6.11) is still meaningful as a definition of the eta invariant if A is invertible. Consider the principal bundle, of the type discussed above,

$$\begin{tikzcd} G^{-\infty}_{\mathrm{sus}}(M/B;E) \arrow[r, dash] & \mathcal{A} \arrow[d, "\nu"] \\ & B \end{tikzcd} \tag{8.1}$$

with fibre

$$\mathcal{A}_b = \Big\{ A + Q; Q \in \Psi^{-\infty}_{\mathrm{sus}}(Z_b; E_b, F_b),\ (A+Q)^{-1} \in \Psi^{-m,-m'}_{\mathrm{psus}}(Z_b; F_b, E_b) \Big\}. \tag{8.2}$$

Proposition 8.1. *The eta invariant, defined by* (6.11)*, is a smooth function on* $\mathcal{A}$ *such that for the fibre action of the structure group at each point*

$$\eta(A(\mathrm{Id}+L)) = \eta(A) + 2\,\mathrm{ind}(\mathrm{Id}+L) \tag{8.3}$$

so projects to

$$\tau = e^{i\pi\eta} : B \longrightarrow \mathbb{C}^* \tag{8.4}$$

which represents the first odd Chern class of the index bundle of the family A.

In particular this result applies to an elliptic, self-adjoint, family of pseudodifferential operators of order 1 by considering the spectral family.

Proof. That $\eta : \mathcal{A} \longrightarrow \mathbb{C}$ is well-defined follows from the discussion above as does the multiplicativity (8.3). Thus, τ is well-defined as a function on B and it only remains to check the topological interpretation.

Note that the fibre of $\mathcal{A}$ is non-empty at each point of the base. In fact it is always possible to find a global smoothing perturbation to make the family invertible, although only when the families index vanishes is this possible with a smoothing perturbation of compact support in the parameter space. Thus, in complete generality, it is possible to choose a smooth map

$$\begin{gathered} Q_+ : \mathbb{R} \longrightarrow \Psi^{-\infty}(M/B; E, F) \text{ such that} \\ Q_+(t) = 0 \text{ for } t << 0,\ Q_+(t) = Q_+(T) \text{ for } t \ge T >> 0, \\ (A(t) + Q(t))^{-1} \in \Psi^{-m,-m'}(M/B; E, F)\ \forall\ t \in \mathbb{R}. \end{gathered} \tag{8.5}$$

This follows directly from the fact that the index bundle, over $\mathbb{R} \times B$, is trivial for $t << 0$ and so is trivial over $\mathbb{R} \times B$ but defines a generally non-trivial index class in $K^1(B)$. In fact the index class of the family A is represented by the map

$$(A(T) + Q_+(T))^{-1} A(T) \in G^{-\infty}(M/B; E). \tag{8.6}$$

Namely, if this family is deformable to the identity in this bundle of groups then there is a perturbation of compact support in t making the original family invertible.

The existence of Q_+ may be directly related to a larger principal bundle with bundle of structure groups $\mathcal{G}^{-\infty}_{\mathrm{sus},+}(M/B; E)$ with fibre

$$\begin{aligned} \big\{ \mathrm{Id}+Q, \mathrm{Id}+Q_b \in \mathcal{C}^\infty(\mathbb{R}; G^{-\infty}(Z_b; E_b), \rho(t)Q(t) \in \mathcal{S}(\mathbb{R}; \Psi^{-\infty}(Z_b; E_b), \\ \exists\ Q_0 \in \Psi^{-\infty}(Z_b; E_b),\ (1-\rho(t))(Q(t) - Q_0) \in \mathcal{S}(\mathbb{R}; \Psi^{-\infty}(Z_b; E_b))\big\}. \end{aligned} \tag{8.7}$$

Here $\rho(t) \in \mathcal{C}^\infty(\mathbb{R})$ is equal to 1 in $t < -1$ and vanishes in $t > 1$. Thus the short exact sequence of groups

$$G^{-\infty}_{\mathrm{sus}}(M/B; E) \longrightarrow G^{-\infty}_{+,\mathrm{sus}}(M/B; E) \xrightarrow{\pi_\infty} G^{-\infty}(M/B; E) \tag{8.8}$$

is the 'delooping sequence' for $G^{-\infty}(M/B; E)$. In particular the central group is weakly contractible and we may consider the enlarged principal bundle

$$\begin{tikzcd} G^{-\infty}_{+,\mathrm{sus}}(M/B; E) \arrow[r, dash] & \mathcal{A}_+ \arrow[d] \\ & B \end{tikzcd} \tag{8.9}$$

defined by replacing $G^{-\infty}_{\mathrm{sus}}$ above by $G^{-\infty}_{+,\mathrm{sus}}$.

