PSEUDODIFFERENTIAL OPERATORS, CORNERS AND SINGULAR LIMITS

Richard B. Melrose

In the first part of my talk I shall describe some of the properties one should expect of a calculus of pseudodifferential operators which corresponds to the microlocalization of a Lie algebra of vector fields. This is not intended to be a formal axiomatic program but it leads one to consider conditions on the Lie algebra for such microlocalization to be possible. The symbolic structure of the calculus also shows how it can be applied in the solution of analytic questions related to the Lie algebra, especially to the inversion of elliptic elements of the enveloping algebra.

In the second part I shall consider several such analytic question which arise in various differential–geometric settings and are, or appear to be, amenable to the application of these pseudodifferential techniques. For those examples which have already been analyzed the Lie algebra is identified and then a specific question is discussed using the calculus of pseudodifferential operators which arises from it.

Finally in the third part of the talk I shall briefly outline a general strategy for the construction of 'the' calculus of pseudodifferential operators which microlocalizes a given Lie algebra satisfying conditions which make it a boundary–fibration structure. This construction applies to most of the situations described in the second part of the talk and, I conjecture, can be extended to apply to the others.

I. Pseudodifferential operators

Even though many of the examples of interest here arise on singular or non–compact spaces it is very convenient from an analytic point of view to work always in a fixed category of spaces. The simplest class of spaces which seems to be sufficiently wide to allow many different types of problems to be attacked is that of compact manifolds with corners; the morphisms are the b–maps, discussed below. Such a space is a compact topological manifold with boundary, X, with a $\mathcal{C}^\infty$ structure locally modeled on the spaces

$$\mathbb{R}^n_k = [0,\infty)^k \times \mathbb{R}^{n-k}. \tag{1}$$

This fixes the space $\mathcal{C}^\infty(X)$ of smooth functions on X; locally these are just the functions on $\mathbb{R}^n_k$ which are the restrictions of $\mathcal{C}^\infty$ functions on $\mathbb{R}^n$. For simplicity we also insist that all the boundary hypersurfaces of X are embedded. If we let $M_1(X)$ be the set of boundary hypersurfaces then this means that each $H \in M_1(X)$ has a defining function

$$\rho_H \in \mathcal{C}^\infty(X) \text{ s.t. } H = \{\rho_H = 0\},\ \rho_H \geq 0,\ d\rho_H \neq 0 \text{ at } H. \tag{2}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

We denote by $M_{(1)}(X)$ the set of true boundary faces, the connected closed submanifolds locally given by intersection of the boundary hypersurfaces and set $M(X) = M_{(1)}(X) \cup \{X\}$; all these submanifolds are embedded. Limiting the analysis to compact manifolds with corners implies that in many cases a preliminary geometric step involving a compactification or blow–up procedure is required.

It is always possible to embed a compact manifold with corners as a subset of a compact manifold, $\widetilde{X}$, without boundary:

$$\iota_X : X \hookrightarrow \widetilde{X},\ \mathcal{C}^\infty(X) = \iota_X^* \left(\mathcal{C}^\infty(\widetilde{X})\right). \tag{3}$$

This allows one to readily discuss any of the usual structures on a $\mathcal{C}^\infty$ manifold, by restriction from $\widetilde{X}$, provided only that such structures are invariant under diffeomorphisms of $\widetilde{X}$ which fix each point of X. In particular $\mathcal{C}^\infty$ vector fields, the tangent bundle, cotangent bundle, form bundles etc. can all be defined in this way; of course they also have more intrinsic definitions.

In most cases the structure we are seeking is most readily expressed in terms of a space of real $\mathcal{C}^\infty$ vector fields:

$$\mathcal{V}(X) \subset \mathcal{C}^\infty(X; TX). \tag{4}$$

The two most basic properties we insist upon are:

$$\mathcal{V}(X) \text{ is a Lie algebra} \tag{5}$$

under the commutation bracket and

$$\mathcal{V}(X) \text{ is a } \mathcal{C}^\infty(X)\text{–module.} \tag{6}$$

Of course the space of all smooth vector fields, $\mathcal{C}^\infty(X; TX)$, has these two properties. However, in order that the vector fields in $\mathcal{V}(X)$ correspond to infinitesmal diffeomorphisms of X we also require that

$$\text{each } V \in \mathcal{V}(X) \text{ is tangent to each } H \in M_1(X). \tag{7}$$

The simplest example is just the space $\mathcal{V}_b(X)$ of all $\mathcal{C}^\infty$ vector fields on X satisfying (7). Thus we can summarize these properties as requiring

$$\mathcal{V}(X) \subset \mathcal{V}_b(X) \text{ is a Lie subalgebra and } \mathcal{C}^\infty(X)\text{–submodule.} \tag{8}$$

Near any point $\bar{x} \in X$ there are local coordinates $x_1, \dots, x_n$ in which $\bar{x}$ is mapped to the origin of $\mathbb{R}^n_k$, for some k. Then $\mathcal{V}_b(X)$ is locally the span, as a $\mathcal{C}^\infty$–module, of the vector fields

$$x_1\partial_{x_1}, \dots, x_k\partial_{x_k}, \partial_{x_{k+1}}, \dots, \partial_{x_n}. \tag{9}$$

There are no relations between these vector fields so there is a natural vector bundle, which we denote ${}^bTX$, of which $\mathcal{V}_b(X)$ forms the space of all sections. Since these sections are vector fields there is a natural vector bundle map

$${}^bTX \longrightarrow TX \tag{10}$$

which is an isomorphism over the interior, but has corank k at a boundary face of codimension k, since the first k elements in (9) are mapped to zero. In general we insist that the Lie algebra $\mathcal{V}(X)$ have a similar structure. Thus we require that there be a vector bundle ${}^{\mathcal{V}}TX$ and vector bundle maps

$$ {}^{\mathcal{V}}TX \longrightarrow {}^{b}TX \longrightarrow TX \tag{11}$$

where the overall map is $\iota_{\mathcal{V}}$ and

$$\mathcal{V}(X) = \iota_{\mathcal{V}} \circ \mathcal{C}^{\infty}(X; {}^{\mathcal{V}}TX). \tag{12}$$

We denote by ${}^{\mathcal{V}}T^*X$ the dual bundle to ${}^{\mathcal{V}}TX$, and by ${}^{\mathcal{V}}\!\Lambda^k X$ the associated form bundles.

Since $\mathcal{V}$ is a Lie algebra of operators acting on $\mathcal{C}^{\infty}(X)$ the enveloping algebra is naturally a filtered ring of operators, denoted $\operatorname{Diff}^m_{\mathcal{V}}(X)$, acting on $\mathcal{C}^{\infty}(X)$. In fact, since the elements of $\mathcal{V}$ are local diffeomorphisms, twisted versions, $\operatorname{Diff}^m_{\mathcal{V}}(X; E, F)$, of these spaces of operators are defined between sections of any two vector bundles, E and F, over X.

One easy way to see why we might be interested in these structures is the following elementary result.

