FREDHOLM REALIZATIONS OF ELLIPTIC SYMBOLS ON MANIFOLDS WITH BOUNDARY II: FIBERED BOUNDARY

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Abstract.

Contents

Introduction

If the boundary of a manifold is the total space of a fibration

$$
\begin{tikzcd}
Z \arrow[r, dash] & \partial X \arrow[d, "\Phi"] \\
& Y
\end{tikzcd}
\tag{1}
$$

one can quantize an invertible symbol

$$\sigma \in \mathcal{C}^\infty(S^*X; \hom(\pi^*E, \pi^*F))$$

as an elliptic pseudodifferential operator in the Φ-b or edge calculus, $\Psi^0_{\Phi\text{-b}}(X; E, F)$, introduced in [7] or alternately as an elliptic operator in the Φ-c or ϕ calculus, $\Psi^0_{\Phi\text{-c}}(X; E, F)$, introduced in [8]. As on a closed manifold, either of these operators will induce a bounded operator acting between natural L^2-spaces of sections but, in contrast to closed manifolds, these operators need not be Fredholm.

A well-known analysis of Atiyah and Bott established that in order to quantize σ as an operator on X for which Fredholm local boundary conditions exist, it is necessary and sufficient that the Atiyah-Bott obtruction of σ vanish, i.e.,

$$[\sigma] \in \ker\left(K_c(T^*X) \to K^1_c(T^*\partial X)\right).$$

For pseudodifferential calculi associated to a fibration ϕ as above, with $Z = \{pt\}$, namely for Φ-b the zero calculus of [6] and for Φ-c the scattering calculus of [11], the vanishing of the Atiyah-Bott obstruction is a necessary an sufficient condition for the existence of a Fredholm quantization of σ. At the other extreme, when $Y = \{pt\}$, the Φ-b calculus is the b-calculus of [10] and the Φ-c calculus is the cusp calculus of [8] and in either case there is no obstruction to finding a Fredholm realization of σ. The case of general ϕ lies in between these two extremes. Indeed, it was established in [13] that one can use Φ and the families index to induce a map $K_c(T^*X) \to K_c^1(T^*Y)$ (see §4) and that σ has a Fredholm quantization in the Φ-c calculus if and only if

$$[\sigma] \in \ker\left(K_c(T^*X) \to K_c^1(T^*Y)\right).$$

One of our main goals in this paper we establish the corresponding result for the edge calculus.

To this end we consider the even 'smooth K-theory' group of the edge calculus $\mathcal{K}_{\Phi\text{-b}}(X)$ consisting of equivalence classes of Fredholm edge operators where two operators are equivalent if they can be joined by bundle stabilization, bundle isomorphims, or smooth homotopy (see §1). This has a natural subgroup $\mathcal{K}_{\Phi\text{-b},-\infty}(X)$ of equivalence classes whose prinicipal symbol is the identity, and there are analogous odd smooth K-theory groups defined by suspension. These groups are defined and identified for the Φ-c calculus in [13].

The analysis of these groups for the Φ-c calculus is simpler for two related reasons. The first is that the Φ-c calculus is 'asymptotically commutative' whereas the Φ-b calculus is 'asymptotically non-commutative'. More precisely, the behavior of a Φ-c operator near the boundary is modeled by a family of operators, parametrized by Y, acting on the Lie group $\mathbb{R}^{h+1}$ times the closed manfiold Z. On the other hand, the behavior of a Φ-b operator near the boundary is modeled by a family of operators, parametrized by Y, acting on the Lie group $\mathbb{R}^+ \ltimes \mathbb{R}^h$ times the closed manifold Z. To see the effect of this difference on the analysis of these calculi, one can compare the relative simplicity of identifying $\mathcal{K}_{\mathrm{sc}}(X)$ in [13, §2] versus the corresponding identification of $\mathcal{K}_0(X)$ in [1].

The other simplification is that the Φ-c calculus is closed under the smooth functional calculus, while the Φ-b calculus is not. This means that for the Φ-c calculus one can study the smooth K-theory groups using much the same constructions one would use to study the K-theory of its C^*-algebra (see for instance the constructions used in [13, §4] to define KK-classes). One could remove this difficulty by passing to a C^*-closure of the Φ-b calculus, but at the cost of losing much of the structure of the Φ-b calculus.

Instead, we will study the smooth K-theory groups of the Φ-b-calculus by constructing an 'adiabatic' calculus of pseudo-differential operators that interpolates between the Φ-b and Φ-c calculi and induces maps, labeled ad, between their smooth K-theory groups. We work more generally in the context of a fibration $X - M \xrightarrow{\phi} B$ where the fibers are manifolds with fibered boundaries, thus altogether we have

(2)
$$\begin{tikzcd}
 & & X \arrow[r, dash] & M \arrow[d, "\phi"] \\
Z \arrow[r, dash] & \partial X \arrow[ur, hook] \arrow[r, dash] \arrow[d] & \partial M \arrow[ur, hook] \arrow[d, "\Phi"] & B \\
 & Y \arrow[r, dash] & D \arrow[ur] &
\end{tikzcd}$$

Using our analysis of the smooth K-theory groups of the zero calculus in [1], we establish the following theorem.

Theorem 1. *The maps*

$$\mathcal{K}_{\Phi\text{-c}}(\phi) \xrightarrow{\mathrm{ad}} \mathcal{K}_{\Phi\text{-b}}(\phi), \quad \mathcal{K}^1_{\Phi\text{-c}}(\phi) \xrightarrow{\mathrm{ad}} \mathcal{K}^1_{\Phi\text{-b}}(\phi)$$

are isomorphisms, and restrict to isomorphisms

$$\mathcal{K}_{\Phi\text{-c},-\infty}(\phi) \xrightarrow{\mathrm{ad}} \mathcal{K}_{\Phi\text{-b},-\infty}(\phi), \quad \mathcal{K}^1_{\Phi\text{-c},-\infty}(\phi) \xrightarrow{\mathrm{ad}} \mathcal{K}^1_{\Phi\text{-b},-\infty}(\phi).$$

Furthermore, these groups fit into a commutative diagram

$$\begin{tikzcd}
\mathcal{K}^0_{\Phi\text{-c},-\infty}(\phi) \arrow[rr] \arrow[dr, "ad"] & & \mathcal{K}^0_{\Phi\text{-c}}(\phi) \arrow[rr] \arrow[d, "ad"] & & K^0_c(T^*M/B) \arrow[ddd] \\
& \mathcal{K}^0_{\Phi\text{-b},-\infty}(\phi) \arrow[r] & \mathcal{K}^0_{\Phi\text{-b}}(\phi) \arrow[r] & K^0_c(T^*M/B) \arrow[d] \arrow[ur, leftrightarrow, "="'] & \\
& K^1_c(T^*M/B) \arrow[u] & \mathcal{K}^1_{\Phi\text{-b}}(\phi) \arrow[l] & \mathcal{K}^1_{\Phi\text{-b},-\infty}(\phi) \arrow[l] & \\
K^1_c(T^*M) \arrow[uuu] \arrow[ur, leftrightarrow, "="] & & \mathcal{K}^1_{\Phi\text{-c}}(\phi) \arrow[ll] \arrow[u, "ad"] & & \mathcal{K}^1_{\Phi\text{-c},-\infty}(\phi) \arrow[ll] \arrow[ul, "ad"']
\end{tikzcd}$$

wherein the inner and outer six term sequences are exact.

In [13], the groups $\mathcal{K}^q_{\Phi\text{-c}}(\phi)$ are identified in terms of the KK-theory of the $C(B)$-module

$$\mathcal{C}_\Phi(M) = \{f \in \mathcal{C}(M) : f\big|_{\partial M} \in \Phi^*\mathcal{C}(D)\}$$

and the $\mathcal{K}^q_{\Phi\text{-c},-\infty}(\phi)$ groups are identified with the K-theory of the vertical cotangent bundle T^*D/B. Below, we establish the analogue of the latter directly and deduce the analogue of the former from Theorem 1.

Corollary 2. *The groups $\mathcal{K}^q_{\Phi\text{-b}}(\phi)$ are given by*

$$\mathcal{K}^q_{\Phi\text{-b}}(\phi) \cong KK^q_B(\mathcal{C}_\Phi(M), \mathcal{C}(B))$$

and the groups $\mathcal{K}^q_{\Phi\text{-b},-\infty}(\phi)$ are given by

$$\mathcal{K}^q_{\Phi\text{-b},-\infty}(\phi) \cong K^q_c(T^*D/B).$$

Also, we can read off from the six-term exact sequence and this corollary the topological obstruction to realizing an elliptic symbol via a family of Fredholm Φ-b operators.