The existence of Q_+ shows that this bundle is trivial, i.e. has a global section

$$q : B \longrightarrow \mathcal{A}_+$$

which induces a 'classifying bundle map'

$$\tilde{q} : \mathcal{A} \longrightarrow \mathcal{G}^{-\infty}_{+,\mathrm{sus}}(M/B;E), \tilde{q}(A_b + Q_b) = (A_b + Q_{+,b})^{-1}(A_b + Q_b) \in \mathcal{G}^{-\infty}_{+,\mathrm{sus}}(Z_b;E_b).$$

Now, the definition and basic properties of the eta invariant given by (6.11) are quite insensitive to the enlargment of $\mathcal{A}$ to $\mathcal{A}_+$ and so still define a smooth function $\eta^+ : \mathcal{A}_+ \longrightarrow \mathbb{C}$. The same is true for the group $\mathcal{G}^{-\infty}_{+,\mathrm{sus}}(M/B;E)$, defining the corresponding function $\tilde{\eta} : \mathcal{G}^{-\infty}_{+,\mathrm{sus}}(M/B;E) \longrightarrow \mathbb{C}$ and the discussion of multiplicativity shows that

$$\eta = \eta^+ \circ q \circ \nu + \tilde{\eta} \circ \tilde{q}. \tag{8.10}$$

From the fundamental theorem of calculus,

$$i\pi d\tilde{\eta} = \pi_\infty^* d \log \det \tag{8.11}$$

so we conclude from (8.10) that

$$\tau = e^{i\pi\eta} = e^{i\pi\eta^+ \circ q \circ \pi} \; (\pi_\infty \tilde{q})^* \det \tag{8.12}$$

defines the same cohomology class as the determinant on the classifying group, i.e. the first odd Chern class of the index bundle. □

9. Adiabatic η

We may further extend the discussion above by replacing the once-product-suspended spaces by $(2n+1)$-times product-suspended spaces using the isotropic quantization in $2n$ of the variables, as in Theorem 5 applied to a decomposition $\mathbb{R}^{2n+1} = \mathbb{R} \times \mathbb{R}^{2n}$ with the standard symplectic form used on $\mathbb{R}^{2n}$. Let $\mathcal{A}[[\epsilon]]$ be the principal bundle of invertible perturbations for the family A with respect to the star product from (C.3).

Proposition 9.1. *If $A \in \Psi^{m,m'}_{\mathrm{psus}(2n+1)}(M/B;E,F)$ is a fully elliptic family and (6.11) is used, with the product interpreted as the parameter-dependent product of Theorem 5 for the symplectic form on $\mathbb{R}^{2n}$ then the resulting eta invariant on the bundle of smoothing pertubations has an asymptotic expansion as $\epsilon \downarrow 0$ which projects to*

$$\eta_\epsilon : \mathcal{A}[[\epsilon]] \longrightarrow \epsilon^{-n}\mathbb{C}[[\epsilon]] \tag{9.1}$$

which has constant term the adibatic eta invariant

$$\eta_{\mathrm{a(n)}} : \mathcal{A}[[\epsilon]] \longrightarrow \mathbb{C} \tag{9.2}$$

which generates the first odd Chern class of the index bundle.

Proof. This is essentially a notational extension of the results above. □

In particular (4.2) is a consequence of this result and Bott periodicity. Namely, given an $2n$ product-suspended family we may always choose a smoothing family, analogous to Q_+ in (8.5) which is Schwartz in the second $2n-1$ variables and in the first is Schwartz at $-\infty$ and of the form $Q_0 + Q'$ with Q' Schwartz at $+\infty$ and Q_0 constant in the first variable (and Schwartz in the remainder). By Bott periodicity, the even index of the family is the odd K-class on $\mathbb{R}^{2n-1} \times B$ given by the product

$(A(t)+Q_0)A(t)^{-1}$ for t large. Then (4.2) follows by an elementary computation and the proof of Lemma 4.2 follows directly.

Appendix A. Symbols and products

By choice of a quantization map, spaces of pseudodifferential operators on a compact manifold can be identified, modulo smoothing operators, with the appropriate spaces of symbols on the cotangent bundle as in (2.3). It is important to discuss, and carefully distinguish between, several classes of such symbols and operators. To prepare for this we describe here classes of product-type symbols for a pair of vector spaces; subsequently this is extended to the case of vector bundles.

For a real vector space V, the space of classical symbols of order 0 on V is just $\mathcal{C}^\infty(\overline{V})$, the space of smooth functions on the radial compactification. In terms of any Euclidean metric on V, $\rho(v) = (1+|v|^2)^{-\frac{1}{2}}$ is a defining function for the boundary of $\overline{V}$ and the space of symbols of any complex order z on V is

$$S^z(V) = \rho^{-z}\mathcal{C}^\infty(\overline{V}). \tag{A.1}$$

If W is a second real vector space then we may consider the radial compactification $\overline{V\times W}$ and corresponding symbol spaces $S^z(V\times W)$. The natural projection $\pi_W : V\times W \longrightarrow W$ does not extend to a map from $\overline{V\times W}$ to $\overline{W}$ and correspondingly classical symbols on W do not generally lift to be classical symbols on $V\times W$. Rather $\overline{V} \hookrightarrow \overline{V\times W}$ may be considered as an embedded submanifold, simply the closure (of the preimage in $\overline{V\times W}$) of $V\times\{0\}$. On the other hand there is certainly a smooth projection from $\overline{V}\times\overline{W}$ to $\overline{W}$; the smooth functions,

$$S^0(V;S^0(W)) = S^0(W;S^0(V)) = \mathcal{C}^\infty(\overline{V}\times\overline{W})) \tag{A.2}$$

on this space are symbols on V with values in the symbols on W (or vice-versa).

