FAMILIES INDEX FOR PSEUDODIFFERENTIAL OPERATORS ON MANIFOLDS WITH BOUNDARY

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Abstract. An analytic families index is defined for (cusp) pseudodifferential operators on a fibration with fibres which are compact manifolds with boundaries. This provides an extension to the boundary case of the setting of the (pseudodifferential) Atiyah-Singer theorem and to the pseudodifferential case of the families Atiyah-Patodi-Singer index theorem for Dirac operators due to Bismut and Cheeger and to Piazza and the first author. In showing that any elliptic family of symbols has a realization as an invertible family of pseudodifferential operators, which is a form of the cobordism invariance of the index, a crucial role is played by the weak contractibility of the group of cusp smoothing operators on a compact manifold with non-trivial boundary and the associated exact sequence of classifying spaces of odd and even K-theory.

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Introduction

For a smooth fibration of compact manifolds

$$\begin{array}{ccc} Z & \longrightarrow & M \\ & & \Big\downarrow \phi \\ & & B \end{array} \tag{1}$$

with model fibre Z, with boundary, we show that any elliptic symbol on the fibres, for bundles E, F over the total space, can be quantised to an invertible family of cusp pseudodifferential operator. This is the cobordism invariance

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of the index in this context, amounting to the fact that for any elliptic family $P \in \Psi^m_{\text{cu}}(M/B; E, F)$ the associated indicial family, which is an elliptic loop $I(P, s) \in \mathcal{C}^\infty(\mathbb{R}; \Psi^m(\partial M/B; E, F))$, defines the trivial class in $K^1(B)$. The homotopy classes of Fredholm perturbations $P + Q$, with $Q \in \Psi^{-\infty}_{\text{cu}}(M/B; E, F)$, form a model for $K^0(B)$, with the group structure given by a relative index formula in terms of the indicial families

$$\operatorname{ind}(P + Q_2) - \operatorname{ind}(P + Q_1) = [I(P + Q_2)I(P + Q_1)^{-1}] \in K^0(B). \tag{2}$$

The result that proves crucial is the weak contractibility of the group of the invertible operators of the form $\operatorname{Id} + A$ with A a cusp operator of order $-\infty$.

The index theorem for families of pseudodifferential operators, of Atiyah and Singer, may be viewed as a topological description of the analytic index. For a fibration of compact manifolds, as in (1) but without boundary, an elliptic family of (classical) pseudodifferential operators on the fibres, acting between sections of vector bundles, $P \in \Psi^m(M/B; E, F)$ has an associated symbol, $\sigma(P) \in \mathcal{C}^\infty(S^*(M/B); E, F)$, which is invertible precisely by the definition of ellipticity. As a family of operators from $\mathcal{C}^\infty(M; E)$ to $\mathcal{C}^\infty(M; F)$, as infinite rank bundles over B, P is Fredholm. It may be perturbed, by an element of $\Psi^{-\infty}(M/B; E, F)$, to have null space of constant rank, and this defines a K-class on the base

$$\operatorname{ind}_{\text{a}}(P) = [\operatorname{null}(P) \ominus \operatorname{null}(P^*)] \in K^0(B) \tag{3}$$

which is independent of the choice of perturbation. In particular in this boundaryless case, the space of Fredholm perturbations $P + Q$ with $Q \in \Psi^{-\infty}(M/B; E, F)$ is contractible and indeed the index class only depends on the symbol $\sigma(P)$. The symbol itself defines an element of $K^0_c(T^*(M/B))$, which may be identified with the relative group for the radial compactification of the fibres of the fibre cotangent bundle, in which case the fibre cosphere bundle may be identified with its boundary:

$$[\sigma(P)] \in K^0_c(T^*(M/B)) = K^0(\overline{T}^*(M/B); S^*(M/B)). \tag{4}$$

The topological index is defined via a Gysin map

$$\operatorname{ind}_{\text{t}} : K^0_c(T^*(M/B)) \longrightarrow K^0(B) \tag{5}$$

and the equality of the notions is a principal result of [2].

Dirac operators, associated to Hermitian Clifford modules, are an important special case of the Atiyah-Singer theorem. Bismut and Cheeger, [3], [4], generalised the families index theorem in this case to manifolds with boundary, following the theorem of Atiyah, Patodi and Singer in the case of a single operator. Bismut and Cheeger proceeded under the assumption that the family of Dirac operators induced on the boundary had null space of constant rank. Topologically, this is not an unreasonable assumption since it is stably equivalent to the vanishing of the class in $K^1(B)$ defined by the (self-adjoint) boundary family; this vanishing is an expression of the cobordism invariance of the index. However, analytically it is not reasonable, since these null spaces have no inherent stability, and it was removed in [10], [11]. In terms of the original context of Atiyah, Patodi and Singer, and also of Bismut and Cheeger, it was shown there that for any family of Dirac operators, $\eth$, associated to a Hermitian Clifford module E on the fibres of a fibration with model fibre Z, a (connected) compact manifold with non-trivial boundary, there is a smooth family of self-adjoint pseudodifferential projections $\Pi \in \Psi^0(\partial M/B; E)$ such that for each $b \in B$, $\Pi_b - \Pi_+$ is spectrally finite for the family of boundary

Dirac operators $\eth_0$. Here Π_+ is the Atiyah-Patodi-Singer projection onto the span of the eigensections of $\eth_0$ with positive eigenvalue. It follows that the family of operators

$$\eth^+ : \{u \in \mathcal{C}^\infty(M/B; E^+); \Pi(u\big|_{\partial M}) = 0\} \longrightarrow \mathcal{C}^\infty(M/B; E^-) \tag{6}$$

is Fredholm. Thus the Fredholm operator depends on a choice of projection Π. The existence of such a smooth family of projections is shown in [10] to be equivalent to the (known) vanishing of the odd index, $[\eth_0] \in K^1(B)$.

The analysis of the index in [10] is carried out in the context of the b-calculus of pseudodifferential operators. This corresponds to the addition of a 'cylindrical end' to the compact manifold with boundary, the method used by Atiyah, Patodi and Singer to obtain their formula in the case of a single Dirac operator. The b-calculus actually corresponds to the asymptotic $\mathbb{R}_+$-invariance of any compact manifold with boundary near its boundary. That is, rather than considering a 'cylindrical end' as a non-compact manifold it is considered as a compact manifold with boundary, obtained in effect by the exponential compactification of the end, $\tilde{x} = e^{-t}$ where t is the translation variable on the end. The effect of this compactification is to replace the fully elliptic Dirac operator (of product type) by a (family of) degenerate 'b-elliptic' Dirac operator associated to a b-metric, meaning one of the form

$$g = \frac{d\tilde{x}^2}{\tilde{x}^2} + h(\tilde{x}) \tag{7}$$

near the boundary, with h smooth and inducing a metric on the boundary. Near the boundary then, an associated Dirac operator is elliptic 'as a function of' the tangent vector fields $\tilde{x}\partial_{\tilde{x}}$, ∂_{y_j} where y_i are boundary coordinates. In the single operator case this is discussed extensively in [8].

In the families case in [10, 11], the Fredholm family for which the index is actually computed is obtained by perturbation, namely $\eth + B$ where B is a b-pseudodifferential operator of order $-\infty$. The non-compactness of such a perturbation is captured by the non-triviality of its indicial family and the perturbation is chosen precisely so that the indicial family of $\eth + B$ is invertible. We make a similar analysis here but in the pseudodifferential case.