Corollary 3. *An elliptic symbol $\sigma \in \mathcal{C}^\infty(S^*M/B; \hom(\pi^*E, \pi^*F))$ can be quantized as a Fredholm family of* Φ-b *operators if and only if its K-theory class satisfies*

$$[\sigma] \in \ker\left(K_c(T^*M/B) \to K^1_c(T^*D/B)\right).$$

Finally, in [13] the second author and Frédéric Rochon defined a topological index as a map $\mathcal{K}_{\Phi\text{-c}}(\phi) \to K(B)$ and showed that it coincided with the analytic index. Thus the group $\mathcal{K}_{\Phi\text{-b}}(\phi)$ inherits a topological index map from its isomorphism with $\mathcal{K}_{\Phi\text{-c}}(\phi)$ and, since the adiabatic limit commutes with the analytic index, we also have the following K-theoretic index theorem.

Corollary 4. *The analytic and topological indices coincide as maps*

$$\mathcal{K}_{\Phi\text{-b}}(\phi) \to K(B).$$

In section 1, we review the Φ-b and Φ-c algebras of pseudodifferential operators and the definition of the smooth K-theory groups. In section 2, we set up the adiabatic calculus and prove that it is closed under composition in an appendix. In section 3 we use an excision lemma and the analysis of the smooth K-theory of the zero calculus from [1] to identify the groups $\mathcal{K}^q_{\Phi\text{-b},-\infty}(\phi)$. Finally, in section 4, we establish the six term exact sequence and finish the proof of Theorem 1.

1. Φ-b and Φ-c algebras of pseudodifferential operators

We recall some of the outstanding features of these calculi and refer the reader to, e.g., [10], [7], and [8] for more details. For the moment we restrict attention to the case of a single operator (i.e., $B = \{pt\}$).

We start by describing the vector fields that act as differential operators in the two calculi. Let $\{x, y_1, \ldots, y_h, z_1, \ldots, z_v\}$ be local coordinates near the boundary with y_i horizontal with respect to the fibration (1) and z_i vertical. Here x is a boundary defining function or bdf, i.e. a non-negative function on $\overline{X}$ with $\{x = 0\} = \partial X$ and $dx \neq 0$ on the boundary. We will always assume that the boundary is embedded which implies the existence of a globally defined bdf. The fibered cusp structure depends mildly on the choice of bdf.

The Lie algebra, $\mathcal{V}_{\Phi\text{-b}}$, of vector fields tangent to the fibers of the fibration over the boundary is locally spanned by the vector fields

$$\langle x\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_h}, \partial_{z_1}, \ldots \partial_{z_v}\rangle.$$

Fibered boundary differential operators are polynomials in these vector fields. That is, any $P \in \mathrm{Diff}^k_{\Phi\text{-b}}(X)$ can be written locally as

$$P = \sum_{j+|\alpha|+|\beta|\leq k} a_{j,\alpha,\beta}(x,y,z)\,(x\partial_x)^j\,(x\partial_y)^\alpha\,(\partial_z)^\beta\,.$$

There is a vector bundle ${}^{\Phi\text{-b}}TX$, the Φ-b-tangent bundle, whose space of smooth sections is precisely $\mathcal{V}_{\Phi\text{-b}}$. It plays the role of the usual tangent bundle in the study of the Φ-b calculus. For instance, Φ-b one forms are elements of the dual bundle, ${}^{\Phi\text{-b}}T^*X$, and a Φ-b metric is a metric on ${}^{\Phi\text{-b}}TX$, e.g. locally

$$g_{\Phi\text{-b}} = \frac{dx^2}{x^2} + \frac{\Phi^* g_Y}{x^2} + g_Z. \tag{1.1}$$

'Extreme' cases of fibered boundary calculi are the b-calculus (where Y is a point) and the 0-calculus (where Z is a point). The b-calculus models non-compact manifolds with a cylindrical end. It was used in [10] to prove the APS-index theorem. The 0-calculus models non-compact manifolds that are asymptotically hyperbolic. It has applications to conformal geometry through the Fefferman-Graham construction and to physics (e.g. holography) through the AdS/CFT correspondence [4].

There is also a Φ-c-tangent bundle, whose space of sections $\mathcal{V}_{\Phi\text{-c}}$ is locally spanned by

$$\langle x^2\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_h}, \partial_{z_1}, \ldots \partial_{z_v}\rangle.$$

Thus a fibered cusp differential operator $P \in \mathrm{Diff}^k_{\Phi\text{-c}}(X)$ can be written locally as

$$P = \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(x,y,z)\,\left(x^2\partial_x\right)^j\,(x\partial_y)^\alpha\,(\partial_z)^\beta\,.$$

The ‘extreme’ cases of fibered cusp calculi are known as the cusp-calculus (where Y is a point) and the scattering-calculus (where Z is a point). The cusp-calculus models the same geometric situation (asymptotically cylindrical manifolds) as the b-calculus, and indeed these calculi are very closely related (see [1, §3]). The scattering-calculus models non-compact, asymptotically locally Euclidean manifolds. Indeed, if one compactifies $\mathbb{R}^n$ radially to a half-sphere, the metric near the boundary takes the form

$$\frac{dx^2}{x^4} + \frac{h_x}{x^2},$$

and so defines a metric on the scattering tangent bundle.

As an illustration of the difference between the Φ-b and Φ-c calculi we point out that the scattering Lie algebra is asymptotically commutative in the sense that

$$[\mathcal{V}_{sc}, \mathcal{V}_{sc}] \subset x\mathcal{V}_{sc},$$

while for the zero calculus,

$$[\mathcal{V}_0, \mathcal{V}_0] \subset \mathcal{V}_0.$$

This asymptotic commutativity of horizontal vector fields in the Φ-c calculus lies behind the simplifications over the Φ-b calculus.

As on a closed manifold, certain interesting operations (e.g. powers, parametrices, or inverses) require passing to a larger calculus of pseudodifferential operators. Pseudodifferential operators mapping sections of a bundle E to sections of a bundle F are denoted $\Psi^*_{\Phi\text{-b}}(X; E, F)$ and $\Psi^*_{\Phi\text{-c}}(X; E, F)$ respectively. Operators in these calculi act by means of an distributional integral kernel as in the Schwartz kernel theorem, i.e.

$$Pf(\zeta) = \int_X \mathcal{K}_P(\zeta, \zeta') f(\zeta').$$

These integral kernels have singularities along the diagonal and when $\zeta, \zeta' \in \partial X$. The latter can be resolved by lifting the kernel to an appropriate blown-up space, denoted respectively $X^2_{\Phi\text{-b}}$ and $X^2_{\Phi\text{-c}}$. We will describe these spaces below, in §2, as part of the construction of the adiabatic calculus.

These pseudodifferential calculi posses two symbol maps. On a closed manifold, It is well-known that the highest order terms in the local expression of a differential operator define invariantly a function on the cosphere bundle of the manifold. This same construction yields the principal symbol of a differential (and more generally pseudodifferential) Φ-b operator as a function on the Φ-b cosphere bundle, i.e., the bundle of unit vectors in ${}^{\Phi\text{-b}}T^*X$. This interior symbol fits into the short exact sequence,

$$\Psi^{k-1}_{\Phi\text{-b}}(X; E, F) \hookrightarrow \Psi^k_{\Phi\text{-b}}(X; E, F) \xrightarrow{\sigma} C^\infty\left({}^{\Phi\text{-b}}S^*X; (N^*X)^k \otimes \hom(\pi^*E, \pi^*F)\right),$$

where $(N^*X)^k$ denotes a line bundle whose sections have homogeneity k and π is the projection map ${}^{\Phi\text{-b}}S^*X \to X$. The interior symbol in the fibered cusp calculus works in much the same way, and the corresponding sequence

$$\Psi^{k-1}_{\Phi\text{-c}}(X; E, F) \hookrightarrow \Psi^k_{\Phi\text{-c}}(X; E, F) \xrightarrow{\sigma} C^\infty\left({}^{\Phi\text{-c}}S^*X; (N^*X)^k \otimes \hom(\pi^*E, \pi^*F)\right),$$

is also exact. The interior symbol is used to define ellipticity: an operator is elliptic if and only if its interior symbol is invertible.