The main space we wish to consider here has some properties between these two compactifications of $V\times W$. Namely, in terms of radial (real) blow-up, we set

$$^{V}\overline{V\times W} = [\overline{V\times W}; \partial(\overline{V}\times\{0\})]. \tag{A.3}$$

This manifold with corners has two boundary faces (unless one or both of the factors is one-dimensional in which case either or both of the boundary hypersurfaces may have two components). We use a superscript V to refer to the new boundary hypersurface produced by the blow-up in (A.3).

Lemma A.1. *The projection $\pi_W : V\times W \longrightarrow W$ extends to a smooth map*

$$\overline{\pi}_W : {}^{V}\overline{V\times W} \longrightarrow \overline{W}$$

which is a fibration (with fibres which are manifolds with boundary) and in terms of Euclidean metrics on V and W the functions

$$\rho_V(v,w) = \left(\frac{1+|w|^2}{1+|v|^2+|w|^2}\right)^{\frac{1}{2}} \text{ and } \rho_r(v,w) = (1+|w|^2)^{-\frac{1}{2}}$$

extend from $V\times W$ to be smooth functions on $^{V}\overline{V\times W}$ and are defining functions for the two boundary faces.

Proof. To check the first statement of the lemma, notice that the projection $V\times W \to V$ has a smooth extension

$$p_W : \overline{V\times W}\setminus\partial(\overline{V}\times\{0\}) \to \overline{W}$$

which is a fibration with typical fibre given by V. Blowing up the submanifold $\partial(\overline{V}\times\{0\})$ in $\overline{V\times W}$ exactly allows us to extend p_W to a fibration

$$\overline{\pi}_W : {}^V\overline{V\times W} \longrightarrow \overline{W}$$

with typical fibre given by $\overline{V}$. Indeed, in $\overline{V\times W}$ near the submanifold $\partial(\overline{V}\times\{0\})$, we can consider the generating functions (i.e. everywhere containing a coordinate system)

$$\hat{v} = \frac{v}{|v|},\ \sigma_V = \frac{1}{(1+|v|^2)^{\frac{1}{2}}},\ \widetilde{w} = \frac{w}{(1+|v|^2)^{\frac{1}{2}}} = \sigma_V w.$$

The blow up amounts to introducing polar coordinates

$$r_V = (\sigma_V^2 + \widetilde{w}^2)^{\frac{1}{2}},\ (\varphi,\hat{\theta}) = (\frac{\sigma_V}{r_V}, \frac{\widetilde{w}}{r_V})$$

so that the blow-down map is given locally by

$${}^V\overline{V\times W} = [\overline{V\times W}; \partial(\overline{V}\times\{0\})] \ni (\hat{v}, r_V, \varphi, \hat{\theta}) \longmapsto (\hat{v}, \sigma_V = r_V\varphi, \widetilde{w} = r_V\hat{\theta}).$$

In these polar coordinates, and for $r_V > 0$, the fibration p_W is given by

$$p_W(\hat{v}, r_V, \varphi, \hat{\theta}) = \left(\frac{\varphi}{(\varphi^2+|\hat{\theta}|^2)^{\frac{1}{2}}}, \frac{\hat{\theta}}{(\varphi^2+|\hat{\theta}|^2)^{\frac{1}{2}}}\right) \in \overline{W} \tag{A.4}$$

where we have used the identification of $\overline{W}$ with the upper half-sphere which is the closure of the image

$$W \ni w \longmapsto (\frac{1}{(1+|w|^2)^{\frac{1}{2}}}, \frac{w}{(1+|w|^2)^{\frac{1}{2}}}) \in \{(a,b)\in\mathbb{R}\times W;\quad a\geq 0,\ \ a^2+|b|^2=1\}.$$

Thus, p_W extends to $r_V = 0$ to give the desired fibration.