Proposition. *(13) Exterior differentiation defines a complex*

$$\mathcal{C}^{\infty}(X) \stackrel{d}{\longrightarrow} \mathcal{C}^{\infty}(X; {}^{\mathcal{V}}\!\Lambda^1) \stackrel{d}{\longrightarrow} \dots \stackrel{d}{\longrightarrow} \mathcal{C}^{\infty}(X; {}^{\mathcal{V}}\!\Lambda^q) \tag{14}$$

where q is the fibre dimension of ${}^{\mathcal{V}}TX$. If g is a fibre metric on ${}^{\mathcal{V}}TX$ then the associated Laplacian is an element of $\operatorname{Diff}^2_{\mathcal{V}}(X; {}^{\mathcal{V}}\!\Lambda^k)$ for each k, $0 \le k \le q$.

The assumption (12) means that the filtration of the ring $\operatorname{Diff}^*_{\mathcal{V}}(X)$ is given by symbol maps, $\sigma_{k,\mathcal{V}}$, giving a short exact sequence

$$0 \longrightarrow \operatorname{Diff}^{k-1}_{\mathcal{V}}(X; E, F) \hookrightarrow \operatorname{Diff}^{k}_{\mathcal{V}}(X; E, F) \stackrel{\sigma_{k,\mathcal{V}}}{\longrightarrow} P^k({}^{\mathcal{V}}T^*X; E, F) \longrightarrow 0, \tag{15}$$

where $P^k({}^{\mathcal{V}}T^*X; E, F)$ is the space of $\mathcal{C}^{\infty}$ homomorphisms of the lifts to ${}^{\mathcal{V}}T^*X$ of the vector bundles E, F which are homogeneous polynomials of degree k on the fibres. An operator is ($\mathcal{V}$–) elliptic if its symbol in this sense is invertible off the zero section of ${}^{\mathcal{V}}T^*X$. The symbol of the Laplacian is always given by multiplication by the square of the length, so it is always elliptic as an element of $\operatorname{Diff}^k_{\mathcal{V}}(X; {}^{\mathcal{V}}\!\Lambda^k)$.

Now by a *small* calculus of $\mathcal{V}$–pseudodifferential operators we mean linear spaces of operators

$$\Psi^m_{\mathcal{V}}(X; E, F) \ni A : \mathcal{C}^{\infty}(X; E) \longrightarrow \mathcal{C}^{\infty}(X; F),\ m \in \mathbb{R}, \tag{16}$$

with (at least) the following additional properties. First these spaces should extend the ring of $\mathcal{V}$–differential operators:

$$\operatorname{Diff}^k \mathcal{V}(X; E, E') \subset \Psi^k_{\mathcal{V}}(X; E, E') \tag{17}$$

$$\Psi^m_{\mathcal{V}}(X; E, E') \circ \Psi^{m'}_{\mathcal{V}}(X; E', E'') \subset \Psi^{m+m'}_{\mathcal{V}}(X; E, E''). \tag{18}$$

Moreover we require that the filtration be fixed by symbol maps giving short exact sequences analogous to (15):

$$0 \longrightarrow \Psi_{\mathcal{V}}^{m-1}(X;E,F) \hookrightarrow \Psi_{\mathcal{V}}^{m}(X;E,F) \xrightarrow{\sigma_{m,\mathcal{V}}} S^{[m]}({}^{\mathcal{V}}T^*X;E,F) \longrightarrow 0, \tag{19}$$

where now $S^{[m]}({}^{\mathcal{V}}T^*X;E,F)$ is the quotient of the space of homomorphisms, as before, but now only required to be symbols of order m, (see (79)) by the subspace of symbols of order $m-1$. In fact we require that (19) should split (continuously), i.e. there should be a $\mathcal{V}$–quantization map:

$$q_{\mathcal{V}} : S^{m}({}^{\mathcal{V}}T^*X;E,F) \longrightarrow \Psi_{\mathcal{V}}^{m}(X;E,F) \tag{20}$$

which splits (19) and which is surjective modulo the residual algebra

$$\Psi_{\mathcal{V}}^{-\infty}(X;E,F) = \bigcap_m \Psi_{\mathcal{V}}^{m}(X;E,F). \tag{21}$$

Naturally the map (19) is to be consistent with (15) as is the product formula:

$$\sigma_{m+m',\mathcal{V}}(A \circ B) = \sigma_{m,\mathcal{V}}(A) \circ \sigma_{m',\mathcal{V}}(B). \tag{22}$$

Finally we require the algebra to be asymptotically complete with respect to the filtration. Thus if $A_j \in \Psi_{\mathcal{V}}^{m-j}(X;E,F)$ then there must exist $A \in \Psi_{\mathcal{V}}^{m}(X;E,F)$ such that

$$A \sim \sum_j A_j, \text{ i.e. } A - \sum_{j<N} A_j \in \Psi_{\mathcal{V}}^{m-N}(X;E,F) \quad \forall\ N \in \mathbb{N}. \tag{23}$$

Then A is determined up to the addition of an element of the residual algebra.

These conditions hold for the calculus of pseudodifferential operators on a compact manifold without boundary. Various of the standard theorems in that case can be extended to any algebra with these properties. For example

Proposition. *(24)If $A \in \Psi_{\mathcal{V}}^{m}(X;E,F)$ is elliptic, in the sense that its symbol has a representative which is invertible outside a compact set, and the conditions* (16)–(21) *hold then there exists $B \in \Psi_{\mathcal{V}}^{-m}(X;F,E)$ such that*

$$A \circ B - \mathrm{Id} \in \Psi_{\mathcal{V}}^{-\infty}(X;F,F),\ B \circ A - \mathrm{Id} \in \Psi_{\mathcal{V}}^{-\infty}(X;E,E). \tag{25}$$

In the familiar case, that is $\partial X = \emptyset$ and $\mathcal{V} = \mathcal{C}^\infty(X;TX)$, the residual calculus consists of smoothing operators, which are compact operators on any Sobolev space. Then (25) is a rather strong conclusion. In other cases, such as those discussed below, the residual calculus does not consist of compact operators. Then for most applications (25) must be improved. The proof of (25) itself is straightforward. Using the surjectivity of the symbol map in (19), the composition formula (22) and the exactness in (19) one can make an initial choice B_1 for B which satisfies the weaker form of (25):

$$A \circ B_1 - \mathrm{Id} = R_1 \in \Psi_{\mathcal{V}}^{-1}(X;F). \tag{26}$$

The space $\Psi_{\mathcal{V}}^{-1}(X;E)$ is an ideal in $\Psi_{\mathcal{V}}^{0}(X;E)$ so, again using (18) the Neumann series for $(\mathrm{Id}+R_1)^{-1}$ converges asymptotically in the sense of (23), allowing $B \sim B_1(\mathrm{Id}+R_1)^{-1}$ to be constructed. To improve (25) we should therefore look for other ideals which allow finer approximations to an inverse to be constructed.