The second symbol map, known as the normal operator, models the behavior of the operator near the boundary. Following [2], at any point $p \in Y$ let $\mathcal{I}_{\Phi^{-1}(p)} \subset \mathcal{V}_{\Phi\text{-b}}$

be the subspace of Φ-b vector fields that vanish along the fiber $\Phi^{-1}(p)$ (as Φ-b vector fields). These vector fields form an ideal and the quotient ${}^{\Phi\text{-b}}T_pX$ is a Lie algebra. The projection

$$\mathcal{V}_{\Phi\text{-b}} \to {}^{\Phi\text{-b}}T_pX$$

lifts to a map of enveloping algebras

$$\mathrm{Diff}^k_{\Phi\text{-b}}(X) \to \mathcal{D}\left({}^{\Phi\text{-b}}T_pX\right),$$

and the image of $P \in \mathrm{Diff}^k_{\Phi\text{-b}}(X)$ is known as the normal operator of P at p, $N_{\Phi\text{-b},p}(P)$. The normal operator extends to pseudodifferential operators and can be realized either globally (its kernel is obtained by restricting the kernel of P to a certain boundary face in $X^2_{\Phi\text{-b}}$) or locally by means of an appropriate rescaling as follows.

Assume that we have chosen a product neighborhood of the boundary and, in a neighborhood $\mathcal{U}_Y$ of a point $p \in Y$, we have chosen a local trivialization of Φ, so that $\mathcal{U} \subset X$ looks like

$$\mathcal{U} \cong [0,\varepsilon) \times \mathcal{U}_Y \times Z \xrightarrow{\Phi} \mathcal{U}_Y, \tag{1.2}$$

with coordinates $\{x\}$, $\{y_i\}$, and $\{z_i\}$ on the respective factors, and a corresponding decomposition of the tangent bundle

$${}^{\Phi\text{-b}}TX = {}^{\Phi\text{-b}}N\partial X \oplus V\partial X.$$

Using (1.2) we can define the dilation

$$(x, y_i, z_j) \overset{M_\delta}{\mapsto} (\delta x, \delta y_i, z_j).$$

If $V \in \mathcal{V}_{\Phi\text{-b}}$ then in local coordinates

$$V_\delta := \left(M_\delta^{-1}\right)^* V$$

is a smooth vector field defined in a neighborhood of zero (increasing as δ decreases). The map

$$\begin{aligned} \mathcal{V}_{\Phi\text{-b}} &\to T\left({}^{\Phi\text{-b}}N\partial Z\right) \oplus V\partial Z \\ V &\mapsto \lim_{\delta\to 0} V_\delta \end{aligned}$$

has kernel $\mathcal{I}_{p,\Phi\text{-b}}$ and realizes ${}^{\Phi\text{-b}}T_pZ$ as a Lie algebra of smooth vector fields on T_pZ. More generally, Let $L_{\mathcal{U}}$ be a chart at p mapping into N_p^+Y, and define the normal operator of $P \in \Psi^0_{\Phi\text{-b}}(X)$ at p to be

$$N_{p,\Phi\text{-b}}(P)\,u := \lim_{\delta\to 0} (M_\delta)^* L_{\mathcal{U}}^* P \left(L_{\mathcal{U}}^{-1}\right)^* \left(M_{\frac{1}{\delta}}\right)^* u.$$

The Lie group structure on ${}^{\Phi\text{-b}}N\partial X$ at the point $p \in Y$ is that of $\mathbb{R}^+ \ltimes \mathbb{R}^h$, i.e.

$$(s,u)\cdot(s',u') = (ss', u + su').$$

The kernel of the normal operator at the point $p \in Y$ is invariant with respect to this action, so the normal operator acts by convolution in the associated variables. We refer to this as a non-commutative suspension, and denote these operators by $\Psi^*_{N\mathrm{sus}}\left({}^{\Phi\text{-b}}N\partial X; E, F\right)$. They fit into a short exact sequence

$$0 \to x\Psi^k_{\Phi\text{-c}}(X;E,F) \to \Psi^k_{\Phi\text{-c}}(X;E,F) \xrightarrow{N_{\Phi\text{-b}}} \Psi^k_{N\mathrm{sus}}\left({}^{\Phi\text{-b}}N\partial X; E, F\right) \to 0.$$

The same construction yields a model operator at every point $p \in Y$ for the fibered cusp calculus. In this case the Lie group is commutative and isomorphic to $\mathbb{R} \times \mathbb{R}^h$; the resulting calculus is referred to as the suspended calculus. The corresponding short exact sequence is

$$0 \to x\Psi^k_{\Phi\text{-c}}(X;E,F) \to \Psi^k_{\Phi\text{-c}}(X;E,F) \xrightarrow{N_{\Phi\text{-c}}} \Psi^k_{\mathrm{sus}}\left({}^{\Phi\text{-c}}N\partial X;E,F\right) \to 0.$$

The volume form of a Φ-b metric (e.g. (1.1)), together with Hermitian metrics on E and F, defines a space of L^2 sections. An operator $P \in \Psi^k_{\Phi\text{-b}}(X;E,F)$ acts linearly on L^2 sections of E; boundedly if $k \le 0$ and as an unbounded operator otherwise. The domains of elliptic operators define a scale of Φ-b Sobolev spaces, $H^s_{\Phi\text{-b}}(X;E)$, and an element $P \in \Psi^k_{\Phi\text{-b}}(X;E,F)$ defines a bounded linear operator

$$H^s_{\Phi\text{-b}}(X;E) \xrightarrow{P} H^{s-k}_{\Phi\text{-b}}(X;F).$$

This operator is Fredholm if and only if both symbol maps $\sigma(P)$ and $N_{\Phi\text{-b}}(P)$ are invertible operators, in which case P is said to be *fully elliptic.*

In order to carry out our analysis below, we will need a good understanding of the normal operators of elements of $\Psi_0^{-\infty}$. A construction introduced in [5] realizes normal operators of the zero calculus as families of 'b,c-calculus' on the interval, referred to as *reduced normal operators.* Operators in this calculus behave like b-operators at one end of the interval and like cusp operators at the other end. The reduced normal operator of an operator in $\Psi_0^{-\infty}(X;E)$ lies inside the space

$$\Psi^{-\infty,-\infty}_{b,c}([0,1],\pi^*E)$$

of operators that behave like b-operators of order $-\infty$ near 0 and vanish to infinite order order near 1. In the following result from [5], π denotes the canonical projection $S^*\partial X \to \partial X$.

Proposition 1.1 ([5], Prop. 4.4.1)**.** *The reduced normal operators of elements of $\Psi_0^{-\infty}(X;E)$ are precisely those functions*

$$\begin{aligned}\mathcal{N}(y',\eta;\tau,\rho) \in C^\infty(S^*\partial X)\,\widehat{\otimes}_\pi \dot{C}^\infty\left([-1,1],\Omega^{1/2}\right)\widehat{\otimes}_\pi \mathcal{S}\left(\mathbb{R}_+,{}^b\Omega^{1/2}\right)\\ \otimes_{C^\infty(\widetilde{\mathcal{I}}^2)} C^\infty\left(\widetilde{\mathcal{I}}^2,\beta^*\left(\mathrm{Hom}\left(\pi^*E\otimes{}^{b,c}\Omega^{-1/2}\right)\right)\right)\end{aligned} \tag{1.3}$$

with $\mathcal{N}(y',\eta;\cdot,\rho)$ the lift of a density on $T^*\partial X$, and which for each $\eta \in S^*\partial X$ extend to an element of $\Psi^{-\infty,-\infty}_{b,c}([0,1],\pi^*E)$ in such a way that

$$\mathcal{N} \in C^\infty\left(S^*\partial X, \Psi^{-\infty,-\infty}_{b,c}([0,1],\pi^*E)\right). \tag{1.4}$$

Remark. Notice that the fact that $\mathcal{N}(y',\eta;\cdot,\rho)$ is the lift of a density on $T^*\partial X$ implies in particular that the b-normal operator is a family defined on $S^*\partial X$ but actually only depending on ∂X.