It follows from this that a defining function for the boundary of $\overline{W}$ such as $(1+|w|^2)^{-\frac{1}{2}}$ lifts from $\overline{W}$ to be smooth and to define the 'old' boundary hypersurface, the one not produced by the blow up. Now $(1+|w|^2+|v|^2)^{\frac{1}{2}}$ is a smooth boundary defining function on $\overline{V\times W}$. It therefore lifts under the blow up in (A.3) to be the product of defining functions for both boundary hypersurfaces and so

$$\rho_V(v,w) = \left(\frac{1+|w|^2}{1+|v|^2+|w|^2}\right)^{\frac{1}{2}}$$

is a boundary defining function for the new boundary produced by the blow-up. □

Now we define general spaces of 'partial-product' symbols by

$$S^{z,z'}({}^V\overline{V\times W}) = \rho_r^z\rho_V^{z'}\mathcal{C}^\infty({}^V\overline{V\times W}). \tag{A.5}$$

Directly from this definition,

$$S^{z,z'}({}^V\overline{V\times W})\cdot S^{\zeta,\zeta'}({}^V\overline{V\times W}) = S^{z+\zeta,z'+\zeta'}({}^V\overline{V\times W}). \tag{A.6}$$

Two of the 'remainder' classes have simpler characterizations. Namely

$$\begin{aligned} S^{-\infty,z'}({}^V\overline{V\times W}) &= \dot{\mathcal{C}}^\infty(\overline{W}; S^{z'}(V)) \\ S^{-\infty,-\infty}({}^V\overline{V\times W}) &= \dot{\mathcal{C}}^\infty(\overline{V\times W}) = \mathcal{S}(V\times W). \end{aligned} \tag{A.7}$$

Appendix B. Product Suspended operators

We can now introduce a generalization of the 'suspended' algebra considered in [11] and in [13] (an algebra similar to the suspended algebra was already introduced by Shubin in [19]).

The d-fold suspended pseudodifferential algebra on a compact manifold X may be viewed as a space of smooth maps from $\mathbb{R}^d$ into $\Psi^k(X; E, F)$ in which the parameters (which we think of as the base variables for a fibration) appear as 'symbolic variables'. The inverse Fourier transform identifies the suspended space

$$\check{\Psi}^k_{\mathrm{sus}(d)}(X; E, F) \subset \Psi^k(\mathbb{R}^d \times X; E, F)$$

directly, as is done in [11], with the elements which are translation-invariant in $\mathbb{R}^d$ and have convolution kernels vanishing rapidly at infinity, with all derivatives, in these variables; this space may also be defined directly as in (2.3).

The subspace of smoothing operators is

$$\Psi^{-\infty}_{\mathrm{sus}(d)}(X; E, F) = \mathcal{S}(\mathbb{R}^d \times X^2; \mathrm{Hom}(E, F) \otimes \Omega_R)$$

in terms of the Schwartz space. Then the finite-order operators may be specified, up to smoothing terms, by Weyl quantization as

$$\text{(B.1)} \quad q_g : \rho^{-k}\mathcal{C}^\infty(\overline{\mathbb{R}^d \times T^*X}; \pi^* \hom(E, F)) \ni a \longmapsto \\ (2\pi)^{-n} \int_{T^*X} \chi e^{iv(x,y)\cdot\xi} a(m(x, y), \zeta, \xi) d\xi dg \in \Psi^k_{\mathrm{sus}(d)}(X; E, F)$$

where the symbol space is compactified in the joint fibre $\mathbb{R}^d \times T^*_x X$. The resulting full symbol sequence is as in (2.4) except that the formal power series have coefficients on the sphere bundle of $\mathbb{R}^p \times T^*X$; the parameters do not affect the operators B_j, acting on T^*X, appearing in the product.

If $A \in \Psi^1(X; E)$ is a first order pseudodifferential operator and τ is the suspension variable for $\Psi^1_{\mathrm{sus}(1)}(X; E)$, then $A + i\tau$ is not in general an element of $\Psi^1_{\mathrm{sus}(1)}(X; E)$. In fact, $A + i\tau \in \Psi^1_{\mathrm{sus}(1)}(X; E)$ if and only if A is a *differential* operator. Similarly, for $A \in \Psi^1(X; E, F)$, the operator

$$\text{(B.2)} \qquad \begin{pmatrix} it + \tau & A^* \\ A & it - \tau \end{pmatrix}$$

is in $\Psi^1_{\mathrm{sus}(2)}(X; E \oplus F)$ if and only if A is a differential operator.

This restriction to differential operators is unfortunate since the operator $A + i\tau$ arises in the alternative definition of the eta invariant as described in Appendix 6, while in section 5 the operator (B.2) is used to implement Bott periodicity for determinant line bundles. For these reasons, and others, we pass to the wider context of product-suspended operators.