There is another compactification of the cylindrical end, which we use here and which achieves essentially the same effect. Namely one can compactify by radial inversion, introducing $x = 1/t$ where t is the translation variable along the cylindrical end. Thus x and $\tilde{x}$ are related transcendentally by $x = 1/\log(1/\tilde{x})$. The asymptotically translation-invariant metric on the cylindrical end then takes the 'cusp' form

$$g = \frac{dx^2}{x^4} + h(x) \tag{8}$$

in place of (7). Just as the (intrinsic) b-structure on a compact manifold with boundary is associated to an algebra of pseudodifferential operators, $\Psi^*_{\mathrm{b}}(Z)$, there is an algebra of cusp operators $\Psi^*_{\mathrm{cu}}(Z)$ which however is not quite intrinsic but depends on the choice of a boundary defining function x. Another choice x' yields the same structure if $x' = cx + O(x^2)$ where $c > 0$ is constant. This can also be identified as a choice of trivialization of the normal bundle to the boundary

$$N(\partial Z) \cong \partial Z \times L \tag{9}$$

where L is a 1-dimensional real oriented vector space.

Our approach depends crucially on the structure of the cusp algebra; this is a special case of the class of algebras discussed in [6]. For any bundles E and F over a compact manifold with boundary it associates a space of operators

$$\Psi^m_{\mathrm{cu}}(Z;E,F) \ni A : \mathcal{C}^\infty(Z;E) \longrightarrow \mathcal{C}^\infty(Z;F), \tag{10}$$

where the choice of a cusp structure is suppressed in the notation. These operators also map $\dot{\mathcal{C}}^\infty(Z;E)$ into $\dot{\mathcal{C}}^\infty(Z;F)$ where $\dot{\mathcal{C}}^\infty(Z;E) \subset \mathcal{C}^\infty(Z;E)$ is the subspace of elements vanishing at the boundary in the sense of Taylor series. The algebra is $*$-invariant, so the elements also act naturally on the dual spaces, consisting of extendible and supported distributional sections and restrict to act on the appropriate Sobolev spaces. In particular

$$P \in \Psi^m_{\mathrm{cu}}(Z;E,F) \Longrightarrow P : H^M_{\mathrm{cu}}(X;E) \longrightarrow H^{M-m}_{\mathrm{cu}}(Z;F),\ \forall\ M \in \mathbb{R}. \tag{11}$$

Over the interior a cusp pseudodifferential operators reduces to a pseudodifferential operator in the usual sense. The symbol map is therefore defined over the interior and it extends by continuity to define the 'cusp' symbol map which gives a well-defined short exact sequence

$$\Psi^{m-1}_{\mathrm{cu}}(Z;E,F) \longrightarrow \Psi^m_{\mathrm{cu}}(Z;E,F) \xrightarrow{\sigma_m} \mathcal{C}^\infty({}^{\mathrm{cu}}S^*Z; \hom(E,F) \otimes R^m). \tag{12}$$

Here ${}^{\mathrm{cu}}S^*Z = {}^{\mathrm{cu}}T^*Z \backslash 0/\mathbb{R}_+$ is the cusp cosphere bundle, the intrinsic sphere bundle of the cusp cotangent bundle. The latter is a bundle canonically associated to the cusp structure. It is isomorphic to the usual cotangent bundle, but not naturally so. The cusp tangent bundle, of which ${}^{\mathrm{cu}}T^*Z$ is the dual, is defined precisely so that a metric (8) is a non-degenerate smooth fibre metric on it. The trivial line bundle R^m in (11) is the bundle over ${}^{\mathrm{cu}}S^*Z$ with sections which are functions over ${}^{\mathrm{cu}}S^*Z \setminus 0$ which are positively homogeneous of degree m. The symbol map is multiplicative

$$\begin{aligned} A \in \Psi^m_{\mathrm{cu}}(Z;E,F),\ B \in \Psi^{m'}_{\mathrm{cu}}(Z;F,G) \Longrightarrow \\ B \circ A \in \Psi^{m+m'}_{\mathrm{cu}}(Z;E,G),\ \sigma_{m+m}(B\circ A) = \sigma_{m'}(B) \circ \sigma_m(A). \end{aligned} \tag{13}$$

For (symbolically) elliptic operators the Fredholm condition is captured by the normal operator, or equivalently the indicial family. For simplicity consider this initially under the assumption that the boundary is connected. For a cusp structure on a compact manifold with boundary, the normal operator is a pseudodifferential operator on the 'model space' at the boundary $L \times \partial Z$. Corresponding to the (intrinsic) 'asymptotic translation invariance' of elements of the cusp calculus, it is invariant under translations in L; in fact its kernel, as a partial convolution operator, is also rapidly decreasing at infinity in L. The algebra of such pseudodifferential operators is the 'L-suspended' algebra, discussed for example in [9]. In the setting of a fibration by compact manifolds with boundary the normal operator corresponds to a short exact sequence with the corresponding bundle of algebras as image

$$x\Psi^m_{\mathrm{cu}}(M/B;E,F) \longrightarrow \Psi^m_{\mathrm{cu}}(M/B;E,F) \xrightarrow{N} \Psi^m_{\mathrm{sus}(L)}(\partial M/B;E,F). \tag{14}$$

The normal map is multiplicative and if P is an elliptic family the existence of an inverse $N(P)^{-1} \in \Psi^{-m}_{\mathrm{sus}(L)}(\partial M/B;F,E)$ to $N(P)$ implies that it is a Fredholm family as a map from the bundle $\dot{\mathcal{C}}^\infty(M;E)$ to $\dot{\mathcal{C}}^\infty(M;F)$ over B. For the action on appropriate Sobolev spaces this condition is necessary and sufficient. In general there is a normal operator associated to each boundary hypersurface and (see the Appendix) smoothing terms between them.

The translation-invariance of $N(P)$ allows it to be realized, by Fourier transform in L, as a 1-parameter family of pseudodifferential operators on the fibres of ∂M; this is the indicial family $I(P,s)$, $s \in \mathbb{R}$. Given the ellipticity of P, the invertibility of $N(P)$ in the suspended algebra is equivalent to the existence of an inverse $I(P,s)^{-1} \in \Psi^{-m}(\partial M/B; F, E)$ for each $s \in \mathbb{R}$. Note that $I(P,s)$ depends on the trivialization of L and, once a corresponding boundary defining function has been chosen, can be defined directly from P

$$I(P,s)v = \left(e^{-is/x}P(e^{is/x}\tilde{v})\right)\big|_{\partial M} \in \mathcal{C}^\infty(\partial M; F),\\ \forall\ v \in \mathcal{C}^\infty(\partial M; E) \text{ with } \tilde{v} \in \mathcal{C}^\infty(M;E),\ \tilde{v}\big|_{\partial M} = v. \tag{15}$$

We denote the set of boundary hypersurfaces of a manifold with boundary X by $M_1(X)$ and by $I_{HG}(P,s)$ the indicial family acting between the boundary hypersurfaces H and G. The full indicial family, $I(P,s)$ is an $N \times N$ matrix of operators between the N components of the boundary.

A general elliptic family $P' \in \Psi^m_{\mathrm{sus}(V)}(M'/B'; E, F)$, for a fibration $\phi : M' \longrightarrow B'$ of compact manifolds without boundary, defines a class in $K^1(B')$ which is precisely the obstruction to the existence of a perturbation $Q' \in \Psi^{-\infty}_{\mathrm{sus}(V)}(M'/B'; E, F)$ to an invertible family with $(P' + Q')^{-1} \in \Psi^{-m}_{\mathrm{sus}(V)}(M'/B'; F, E)$. The cobordism invariance of the index in this case takes the form of the vanishing of the K^1 class for the boundary family of any elliptic family of cusp operators. This is shown in Theorem 1 below. A brief discussion of cusp pseudodifferential operators can be found in the Appendix. In particular there are always Fredholm perturbations of this type and in Theorem 2 we compute the difference of the families index for two such perturbations and show that the homotopy classes of such perturbations form a model for $K^0(B)$. As shown in Theorem 3, there is an invertible perturbation which defines the origin in this space. The analogous 'odd' families theorems (in this case for the corresponding suspended algebras) are also discussed. In a future publication we shall give a formula for the Chern character of the families index, involving eta forms on the boundary and analyse the determinant bundle.