Next, we define K-theory groups of Φ-b operators. Following [13, Definition 2], define

$$A_{\Phi\text{-b}}(M;E) = \{(\sigma(A),\mathcal{N}(A)) : A \in \Psi^0_{\Phi\text{-b}}(M;E) \text{ Fredholm}\}$$

so that $\mathcal{K}^0_{\Phi\text{-b}}(M)$ consists of equivalence classes of elements in $A_{\Phi\text{-b}}(M;E)$, where two elements are equivalent if there is a finite chain consisting of the following:

$$(\sigma,N) \in A_{\Phi\text{-b}}(E) \sim (\sigma',N') \in A_{\Phi\text{-b}}(E') \tag{1.5}$$

if there exists a bundle isomorphism $E \to E'$ over M, that intertwines (σ, N) and (σ', N'),

$$(\sigma, N) \in A_{\Phi\text{-b}}(E) \sim \left(\widetilde{\sigma}, \widetilde{N}\right) \in A_{\Phi\text{-b}}(E) \tag{1.6}$$

if there exists a homotopy of Fredholm operators A_t in $\Psi^0_{\Phi\text{-b}}(M; E)$, with $(\sigma, N) = (\sigma(A_0), \mathcal{N}(A_0))$ and $\left(\widetilde{\sigma}, \widetilde{N}\right) = (\sigma(A_1), \mathcal{N}(A_1))$, and

$$(\sigma, N) \in A_{\Phi\text{-b}}(E) \sim (\sigma \oplus \mathrm{Id}_F, N \oplus \mathrm{Id}_F) \in A_{\Phi\text{-b}}(E \oplus F). \tag{1.7}$$

Similarly, we define $\mathcal{K}^1_{\Phi\text{-b}}(M)$ as equivalence classes of elements in the space of based loops

$$\Omega A_{\Phi\text{-b}}(E) = \{s \in C^\infty\left(\mathbb{S}^1, A_{\Phi\text{-b}}(E)\right) : s(1) = \mathrm{Id}\},$$

where the equivalences are finite chains of (1.5), (1.6), (1.7) with bundle transformations and homotopies required to be the identity at $1 \in \mathbb{S}^1$.

Analogously, we can describe $\mathcal{K}^0_{\Phi\text{-b},-\infty}(M)$ and $\mathcal{K}^1_{\Phi\text{-b},-\infty}(M)$ as equivalence classes of elements in

$$A_{\Phi\text{-b},-\infty}(E) = \{N(\mathrm{Id} + A) : A \in \Psi^{-\infty}_{\Phi\text{-b}}(M; E), \mathrm{Id} + A \text{ Fredholm}\},$$

and

$$\Omega A_{\Phi\text{-b},-\infty}(E) = \{s \in C^\infty\left(\mathbb{S}^1, A_{\Phi\text{-b},-\infty}(E)\right) : s(1) = \mathrm{Id}\}$$

respectively.

2. Adiabatic limit of edge to ϕ calculi

For a compact manifold with boundary, with a specified fibration of the boundary, we show that there is an adiabatic limit construction passing from the fibered boundary (for $\varepsilon > 0$) calculus to the fibered cusp calculus in the limit. Every elliptic Fredholm (i.e. totally elliptic) operator in each of the calculi occurs in a totally elliptic family in the adiabatic calculus, so with constant index. This allows the K-theory of the two algebras to be identified and the (families) index map for one to be reduced to that of the other.

Let X be the compact manifold with boundary, with boundary fibration $Z \longrightarrow \partial X \longrightarrow Y$. We construct a resolution of $X^2 \times [0, \varepsilon_0]$ which carries the adiabatic calculus. First we blow up the corner and consider

$$X^2_{(1)} := \left[X^2 \times [0, \varepsilon_0]; (\partial X)^2 \times \{0\}\right] \xrightarrow{\beta_{(1)}} X^2 \times [0, \varepsilon_0].$$

Then we blow-up the two sides to get

$$X^2_{(2)} := \left[X^2_{(1)}; X \times \partial X \times \{0\}; \partial X \times X \times \{0\}\right] \xrightarrow{\beta_{(2)}} X^2_{(1)}$$

the lifts of these manifolds are disjoint in $X^2_{(1)}$ and commutativity of blow-ups shows that there is a smooth map which is in fact a b-fibration, to the rescaled single space

$$X_\varepsilon := [X \times [0, \varepsilon_0]; \partial X \times \{0\}]$$

in either factor:

$$X^2_{(2)} \underset{\pi_{R,\varepsilon}}{\overset{\pi_{L,\varepsilon}}{\rightrightarrows}} X_\varepsilon.$$

Understanding this will lead to the action of the adiabatic operators on $\mathcal{C}^\infty(X_\varepsilon)$.

The choice of the fibered-cusp structure on X corresponds to the singling out of a class of boundary defining functions which all induce the same trivialization of the normal bundle to the boundary along the fibres of Φ. Let x be such a boundary defining function, on the left factor of X in X^2 and let x' denote the same function on the right factor. The hypersurface $x = x'$ is smooth near $(\partial X)^2 \subset X^2$ and lifts to be smooth near the front face of $X^2_{(2)}$, i.e. the lift of $(\partial X)^2 \times \{0\}$. In fact it meets the front face in a smooth hypersurface; we denote by $\mathfrak{F}$ its intersection with the fibre diagonal

$$\mathfrak{F} \subset \mathrm{ff}\left(X^2_{(2)}\right)$$

(in fact it is already smooth in $X^2_{(1)}$ and does not meet the side faces blown up to produce $X^2_{(2)}$).

Note that $\mathfrak{F}$ is not a p-submanifold because of its intersection with the lift of $(\partial X)^2 \times [0, \varepsilon_0]$ where it meets the boundary of the fibre diagonal

$$\mathfrak{B} \subset X^2_{(2)}$$

(really of course the lift of the fibre diagonal inside $(\partial X)^2 \times [0, \varepsilon_0]$ to $X^2_{(2)}$). The blow-up of $\mathfrak{B}$ resolves $\mathfrak{F}$ to a p-submanifold, so we can set

$$X^2_{\varepsilon\Phi} := \left[X^2_{(2)}; \mathfrak{B}; \mathfrak{F}\right].$$

Proposition 2.1. *The diagonal,* $\mathrm{Diag} \times [0, \varepsilon_0] \subset X^2 \times [0, \varepsilon_0]$ *lifts to an interior p-submanifold of* $X^2_{\varepsilon\Phi}$ *and the projections lift to b-fibrations*

$$X^2_{\varepsilon\Phi} \underset{\pi_R}{\overset{\pi_L}{\rightrightarrows}} X_\varepsilon$$

which are transverse to the lifted diagonal.

For any $\mathbb{Z}_2$-graded vector bundle $\mathbb{E} = (E_+, E_-)$ over X the space $\Psi^k_{\varepsilon\Phi}(X; \mathbb{E})$ of adiabatic-Φ pseudodifferential operators on X, between sections of E_+ and sections of E_-, is defined to be

$$\Psi^k_{\varepsilon\Phi}(X; E_+, E_-) = \big\{A \in I^{k-\frac14}\left(X^2_{\varepsilon\Phi}, \mathrm{Diag} \times [0, \varepsilon_0]; \mathrm{Hom}(E_+, E_-) \otimes \Omega_{\varepsilon\Phi}\right) :$$
$$A \equiv 0 \text{ at the lifts of } \partial X \times X \times [0, \varepsilon_0], X \times \partial X \times [0, \varepsilon_0], \partial X \times X \times \{0\}, X \times \partial X \times \{0\}\big\}.$$

Proposition 2.2. *These kernels define continuous linear operators*

$$\mathcal{C}^\infty(X_\varepsilon; E_+) \to \mathcal{C}^\infty(X_\varepsilon; E_-).$$

Proof. This follows from the push-forward and pull-back theorems of [9] together with proposition 2.1. Indeed, given a section $f \in \mathcal{C}^\infty\left(X_\varepsilon, {}^{\varepsilon\Phi}\Omega^{1/2}\right)$ and an operator $A \in \Psi^k_{\varepsilon\Phi}\left(X; {}^{\varepsilon\Phi}\Omega^{1/2}\right)$ we have

$$Af = (\pi_L)_*(A \cdot \pi_R^* f).$$

Since A vanishes to infinite order at all faces of $X^2_{\varepsilon\Phi}$ not meeting the diagonal, and each boundary face meeting the diagonal maps down to a unique boundary face in X_ε it follows directly that smooth functions are mapped to smooth functions. In the same way, if $\mathcal{E}$ is a smooth index set for X_ε and $\mathcal{A}^{\mathcal{E}}$ is the associated set of polyhomogeneous functions then

$$\mathcal{A}^{\mathcal{E}} \xrightarrow{A} \mathcal{A}^{\mathcal{E}}.$$

The same argument holds with bundle coefficients with only notational differences. $\square$

We will give a similar geometric proof of the composition

$$\Psi^{k}_{\varepsilon\Phi}\left(X;G,H\right)\circ\Psi^{k'}_{\varepsilon\Phi}\left(X;E,G\right)\subset\Psi^{k+k'}_{\varepsilon\Phi}\left(X;E,H\right).$$

in the appendix (Proposition A.2), after we describe the corresponding triple space.