We first need to enlarge the space of symbols as in Appendix A. Identifying X with the zero section of T^*X, consider the blown-up space

$$\text{(B.3)} \qquad {}^X\overline{\mathbb{R}^d \times T^*X} = [\overline{\mathbb{R}^d \times T^*X}; \partial\overline{\mathbb{R}^d} \times X]$$

where $\overline{\mathbb{R}^d \times T^*X}$ is the radial compactification of $\mathbb{R}^d \times T^*X$ fibre by fibre and

$$\overline{\mathbb{R}^d} \times X \subset \overline{\mathbb{R}^d \times T^*X}$$

is the closure of $\mathbb{R}^d \times X$ in $\overline{\mathbb{R}^d \times T^*X}$. In terms of a Riemannian metric g and the Euclidean metric on $\mathbb{R}^d$. Lemma A.1 generalizes directly to

Lemma B.1. *The projection $\mathbb{R}^d \times T^*X \to T^*X$ extends to a smooth map*

$$\overline{\pi}_{T^*X} : {}^X\overline{\mathbb{R}^d \times T^*X} \longrightarrow \overline{T^*X}$$

which is a fibration with typical fibre $\overline{\mathbb{R}^d}$ and the smooth functions

$$\rho_{\mathrm{sus}}(v,w) = \frac{(1+|w|^2)^{\frac12}}{(1+|v|^2+|w|^2)^{\frac12}},\ \rho_r(v,w) = (1+|w|^2)^{-\frac12},\ v\in\mathbb{R}^d,\ w\in T^*X,$$

define the two boundary faces.

Proof. This results from the invariance of the construction in Appendix A under those linear transformations of $V \times W$ which leave V invariant, so Lemma A.1 extends to the case of a vector bundle. □

For $z, z' \in \mathbb{C}$, the space of (partially) product-type symbols with values in a vector bundle over X is then

$$\mathcal{S}^{z,z'}({}^X\overline{\mathbb{R}^d \times T^*X}; U) = \rho_r^{-z}\rho_{\mathrm{sus}}^{-z'}\mathcal{C}^\infty({}^X\overline{\mathbb{R}^d \times T^*X}; U). \tag{B.4}$$

On $\overline{\mathbb{R}^d} \times X \times X$, consider the boundary defining function $\rho_\tau(\tau) = (1+|\tau|^2)^{-\frac12}$. Let E and F be smooth complex vector bundles on X. For $z' \in \mathbb{C}$ set

$$\Psi^{-\infty,z'}_{\mathrm{psus}(d)}(X;E,F) = \rho_\tau^{-z'}\mathcal{C}^\infty(\overline{\mathbb{R}^d} \times X \times X; \mathrm{Hom}(E,F)\otimes\Omega_R X), \tag{B.5}$$

where $\Omega_R X = \pi_3^*\Omega X$, π_3 being the projection on the third factor, and ΩX being the bundle of densities on X. This is the space of smoothing operators (defined as usual through their kernels) on X depending symbolically on d parameters; which we identify as the product-suspended operators of order $-\infty$ on X.

Definition B.2. The general spaces of product d-suspended pseudodifferential operators of order $k, k' \in \mathbb{Z}$ acting from $\mathcal{S}(\mathbb{R}^d \times X; E)$ to $\mathcal{S}(\mathbb{R}^d \times X; F)$ is

$$\Psi^{k,k'}_{\mathrm{psus}(d)}(X;E,F) = q_g(\mathcal{S}^{k,k'}({}^X\overline{\mathbb{R}^d \times T^*X}; \hom(E,F))) + \Psi^{-\infty,k'}_{\mathrm{psus}(d)}(X;E,F)$$

where q_g is the Weyl quantization (B.1) applied to these more general symbol spaces.

We limit attention to integral orders here only because it is all that is needed.

Pseudodifferential operators are included in the product-suspended operators

$$\Psi^k(X;E,F) \subset \Psi^{k,0}_{\mathrm{psus}(d)}(X;E,F),$$

being independent of the parameters. For integers $l \le k$, $l' \le k'$, there are inclusions

$$\Psi^{l,l'}_{\mathrm{psus}(d)}(X;E,F) \subset \Psi^{k,k'}_{\mathrm{psus}(d)}(X;E,F).$$

Furthermore, as we will see below in Theorem 5, product d-suspended operators compose in the expected way

$$\Psi^{k,k'}_{\mathrm{psus}(d)}(X;E,F)\circ\Psi^{l,l'}_{\mathrm{psus}(d)}(X;G,E) \subset \Psi^{k+l,k'+l'}_{\mathrm{psus}(d)}(X;G,F).$$

Suspended operators are particular instances of product suspended operators,

$$\Psi^k_{\mathrm{sus}(d)}(X;E,F) \subset \Psi^{k,k}_{\mathrm{psus}(d)}(X;E,F),\ k\in\mathbb{Z},$$

and

$$\Psi^{-\infty}_{\mathrm{sus}(d)}(X;E,F) = \Psi^{-\infty,-\infty}_{\mathrm{psus}(d)}(X;E,F).$$