To find such ideals in the calculus we need to assume more about the algebra $\mathcal{V}$. Through any point $x \in X$ let $Q = Q(x)$ be the closure of the set of endpoints of integral curves of $\mathcal{V}$ starting at x. We shall demand that this be always a p–submanifold. This is the strongest condition on a submanifold in the category of manifolds with corners, it just means that near each point of Q there is a coordinate system reducing the point to the origin in $\mathbb{R}^n_k$, and such that Q is locally given by the intersection of coordinate planes; the first k of these being the local boundary hypersurfaces. Restriction to Q of the elements of $\mathcal{V}$ defines a Lie algebra of vector fields on Q :

$$\mathcal{V}(X) \longrightarrow \mathcal{W}_{\mathcal{V}}(Q) \subset \mathcal{V}_b(Q). \tag{27}$$

We demand that

$$\mathcal{W}_{\mathcal{V}}(Q) = \mathcal{W}(Q) \text{ span } \mathcal{V}_b(Q) \text{ over the interior of } Q. \tag{28}$$

It follows that the null space of the restriction (27), $\mathcal{N}(\mathcal{V};Q)$, defines a subbundle

$$^{\mathcal{V}}NQ \subset {}^{\mathcal{V}}T_QX. \tag{29}$$

The fibres of this bundle are finite dimensional Lie algebras and

$$\mathcal{W}(Q) = \mathcal{C}^\infty(Q; {}^{\mathcal{V}}T_QX/{}^{\mathcal{V}}NQ)$$

i.e. the structure bundle, ${}^{\mathcal{W}}TQ$, of $\mathcal{W}$ is just the quotient bundle ${}^{\mathcal{V}}T_QX/{}^{\mathcal{V}}NQ$.

We actually require that these conditions on the integral leaves of $\mathcal{V}$ hold locally uniformly in an appropriate sense. To explain this we first consider the notion of a b–map between compact $\mathcal{C}^\infty$ manifolds with corners. A $\mathcal{C}^\infty$ map

$$f : X \longrightarrow X' \tag{30}$$

is a *b–map* if, for each boundary hypersurface $H' \in M_1(X')$ of X' the lift of the ideal of functions, $\mathcal{I}(H') \subset \mathcal{C}^\infty(X')$, vanishing at H' is the product of such ideals in X :

$$f^*\mathcal{I}(H') = \prod_{H \in M_1(X)} \mathcal{I}(H)^{e(H,H')}. \tag{31}$$

In terms of defining functions of the boundary hypersurfaces this just means

$$f^*\rho_{H'} = a_{H'} \prod_{H \in M_1(X)} \rho_H^{e(H,H')},\ 0 < a_{H'} \in \mathcal{C}^\infty(X). \tag{32}$$

The powers $e(H,H')$ are non–negative integers. One important consequence of (32) is that the differential of f extends to the b–tangent space ${}^bTX$, introduced above:

$$^bf_* : {}^bT_xX \longrightarrow {}^bT_{f(x)}X' \quad \forall\ x \in X. \tag{33}$$

We introduce three refinements of the notion of a b–map related to the properties if this b–differential. Namely a b–map is said to be a *b–submersion* if the b–differential, (33), is everywhere surjective. It is said to be a *b–normal* map if

$$\forall\ H \in M_1(X)\ e(H,H') \neq 0 \text{ for at most one } H' \in M_1(X'). \tag{34}$$

Finally a b–map is said to be a *b–fibration* if it is both b–normal and a b–submersion. A fibration of compact manifolds with corners is easily seen to be a b–fibration. The converse statement is not true. For any b–fibration $\phi : X \longrightarrow X'$ the subspace $\mathcal{V}_\phi(X) \subset \mathcal{V}_b(X)$, just the sections of the null space of ${}^b\phi_*$, is a Lie algebra of vector fields satisfying all the conditions imposed so far, as is the weighted version

$$\begin{gathered} \rho^k\mathcal{V}_\phi(X) = \big\{V \in \mathcal{V}_b(X); V = \rho^k W,\ W \in \mathcal{V}_\phi(X)\big\} \\ \rho^k = \prod_{h\in M_1(X)} \rho_H^{k(h)} \end{gathered} \tag{35}$$

for any map $k : M_1(X) \longrightarrow \mathbb{N}_0 = \{0, 1, \dots\}$.

The final condition we wish to impose on $\mathcal{V}$ is that each boundary face $F \in M(X)$ (including X itself) should have a b–fibration

$$\phi_F : F \longrightarrow L(F) \tag{36}$$

such that

$$Q(x) = \phi_F^{-1}(\phi_F(x))\ \forall\ x \text{ in the interior of } F. \tag{37}$$

Moreover we require that the Lie algebra is not too far from being that of the b–fibration corresponding to the interior:

$$\mathcal{V}(X) \supset \rho^k\mathcal{V}_\phi(X) \text{ for some } k : M_1(X) \longrightarrow \mathbb{N}_0 \tag{38}$$

where $\phi = \phi_X$. Although the integral leaves through the interior points of F are just the leaves of ϕ_F through those points this still allows considerable freedom for the Q to change, in dimension, from face to face.

Definition. *(39)A boundary–fibration structure on a compact manifold with corners is a subspace $\mathcal{V}(X) \subset \mathcal{V}_b(X)$ satisfying* (8)*,* (11)*,* (12)*,* (28)*,* (37) *and* (38)*.*

With these additional conditions it follows that for each integral submanifold $Q = Q(x)$

$$\text{each fibre } {}^{\mathcal{V}}N_pQ,\ p \in Q, \text{ is a solvable Lie algebra.} \tag{40}$$

Notice that we are immediately forced to consider a more general notion than that of a boundary–fibration structure, namely the structure induced by a boundary–fibration structure on each of the leaves. In this extended type of structure there are hidden variables, namely those in the bundle, ${}^{\mathcal{V}}NQ$ of solvable Lie algebras. A small calculus of pseudodifferential operators, still denoted $\Psi^m_{\mathcal{V}}(X)$, corresponding to such a structure should behave as convolution operators on the fibres of the associated bundle of Lie groups.

The ideals needed to obtain finer approximate inverses of elliptic operators are associated with the quotients

$$0 \longrightarrow \mathcal{I}_Q(X)\cdot\mathcal{V}(X) \longrightarrow \mathcal{V}(X) \longrightarrow \mathcal{U}(Q) = \mathcal{C}^\infty(Q; {}^{\mathcal{V}}T_Q X) \longrightarrow 0. \tag{41}$$

Since $\mathcal{I}(Q)\cdot\mathcal{V}$ is an ideal this gives rise to a homomorphism of the enveloping algebras, which we call the normal operator at Q :

$$N_Q : \mathrm{Diff}^k_{\mathcal{V}}(X; E, F) \twoheadrightarrow \mathrm{Diff}^k_{\mathcal{U}}(Q; E, F). \tag{42}$$

We require (as a condition on the $\mathcal{V}$–pseudodifferential operators) that there be an extension of this map

$$N_Q : \Psi^m_{\mathcal{V}}(X; E, F) \twoheadrightarrow \Psi^m_{\mathcal{U}}(Q; E, F). \tag{43}$$

The ideal

$$\Psi^{m,1}_{\mathcal{V}}(X; E, F) = \{A \in \Psi^m_{\mathcal{V}}(X; E, F); N_Q(A) = 0 \ \forall\ Q \subset \partial X\} \tag{44}$$

and its powers

$$\Psi^{m,k}_{\mathcal{V}}(X; E, F) = \Psi^{m,1}_{\mathcal{V}}(X; E, F)\cdot\Psi^{0,k-1}_{\mathcal{V}}(X; E) \tag{45}$$

are then of considerable interest and in particular we would expect that

$$\Psi^{-\infty,\infty}_{\mathcal{V}}(X; E, F) \ni A : \mathcal{C}^\infty(X; E) \longrightarrow \dot{\mathcal{C}}^\infty(X; F), \tag{46}$$

the latter space being the subspace of $\mathcal{C}^\infty(X; F)$ consisting of the sections vanishing to all order at ∂X.