Operators in the adiabatic calculus, along with the usual interior symbol, have four 'normal operators' from restricting their kernels to the different boundary faces. These restrictions in turn have natural interpretations as operators in simpler calculi derived from the suspended and non-commutative suspended calculi in which the normal operators of the fibered cusp and fibered boundary calculi respectively lie. All together, we have five surjective homomorphisms

$$\begin{pmatrix}\sigma\\ N_{\mathfrak{B}}\\ N_{\mathfrak{F}}\\ R_0\\ R_1\end{pmatrix}:\Psi^{m,k}_{\Phi\text{-a}}(X;\mathbb{E})\longrightarrow\begin{pmatrix}S^m({}^{\Phi\text{-a}}T^*X,\hom(\mathbb{E}))\\ \mathcal{C}^\infty\left([0,1]_\varepsilon,\Psi^k_{\Phi-Nsus}\left(X;\mathbb{E}\right)\right)\\ \mathcal{C}^\infty\left([-1,1]_\tau,\Psi^k_{\Phi-sus}\left(X;\mathbb{E}\right)\right)\\ \Psi^m_{\Phi\text{-c}}(X;\mathbb{E})\\ \Psi^m_{\Phi\text{-b}}(X;\mathbb{E})\end{pmatrix}$$

with range respectively the homogeneous sections, the noncommutative Φ-suspended calculus, the Φ-suspended calculus, the fibered cusp and fibered boundary calculi respectively.

The model adiabatic calculus, where $N_{\mathfrak{F}}$ takes values, is a bundle of suspended pseudodifferential algebras on $Y\times(0,1)$. Namely the adiabatic front face of $X^2_{\varepsilon\Phi}$ as a bundle over $Y\times[-1,1]=\mathrm{ff}\left(X_\varepsilon\right)/Z$ (where Z is the fiber of Φ) with fiber over $\{y\}\times\{\frac{x-\varepsilon}{x+\varepsilon}\}$ the product of $RC(W)\times Z_y\times Z_y$ with $RC(W)$ the radial compactification of the subspace

$$W=\ker\left({}^{\varepsilon\Phi}TX_\varepsilon\hookrightarrow TX_\varepsilon\right).$$

It is useful to see how the product on the fibers evolves as $\varepsilon\to0$. For simplicity assume we are in the conformally compact case. In projective coordinates away from $\varepsilon=0$,

$$(x,y,s,u,\varepsilon):=\left(x,y,\frac{x'}{x},\frac{y'-y}{x},\varepsilon\right),$$

the product on the fiber of the zero front face over the point $(0,y_0,\varepsilon)$ is given by

$$(s,u)\cdot(s',u')=(ss',u+su').$$

In the perhaps less natural coordinates

$$(x,y,S,U,\varepsilon):=\left(x,y,\frac{x'-x}{x},\frac{y'-y}{x},\varepsilon\right)$$

this product takes the form

$$(S,U)\cdot(S',U')=(S+S'+SS',U+U'+SU').$$

Near the adiabatic front face we can use coordinates

$$(x,y,\varepsilon,\tau,S_\varepsilon,U_\varepsilon):=\left(x,y,\varepsilon,\frac{x-\varepsilon}{x+\varepsilon},\frac{x'-x}{x\left(x+\varepsilon\right)},\frac{y'-y}{x}\right)$$

and obtain a family of products over the point $(0,y_0,0,\tau_0)$

$$(S_\varepsilon,U)\cdot(S'_\varepsilon,U')=(S_\varepsilon+S'_\varepsilon+\varepsilon S_\varepsilon S'_\varepsilon,U+U'+\varepsilon S_\varepsilon U').$$

Note that if $\varepsilon=1$ this product coincides with that on the zero front face, while at the adiabatic front face ($\varepsilon=0$) it coincides with the product on the scattering front face.

Note that this construction extends to families: If $X - M \xrightarrow{\phi} B$ is a fibration with X as above, then carrying out the above blow-ups on $M \times_B M$ (the fibered product of M with itself) we obtain a fibration

$$\begin{tikzcd} X^2_{\varepsilon\Phi} \arrow[r, dash] & M^2_{\varepsilon\Phi} \arrow[d] \\ & B \end{tikzcd}$$

which carries the kernels of families of adiabatic operators, $\Psi^*_{\varepsilon\Phi}(M/B;\mathbb{E})$.

Proposition 2.3 (cf. [13], Proposition C.1)**.** *If $P \in \Psi^m_{\varepsilon\Phi}(M/B;\mathbb{E})$ and the normal operators $\sigma(P)$, $N_{\mathfrak{B}}(P)$ and $N_{\mathfrak{F}}(P)$ are invertible then $R_0(P)$ and $R_1(P)$ are both fully elliptic and have the same families index.*

Proof. The invertibility of $\sigma(P)$ allows us to construct a parametrix $Q_0 \in \Psi^{-m}_{\varepsilon\Phi}(M/B;E_-,E_+)$ with residues

$$\operatorname{Id} - Q_0P \in \Psi^{-\infty}_{\varepsilon\Phi}(M/B;E_+), \qquad \operatorname{Id} - PQ_0 \in \Psi^{-\infty}_{\varepsilon\Phi}(M/B;E_-).$$

Each invertible normal operator allows us to refine to a parametrix whose residues vanish to infinite order at the corresponding boundary face. Thus the hypothesis of the proposition permit us to find $Q \in \Psi^{-m}_{\varepsilon\Phi}(M/B;\mathbb{E}_-)$ with residues

$$\operatorname{Id} - QP \in x^\infty\Psi^{-\infty}_{\varepsilon\Phi}(M/B;E_+), \qquad \operatorname{Id} - PQ \in x^\infty\Psi^{-\infty}_{\varepsilon\Phi}(M/B;E_-).$$

The induced equations for $R_0(P)$ and $R_1(P)$ show that these operators are fully elliptic hence Fredholm.

The residues above are families of compact operators over $[0,1]$ in the intersection of the fibered cusp and fibered boundary calculi hence it is possible to stabilize. That is, we can find an operator

$$A \in x^\infty\Psi^{-\infty}_{\varepsilon\Phi}(M/B;\mathbb{E}) = \mathcal{C}^\infty\left([0,1]_\varepsilon, \dot{\Psi}^{-\infty}(M/B;\mathbb{E})\right)$$

such that the null space of $P + A$ is in $\dot{\mathcal{C}}^\infty(M/B;E_+)$ and forms a trivial smooth bundle over $[0,1]\times B$ (see [13, Lemma 1.1]). It follows that the index bundles for $R_0(P)$ and $R_1(P)$ coincide in $K(B)$. □

Theorem 2.4. *The adiabatic construction above induces a natural map*

$$\mathcal{K}^*_{\Phi\text{-c}}(\phi) \to \mathcal{K}^*_{\Phi\text{-b}}(\phi).$$

This map restricts to

$$\mathcal{K}^*_{\Phi\text{-c},-\infty}(\phi) \to \mathcal{K}^*_{\Phi\text{-b},-\infty}(\phi),$$

and fits into the following commutative diagram

$$\begin{tikzcd} & \mathcal{K}^*_{\Phi\text{-c}}(\phi) \arrow[dl, "\operatorname{ind}_a"'] \arrow[d] \arrow[dr, "\sigma"] & \\ K^*(B) & \mathcal{K}^*_{\Phi\text{-b}}(\phi) \arrow[l, "\operatorname{ind}_a"'] \arrow[r, "\sigma"] & K_c(T^*(X/B)) \end{tikzcd}$$