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1. Reduced index formula

If X is a compact manifold with boundary and E is a smooth vector bundle over it, the elements of $\Psi^{-\infty}_{\mathrm{cu}}(X;E)$ which are compact form the ideal

$$x\Psi^{-\infty}_{\mathrm{cu}}(X;E) \hookrightarrow \Psi^{-\infty}_{\mathrm{cu}}(X;E) \xrightarrow{N} \Psi^{-\infty}_{\mathrm{sus}}(\partial X; E) \tag{1.1}$$

where the quotient is given by the normal operator. The 'reduced' formula computes the index of a Fredholm operator of the form $\mathrm{Id} + A$, $A \in \Psi^{-\infty}_{\mathrm{cu}}(X;E)$. It reappears later as the relative index formula for the full algebra.

Proposition 1. *If* $A \in \Psi^{-\infty}_{\mathrm{cu}}(X;E)$ *then* $\mathrm{Id} + A$ *is Fredholm acting on* $L^2_{\mathrm{cu}}(X;E)$ *(or on any of the spaces* $x^r H^m_{\mathrm{cu}}(X;E)$*,* $r, m \in \mathbb{R}$*) if and only if the indicial family* $\mathrm{Id} + I(A,s)$ *is invertible, acting on* $L^2(\partial X; E)$*, for all* $s \in \mathbb{R}$*, and then the index can be expressed in terms of the winding number of the Fredholm determinant of the indicial family*

$$\mathrm{ind}(\mathrm{Id} + A) = \mathrm{wn}(\det(\mathrm{Id} + I(A,s))). \tag{1.2}$$

Proof. The necessity of the condition on the indicial family is part of the structure theory of the cusp calculus. To prove it, suppose that $\mathrm{Id} + I(A, s_0)$ is not invertible for some $s_0 \in \mathbb{R}$. By the Fredholm alternative, this implies that $(\mathrm{Id} + I(A, s_0))\phi = 0$ for some non-trivial $\phi \in \mathcal{C}^\infty(\partial X; E) = \bigoplus_{H \in M_1(X)} \mathcal{C}^\infty(H; E)$. This can be used to construct a family $u_t \in L^2(X; E)$ with $\|u_t\|_{L^2} = 1$ which is weakly convergent to 0 but is such that $\|(\mathrm{Id} + A)u_t\| \to 0$. It follows from this that $\mathrm{Id} + A$ cannot be Fredholm on L^2.

If $A \in \Psi^{-\infty}_{\mathrm{cu}}(X; E)$ is such that $\mathrm{Id} + I(A, s)$ is invertible for all $s \in \mathbb{R}$. then we can construct a parametrix for the form $\mathrm{Id} + B$, $B \in \Psi^{-\infty}_{\mathrm{cu}}(X; E)$. Here $\mathrm{Id} + I(B, s) = (\mathrm{Id} + I(A, s))^{-1}$ and standard inductive and asymptotic summation arguments are used. It can be arranged that $\mathrm{Id} + B$ is the generalised inverse, so has null space exactly the null space of $\mathrm{Id} + A^*$ and range the orthocomplement to $\mathrm{null}(\mathrm{Id} + A)$.

Then Calderón's formula for the index becomes

$$\mathrm{ind}(\mathrm{Id} + A) = \mathrm{Tr}([\mathrm{Id} + A, \mathrm{Id} + B]) = \mathrm{Tr}([A, B]) = \overline{\mathrm{Tr}}([A, B]). \tag{1.3}$$

Assuming for simplicity of notation that ∂X is connected we can apply the trace-defect formula, (A.27), and the standard formula for the logarithmic derivative of the determinant

$$\begin{aligned} \mathrm{ind}(\mathrm{Id} + A) &= \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{Tr}\left(I(B, s) \frac{d}{ds} I(A, s) \right) ds \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{Tr}\left((\mathrm{Id} + I(B, s)) \frac{d}{ds} I(A, s) \right) ds \\ &= \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{Tr}\left((\mathrm{Id} + I(A, s))^{-1} \frac{d}{ds} I(A, s) \right) ds \\ &= \mathrm{wn}(\det(\mathrm{Id} + I(A, s)). \end{aligned} \tag{1.4}$$

□

Let $\dot{\Psi}^{-\infty}(X; E)$ be the space of smoothing operators with kernels vanishing to infinite order at the boundary.

Proposition 2. *The index map*

$$\mathrm{ind} : \{A \in \Psi^{-\infty}_{\mathrm{cu}}(X; E); \mathrm{Id} + A \text{ is Fredholm on } L^2_{\mathrm{cu}}(X; E)\} \longrightarrow \mathbb{Z}$$

is surjective and the null space is

$$\{A \in \Psi^{-\infty}_{\mathrm{cu}}(X; E);\ \exists\ S \in \dot{\Psi}^{-\infty}(X; E) \text{ with } (\mathrm{Id} + A + S)^{-1} = \mathrm{Id} + B,\ B \in \Psi^{-\infty}_{\mathrm{cu}}(X; E)\}. \tag{1.5}$$

Proof. The surjectivity of the index follows immediately from Proposition 1 above. If $\mathrm{Id} + A$ has index zero then its null space is a finite dimensional subspace of $\dot{\mathcal{C}}^\infty(X; E)$ with the same dimension as another subspace of $\dot{\mathcal{C}}^\infty(X; E)$ which is a complement to its range. Taking a finite rank operator $S \in \dot{\Psi}^{-\infty}(X; E)$ which is an isomorphism between them and is trivial on the orthocomplement of the null space gives the characterisation (1.5) of the set of operators of index zero. □

2. The cusp-smoothing group

The group

$$G^{-\infty}_{\text{cu}}(X;E)=\{A\in\Psi^{-\infty}_{\text{cu}}(X;E);\operatorname{Id}+A\text{ is invertible with }(\operatorname{Id}+A)^{-1}=\operatorname{Id}+B,\ B\in\Psi^{-\infty}_{\text{cu}}(X;E)\}\tag{2.1}$$

plays a crucial role in the analysis below. In particular, we show in this section that it is weakly contractible, i.e. that all its homotopy groups are trivial.

First consider the normal subgroup consisting of invertible smoothing perturbations of the identity.

Proposition 3. *The group of smoothing operators on any compact manifold with boundary*

$$\dot G^{-\infty}(X;E)=\Big\{\operatorname{Id}+A,\ A\in\dot\Psi^{-\infty}(X;E);(\operatorname{Id}+A)^{-1}=\operatorname{Id}+B,\ B\in\dot\Psi^{-\infty}(X)\Big\}\tag{2.2}$$

is a classifying group for odd K-theory so for all $k\in\mathbb{N}$, $\Pi_{2k}(\dot G^{-\infty}(X;E))=\{0\}$, and isomorphism

$$\Pi_{2k-1}(\dot G^{-\infty}(X;E))\longrightarrow\mathbb{Z}\tag{2.3}$$

are defined by pull-back and integration of the basic (or index) classes,

$$\beta^{odd}_k=c^{odd}_k\operatorname{Tr}\left((a^{-1}da)^{2k-1}\right),\ c^{odd}_k=\frac{1}{(2\pi i)^k}\frac{(k-1)!}{(2k-1)!}.\tag{2.4}$$

Proof. Finite rank approximation reduces this to the computation of the stable homotopy groups of $\mathrm{GL}(N,\mathbb{C})$ and of the index classes. The explicit constant c^{odd}_k can be found in Theorem 19.3.1 of the book of Hörmander [5] and in section 2 of the paper of Atiyah, [1]. □