Error terms of the type in (46) would be extremely satisfactory, but in most practical cases it is not possible to obtain them directly. This can be understood from the discussion of (26). In order to get the error in $\Psi^{-1,1}_{\mathcal{V}}(X; F)$, which could be improved by iteration, as in the proof of (25), we need to be able to solve the *model problem*

$$N_Q(A)\cdot N_Q(B) = \mathrm{Id} \tag{47}$$

exactly, for all integral submanifolds $Q \subset \partial X$. This can often be done, and this is precisely the method used in the problems discussed below. However the inverse, $N_Q(B)$, is seldom in the small calculus associated to $\mathcal{U}$. In practice it is therefore necessary to consider *full* calculi which have more off–diagonal singularities. This is discussed in III below.

II. Applications and Examples

Next I shall consider various examples of Lie algebras of vector fields, $\mathcal{V}$, and give some representative results concerning $\mathcal{V}$–differential operators which can be obtained by use of the calculus of $\mathcal{V}$–pseudodifferential operators.

(i). b–structure

The first example is the one already mentioned, namely $\mathcal{V}_b(X)$. This is naturally defined on any compact manifold with corners. For a compact manifold with boundary the associated b–calculus was described in [22], see also [15;Chapter 18], and called the totally characteristic calculus. For the general case of a compact manifold with corners the small calculus is described in [25] and the full calculus in [23]. Certainly $\mathcal{V}_b(X)$ is a boundary–fibration structure and the calculus has all the properties described above. In the case of a compact manifold with boundary the only integral submanifolds are X and the components of the boundary $H \in M_1(X)$. The b–fibrations required in (37) are all the single–leaf fibrations. Consider the normal operator for the calculus, as indicated in (43). Since this is such a natural calculus the normal operator can be realized as an operator on an associated geometric space. Namely the normal bundle to a component of the boundary, $N\{X;H\}$, $H \in M_1(X)$, can be compactified to a bundle of intervals, denoted X_H over H. This has a natural action of $[0,\infty)$. The normal operator takes the form

$$N_H : \Psi_b^m(X) \longrightarrow\!\!\!\!\!\rightarrow \Psi_{b,I}^m(X_H) \tag{48}$$

where the suffix I denotes the $[0,\infty)$–invariant elements. The invertibility properties of these operators can be discussed using the Mellin transform. In fact the Mellin transform of the normal operator gives an entire family of pseudodifferential operators on H, called the indicial family:

$$\mathbb{C} \ni \lambda \longmapsto I(A,\lambda) \in \Psi^m(H). \tag{49}$$

If A is elliptic as a b–pseudodifferential operators this family is elliptic for each λ (in fact the principal symbol is independent of λ) and $I(A,\lambda)$ is invertible except for a discrete set of points:

$$\operatorname{spec}_b(A) = \{\lambda \in \mathbb{C} : I(A,\lambda) \text{ is not invertible}\}. \tag{50}$$

Theorem. *(51) (Melrose and Mendoza* [24]*) An elliptic element of $\Psi_b^m(X)$ is Fredholm as a map on the space $\rho^r\mathcal{C}^\infty(X)$ if and only if*

$$\operatorname{Im}\lambda \neq -r \quad \forall\ \lambda \in \operatorname{spec}_b(A). \tag{52}$$

The formula for the index of such an elliptic b–pseudodifferential operators involves extensions of the work of Atiyah, Patodi and Singer [1] on index theory and the η–invariant.

Metrics on the bundle ${}^bTX$ are related to conic geometry. If X is a compact manifold with boundary then near ∂X it decomposes as a product, since ∂X has a (global) defining function $x \in \mathcal{C}^\infty(X)$. By a *conic metric* (of weight s) on X we mean a Riemann metric on the interior which, near ∂X, takes the form

$$g_b \sim x^{2s}\big[\big(\frac{dx}{x}\big)^2 + h\big(x,y,dx,dy\big)\big] \tag{53}$$

for some defining function x, and with $h(0,y,0,dy)$ a metric on ∂X. This is slightly more special than simply a fibre metric on ${}^bTX$. Cheeger considered such questions as the identity of the L^2 cohomology groups of X with respect to g_b (see [6], [7], [8]). These questions can be treated rather directly by constructing a good parametrix for the Laplacian using the b–calculus, see [23].

Another differential–geometric question involving the b–calculus is suggested by the work of Baum, Douglas and Taylor [3]). Namely to find an analytic construction of Poincaré duality for K–theory (Kasparov [16], [17]) for compact manifolds with boundary, extending the original motivation for K–homology of Atiyah.

Theorem. *(54)(Melrose and Piazza* [25]*) A b-quantization map induces isomorphisms realizing Poincaré duality:*

$$K^i(T^*X) \longleftrightarrow K_i(X, \partial X),\ \ K^i(T^*X, \partial T^*X) \longleftrightarrow K_i(X),\ \ i = 0, 1.$$

This means that the K_0–homology groups can be identified with equivalence classes of elliptic b–pseudodifferential operators on X. The K_1 groups can be identified with such classes on $X \times [0, 1]$, so the calculus on manifolds with corners is useful even if X is a manifold with boundary.

The b–calculus also has other more analytic applications, particularly as related to spaces of conormal distributions.

(ii). 0-structure

A second natural boundary–fibration structure on any compact manifold with boundary is the 0–structure. The Lie algebra of vector fields is

$$\mathcal{V}_0(X) = \{V \in \mathcal{V}_b(X); V = 0 \text{ at } \partial X\}\,. \tag{55}$$

The associated calculus described in [19] has all the properties introduced above. The integral submanifolds are simply X and the individual points of the boundary. The b–fibrations are therefore again trivial, the single–fibre fibration of X and the point–fibre fibration of the boundary. Since each boundary point is a leaf the fibre of the structure bundle ${}^0T_pX$ is a solvable Lie algebra for each $p \in \partial X$; it is just the homogeneous extension of an abelian algebra. The normal operators at each boundary point are convolution operators on the associated solvable Lie group. Again the naturality of the 0–structure means that the normal operators actually act on a geometric space, in this case on the inward–pointing half of the tangent space to X at p.