Proof. Given a fibered cusp pseudo-differential operator, $P \in \Psi^0_{\Phi\text{-c}}(M/B;\mathbb{E})$ we can construct an element of $\Psi^0_{\varepsilon\phi}(M/B;\mathbb{E})$ by following the construction in [13, Proposition 8]. First the kernel of P gives us the normal operator of a putative

$P(\varepsilon)$ at the (lift of) $\varepsilon = 0$. This fixes the boundary value for the face $\mathfrak{F}$, which as we have just seen is fibered over the lifted variable

$$\tau = \frac{x - \varepsilon}{x + \varepsilon} \in [-1, 1],$$

with fiber given by the front face in the fibered cusp calculus. Knowing the value at $\tau = 1$, we extend to the rest of $\mathfrak{F}$ to be independent of τ. This fixes the kernel at the boundary of $\mathfrak{B}$ and we simply extend it as a conormal distribution to the diagonal and vanishing to infinite order at the other boundary faces. Having defined a kernel consistently on the boundary of $M^2_{\varepsilon\phi}$ we can choose an extension which is conormal to the diagonal in the interior, thus obtaining an element $P(\varepsilon) \in \Psi^0_{\varepsilon\phi}(M/B; \mathbb{E})$.

If P is Fredholm then the normal operators $\sigma(P(\varepsilon))$ and $N_{\mathfrak{F}}(P(\varepsilon))$ are invertible as is $N_{\mathfrak{B}}(P(\varepsilon))$ for small enough ε, say $\varepsilon \in [0, \delta]$. Of course we can arrange for $N_{\mathfrak{B}}(P(\varepsilon))$ to be invertible for all ε for instance by rescaling $[0, \delta]$ to $[0, 1]$ or by perturbing $P(\varepsilon)$ with a family $A(\varepsilon)$ such that $N_{\mathfrak{B}}(A(\varepsilon))$ is a finite rank smoothing perturbation vanishing at $\varepsilon = 0$ and making the family $N_{\mathfrak{B}}(P(\varepsilon))$ invertible (as in [13, Remark 1.2]).

The invertibility of these normal operators allows us to apply Proposition 2.3 and conclude that $R_1(P(E))$ is fully elliptic. Thus this construction yields a map (2.1)

$$\{\Psi^0_{\phi\text{-c}}(M/B; \mathbb{E}) \ \text{Fredholm}\} \ni P \xrightarrow{\widetilde{\text{ad}}} R_1(P(\varepsilon)) \in \{\Psi^0_{\Phi\text{-b}}(M/B; \mathbb{E}) \ \text{Fredholm}\},$$

albeit after making various choices. The choice of δ and extension to $\varepsilon = 1$ do not change the homotopy class of $R_1(P(\varepsilon))$. Furthermore, we can always extend a homotopy of P to a homotopy of $P(\varepsilon)$ as well as the other equivalence relations defining the K-groups (stabilization and bundle isomorphism), so we have a map

$$\mathcal{K}^*_{\Phi\text{-c}}(\phi) \ni [P] \xrightarrow{\text{ad}} [R_1(P(\varepsilon))] \in \mathcal{K}^*_{\Phi\text{-b}}(\phi),$$

well-defined independently of choices.

From the construction it is clear that perturbations of the identity by a smoothing operator are preserved as is the interior symbol, and from Proposition 2.3 so is the families index. $\square$

3. Fredholm perturbations of the identity

In this section, we prove that the map

$$\mathcal{K}^*_{\Phi\text{-c},-\infty}(\phi) \to \mathcal{K}^*_{\Phi\text{-b},-\infty}(\phi),$$

induced by the adiabatic construction in Theorem 2.4 is an isomorphism.

We first recall the excision argument from [13] and reduce to the case $\Phi = \mathrm{Id}$, i.e., the scattering and zero calculi. We start by replacing M with a collar neighborhood of its boundary, $\partial M \times [0, 1]_x$. The K-theory groups we are considering are made up of equivalence classes of normal operators, so there is no loss in restricting attention to the quantizations that are supported in this neighborhood, and identically the identity at $x = 1$.

The fibration extends off the boundary into the entire neighborhood $\partial M \times [0,1]_x$, so we have the simpler situation

$$\begin{tikzcd}
Z \times [0,1] \arrow[r, dash] & \partial X \times [0,1] \arrow[r, dash] \arrow[d] & \partial M \times [0,1] \arrow[d] \\
& Y \times [0,1] \arrow[r, dash] & D \times [0,1] \arrow[d, "\widetilde{\phi}"] \\
& & B
\end{tikzcd}$$

where furthermore we have the following reduction.

Lemma 3.1. *There are 'excision' isomorphisms*

$$\mathcal{K}^*_{\Phi\text{-c},-\infty}(\phi) \cong \mathcal{K}^*_{\mathrm{sc},-\infty}(\widetilde{\phi}) \text{ and } \mathcal{K}^*_{\Phi\text{-b},-\infty}(\phi) \cong \mathcal{K}^*_{0,-\infty}(\widetilde{\phi})$$

Proof. Recall [14, Lemma 6.3], [13, proof of Prop. 3.1] that given

$$b \in \Psi^{-\infty}_{\Phi-sus}(\partial M/D; E)$$

such that $\mathrm{Id} + b$ is invertible, we can assume that b acts on a finite-rank sub-bundle W of $\mathcal{C}^\infty(\partial M/D)$ pulled-back to $T^*(D/B) \times \mathbb{R}$ and then think of b as the boundary family of a family in $\Psi^{-\infty}_{\Phi\text{-c}}(M/B; E)$ or in $\Psi^{-\infty}_{\mathrm{sc}}(D \times [0,1]/B; W)$. Doing the same to fibered boundary operators we obtain the isomorphisms above. $\square$

Hence, it suffices to show that the adiabatic homomorphism between $\mathcal{K}^*_{\mathrm{sc},-\infty}(\widetilde{\phi})$ and $\mathcal{K}^*_{0,-\infty}(\widetilde{\phi})$ is an isomorphism. Both the kernel of an operator in $\Psi^{-\infty}_{\mathrm{sc}}(M; E)$ and the kernel of an operator in $\Psi^{-\infty}_0(M; E)$ are, near their respective front faces, translation invariant in the 'horizontal' directions. We can apply the Fourier transform in these directions and restrict to the front face, and obtain in either case an element of

$$\mathcal{S}(T^*\partial M)\,\widehat{\otimes}_\pi \dot{C}^\infty([-1,1])\,\widehat{\otimes}_\pi \operatorname{Hom}(E). \tag{3.1}$$

In the scattering case this is easily recognized as defining a class in $K^{-1}_c(T^*\partial M \times \mathbb{R}) \cong K^{-2}_c(T^*\partial M)$ and respecting the product structure. We show in the following theorem that the zero case defines the same class in $K^0(T^*\partial M)$. The coincidence of the classes defined by the commutative (scattering) product with that defined by the non-commutative (zero) product is a manifestation of Bott periodicity (cf. [3, Proposition 9.9]).

Theorem 3.2. *The map $\mathcal{K}^*_{\mathrm{sc},-\infty}(\widetilde{\phi}) \to \mathcal{K}^*_{0,-\infty}(\widetilde{\phi})$ induced by the adiabatic calculus is an isomorphism and fits into the commutative diagram*

$$\begin{tikzcd}
\mathcal{K}^*_{\mathrm{sc},-\infty}(\phi) \arrow[r, "\cong"] \arrow[d] & K^*_c(T^*\partial M/B) \\
\mathcal{K}^*_{0,-\infty}(\phi) \arrow[ur, "\cong"] &
\end{tikzcd}$$

Before proving this theorem, we quickly review the relevant descriptions of the topological K-theory groups from [12]. Recall that for any manifold X we can describe $K^1(X)$ as stable homotopy classes of maps

$$K^1(X) = \lim_{\to}\,[S; \mathrm{GL}(N)],$$

so also [12, Proposition 3],

$$K^1(X) = \left[X; \dot{G}^{-\infty}(V; E)\right], \tag{3.2}$$

where V is any compact manifold with boundary and

$$\dot{G}^{-\infty}(V; E) = \{\mathrm{Id} + A : A \in \dot{\Psi}^{-\infty}(X; E) \text{ and } (\mathrm{Id} + A) \text{ is invertible}\}.$$