Proposition 4. *For any compact manifold without boundary, Y, the group of suspended smoothing operators*

$$G^{-\infty}_{\text{sus}}(Y;E)=\big\{\operatorname{Id}+A;A\in\mathcal{S}(\mathbb{R};\Psi^{-\infty}(Y;E)),\ (\operatorname{Id}+A)^{-1}=\operatorname{Id}+B,\ B\in\mathcal{S}(\mathbb{R};\Psi^{-\infty}(Y;E))\big\}\tag{2.5}$$

is a classifying group for even K-theory, so $\Pi_{2k-1}(G^{-\infty}_{\text{sus}}(Y;E)=\{0\}$ with the isomorphisms

$$\Pi_{2k}(G^{-\infty}(Y;E))\longrightarrow\mathbb{Z}\tag{2.6}$$

defined by pull-back and integration of the basic classes

$$\beta^{even}_k=c^{even}_k\int_{\mathbb{R}}\operatorname{Tr}\left((a^{-1}da)^{2k}a^{-1}\frac{da}{ds}\right),\ c^{even}_k=\frac{1}{(2\pi i)^{k+1}}\frac{k!}{(2k)!}=\frac{c^{odd}_k}{4\pi i}.\tag{2.7}$$

Remark 1. It is conventional to suppose that a manifold is connected. However this conflicts with the convenient assertion that the boundary of a manifold with boundary is itself a manifold, rather than the more awkward 'finite union of disjoint manifolds'. In Propositions 3 and 4 we can allow the wider, not necessarily connected notion, of a manifold provided operators are allowed *between* the components. Thus, for example, the space of smoothing operators in this case is isomorphic to

$$\mathcal{C}^\infty(Y^2) = \bigoplus_{i,j=1}^{N} \mathcal{C}^\infty(Y_i \times Y_j) \tag{2.8}$$

where $Y = \bigcup_{i=1}^{N} Y_i$ is the decomposition into components. This is our convention in this paper.

Proof. If $f : \mathbb{S}^{2k+1} \to G^{-\infty}(Y;E)$ is a generator of $\pi_{2k+1}(G^{-\infty}(Y;E)) \cong \mathbb{Z}$, it gives rise in a natural way to a generator of $\pi_{2k}(G^{-\infty}_{\mathrm{sus}}(Y;E)) \cong \mathbb{Z}$. In fact, using the connectedness of $G^{-\infty}(Y;E)$, we may deform f so that it is the identity in a neighbourhood of some point $p \in \mathbb{S}^{2k+1}$ and the antipodal point $-p$. This gives rise to a new map $g : \mathbb{R} \times \mathbb{S}^{2k} \to G^{-\infty}(Y;E)$ with $g(s,\omega) = \mathrm{Id}$ for $|s| > 0$ large enough, where $(s,\omega) \in \mathbb{R} \times \mathbb{S}^{2k}$. By (A.6), the map g can also be thought as a map $g : \mathbb{S}^{2k} \to G^{-\infty}_{\mathrm{sus}}(Y;E)$. The argument can be reversed, so this is a generator of $\pi_{2k}(G^{-\infty}_{\mathrm{sus}}(Y;E)) \cong \mathbb{Z}$. It is now easy to relate c_k^{even} to c_k^{odd}. If (s,ω) denotes coordinates for $\mathbb{R} \times \mathbb{S}^{2k}$, we have

$$\begin{aligned}
1 &= \int_{\mathbb{S}^{2k+1}} f^*\beta^{\mathrm{odd}}_{k+1} = \int_{\mathbb{R}\times\mathbb{S}^{2k}} g^*\beta^{\mathrm{odd}}_{k+1} \\
&= \int_{\mathbb{R}\times\mathbb{S}^{2k}} g^* c^{\mathrm{odd}}_{k+1} \operatorname{Tr}\left((a^{-1}da)^{2k+1}\right) \\
&= c^{\mathrm{odd}}_{k+1} \int_{\mathbb{R}\times\mathbb{S}^{2k}} \operatorname{Tr}\left((a^{-1}(s,\omega)\left(\frac{da}{ds}ds + d_\omega a\right))^{2k+1}\right) \\
&= (2k+1)c^{\mathrm{odd}}_{k+1} \int_{\mathbb{R}\times\mathbb{S}^{2k}} \operatorname{Tr}\left(\left(a^{-1}(s,\omega)d_\omega a\right)^{2k} \left(a^{-1}(s,\omega)\frac{da}{ds}ds\right)\right) \\
&= (2k+1)\frac{c^{\mathrm{odd}}_{k+1}}{c^{\mathrm{even}}_k} \int_{\mathbb{S}^{2k}} g^*\beta^{\mathrm{even}}_k = (2k+1)\frac{c^{\mathrm{odd}}_{k+1}}{c^{\mathrm{even}}_k},
\end{aligned} \tag{2.9}$$

so we conclude that $c_k^{\mathrm{even}} = (2k+1)c^{\mathrm{odd}}_{k+1} = \frac{1}{(2\pi i)^{k+1}}\frac{k!}{(2k)!}$. ∎

We define forms on $G^{-\infty}_{\mathrm{cu}}(X;E)$ using the regularized trace functional defined in (A.25) by a choice of boundary defining function and analytic continuation:

$$\gamma_k = c^{\mathrm{odd}}_k \overline{\operatorname{Tr}}\left((a^{-1}da)^{2k-1}\right). \tag{2.10}$$

Proposition 5. *If X is a connected compact manifold with boundary then for each k,*

$$d\gamma_k = I^*(\beta^{even}_k + dT_k), \tag{2.11}$$

where T_k is also a smooth form on $G^{-\infty}_{\mathrm{sus}}(\partial X;E)$ and

$$I : G^{-\infty}_{\mathrm{cu}}(X;E) \longrightarrow G^{-\infty}_{\mathrm{sus}}(\partial X;E) \tag{2.12}$$

is the indicial map.

Proof. Using (2.7) and (A.27), an explicit computation gives

(2.13)

$$\begin{aligned}
d\gamma_k &= c_k^{\mathrm{odd}}\overline{\mathrm{Tr}}\left(d(a^{-1}da)^{2k-1}\right) \\
&= -c_k^{\mathrm{odd}}\overline{\mathrm{Tr}}\left((a^{-1}da)^{2k}\right) \\
&= -\frac{1}{2}c_k^{\mathrm{odd}}\overline{\mathrm{Tr}}\left([(a^{-1}da)^{2k-1}, a^{-1}da]\right) \\
&= \frac{1}{2}c_k^{\mathrm{odd}}\frac{1}{2\pi i}\int_{\mathbb{R}}\mathrm{Tr}\left((a^{-1}da)^{2k-1}(a^{-1}\dot{a}a^{-1}da - a^{-1}d\dot{a})\right)d\tau \\
&= \frac{1}{4\pi i}c_k^{\mathrm{odd}}\left(\int_{\mathbb{R}}\mathrm{Tr}\left((a^{-1}da)^{2k}a^{-1}\dot{a}\right)d\tau - d\int_{\mathbb{R}}\mathrm{Tr}\left((a^{-1}da)^{2k-1}a^{-1}\dot{a}\right)d\tau\right) \\
&= I^*\left(\beta_k^{\mathrm{even}} + d\left(-\frac{1}{4\pi i}c_k^{\mathrm{odd}}\int_{\mathbb{R}}\mathrm{Tr}\left((a^{-1}da)^{2k-1}a^{-1}\dot{a}\right)d\tau\right)\right).
\end{aligned}$$