Fibre metrics on the structure bundle ${}^0TX$ are always of the form

$$g_0 = \frac{h}{x^2} \tag{56}$$

where h is a metric in the ordinary sense on X and x is a defining function for the boundary. The invariant metric on hyperbolic n-space, represented as the ball $\{|x| < 1\}$ in $\mathbb{R}^n$,

$$g_H = \frac{|dx|^2}{(1 - |x|^2)}$$

is clearly an example of such a metric. In fact the normal operator of the Laplacian of a metric (56) is always reducible to the hyperbolic Laplacian. This can be used to show that the spectral theory (indeed the scattering theory) of a 0–metric is similar to that of the hyperbolic Laplacian. For example

Theorem. *(57)(Mazzeo and Melrose* [19]*) If* $0 < \kappa \in \mathcal{C}^\infty(X)$ *and* $\kappa|dx|_h^2 = 1$ *at* ∂X *then the kernel of the normalized resolvent of the metric* (56)*,*

$$[\Delta - \kappa s(n-s)]^{-1}\,, \tag{58}$$

extends to a meromorphic family of 0*–pseudodifferential operators,* $s \in \mathbb{C}$*.*

Mazzeo in [18] used the construction of a parametrix for the Laplacian to find the Hodge cohomology of a 0–metric as in (56). Metrics of this form arise on cuspless, infinite volume, quotients of hyperbolic space by a geometrically finite discrete group (e.g. Schottky groups).

Other problems can be tackled using these operators, in particular the solution operators to elliptic boundary problems lie in this calculus.

(iii). Bergman geometry

Any $\mathcal{C}^\infty$ bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ carries Kähler metrics with Kähler form

$$\omega_\rho = -i\partial\bar{\partial}\log\rho \tag{59}$$

where $-\rho$ is a plurisubharmonic defining function for $\partial\Omega$. The Laplacian of this metric corresponds to a boundary–fibration structure on $X = \Omega_{\frac12}$, the manifold obtained from (and diffeomorphic to) Ω by adjoining $\sqrt{\rho}$ to $\mathcal{C}^\infty(\Omega)$, to give $\mathcal{C}^\infty(\Omega_{\frac12})$. Thus there is a $\mathcal{C}^\infty$ map (with fold singularity at the boundary) $\iota_{\frac12} : X \longrightarrow \Omega$. On X the boundary–fibration structure is determined by the contact form θ on $\partial\Omega$. This form fixes a class of 1-forms on X such that for any representative, Θ,

$$\mathcal{V}_\Theta(X) = \left\{V \in \mathcal{V}_0(X); \Theta(V) = O(r^2)\right\} \tag{60}$$

where $r = (\iota_{\frac12})^*\sqrt{\rho} \in \mathcal{C}^\infty(X)$.

Theorem. *(61)(Epstein, Melrose and Mendoza* [11]*) The resolvent of the Bergman–type Laplacian, of the metric corresponding to* (59)*, has an analytic extension similar to that of the Laplacian on the ball:*

$$[\Delta - s(1-s)]^{-1}$$

extends from $\operatorname{Re} s > \frac12$ *to be a meromorphic family of* Θ*-pseudodifferential operators,* $s \in \mathbb{C}$*.*

Using this same approach, i.e. by studying the Bergman–type Laplacian, but now on $(n,0)$ and $(n,1)$–forms one can provide an alternate solution of Kohn's $\bar{\partial}$–Neumann problem:

$$\begin{gathered}\bar{\partial}u = f,\ f \in \mathcal{C}^\infty(\Omega;\Lambda^{0,1}),\ \bar{\partial}f = 0\\ u \perp \mathcal{H}(\Omega) = \left\{u \in L^2(\Omega),\ \bar{\partial}u = 0\right\}.\end{gathered} \tag{62}$$

Corollary. *(63)On any* $\mathcal{C}^\infty$ *strictly pseudoconvex domain Kohn's* $\bar{\partial}$*–Neumann problem* (62) *is solved,* $u = Nf$*, by a* Θ*–pseudodifferential operator of order* -1*.*

The standard regularity properties, see [13], then follow readily.

(iv). Adiabatic limit of a fibration

Consider a fibration of compact manifolds

$$\begin{array}{ccc} F & \longrightarrow & M \\ & & \big\downarrow \phi \\ & & Y. \end{array}$$

On the total space, M, consider a 1-parameter family of metrics which becomes singular as the parameter, $x \downarrow 0$:

$$g_x = g_\infty + \frac{\phi^* h}{x^2}. \tag{64}$$

Here g_∞ is a metric on M (or just a symmetric cotensor which is non–degenerate on the fibres) and h is a metric on Y.

This is an example of a singular limit. It was considered by Witten ([28]) in a special case to study the η–invariant. In general the behaviour of the adiabatic limit of the η–invariant was discussed by Bismut and Cheeger [2] and by Dai [9]. The analytic torsion or Ray and Singer can be examined by similar methods (see Dai, Epstein and Melrose [10]).

The associated boundary–fibration structure provides a simple way to describe the adiabatic limit of the Hodge cohomology. The manifold with corners on which we work is $X = M \times [0, 1]$ where the extra factor is the parameter x. Then set

$$\begin{aligned} \mathcal{V}_a(X) = \big\{V \in & \mathcal{V}_b(X); Vx \equiv 0 \text{ and} \\ & V \text{ is tangent to the fibres of } \phi \text{ at } x = 0\big\}. \end{aligned} \tag{65}$$

If local coordinates are taken for the fibration, y in the base and z in the fibres then locally

$$\mathcal{V}_a(X) = \operatorname{sp}\{x\partial y, \partial z\} = \mathcal{C}^\infty(X; {}^aTX). \tag{66}$$

The b–fibrations required, in (37), to show that $\mathcal{V}_a(X)$ is a boundary–fibration structure are given by the projection $X \longrightarrow M$ over the interior and $\{x = 1\}$ and by the fibration ϕ over $\{x = 0\}$. The metrics (64) fix a fibre metric on ${}^aTX$ and so, from (13) the Laplacian $\Delta_a \in \operatorname{Diff}^2_a(X; {}^a\Lambda^k)$. Then

$$\mathcal{H}^k = \big\{u \in \mathcal{C}^\infty(X; {}^a\Lambda^k); \Delta_a u = 0\big\}$$

is the space of all $\mathcal{C}^\infty$ sections of a vector bundle over $[0, 1]$ with fibre $H^k_x \subset \mathcal{C}^\infty(X_x; {}^a\Lambda^k)$. The behaviour of these fibres at 0 brings out the Leray spectral sequence for the fibration. Namely

Proposition. *(67)(Mazzeo and Melrose* [20]*) For each k the spaces, defined for each $\ell \in \mathbb{N}_0$,*

$$\begin{aligned} E^{k,\ell} = \big\{u_0 \in \mathcal{C}^\infty(X_0; {}^a\Lambda^k); \exists\ \tilde u \in \mathcal{C}^\infty(X; {}^a\Lambda^k) \\ \text{with } \tilde u\big|_{x=0} = u_0,\ \Delta_a \tilde u = 0(x^{2\ell})\big\} \end{aligned} \tag{68}$$

form a decreasing sequence of vector spaces stabilizing to H^k_0, which has the dimension of the cohomology of M :

$$E^{k,0} \supset E^{k,1} \supset E^{k,2} \supset \cdots \supset E^{k,N} = \cdots = E^{k,\infty} = H^k_0. \tag{69}$$

Here $E^{k,0} = \mathcal{C}^\infty(X_0; {}^a\Lambda^k)$, $E^{1,k}$ is the space of sections of a vector bundle over Y and $E^{k,2}$ is finite dimensional.