Similarly, we can define $K^0(X)$ from $K^1(X)$ by suspension, i.e.,

$$K^0(X) = \varinjlim \left[X; C^\infty\left((\mathbb{S}^1, 1); (\mathrm{GL}(N), \mathrm{Id})\right)\right]$$

and this can also be though of as [12, Proposition 4]

$$K^0(X) = \left[X; G^{-\infty}_{sus}(U; E)\right], \tag{3.3}$$

with U any closed manifold of positive dimension and, using $\mathcal{S}$ to denote Schwartz functions,

$$G^{-\infty}_{sus}(U; E) = \{\mathrm{Id} + A : A \in \mathcal{S}(\mathbb{R}; \Psi^{-\infty}(U; E)) \text{ and } (\mathrm{Id} + A) \text{ is invertible}\}.$$

We define the K-theory of a non-compact manifold as the reduced K-theory of its one-point compactification,

$$K^*_c(X) = \widetilde{K}^*\left(\overline{X}/\{\infty\}\right).$$

Proof (of Theorem 3.2). Recall that in [1] it is shown that the groups $\mathcal{K}^q_{0,-\infty}(\phi)$ are canonically isomorphic to the groups $K^q_c(T^*\partial M/B)$. So, for each $\varepsilon_0 > 0$, we have an isomorphism into $K^0_c(T^*\partial M/B)$ consistent with the product on the $\varepsilon = \varepsilon_0$ slice of the adiabatic calculus. The smoothness of the product in the adiabatic calculus shows that, for any Fredholm operator $\mathrm{Id} + A \in \Psi^{-\infty}_{\varepsilon\phi}$, we obtain for each $\varepsilon_0 > 0$ the same class in $K^0_c(T^*\partial M)$.

Thus we have two homomorphisms from $\mathcal{K}^*_{\mathrm{sc},-\infty}(\phi)$ into $K^0_c(T^*\partial M)$. The first is the isomorphism given by identifying (3.1) with an element of $K^{-1}_c(T^*\partial M \times \mathbb{R}) \cong K^{-2}_c(T^*\partial M)$. The second from extending the given Fredholm scattering operator into a Fredholm adiabatic operator and applying the isomorphism of [1]. The first can be represented by a map

$$\begin{aligned}\left[T^*\partial M \times \mathbb{R}; \mathrm{GL}_N(\mathbb{C})\right] &= \left[S^*\partial M \times \mathbb{R}_+ \times \mathbb{R}; \mathrm{GL}_N(\mathbb{C})\right]_0 \\ &= \left[S^*\partial M; \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathrm{GL}_N(\mathbb{C})\right]_0\end{aligned} \tag{3.4}$$

with the zero subscript denoting that the restriction to zero in the $\mathbb{R}_+$ factor depends only on the base variable in $S^*\partial M$. The second homomorphism comes from quantizing the product in (3.4) into the b, c-calculus with a parameter ε:

$$\left[S^*\partial M; \mathcal{S}(\mathbb{R}_+ \times \mathbb{R}) \otimes \mathrm{GL}_N(\mathbb{C})\right]_0 \xrightarrow{q_\varepsilon} \left[S^*\partial M; \Psi^{-\infty,-\infty}_{b,c}\left([0,1]; \mathbb{C}^N\right)\right]_0.$$

When $\varepsilon = 0$ we have the commutative product and for positive ε, the noncommutative (cf. [3, Proposition 9.9, Theorem 9.5]). Hence the isomorphism between $\mathcal{K}^0_{0,-\infty}(\phi)$ and $K^0_c(T^*\partial M)$ is taken into the isomorphism between $\mathcal{K}^0_{\mathrm{sc},-\infty}(\phi)$ and $K^0_c(T^*\partial M)$ by the adiabatic limit. $\square$

4. Six term exact sequence

The Atiyah-Singer Index Theorem induces natural maps

$$\begin{aligned} K_c^q(T^*M/B) &\xrightarrow{I_q} K_c^{q+1}(T^*D/B)\ , \\ [\sigma] &\longmapsto [\mathrm{ind}_{AS}\, r_{\partial M}^*\sigma] \end{aligned}$$

and, together with the identifications $K_c^{q+1}(T^*D/B) \cong \mathcal{K}^{q+1}_{\Phi\text{-b},-\infty}(\phi)$, these are maps $K_c^q(T^*M/B) \to \mathcal{K}^{q+1}_{\Phi\text{-b},-\infty}(\phi)$. Just as in [13, Thm. 6.3], they fit into a diagram

$$\begin{tikzcd} \mathcal{K}^0_{\Phi\text{-b},-\infty}(\phi) \arrow[r, "i_0"] & \mathcal{K}^0_{\Phi\text{-b}}(\phi) \arrow[r, "\sigma_0"] & K_c^0(T^*M/B) \arrow[d, "I_0"] \\ K_c^1(T^*M/B) \arrow[u, "I_1"] & \mathcal{K}^1_{\Phi\text{-b}}(\phi) \arrow[l, "\sigma_1"] & \mathcal{K}^1_{\Phi\text{-b},-\infty}(\phi) \arrow[l, "i_1"] \end{tikzcd} \tag{4.1}$$

with the maps i_* and σ_* induced by inclusion and the principal interior symbol, which we now show is exact.

Proposition 4.1. *The diagram* (4.1) *is exact.*

Proof. While exactness at $\mathcal{K}^*_{\Phi\text{-b}}(\phi)$ follows directly from the definitions, to establish exactness elsewhere we give a different description of the maps I_q.

Given a class $[\sigma_t] \in K_c^1(T^*M/B)$, it is shown in [13, Proposition 6.2] that there is a family of operators

$$\begin{gathered} P_t \in \Psi^0_{\Phi\text{-c}}(M/B;\mathbb{C}^N) \cap \mathrm{Fredholm},\ \text{s.t.} \\ \sigma(P_t) = \sigma_t, \quad P_0 = \mathrm{Id}, \quad P_1 \in (\mathrm{Id} + \Psi^{-\infty}_{\Phi\text{-c}}(M/B;\mathbb{C}^N)) \cap \mathrm{Fredholm} \end{gathered}$$

and the identification $\mathcal{K}^0_{\Phi\text{-c},-\infty}(\phi) \cong K_c^0(T^*D/B)$ takes $[P_1]$ to $[\mathrm{ind}_{AS}\, r^*_{\partial M}\sigma]$. Applying the map $\widetilde{\mathrm{ad}}$ from (2.1) we obtain a family

$$\begin{gathered} \widetilde{P}_t \in \Psi^0_{\Phi\text{-b}}(M/B;\mathbb{C}^N) \cap \mathrm{Fredholm},\ \text{s.t.} \\ \sigma(\widetilde{P}_t) = \sigma_t, \quad \widetilde{P}_0 = \mathrm{Id}, \quad \widetilde{P}_1 \in (\mathrm{Id} + \Psi^{-\infty}_{\Phi\text{-b}}(M/B;\mathbb{C}^N)) \cap \mathrm{Fredholm} \end{gathered}$$

and, from Theorem 3.2, the identification $\mathcal{K}_{\Phi\text{-b},-\infty}(\phi) \cong K_c(T^*D/B)$ takes $[\widetilde{P}_1]$ to $[\mathrm{ind}_{AS}\, r^*_{\partial M}\sigma]$. Thus we can think of I_1 as being $[\sigma_t] \mapsto [\widetilde{P}_1]$.