□

Since $\dot{\Psi}^{-\infty}(X;E) \subset \Psi_{\mathrm{cu}}^{-\infty}(X;E)$ is an ideal, $\dot{G}^{-\infty}(X;E) \subset G_{\mathrm{cu}}^{-\infty}(X;E)$ is a normal subgroup. The normal operator is trivial on $\dot{\Psi}^{-\infty}(X;E)$ and so maps the quotient into $G_{\mathrm{sus}}^{-\infty}(\partial X;E)$. Applying Proposition 2 shows that it has range exactly $G_{\mathrm{sus,ind}=0}^{-\infty}(\partial X;E)$. Thus we have a short exact sequence of groups

$$\dot{G}^{-\infty}(X;E) \longrightarrow G_{\mathrm{cu}}^{-\infty}(X;E) \xrightarrow{N} G_{\mathrm{sus,ind}=0}^{-\infty}(\partial X;E)[[x]] \tag{2.14}$$

where the quotient group is precisely the space of formal power series

$$\begin{aligned}
&G_{\mathrm{sus,ind}=0}^{-\infty}(\partial X)[[x]] \\
&\quad = \{L \in \Psi_{\mathrm{sus}}^{-\infty}(\partial X;E)[[x]]; L = \sum_{k=0}^{\infty} L_k x^k,\ \mathrm{Id} + L_0 \in G_{\mathrm{sus,ind}=0}^{-\infty}(\partial X;E)\}.
\end{aligned} \tag{2.15}$$

The product here is a $\star$ product.

The lower order terms in the image group in (2.14) are arbitrary affine terms, so it is contractible to $G_{\mathrm{sus,ind}=0}^{-\infty}(X;E)$. The sequence (2.14) is not quite a fibration, since there is no local splitting. However, we have the following weaker result.

Lemma 1. *The exact sequence of groups* (2.14) *is a Serre fibration, that is, it has the homotopy lifting property for disks.*

Proof. Let $h_t : I^k \longrightarrow G_{\mathrm{sus,ind}=0}^{-\infty}(\partial X;E)[[x]]$, $t \in [0,1]$, be a homotopy such that h_0 has a lift to $G_{\mathrm{cu}}^{-\infty}(X;E)$, so that there exists a map $\tilde{h}_0 : I^k \longrightarrow G_{\mathrm{cu}}^{-\infty}(X;E)$ with the property that $N \circ \tilde{h}_0 = h_0$. Then we must show that we can extend $\tilde{h}_0$ to a homotopy $\tilde{h}_t : I^k \longrightarrow G_{\mathrm{cu}}^{-\infty}(X;E)$ such that $N \circ \tilde{h}_t = h_t$ for all $t \in [0,1]$.

First note that h_t gives rise to a map $H \in \mathcal{C}^\infty(I^{k+1} \times G_{\mathrm{sus,ind}=0}^{-\infty}(\partial X;E)[[x]])$. Using Borel's lemma, it can be lifted to a map $\tilde{H} \in \mathcal{C}^\infty(I^{k+1} \times (\mathrm{Id} + \Psi_{\mathrm{cu}}^{-\infty}(X;E)))$ which gives rise to a homotopy $h_t' : I^k \longrightarrow (\mathrm{Id} + \Psi_{\mathrm{cu}}^{-\infty}(X;E))$ such that $N \circ h_t' = h_t$ for all $t \in [0,1]$. Initially however, nothing ensures us that the image of h_t' lies in $G_{\mathrm{cu}}^{-\infty}(X;E)$.

Nevertheless, by Propositon 2, we know that the image of h_t' is contained in the space of Fredholm operators of index zero. In fact, again by propositon 2, the homotopy h'_t gives rise to a principal bundle

$$\begin{array}{ccc} \dot{G}^{-\infty}(X;E) & \longrightarrow & \mathcal{B} \\ & & \downarrow \\ & & [0,1]\times I^k \end{array} \tag{2.16}$$

where the fibre $\mathcal{B}_{(t,x)}$ above the point $(t,x) \in [0,1]\times I^k$ is given by

$$\mathcal{B}_{(t,x)} = \{(\mathrm{Id} + h'_t(x) + A) \in G^{-\infty}_{\mathrm{cu}}(X;E); A \in \dot{\Psi}^{-\infty}(X;E)\}. \tag{2.17}$$

Here, the group $\dot{G}^{-\infty}(X;E)$ has an obvious left action on the fibre. Since the base space is contractible, this principal bundle must be trivial (see for instance corollary 11.6 in [12]). In particular, it has a global smooth section $s : [0,1]\times I^k \longrightarrow \mathcal{B}$. Without loss of generality, we can assume that this smooth section is such that $s(0,x) = \tilde{h}_0(x)$ for all $x \in I^k$. Finally, looking at the section s as a map $s : [0,1]\times I^k \longrightarrow G^{-\infty}_{\mathrm{cu}}(X;E)$, we see that the homotopy $\tilde{h}_t : I^k \longrightarrow G^{-\infty}_{\mathrm{cu}}(X;E)$ defined by $\tilde{h}_t(x) = s(t,x)$ is the desired lift of h_t. □

It follows that there is a long exact sequence in homotopy theory

$$\begin{aligned} \ldots \longrightarrow \pi_l(\dot{G}^{-\infty}(X;E)) \longrightarrow \pi_l(G^{-\infty}_{\mathrm{cu}}(X;E)) \longrightarrow \\ \pi_l(G^{-\infty}_{\mathrm{sus,ind}=0}(\partial X;E)[[x]]) \xrightarrow{b} \pi_{l-1}(\dot{G}^{-\infty}(X;E))\ldots, \end{aligned} \tag{2.18}$$

all classes being represented by smooth maps.

Proposition 6. *On any compact manifold, X, with non-trivial boundary, and for any complex vector bundle E over X, the group $G^{-\infty}_{\mathrm{cu}}(X;E)$ defined by (2.1) is weakly contractible.*

Proof. Using the long exact sequence (2.18), we see that the weak contractibility of the group $G^{-\infty}_{\mathrm{cu}}(X;E))$ is equivalent to the fact that the maps

$$b : \pi_l(G^{-\infty}_{\mathrm{sus,ind}=0}(\partial X;E)[[x]]) \longrightarrow \pi_{l-1}(\dot{G}^{-\infty}(X;E)) \tag{2.19}$$

are isomorphisms for all $l \in \mathbb{N}_0$.

For $l=0$, this follows from the fact that $G^{-\infty}_{\mathrm{sus,ind}=0}(\partial X;E)[[x]]$ is connected (see theorem 4 in [9]). For $l = 2k+1, k \in \mathbb{N}_0$, this is clearly the case, since $\pi_{2k+1}(G^{-\infty}_{\mathrm{sus,ind}=0}(\partial X;E)[[x]]) = \pi_{2k}(\dot{G}^{-\infty}(X;E)) = \{0\}$. Finally, for $l=2k, k\in\mathbb{N}$, $\pi_{2k}(G^{-\infty}_{\mathrm{sus,ind}=0}(\partial X;E)[[x]]) \simeq \pi_{2k-1}(\dot{G}^{-\infty}(X;E)) \simeq \mathbb{Z}$, so the map b will be an isomorphism if it maps a generator of $\pi_{2k}(G^{-\infty}_{\mathrm{sus,ind}=0}(\partial X;E)[[x]])$ to a generator of $\pi_{2k-1}(\dot{G}^{-\infty}(X;E))$. Therefore, let us assume without loss of generality that $f : \mathbb{S}^{2k} \longrightarrow G^{-\infty}_{\mathrm{sus,ind}=0}(\partial X;E))$ is such that $[f] \in \pi_{2k}(G^{-\infty}_{\mathrm{sus,ind}=0}(\partial X;E)) \simeq \pi_{2k}(G^{-\infty}_{\mathrm{sus,ind}=0}(\partial X;E)[[x]])$ is a generator.