Product d-suspended pseudodifferential operators are intimately related with the algebra of product-type operators introduced in [14]. More precisely, consider the projection

$$\phi : \mathbb{R}^d \times X \to \mathbb{R}^d \tag{B.6}$$

as a fibration. If E and F are smooth complex vector bundles on X, then as discussed in [14], to such a fibration one can associate the space of product-type pseudodifferential operators of order (k, k')

$$\Psi^{k,k'}_{\phi-p}(\mathbb{R}^d \times X; E, F)$$

acting from $\mathcal{C}^\infty_c(\mathbb{R}^d \times X; E)$ to $\mathcal{C}^\infty(\mathbb{R}^d \times X; F)$. Given $\tau \in \mathbb{R}^d$, let

$$T_\tau : \mathbb{R}^d \times X \to \mathbb{R}^d \times X$$

denote the translation in the first factor $T_\tau(t, x) = (t - \tau, x)$. We can consider the product-type pseudodifferential operators which are translation-invariant in the Euclidean variable, that is, satisfying

$$T^*_\tau(Af) = AT^*_\tau f, \forall\, \tau \in \mathbb{R}^d,\ f \in \mathcal{C}^\infty_c(\mathbb{R}^d \times X; E). \tag{B.7}$$

In terms of the Schwartz kernel K_A of A, this means that K_A acts by convolution in the first factor

$$Af(x, t) = \int_{\mathbb{R}^d} \int_X K_A(t - s, x, x')f(x', s)ds$$

where K_A is a density in the x' variable. Now one can ask in addition that this convolution kernel decay to all orders at infinity

$$K_A \in \mathcal{C}^{-\infty}_c(\mathbb{R}^d \times X^2; \operatorname{Hom}(E, F) \otimes \Omega_R X) + \mathcal{S}(\mathbb{R}^d \times X^2; \operatorname{Hom}(E, F) \otimes \Omega_R X). \tag{B.8}$$

This leads to following characterization of product d-suspended operators.

Lemma B.3. *Fourier transformation in the suspension variables*

$$(\hat{A}(\tau)f)(x) = \int_X \int_{\mathbb{R}^d} e^{-it\tau} K_A(t, x, x')f(x')dt, \quad \tau \in \mathbb{R}^d$$

is an isomorphism of the space of translation-invariant product-type pseudodifferential operators satisfying (B.8) *onto the d-parameter product-suspended pseudodifferential operators; it preserves products.*

Proof. Modulo small changes of notation, this is the same as for suspended operators. □

One advantage of the alternative definition through Lemma B.3 is that the Fredholm theory for product d-suspended operators follows almost immediatly from the corresponding Fredholm theory for product-type operators. Indeed, the principal symbol map and the base family map for product-type operators gives via the inclusion (using the inverse Fourier transform) $\check{\Psi}_{\mathrm{psus}(d)}(X; E, F) \subset \Psi^{k,k'}_{\phi-p}(\mathbb{R}^d \times X; E, F)$ a corresponding symbol map and base family map for product d-suspended operators. For the convenience of the reader, we will define these directly without refering to product-type operators.

Of the two boundary faces of ${}^X\overline{\mathbb{R}^p \times T^*X}$, the 'old' boundary, or really its blow-up,

$$B_\sigma = [S(\mathbb{R}^d \times T^*X); S(\mathbb{R}^d) \times X]$$

with X being the zero section of T^*X, carries the replacement for the usual principal symbol. In terms of a quantization map as above, this is given by the restriction of the full symbol $a \in \mathcal{S}^{m,m'}(\mathbb{R}^d \ltimes T^*X; E, F)$ of an operator $A = q_g(a)$ to this boundary face,

$$\sigma_{m,m'} : \Psi^{m,m'}_{\mathrm{psus}(d)}(X; E, F) \longrightarrow \mathcal{S}^{m,m'}_{\mathrm{psus}(d)}(X; E, F) \tag{B.9}$$

with

$$\mathcal{S}^{m,m'}_{\mathrm{psus}(d)}(X; E, F) = \mathcal{C}^\infty(B_\sigma; \hom(E, F) \otimes N^{-m} \otimes N_{\mathrm{ff}}^{-m'})$$

where N is the normal bundle to B_σ and and N_{ff} is the normal bundle of the 'new' boundary, which is canonically identified with the normal bundle to the boundary of B_σ. Both are trivial bundles. This corresponds to the multiplicative short exact sequence

$$0 \longrightarrow \Psi^{m-1,m'}_{\mathrm{psus}(d)}(X; E, F) \longrightarrow \Psi^{m,m'}_{\mathrm{psus}(d)}(X; E, F) \stackrel{\sigma_{m,m'}}{\longrightarrow} \mathcal{S}^{m,m'}_{\mathrm{psus}(d)}(X; E, F) \to 0. \tag{B.10}$$

A product d-suspended operator $A \in \Psi^{k,k'}_{\mathrm{psus}(d)}(X; E, F)$ is *elliptic* if its principal symbol $\sigma_{m,m'}(A)$ is invertible.

Ellipticity alone does imply that the family is Fredholm for each value of the parameter but, as for product-type operators, is does not suffice to allow the construction of a parametrix modulo Schwartz-smoothing errors. There is a second symbol map which takes into account the behavior of the operator for large values of the suspension parameters.