(v). Töplitz Correspondence

Any compact $\mathcal{C}^\infty$ manifold, Y, can be embedded as a totally real submanifold of a complex manifold, Ω, with $\dim_{\mathbb{C}} \Omega = \dim_{\mathbb{R}} Y$, (Bruhat and Whitney [5]). There exists $\rho \in \mathcal{C}^\infty(\Omega)$ with

$$\rho \geq 0,\ Y = \{\rho = 0\}$$

and a non-degenerate minimum at, i.e. non-degenerate Hessian normal to, Y. Then the tubes around Y

$$\Omega_\epsilon = \left\{z \in \Omega; \rho(z) \leq \epsilon^2\right\}$$

are, for $\epsilon > 0$ small enough, strictly pseudoconvex neighbourhoods of Y. A fibration of Ω near Y and transversal to it can be chosen so that $\operatorname{Im} \partial\rho$ vanishes on the fibres. The Töplitz correspondence is the map obtained by fibre–integration of holomorphic $(n, 0)$ forms:

$$T_\epsilon : \left\{u \in \mathcal{C}^\infty(\Omega_\epsilon; \Lambda^{n,0});\ \bar\partial u = 0\right\} \xrightarrow[\text{fibre}]{\int} \mathcal{C}^\infty(Y). \tag{70}$$

Boutet de Monvel and Guillemin [4] showed that this map is Fredholm and conjectured the following

Theorem. *(71)(Epstein and Melrose* [12]*) There exists* $\epsilon_0 > 0$ *such that for* $0 < \epsilon < \epsilon_0$ *the Töplitz correspondence* (70) *is an isomorphism.*

The main step in the proof is the uniform solution of the $\bar\partial$–Neumann problem in Ω_ϵ as $\epsilon \downarrow 0$. For this we start with the manifold $M \simeq \Omega_{\epsilon_0} \times [0, \epsilon_0]$, obtained by blowing up the singular set in the cone formed by the shrinking tubes,

$$Y \times \{0\} \subset \left\{(z, \epsilon); \rho(z) \leq \epsilon^2,\ 0 \leq \epsilon \leq \epsilon_0\right\}.$$

Thus M is a manifold with corners, it has three boundary hypersurfaces. The blow–up procedure introduces an 'adiabatic' boundary H_a, and this has a fibration, $\phi_a : H_a \longrightarrow Y$. There is a trivial boundary at $\{\epsilon = \epsilon_0\}$ and the θ–boundary, H_θ. The θ–boundary corresponds to the boundary of Ω and has a contact bundle defined on it. Let X be the manifold with the square–root $\mathcal{C}^\infty$ structure introduced at H_θ, there is a conformal class of 1–forms, with representative Θ, defined at the new H_Θ. On X the α–structure is obtained by combining the Θ–structure of (iii) with the adiabatic structure of (iv):

$$\begin{aligned}\mathcal{V}_\alpha(X) = \big\{V \in &\mathcal{V}_b(X); Vx \equiv 0,\\ &V \text{ is tangent to the fibres of } \phi_a \text{ and}\\ &V \text{ vanishes at } H_\Theta \text{ with } \Theta(V) = O(\rho_\Theta^2)\big\}.\end{aligned} \tag{72}$$

Here ρ_Θ is a defining function for H_Θ; $\mathcal{V}_\alpha(X)$ is a boundary–fibration structure. The adiabatic Bergman–type metric with Kähler form

$$-i\partial\bar\partial \log\big(\frac{\rho}{\epsilon^2} - 1\big)$$

is a fibre metric on the structure bundle ${}^\alpha TX$ of $\mathcal{V}_\alpha$. Thus, by Proposition (13) the Laplacian is an elliptic element of $\operatorname{Diff}^2_\alpha(X; {}^\alpha\Lambda^{p,q})$. One of the crucial steps in proving Theorem (71) is to show that the Bergman projection, onto holomorphic functions on the tube of radius ϵ, is an α–pseudodifferential operator.

(vi). Analytic Surgery

Suppose M is a compact manifold without boundary and $H \subset M$ is an embedded, oriented closed hypersurface. If ρ is a defining function for H and x is a parameter consider the 1-parameter family of metrics

$$g_x = a\, d\rho^2 + (\rho^2 + x^2)h.$$

Here $a > 0$ is a $\mathcal{C}^\infty$ function in (ρ, x)-polar coordinates, i.e. on the manifold, $X = \big[M \times [0,1]; H \times \{0\}\big]$, obtained by blowing up $H \times \{0\}$ in $M \times [0,1]$ and h is a metric on M. As $x \downarrow 0$ the metric g_x degenerates to a conic metric (incomplete) on the manifold with boundary obtained by closing $M \backslash H$ with two copies of H. The boundary–fibration structure on X is

$$\mathcal{V}_\nu = \{V \in \mathcal{V}_b(X); Vx \equiv 0\}\,. \tag{73}$$

McDonald in his PHD thesis [21] has constructed the ν–calculus which contains the inverse of the Laplacian of g_x. This allows one to describe, somewhat as in (iv), the behaviour of the Hodge cohomology as $x \downarrow 0$. Partial results of this type have been obtained by Seeley and Singer [27] and more recently extended by Seeley [26].

(vii). Projective algebraic varieties

One rather intriguing question is whether the notion of boundary–fibration in Definition (39) applies to a singular projective algebraic variety M. More precisely let X be a compact manifold with corners obtained from M in two steps. First let $F : \widetilde{M} \longrightarrow M$ be a resolution of M, so the preimage of the singular locus of M is a finite union of hypersurfaces in $\widetilde{M}$ with only normal intersections. From the real point of view these are embedded submanifolds of codimension two, let X be obtained by real blow–up of these submanifolds. Now we ask:

Question. *(74)Is there are boundary–fibration structure on X (for an appropriate choice of $\widetilde{M}$) such that the induced (Fubini–Study) metric on the regular part of M is a weighted fibre metric on the structure bundle?*

A weighted metric is as in (53), i.e. a fibre metric with a conformal factor which is a product of powers of defining functions for boundary hypersurfaces. If such a boundary–fibration structure exists then the construction of the corresponding $\mathcal{V}$–pseudodifferential calculus, as outlined next, should allow a rather direct treatment of the Hodge cohomology of the Laplacian.