We proceed to construct $b([f])$. First, lift f to the 1-point blow-up

$$\beta_q : [\mathbb{S}^{2k}, \{q\}] \longrightarrow \mathbb{S}^{2k}.$$

Thus, $f\circ\beta_q$ is a map from the $2k$-ball and consequently has a lift $\tilde{f} : [\mathbb{S}^{2k},\{q\}] \longrightarrow G^{-\infty}_{\mathrm{cu}}(X;E)$. Since $I(\partial[\mathbb{S}^{2k};\{q\}]) = q$, $\tilde{f}$ has its image in a fibre when restricted to $\partial[\mathbb{S}^{2k};\{q\}]$, so it is of the form $\tilde{f}(s) = g(s)\circ\tilde{f}(q)$ for $s \in \partial[\mathbb{S}^{2k};\{q\}] = \mathbb{S}^{2k-1}$ with $g : \mathbb{S}^{2k-1} \longrightarrow \dot{G}^{-\infty}(X;E)$. This is just Serre's construction, see for instance [12], of $b([f]) = [g] \in \pi_{2k-1}(\dot{G}^{-\infty}(X;E))$. Hence, what we need to show is that $[g]$ is a generator of $\pi_{2k-1}(\dot{G}^{-\infty}(X;E))$. By Proposition 3, this amounts to showing that $\int_{\mathbb{S}^{2k-1}} g^*\beta_k^{\mathrm{odd}} = \pm 1$. By Proposition 4, Proposition 5 and Stokes' theorem

$$\begin{aligned}\int_{\mathbb{S}^{2k-1}} g^*\beta_k^{odd} = & \int_{\partial[\mathbb{S}^{2k},\{q\}]} \tilde{f}^*\gamma_k = \int_{[\mathbb{S}^{2k},\{q\}]} \tilde{f}^* d\gamma_k \\ = & \int_{\mathbb{S}^{2k}} \tilde{f}^*(I^*(\beta_k + dT_k)) = \int_{\mathbb{S}^{2k}} f^*\beta_k = 1\end{aligned} \tag{2.20}$$

since f was assumed to be a generator of $\pi_{2k}(G^{-\infty}_{\mathrm{sus,ind}=0}(\partial X;E))$. Therefore, $b([f]) = [g]$ is a generator of $\pi_{2k-1}(\dot{G}^{-\infty}(X;E))$, which implies that the map b is an isomorphism and we conclude that the group $G^{-\infty}_{\mathrm{cu}}(X;E)$ is weakly contractible. □

3. Perturbations of elliptic cusp operators

Proposition 7. *For any elliptic operator $P \in \Psi^m_{\mathrm{cu}}(X;E,F)$ there exist perturbations $Q \in \Psi^{-\infty}_{\mathrm{cu}}(X;E,F)$ such that $P+Q$ is Fredholm as a map from $\dot{\mathcal{C}}^\infty(X;E)$ to $\dot{\mathcal{C}}^\infty(X;F)$, or from the cusp Sobolev space $H^m_{\mathrm{cu}}(X;E)$ to $L^2_{\mathrm{cu}}(X;F)$.*

Proof. From the definition of $\Psi^m_{\mathrm{cu}}(X;E)$ we may cut the kernel of P off near the boundary of the front face corresponding to each pair of boundary components and so arrange that the normal operator has compact support. This only changes P by an element of $\Psi^{-\infty}_{\mathrm{cu}}(X;E)$ and ensures that the indicial family is an entire family of elliptic elements $I(P,s) \in \Psi^m(\partial X;E)$. In fact this family is necessarily invertible as $|\operatorname{Re} z| \to \infty$ with $\operatorname{Im} z$ bounded. We may perturb it to make it is invertible on the real axis, which implies that the resulting operator is Fredholm. To see this, note that there can only be a finite number of points on the real axis at which $I(P,s)$ has non-trivial kernel. Near such a point s_0, we may divide the space, either $\dot{\mathcal{C}}^\infty(\partial X;E)$ or $H^m(\partial X;E)$, into the finite dimensional null space and its L^2 orthocomplement. With a similar decomposition of the range in terms of the adjoint, $I(P,s)$ becomes a matrix of the form

$$\begin{pmatrix} P' & (s-s_0)Q \\ (s-s_0)L & (s-s_0)A \end{pmatrix}. \tag{3.1}$$

The matrix $M(s) = \begin{pmatrix} P' & 0 \\ 0 & G \end{pmatrix}$, where G is an isomorphism from the null space to the null space of the adjoint, is invertible. Composing on the left with the inverse gives a matrix

$$\begin{pmatrix} \mathrm{Id} & (s-s_0)Q' \\ (s-s_0)L' & (s-s_0)A' \end{pmatrix}. \tag{3.2}$$

This matrix is of the form $\mathrm{Id} + R(s)$ with $R(s)$ of finite rank, and smoothing. Moreover the determinant vanishes only to finite order at $s = s_0$ so there is a smoothing perturbation supported arbitrarily close to s_0 making it invertible in a fixed neighbourhood of s_0. Doing this at each point at which there is non-trivial null space gives a perturbation of $\mathrm{Id} + R(s)$ making it invertible. Composing on the left with $M(s)$ gives a perturbation of $I(P,s)$ making it invertible for all $s \in \mathbb{R}$. This perturbation is the indicial family of an element $Q \in \Psi^{-\infty}_{\mathrm{cu}}(X;E,F)$ which therefore gives the desired perturbation of P. □

We may strengthen Proposition 7 further.

Corollary 1. *Under the assumptions of Proposition 7 there exists a perturbation* $Q \in \Psi^{-\infty}_{\mathrm{cu}}(X; E, F)$ *such that* $P + Q$ *is an isomorphism, in either sense.*

Proof. Having perturbed P by a term $Q_1 \in \Psi^{-\infty}_{\mathrm{cu}}(X; E, F)$ so that it is Fredholm, suppose its index is $k \in \mathbb{Z}$. Multiplying on the right by an operator of the form $\mathrm{Id} + A$, $A \in \Psi^{-\infty}_{\mathrm{cu}}(X; E)$ with index $-k$ gives an operator $P + Q_2$, still with $Q_2 \in \Psi^{-\infty}_{\mathrm{cu}}(X; E, F)$, which is Fredholm of index zero. The null space is in $\dot{\mathcal{C}}^\infty(X; E)$ and the null space of the adjoint, with respect to a choice of fibre metric, is in $\dot{\mathcal{C}}^\infty(X; F)$ and has the same dimension. Thus, adding a finite rank smoothing operator in $\dot{\Psi}(X; E, F)$ gives an invertible operator $P + Q$ with Q as claimed. □

4. Perturbation of Elliptic Families

Now consider a smooth fibration of compact manifolds (1) where the model fibre Z is a manifold with boundary. If $P \in \Psi^m_{\mathrm{cu}}(M/B; E, F)$ is a smooth family of elliptic cusp operators on the fibres acting between sections of vector bundles over the total space M, then for $b \in B$ we may consider

$$\mathcal{P}_b = \{P_b + Q_b; Q_b \in \Psi^{-\infty}_{\mathrm{cu}}(Z_b; E_b, F_b),\ \exists\ (P_b + Q_b)^{-1} \in \Psi^{-m}_{\mathrm{cu}}(Z_b; F_b, E_b)\}. \tag{4.1}$$

Here $Z_b = \phi^{-1}(b)$ is the fibre and E_b, F_b are the restrictions of the bundles to it.