Let $B_{\mathrm{sus}} \subset {}^X\overline{\mathbb{R}^d \times T^*X}$ denote the 'new' boundary, which is the 'front face' produced by the blow up. The fibration of Lemma B.1 gives a canonical identification of B_{sus} with $S(\mathbb{R}^d) \times \overline{T^*X}$. Thus, the restriction map (using a boundary defining function ρ_{sus} for B_{sus}) becomes

$$\begin{aligned} R : \mathcal{S}^{k,k'}(\mathbb{R}^d \ltimes X; \hom(E, F) \ni a \longmapsto \\ \rho^{m'}_{\mathrm{sus}} a\big|_{B_{\mathrm{sus}}} \in \mathcal{C}^\infty(S(\mathbb{R}^d); \mathcal{S}^{k,k'}(\overline{T^*X}; \hom(E, F))) \end{aligned} \tag{B.11}$$

Given any element $A = q_g(a_1) + A_2 \in \Psi^{k,k'}_{\mathrm{psus}(d)}(X; E, F)$ with $a_1 \in \mathcal{S}^{k,k'}(\mathbb{R}^d \ltimes X; \hom(E, F)$ and $a_2 \in \Psi^{-\infty,k'}_{\mathrm{psus}(d)}(X; E, F)$, the *base family* is defined by

$$L(A) = q_g(R(a_1)) + \rho^{k'}_{\mathrm{sus}} A_2)\big|_{B_{\mathrm{sus}}} \in \mathcal{C}^\infty(S(\mathbb{R}^d); \Psi^m(X; E, F)) \tag{B.12}$$

Proposition B.4. *The base family* (B.12) *is independent of choices and corresponds to the multiplicative short exact sequence*

$$0 \longrightarrow \Psi^{k,k'-1}_{\mathrm{psus}(d)}(X; E, F) \longrightarrow \Psi^{k,k'}_{\mathrm{psus}(d)}(X; E, F) \stackrel{L}{\longrightarrow} \mathcal{C}^\infty(S(\mathbb{R}^d); \Psi^k(X; E, F)) \longrightarrow 0, \tag{B.13}$$

so

$$L(A \circ B) = L(A) \circ L(B),\ A \in \Psi^{m,m'}_{\mathrm{psus}(d)}(X; E, F),\ B \in \Psi^{k,k'}_{\mathrm{psus}(d)}(X; G, E).$$

Proof. The fact that there is a short exact sequence is essentially by definition of L. The fact that L is a homomorphism follows by very simple 'oscillatory testing'. Namely, if $u \in \mathcal{C}^\infty(X; E)$ and $A \in \Psi^{k,k'}_{\mathrm{psus}(p)}(X; E, F)$ then

$$Au \in \rho_\tau^{-k'} \mathcal{C}^\infty(\overline{\mathbb{R}^p} \times X; F) \text{ and } L(A)u = \rho_\tau^{k'} Au\big|_{\partial\overline{\mathbb{R}^p}} \in \mathcal{C}^\infty(\mathbb{S}^{p-1} \times X; F). \tag{B.14}$$

□

Definition B.5. The *joint symbol* $J(A)$ of an operator $A \in \Psi^{k,k'}_{\mathrm{psus}(d)}(X; E, F)$ is the combination of its principal symbol and its base family

$$J(A) = (\sigma(A), L(A)) \text{ where } \sigma(L(A)) = \sigma(A)\big|_{B_\sigma}.$$

An operator A is said to be *fully elliptic* if its joint symbol is invertible.

The important feature that motivates the introduction of product-suspended operators (as opposed to suspended operators) is the following lemma.

Lemma B.6. *If $A \in \Psi^1(X; E)$ then the one-parameter family $\tau \longmapsto A + i\tau \in \Psi^{1,1}_{\mathrm{psus}(1)}(X; E)$ and if $B \in \Psi^1(X; E, F)$, then the two-parameter family*

$$(t, \tau) \longmapsto \hat{B}(t, \tau) = \begin{pmatrix} it + \tau & B^* \\ B & it - \tau \end{pmatrix} \in \Psi^{1,1}_{\mathrm{psus}(2)}(X; E \oplus F).$$

Moreover if A is self-adjoint and elliptic (respectively B is elliptic) then $A + i\tau$ (respectively $\hat{B}$) is fully elliptic.

In fact, it suffices that all the eigenvalues of the symbol of A have a nonvanishing real part for $A + i\tau$ to be fully elliptic.