III. Microlocalization

In the final part of my talk I wish to give a brief description of a general procedure to construct a calculus of $\mathcal{V}$–pseudodifferential operators. For simplicity I shall only consider the cases involving no parameters, i.e. when $\mathcal{V}$ spans the tangent space over the interior of X. This description therefore only applies to cases (i)–(iii) although the general case is quite similar, simply more complicated. Also for simplicity of presentation I shall suppress various line bundles, typically weighted density bundles, which occur during the construction by describing their sections as 'distributional densities'.

Schwartz' kernel theorem asserts that the space of continuous linear maps

$$A : \dot{\mathcal{C}}^\infty(X) \longrightarrow \mathcal{C}^{-\infty}(X), \tag{75}$$

where the range space is the space of extendible distributions on a compact manifold with corners, can be identified with a space distributional densities over the product $X^2 = X \times X$; namely the sections of the 'kernel density bundle' KD. Thus k_A is the kernel if A if

$$A\psi(x) = \int_X k_A(x,x')\psi(x'). \tag{76}$$

If $\partial X = \emptyset$ then under this identification

$$\Psi^m(X) \longleftrightarrow I^m(X^2,\Delta;\mathrm{KD}). \tag{77}$$

The notation for the space of conormal distributions on the right, on X^2 associated to the diagonal, $\Delta = \{(x,x') \in X^2; x = x'\}$, is that of Hörmander ([14]). The differential operators on X are precisely the local pseudodifferential operators so:

$$\mathrm{Diff}^k(X) \longleftrightarrow \left\{k \in I^k(X^2,\Delta;\mathrm{KD}); \mathrm{supp}(k) \subset \Delta\right\}. \tag{78}$$

This means that the kernels of differential operators are Dirac sections of KD.

The conormal distributions in (77) are those distributions which are singular only at Δ and in any local coordinates have Fourier transform in directions transversal to Δ a symbol:

$$\begin{gathered} a(x,\xi) = \int k(x,x')e^{i(x-x')\cdot\xi}dx' \in S^m \text{ i.e. satisfies} \\ |\partial_x^\alpha\partial_\xi^\beta a(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-|\beta|} \quad \forall\ \alpha,\beta. \end{gathered} \tag{79}$$

A function which is smooth in x and a polynomial in ξ certainly satisfies these estimates. The differential operators are pseudodifferential operators for which a is a polynomial. Thus if we think of the algebra extension

$$\mathrm{Diff}^*(X) \hookrightarrow \Psi^*(X) \tag{80}$$

as microlocalization of $\mathcal{C}^\infty(X;TX)$ it consists in the replacement of all polynomials by all symbols in (79), or all Dirac sections smooth along Δ in (76) by all conormal sections. Then one can think of pseudodifferential operators as non–commutative symbolic functions of vector fields, just as differential operators are non–commutative polynomials in vector fields.

Algebraically it is clear that this construction should be extendible to boundary–fibration structures. Indeed the assumption (12) means that the elements of $\mathcal{V}$ are just linear functions on the fibres of the bundle ${}^{\mathcal{V}}T^*X$. Thus one can think of the elements of $\mathrm{Diff}^*_{\mathcal{V}}(X)$ as corresponding to all polynomials on ${}^{\mathcal{V}}T^*X$. However we need to realize this in a more geometric fashion to generalize (77). Note that $\mathrm{Diff}^k_{\mathcal{V}}(X)$ cannot correspond to all the Dirac sections at the diagonal, since the full algebra of differential operators does!

So we look for a new manifold with corners, the $\mathcal{V}$–stretched product $X^2_{\mathcal{V}}$, which resolves the diagonal from this point of view. More precisely there should be a b–map

$$\beta_{\mathcal{V}} : X^2_{\mathcal{V}} \longrightarrow\!\!\!\!\!\rightarrow X^2 \tag{81}$$

which is a diffeomorphism of the interior of $X^2_{\mathcal{V}}$ to the interior of X^2 and a p–submanifold $\Delta_{\mathcal{V}} \subset X^2_{\mathcal{V}}$ such that

$$\beta_{\mathcal{V}} : \Delta_{\mathcal{V}} \longleftrightarrow \Delta. \tag{82}$$

Moreover $X^2_{\mathcal{V}}$ should be constructed in such a way that each element of $\mathcal{V}$, lifted to X^2 acting on the left factor of X, is $\beta_{\mathcal{V}}$–related to a vector field on $X^2_{\mathcal{V}}$, i.e. the Lie algebra lifts from the left factor, and so that the lifted algebra is transversal to $\Delta_{\mathcal{V}}$, meaning it spans the normal bundle. It is also natural to demand symmetry of the construction, so that the reflection giving interchange of the factors,

$$\tau : X^2 \ni (x, x') \longmapsto (x', x) \in X^2$$

should lift to diffeomorphism $\tau_{\mathcal{V}}$ of $X^2_{\mathcal{V}}$.

Once we find such a resolution we can define a corresponding class of operators. The fact the $\beta_{\mathcal{V}}$ is a b–map which is an isomorphism over the interior means that

$$\beta^*_{\mathcal{V}} : \mathcal{C}^{-\infty}(X^2) \longleftrightarrow \mathcal{C}^{-\infty}(X^2_{\mathcal{V}}). \tag{83}$$

Thus the kernel of an operator can just as well be considered as a distributional density on $X^2_{\mathcal{V}}$ as on X^2. Next we need to find a replacement for the space on the right in (77). This is also easy since the assumption that $\Delta_{\mathcal{V}}$, the lifted diagonal, is a p–submanifold means that it is natural just to take $I^m(X^2_{\mathcal{V}}, \Delta_{\mathcal{V}})$ to be the set of restrictions to $X^2_{\mathcal{V}}$ of the conormal distributions with respect to an extension of $\Delta_{\mathcal{V}}$ in an extension of X^2 as in (3). Away from $\Delta_{\mathcal{V}}$ these distributions are $\mathcal{C}^\infty$ so we can impose additional conditions to get

$$\begin{aligned} I^m_{\mathrm{sm}}(X^2_{\mathcal{V}}, \Delta_{\mathcal{V}}) = \big\{\kappa \in & I^m(X^2_{\mathcal{V}}, \Delta_{\mathcal{V}}); \kappa \equiv 0 \text{ at all} \\ & H \in M_1(X^2_{\mathcal{V}}) \text{ s.t. } H \cap \Delta_{\mathcal{V}} = \emptyset\big\}. \end{aligned} \tag{84}$$

This allows us to define the small calculus by analogy with (77):

$$\Psi^m_{\mathcal{V}}(X) \longleftrightarrow I^m_{\mathrm{sm}}(X^2_{\mathcal{V}}, \Delta_{\mathcal{V}}; \mathrm{KD}). \tag{85}$$

In all the cases (i)–(vi) (and various others) which have been analyzed the stretched product can be defined by a process of blowing up the integral submanifolds Q of $\mathcal{V}$ in ∂X as submanifolds of the boundary of the diagonal in X^2.