Proposition 8. *For an elliptic family* $P \in \Psi^m_{\mathrm{cu}}(M/B; E, F)$ *the spaces in* (4.1) *form a smooth bundle of principal* $G^{-\infty}_{\mathrm{cu}}(Z_b; E)$*-spaces over* B,

$$\begin{array}{ccc} G^{-\infty}_{\mathrm{cu}}(Z_b; E) & \text{———} & \mathcal{P} \\ & & \Big\downarrow \phi \\ & & B. \end{array} \tag{4.2}$$

Proof. By Corollary 1 the spaces in (4.1) are non-empty. Moreover, the notion of a local smooth section is well defined. Furthermore, any two points in $\mathcal{P}$ corresponding to invertible perturbations by Q_1 and Q_2 are related by composition on the right with $(P_b + Q_2)^{-1}(P_b + Q_1) \in G^{-\infty}_{\mathrm{cu}}(Z_b; E_b)$ and conversely. □

Using the contractibility of the fibres, it is now easy to show the existence of a global section of the bundle $\mathcal{P}$.

Theorem 1. *Let* P *be a smooth family of elliptic cusp pseudodifferential operators of order* m *on the fibres of a fibration of a compact manifold with boundary. Then there exists a smooth family* Q *of cusp pseudodifferential operators of order* $-\infty$ *such that* $P + Q : \dot{\mathcal{C}}^\infty(X; E) \longrightarrow \dot{\mathcal{C}}^\infty(X; F)$ *is invertible.*

Proof. Given $p \in B$, there exists an open neighbourhood $\mathcal{U}$ of p and a local trivialization of the bundle M,

$$h : \mathcal{U} \times Z \longrightarrow \phi^{-1}(\mathcal{U}), \quad \textit{with} \quad \phi \circ h(u, z) = u\ \forall\ (u, z) \in \mathcal{U} \times Z. \tag{4.3}$$

This gives rise to a local trivialization of the bundle $\mathcal{P}$, and in fact to a local structure of principal $G^{-\infty}_{\mathrm{cu}}(Z; E)$-bundle

$$H : \mathcal{U} \times G^{-\infty}_{\mathrm{cu}}(Z; E) \longrightarrow \phi^{-1}(\mathcal{U})\ . \tag{4.4}$$

Notice however that this local structure of principal $G^{-\infty}_{\mathrm{cu}}(Z; E)$-bundle depends on the choice of the trivialization in (4.3), so is not canonical. In particular, this means that there is no natural way of defining a global principal bundle structure with fixed structure group. Nevertheless this suffices to construct a global continuous section of the bundle $\mathcal{P}$. Indeed, consider a triangulation of the manifold B such that each simplex is contained in some open set $\mathcal{U}$ as in (4.4). Then, assigning arbitrary values on the vertices of this triangulation to the section $s : B \longrightarrow \mathcal{P}$ we want to construct, we see, using Proposition 6 and the discussion of §29.1 in [12], that it is possible to extend the section s continuously to all of B. Using a partition of unity and local smoothing we may perturb this continuous section to be smooth. □

5. Relative index theorem

Under the same, ellipticity, hypothesis as in Theorem 1 we now consider all the Fredholm perturbations in the cusp algebra.

Theorem 2. *If $P \in \Psi^m_{\mathrm{cu}}(M/B; E, F)$ is a smooth family of elliptic cusp pseudo-differential operators of order m on the fibres of a fibration of a compact manifold with boundary then*

$$\begin{gathered}\pi_0(\mathcal{P}_F) \simeq K^0(B) \\ \mathcal{P}_F = \left\{Q \in \Psi^{-\infty}_{\mathrm{cu}}(M/B; E, F); P + Q \text{ is a family of Fredholm operators}\right\}\end{gathered} \tag{5.1}$$

with the isomorphism given by the relative index formula

$$\operatorname{ind}(P + Q) = [I(P + Q)I(P + Q_0)^{-1}] \text{ in } K^0(B) \tag{5.2}$$

where $P + Q_0$ is invertible.

Proof. We have shown above that the bundle $\mathcal{P}$ in (4.2) has a global section consisting of invertible operators; $P + Q_0$ in (5.2) is such a choice. The space of Fredholm perturbations, which appears in (5.1) consists precisely of those Q such that $I(P+Q) \in \Psi^m_{\mathrm{sus}}(\partial M/B; E, F)$ is invertible. Since $I(P+Q_0)$ is invertible this defines a map

$$\mathcal{P}_F \ni Q \longmapsto I(P + Q)I(P + Q_0)^{-1} \in G^{-\infty}_{\mathrm{sus}}(\partial M/B; F). \tag{5.3}$$

Moreover the fibres of this map are contractible, since if $I(P + Q') = I(P + Q)$ with $Q, Q' \in \mathcal{P}_F$ then $sQ' + (1 - s)Q \in \mathcal{P}_F$. Thus the families index only depends on this normalised indicial family. Furthermore, (5.3) is surjective so

$$\pi_0(\mathcal{P}_F) \simeq \pi_0(G^{-\infty}_{\mathrm{sus}}(\partial M/B; F)) \simeq K^0(B). \tag{5.4}$$

□

6. Odd case

The suspended algebra of cusp operators can be defined by close analogy with the suspension of the usual algebra. That is, for a compact manifold with boundary Z and vector bundles E, F, consider the space of cusp operators $\Psi^m_{\mathrm{cu}}(\mathbb{R} \times Z; E, F)$ on $\mathbb{R} \times Z$ corresponding to the lift of a cusp structure from Z. Thus the algebra is translation-invariant and within it consider the subspace of translation-invariant operators. Away from the diagonal the kernels of these operators are smooth functions on

$$(\mathbb{R} \times Z)^2_{\mathrm{cu}} = \mathbb{R} \times \mathbb{R} \times Z^2_{\mathrm{cu}}. \tag{6.1}$$

Thus we may further consider the subspace of kernels which are Schwartz near infinity. This subspace is closed under composition and we denote the resulting filtered algebra $\Psi^m_{\text{s-cu}}(Z; E, F)$.

The symbol sequence for these operators

$$\Psi^{m-1}_{\text{s-cu}}(Z; E, F) \longrightarrow \Psi^m_{\text{s-cu}}(Z; E, F) \longrightarrow \mathcal{C}^\infty(S(\mathbb{R} \times T^*_{\text{cu}} Z); \hom(E, F) \otimes R^m) \tag{6.2}$$

has the usual multiplicative properties as does the boundary sequence

$$x\Psi^m_{\text{s-cu}}(Z; E, F) \longrightarrow \Psi^m_{\text{s-cu}}(Z; E, F) \longleftrightarrow \Psi^m_{\text{sus}(\mathbb{R}\times L)}(\partial Z; E, F) \tag{6.3}$$

where $\Psi^m_{\text{sus}(\mathbb{R}\times L)}(\partial Z; E, F)$ is the doubly-suspended algebra of pseudodifferential operators on the boundary.

The 'odd' families index theorem can be viewed as the families index theorem for this algebra. In essence this is Theorem 1 except that we are dealing with a special family.

Theorem 3. *For any elliptic suspended family on the fibres of a fibration of a compact manifold with boundary, $P \in \Psi^m_{\text{s-cu}}(M/B; E)$, there is a perturbation $Q \in \Psi^{-\infty}_{\text{s-cu}}(M/B; E)$ such that $P+Q$ is invertible with inverse in $\Psi^{-m}_{\text{s-cu}}(M/B; E)$ and the set of homotopy classes of Fredholm perturbations $Q \in \Psi^{-\infty}_{\text{s-cu}}(M/B; E)$ is naturally isomorphic to $K^1(B)$.*

Proof. Taking the Fourier transform in the suspension variable gives a 1-parameter family of suspended operators. If the family is elliptic this is automatically invertible for large enough values of the parameter. Restricting the parameter space to $[-R, R] \times B$ for R large enough, we may apply Proposition 7 to find a perturbation $Q' \in \Psi^{-\infty}_{\text{s-cu}}(M/B; E, F)$ making it invertible over $[-R, R] \times B$. Restricting the parameter to values $\pm R$ gives, in each case, two invertible families over B, namely $P + Q$ and P. Since

$$(P + Q)P^{-1} = \operatorname{Id} + R, \ R \in \Psi^{-\infty}_{\text{cu}}(\partial M/B; F) \tag{6.4}$$

and we know weak contractibility of this group from Proposition 6, we may modify the family of perturbations Q' to vanish outside $[-R - 1, R + 1] \times B$ and still have $P + Q'$ everywhere invertible. Taking the inverse Fourier transform in the suspending variable gives the invertible perturbation.