Proof. Fix a quantization q_g. In the first case $a \in \rho^{-1}\mathcal{C}^\infty(\overline{T^*X}; \pi^* \hom(E))$ exists such that $(A - q_g(a)) \in \Psi^{-\infty}(X; E)$. Then

$$a + i\tau \in \mathcal{S}^{1,1}(\mathbb{R} \ltimes T^*X; E) \text{ and } A + i\tau - q_g(a + i\tau) \in \Psi^{-\infty,1}_{\mathrm{psus}(1)}(X; E),$$

which shows that $A + i\tau \in \Psi^{1,1}_{\mathrm{psus}(1)}(X; E)$. The symbol of $A + i\tau$ is invertible if $\sigma(A)$ has no eigenvalues in $i\mathbb{R}$ and its base family is $\pm i \,\mathrm{Id}$ at the two components of $\partial\overline{\mathbb{R}_\tau}$. Thus $A + i\tau$ is fully elliptic.

In the second case, choose $b \in \rho^{-1}\mathcal{C}^\infty(\overline{T^*X}; \pi^* \hom(E, F))$ such that

$$B - q_g(b) \in \Psi^{-\infty}(X; E).$$

Then

$$\hat{b} = \begin{pmatrix} it + \tau & b^* \\ b & it - \tau \end{pmatrix} \in \mathcal{S}^{1,1}(\mathbb{R}^2 \ltimes T^*X; E, F)$$

and $\hat{B} - q_g(\hat{b}) \in \Psi^{-\infty,1}_{\mathrm{psus}(2)}(X; E, F)$, which shows that $\hat{B} \in \Psi^{1,1}_{\mathrm{psus}(1)}(X; E, F)$. To see that $\hat{B}$ is fully elliptic when B is elliptic, consider the invertible operator

$$Q = \hat{B}^*\hat{B} + 1 = \begin{pmatrix} B^*B + t^2 + \tau^2 + 1 & 0 \\ 0 & BB^* + t^2 + \tau^2 + 1 \end{pmatrix} \in \Psi^{2,2}_{\mathrm{psus}(2)}(X; E \oplus F).$$

Then

$$(Q^{-1}\hat{B}^*)\hat{B} - \mathrm{Id}_{E \oplus F} = -Q^{-1} \in \Psi^{-2,-2}_{\mathrm{psus}(2)}(X; E \oplus F),$$

so that $J(\hat{B})^{-1} = J(Q^{-1}\hat{B}^*)$ exists, which shows that $\hat{B}$ is fully elliptic. □

Appendix C. Mixed isotropic operators

Next we proceed to the 'parameter quantization' of these spaces of product suspended operators. That is we introduce a new product depending on the choice of an antisymmetric form on $\mathbb{R}^p$. These products are used above in the identification of the determinant bundle, as constructed in the product $2n$-suspended case, with the determinant bundle as introduced by Quillen. To do so we use an adiabatic limit, with a parameter which passes from the quantized to the unquantized case discussed above; for the isotropic algebra itself such degenerations are treated in [7] and as shown there implements Bott periodicity. So, to introduce these spaces we simply combine (2.3) and its Euclidean analogue (2.20). Note that the quantization map will be global in the Euclidean variables but can only be local near the diagonal in the manifold. In defining these spaces we use the formula for the action of an operator by Weyl quantization in (2.21).

Proposition C.1. *Let X be a compact manifold E and F complex bundles over X then for any $p \in \mathbb{N}$ combining* (2.21) *with the operator product gives a smooth family of associative products*

$$\Lambda^2(\mathbb{R}^n) \times \Psi^{m_1,m_1'}_{\mathrm{psus}(2n)}(X;F,G) \times \Psi^{m_2,m_2'}_{\mathrm{psus}(2n)}(X;E,F) \longrightarrow \Psi^{m_1+m_2,m_1'+m_2'}_{\mathrm{psus}(2n)}(X;E,G). \tag{C.1}$$

This follows by combining essentially standard treatments of the composition of pseudodifferential operators with those of the 'isotropic' operators on $\mathbb{R}^n$.

We are especially interested in the 'adiabatic limit' where the general ω is replaced by $\epsilon\omega$ for a fixed antisymmetric form. The cases which occur above are where p is even and ω is non-degenerate, or where p is odd and ω has maximal rank. In this case we state the corresponding corollary of the result above (see also [10] and [7]).

Theorem 5. *For any fixed antisymmetric form on $\mathbb{R}^p$, the composition* (C.1) *induces a smooth 1-parameter family of quantized products*

$$[0,1]_\epsilon \times \Psi^{k,k'}_{\mathrm{psus}(p)}(X;F,G) \times \Psi^{l,l'}_{\mathrm{psus}(p)}(X;E,F) \longrightarrow \Psi^{k+l,k'+l'}_{\mathrm{psus}(p)}(X;E,G) \tag{C.2}$$

and as $\epsilon \downarrow 0$ there is a Taylor series expansion

$$A \circ_\epsilon B(u) \sim \sum_{k=0}^{\infty} \frac{(-i\epsilon)^k}{2^k k!} \omega(D_v, D_w)^k A(v)B(w)\big|'_{v=w=u} \tag{C.3}$$

in particular, when $\epsilon = 0$ the product reduces to the usual parameterized product of suspended operators.

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