An element $[\sigma_t]$ is in the null space of I_1 if $\widetilde{P}_1 \sim \mathrm{Id}$, i.e., precisely when $[\sigma_t]$ is in the image of the map $K^1_{\Phi\text{-b}}(\phi) \xrightarrow{\sigma_1} K_c^1(T^*M/B)$. On the other hand the homotopy $\widetilde{P}_t$ shows that $[\widetilde{P}_1]$ is automatically trivial as an element $K^0_{\Phi\text{-b}}(\phi)$, that is $\mathrm{Im}\, I_1 \subseteq \mathrm{null}\, i_0$, and conversely, if $[Q] \in \mathrm{null}\, i_0$, there is (after stabilization) a homotopy of operators from the identity to Q, the symbol of this homotopy defines an element in $K_c^1(T^*M/B)$ that is sent by I_1 to $[Q]$. Thus we see that the diagram is exact at $K_c^1(T^*M/B)$ and at $\mathcal{K}^1_{\Phi\text{-b}}(\phi)$. Just as in [13, Theorem 6.3] the same arguments prove exactness at $K_c^1(T^*M/B)$ and $\mathcal{K}^1_{\Phi\text{-b}}(\phi)$ using Bott periodicity. □

As explained in the proof of this proposition, the diagrams

$$\begin{tikzcd} K^q_c(T^*M/B) \arrow[r, "I_q"] \arrow[dr, "I_q"'] & \mathcal{K}^{q+1}_{\Phi\text{-c}}(\phi) \arrow[d, "\mathrm{ad}"] \\ & \mathcal{K}^{q+1}_{\Phi\text{-b}}(\phi) \end{tikzcd}$$

are commutative. Hence the six term exact sequences for the smooth K-theory of Φ-c and Φ-b operators make up a commutative diagram:

$$\begin{tikzcd}
\mathcal{K}^0_{\Phi\text{-c},-\infty}(\phi) \arrow[rr] \arrow[dr, "ad"] & & \mathcal{K}^0_{\Phi\text{-c}}(\phi) \arrow[rr] \arrow[d, "ad"] & & K^0_c(T^*M/B) \arrow[dl, leftrightarrow, "="] \arrow[ddd] \\
& \mathcal{K}^0_{\Phi\text{-b},-\infty}(\phi) \arrow[r] & \mathcal{K}^0_{\Phi\text{-b}}(\phi) \arrow[r] & K^0_c(T^*M/B) \arrow[d] & \\
& K^1_c(T^*M/B) \arrow[u] \arrow[dl, leftrightarrow, "="'] & \mathcal{K}^1_{\Phi\text{-b}}(\phi) \arrow[l] & \mathcal{K}^1_{\Phi\text{-b},-\infty}(\phi) \arrow[l] & \\
K^1_c(T^*M) \arrow[uuu] & & \mathcal{K}^1_{\Phi\text{-c}}(\phi) \arrow[ll] \arrow[u, "ad"] & & \mathcal{K}^1_{\Phi\text{-c},-\infty}(\phi) \arrow[ll] \arrow[ul, "ad"']
\end{tikzcd}$$

and, since we have already shown that $\mathcal{K}^q_{\Phi\text{-c}}(\phi) \xrightarrow{\mathrm{ad}} \mathcal{K}^q_{\Phi\text{-b}}(\phi)$ is an isomorphism, the five-lemma implies the following.

Theorem 4.2. *The adiabatic calculus induces an isomorphism of the smooth K-theory groups of the* Φ-c *calculus and those of the* Φ-b *calculus,*

$$\mathcal{K}^q_{\Phi\text{-c}}(\phi) \underset{\cong}{\xrightarrow{\mathrm{ad}}} \mathcal{K}^q_{\Phi\text{-b}}(\phi).$$

Appendix A. Triple adiabatic space

In this section we prove the composition formula for the adiabatic calculus. Roughly, the kernel of the composition is given by

$$\mathcal{K}_{A\circ B}(\zeta,\zeta',\varepsilon) = \int_{M^2_{\varepsilon\phi}} \mathcal{K}_A(\zeta,\zeta'',\varepsilon)\,\mathcal{K}_B(\zeta'',\zeta',\varepsilon)\,.$$

Our method is geometric, we construct a space $M^3_{\varepsilon\phi}$ with three b-fibrations 'first', 'second', and 'composite' down to $M^2_{\varepsilon\phi}$. The kernel of the composite is then correctly given by

$$(\pi_C)_*\left(\pi_F^* A \cdot \pi_S^* B\right).$$

The basic triple space is $M^3 \times [0,\epsilon_0]$ where $\epsilon_0 > 0$ and M is a compact manifold with boundary and a specified fibration $\phi : \partial M \longrightarrow Y$ of the boundary. We first have to perform the 'boundary adiabatic' blow-ups to be able to map back to the single and double spaces

$$\begin{gathered} M^3_{(1)} = [M^3 \times [0,\epsilon_0], A_T, A_F, A_C, A_S, A_R, A_M, A_L], \\ A_T = (\partial M)^3 \times \{0\},\ A_F = M \times \partial M \times \partial M \times \{0\}, \\ A_S = \partial M \times \partial M \times M \times \{0\},\ A_C = \partial M \times M \times \partial M \times \{0\}, \\ A_R = M \times M \times \partial M \times \{0\},\ A_M = M \times \partial M \times M \times \{0\}, \\ A_L = \partial M \times M \times M \times \{0\}. \end{gathered} \tag{A.1}$$

Then consider the lifts of the triple and double fibre diagonals, lying within the lifts of the faces $(\partial M)^3 \times [0, \epsilon_0]$ and $(\partial M)^2 \times M \times [0, \epsilon_0]$ and its cyclic images. We can denote these as $\mathfrak{B}_T$, the triple boundary fibred diagonal and $\mathfrak{B}_S$, $\mathfrak{B}_C$ and $\mathfrak{B}_F$, the double fibred diagonals. These are all p-submanifolds, meeting in the standard way for diagonals so we may define

$$M^3_{(2)} = [M^3_{(1)}, \mathfrak{B}_T, \mathfrak{B}_S, \mathfrak{B}_C, \mathfrak{B}_F]. \tag{A.2}$$

The double and triple fibred-cusp diagonal faces are not p-submanifolds in $M_{(1)}$ but lift to be p-submanifolds, which we denote $\mathfrak{F}_T$, $\mathfrak{F}'_F$, $\mathfrak{F}'_C$, $\mathfrak{F}'_S$ and $\mathfrak{F}''_F$, $\mathfrak{F}''_C$ and $\mathfrak{F}''_F$. Here, $\mathfrak{F}'_O$ is the lift of the corresponding double diagonal in the face produced by the blow-up of A_O, $O = S, C, F$ and $\mathfrak{F}''_O$ is the lift of the intersection of this submanifold under the blow-up of A_T. The intersection properties of these submanifolds is essentially the same as for the fibred-cusp (or indeed the cusp) setting itself. Thus we complete the definition by blowing up in 'the usual' order

$$M^3_{\epsilon\phi} = [M^3_{(2)}, \mathfrak{F}_T, \mathfrak{F}''_S, \mathfrak{F}''_C, \mathfrak{F}''_F, \mathfrak{F}'_S, \mathfrak{F}'_C, \mathfrak{F}'_F]. \tag{A.3}$$

Proposition A.1. *Each of the projections, dropping one or two factors of M, lift to b-fibrations*

$$\begin{aligned} &\begin{pmatrix} \pi_F \\ \pi_C \\ \pi_S \end{pmatrix} : M^3_{\epsilon\phi} \rightrightarrows M^2_{\epsilon\phi}, \\ &\begin{pmatrix} \pi_L \\ \pi_M \\ \pi_R \end{pmatrix} : M^3_{\epsilon\phi} \rightrightarrows M_{\epsilon}. \end{aligned} \tag{A.4}$$

Proof. This is the usual commutativity of blow-ups argument. Rearranging the 'fibred-cusp' faces works just as for the (fibred-) cusp calculus. After these triple and two double faces have been blown down, the corresponding fibred-boundary faces can be blown down, as can the pure adiabitic faces. □

Proposition A.2.

$$\Psi^k_{\varepsilon\phi}(M; G, H) \circ \Psi^{k'}_{\varepsilon\phi}(M; E, G) \subset \Psi^{k+k'}_{\varepsilon\phi}(M; E, H)$$

Proof. This follows from Proposition A.1 via the push-forward and pull-back theorems of [9]. Indeed, given two operators (acting on half-densities for simplicity) A and B the kernel of their composition is given by

$$A \circ B = (\pi_C)_* (\pi_S^* A \cdot \pi_F^* B).$$

As the maps are b-fibrations and transversal to the diagonals this yields a distribution conormal to the diagonal of the appropriate degree. Since the kernels of A and B vanish to infinite order at every face not meeting the diagonal, the product of their lifts will vanish to infinite order at every face not meeting the triple diagonal (as this is the intersection of the lift of the two diagonals). Hence only the faces coming from $\operatorname{diag}(M^3) \times \{0\}$, and the blow-ups of $\mathfrak{B}_T$ and $\mathfrak{F}_T$ potentially contribute to the push-forward. It follows that the resulting distribution is smooth down to the each of the boundary faces. □

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