This definition automatically gives the inclusion (17) and a symbol map as in (19). The product formula (18) is not so immediate. Before discussing this briefly let me note where the 'full' calculus, alluded to above, comes from. The kernels in (85) are singular just at the diagonal. However it is rather natural to expect similar conormal singularities to arise at the other naturally defined submanifolds of $X^2_{\mathcal{V}}$, namely the boundary hypersurfaces. Thus we can consider more general spaces of conormal distributions, denoted $I^{m,\mathfrak{m}}(X^2_{\mathcal{V}}, \Delta)$, where $\mathfrak{m} : M_1(X) \longrightarrow \mathbb{R}$ associates an order to each boundary face. Then

$$\Psi^{m,\mathfrak{m}}_{\mathcal{V}}(X) \longleftrightarrow I^{m,\mathfrak{m}}(X^2_{\mathcal{V}}, \Delta_{\mathcal{V}}; \mathrm{KD}). \tag{86}$$

In practice one should consider polyhomogeneous spaces, with elements having expansions in terms of powers of the defining functions at the boundary hypersurfaces and the diagonal. The small calculus then just corresponds to kernels of order $+\infty$ (meaning rapidly vanishing) at hypersurfaces not meeting $\Delta_{\mathcal{V}}$ and smooth up to the other hypersurfaces (having expansions with non–negative integral powers only).

The most important outstanding point is the proof of a product formula, (18), or more generally for the full calculus (although this is not usually an algebra so the orders must satisfy appropriate bounds for operators to compose). This proof is also carried out geometrically in these cases. The crucial construction is that of the $\mathcal{V}$–triple product, $X^3_{\mathcal{V}}$. This is related to $X^2_{\mathcal{V}}$ in essentially the same way that X^3 is related to X^2. Namely there are three maps

$$\pi^3_{o,\mathcal{V}} : X^2_{\mathcal{V}} \longrightarrow X^2_{\mathcal{V}} \tag{87}$$

for $o = f, c, s$ corresponding to projection off the left, the central and the right factor of X. The notation corresponds to the fact that in the product formula

$$C = A \cdot B \tag{88}$$

the kernel of C, the composite operator, can be computed from those of A, the second operator, and B, the first operator, by pull–back, product and push–forward operations:

$$\kappa_C = (\pi^3_{c,\mathcal{V}})_* \left[(\pi^3_{s,\mathcal{V}})^* \kappa_A \cdot (\pi^3_{f,\mathcal{V}})^* \kappa_B \right]. \tag{89}$$

Here of course one needs to interpret the operations appropriately in terms of distributional densities. Thus the functorial properties of conormal distributions are of primary importance. In particular it is crucial that the stretched projections (87) be b–fibrations since the push–forward and pull–back of polyhomogeneous conormal distributions under b–fibrations are polyhomogeneous conormal, see [23].

References

1. Atiyah, M.F., Patodi, V.K., and Singer, I.M, *Spectral Asymmetry and Riemannian Geometry.*, Math. Proc. Camb. Phil. Soc. **77** (1975), 43–69.
2. Bismut, J.-M., and Cheeger, J., *Invariants êta et indices des familles pour les variétés à bord. C.R. Acad. Sci. Paris*, **305** (1987), 127–130.
3. Baum, P., Douglas, R.G., and Taylor, M.E., *Cycles and Relative Cycles in Analytic K–theory.*, J. Diff. Geom. **30** (1989), 761–804.
4. Boutet de Monvel, L., and Guillemin, V.W., *The Spectral Theory of Toeplitz Operators.*, Princeton Univ. Press, Princeton, 1981.
5. Bruhat, F. and Whitney, H., *Quelques propriétés fondamentales des ensembles analytiques–réels.*, Commentarii Math. Helv. **33** (1959), 132–160.
6. Cheeger, J., *On the Hodge Theory of Riemannian Pseudomanifolds.*, Proc. Sympos. Pure Math. **36** (1980), 91–146.
7. Cheeger, J., *Spectral Geometry of Singular Riemannian Spaces.*, J. Diff. Geom. **18** (1983), 575–657.
8. Cheeger, J., *Eta Invariants, the Adiabatic Approximation and Conical Singularities.*, J. Diff. Geom. **26** (1987), 175–221.
9. Dai, X., *Adiabatic Limits, Non–multiplicativity of the Signature and the Leray Spectral Sequence.*, PhD Thesis, S.U.N.Y. Stony Brook, (1989).
10. Dai, X., Epstein, C.L., and Melrose, R.B., *Adiabatic Limit of the Ray–Singer Analytic Torsion. (In preparation).*
11. Epstein, C.L., Melrose, R.B., and Mendoza, G., *Resolvent of the Laplacian on Strictly Pseudoconvex Domains. Acta math. (to appear).*

12. Epstein, C.L., and Melrose, R.B., *Shrinking Tubes and the $\bar{\partial}$–Neumann Problem. Preprint (1990).*
13. Folland, G.B., and Kohn, J.J., *The Neumann Problem for the Cauchy–Riemann Complex. Ann. of Math. Studies, No. 75, Princeton Univ. Press, Princeton, 1972.*
14. Hörmander, L., *Fourier Integral Operators I.*, Acta Math. **127** (1971), 79–183.
15. Hörmander, L., *The Analysis of Linear Partial Differential Operators III. Springer–Verlag, Berlin Heidelberg New York Tokyo, 1985.*
16. Kasparov, G.G., *Topological invariants of elliptic operators, I: K–homology. Math. USSR Izvestija* **9** (1975), 751–792.
17. Kasparov, G.G., *Equivariant KK–theory and the Novikov conjecture.*, Invent. Math. **91** (1988), 147–201.
18. Mazzeo, R., *Hodge Cohomology of a Conformally Compact Metric.*, J. Diff. Geom. **28** (1988), 309–339.
19. Mazzeo, R. and Melrose, R.B., *Meromorphic Extension of the Resolvent on Complete Spaces with Asymptotically Negative Curvature.*, J. Funct. Anal. **75** (1987), 260–310.
20. Mazzeo, R., and Melrose, R.B., *The Adiabatic Limit, Hodge Cohomology and Leray's Spectral Sequence for a Fibration.*, J. Diff. Geom. **31** (1990), 185–213.
21. McDonald, P.T., *The Laplacian for Spaces with Cone–Like Singularities.*, PhD Thesis, M.I.T. (1990).
22. Melrose, R.B., *Transformation of Boundary Problems. Acta Math.* **147** (1981), 149–236.
23. Melrose, R.B., (In preparation).
24. Melrose, R.B. and Mendoza, G., *Elliptic Operators of Totally Characteristic Type. MSRI Preprint (1983).*
25. Melrose, R.B. and Piazza, P., *Analytic K-theory on Manifolds with Corners. Adv. in Math. To appear.*
26. Seeley, R., *Conic Degeneration of the Dirac Operator. Preprint.*
27. Seeley, R., and Singer, I.M., *Extending $\bar{\partial}$ to Singular Riemann Surfaces. J. Geom. and Phys.* **5** (1988), 121–136.
28. Witten, E., *Global Gravitational Anomalies.*, Comm. Math. Phys. **100** (1985), 197–229.