If we consider perturbations making the family everywhere Fredholm, i.e. so that the suspended indicial family (so doubly-suspended families over the boundary) is everywhere invertible, we get an odd index, with values in $K^1(B)$. Two different perturbations of this type correspond to a family of double-suspended invertible operators in $G^{-\infty}(\partial Z)$ and hence to an element of $K^1(B)$ by Bott periodicity. This is our relative index theorem, which shows that the set of components of the space of everywhere Fredholm perturbations is isomorphic to $K^1(B)$. □

Appendix:Cusp algebra

On a compact manifold with boundary there are various distinct classes of pseudodifferential operators. Here we use the algebra of 'cusp' operators. These operators can be thought of as 'asymptotically translation-invariant' when a neighbourhood of the boundary is identified, through inversion of a defining function, with a 'cylindrical end'. The definition and development of the basic properties of these operators is due to Rafe Mazzeo and the first author; more details (and a more general class of operators) can be found in [6]. Notice however, that the definition here is slightly different to that in [6] in that we use the 'overblown' cusp calculus, which admits terms relating the different boundary components.

First consider the appropriate class of translation-invariant operators. Thus, if Y is a compact manifold without boundary we may consider pseudodifferential operators on $\mathbb{R}\times Y$, acting from sections of one vector bundle, E over Y to another, F. If $A\in\Psi^m(\mathbb{R}\times Y;E,F)$ is such an operator which is invariant under translation in $\mathbb{R}$ then it may be considered as a convolution operator on $\mathbb{R}$ and hence has as Schwartz kernel a distribution, $A\in\mathcal{C}^{-\infty}(\mathbb{R}\times Y^2;\operatorname{Hom}(E,F)\otimes\pi_R^*\Omega(Y))$ which is smooth away from the 'diagonal' $\{0\}\times\operatorname{Diag}_Y$ and has a classical conormal singularity at the diagonal. As in [9], we consider the subspace of those operators which have kernels which are rapidly decreasing at infinity with all derivatives. These operators, the space of which we denote $\Psi^m_{\text{sus}}(Y;E,F)$, map $\mathcal{S}(\mathbb{R}\times Y;E)$ into $\mathcal{S}(\mathbb{R}\times Y;F)$. They have the natural composition property

$$\Psi^m_{\text{sus}}(Y;F,G)\circ\Psi^{m'}_{\text{sus}}(Y;E,F)=\Psi^{m+m'}_{\text{sus}}(Y;E,G). \tag{A.5}$$

Fourier transformation on $\mathbb{R}$ converts $A\in\Psi^m_{\text{sus}}(Y;E,F)$ into a family of pseudodifferential operators on Y :

$$A(e^{it\tau}u)=e^{it\tau}\hat{A}(\tau)u,\ u\in\mathcal{C}^\infty(Y;E). \tag{A.6}$$

The family $\hat{A}:\mathbb{R}\longrightarrow\Psi^m(Y;E,F)$ determines A but is not an arbitrary family of pseudodifferential operators on Y depending smoothly on $\tau\in\mathbb{R}$. Rather, the families arising this way may be characterised as having full symbols, in terms of some global quantisation, which are (classical) symbols on $\mathbb{R}_\tau\times T^*Y$.

If Y_1 and Y_2 are different manifolds then the discussion above extends directly to define the spaces $\Psi^{-\infty}_{\text{sus}}(Y_1,Y_2;E_1,E_2)$ of suspended smoothing operators between sections of bundles E_i over Y_i, $i=1,2$. In general there is no analogous space of positive order pseudodifferential operators. These smoothing operators form modules over the spaces of suspended pseudodifferential operators:

$$\begin{gathered}\Psi^{-\infty}_{\text{sus}}(Y_1,Y_2;E_1,E_2)\circ\Psi^m_{\text{sus}}(Y_1;E_0,E_1)\subset\Psi^{-\infty}_{\text{sus}}(Y_1,Y_2;E_0,E_2),\\ \Psi^m_{\text{sus}}(Y_2;E_2,E_3)\circ\Psi^{-\infty}_{\text{sus}}(Y_1,Y_2;E_1,E_2)\subset\Psi^{-\infty}_{\text{sus}}(Y_1,Y_2;E_1,E_3),\\ \Psi^{-\infty}_{\text{sus}}(Y_2,Y_3;E_2,E_3)\circ\Psi^{-\infty}_{\text{sus}}(Y_1,Y_2;E_1,E_2)\subset\Psi^{-\infty}_{\text{sus}}(Y_1,Y_3;E_1,E_3).\end{gathered} \tag{A.7}$$

If Y is a 'disconnected manifold', meaning the union of a finite number of disjoint smooth manifolds (of fixed dimension), then we will denote by $\Psi^m_{\text{sus}}(Y;E,F)$ the space of matrices, with entries parameterized by $\pi_0(Y)$ of the type discussed above between the components of Y. The entries between different components are necessarily smoothing operators.

Now, let X be a compact manifold with boundary, and suppose E and F are vector bundles over X. We will assume initially that the boundary of X is connected. The most basic class of operators we use below, which we denote by $\dot{\Psi}^{-\infty}(X;E,F)$, consists of those operators which have Schwartz kernels in $\dot{\mathcal{C}}^\infty(X^2;\operatorname{Hom}(E,F)\otimes\pi_R^*\Omega(X))$, i.e. these are smoothing operators with kernels vanishing to infinite order at all boundary points of X^2. They form an algebra which is isomorphic to the algebra of smoothing operators on any manifold without boundary (provided it is non-trivial, connected and of positive dimension).

If $x\in\mathcal{C}^\infty(X)$ is a boundary defining function and $F:\{x<\epsilon\}\longrightarrow[0,\epsilon)\times\partial X$ is an associated product decomposition near the boundary, for some $\epsilon>0$, then we may allow elements of $\Psi^m_{\text{sus}}(\partial X;E,F)$ to act on X by choosing a cut-off function

$\chi \in \mathcal{C}^\infty_c([0,\epsilon))$, $\chi(x) = 1$ in $x \leq \frac{\epsilon}{2}$, and setting

$$A'u = F^*(\chi(x)(Av)(1/x,\cdot)),\ v(t,\cdot) = \chi(1/t)((F^{-1})^*u)(1/t,\cdot),\\ A \in \Psi^m_{\text{sus}}(\partial X; E, F),\ u \in \dot{\mathcal{C}}^\infty(X;E). \tag{A.8}$$

Here it should be observed that $v \in \mathcal{S}(\mathbb{R}_t \times \partial X)$. This action is not multiplicative.

Now the cusp pseudodifferential operators $\Psi^m_{\text{cu}}(X;E,F) \subset \Psi^m(\operatorname{int} X; E, F)$ are operators $B : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;F)$ such that $gBg' \in \dot{\Psi}^{-\infty}(X;E,F)$, if $g, g' \in \mathcal{C}^\infty(X)$ have supports with disjoint interiors and such that there exist elements $A_j \in \Psi^m_{\text{sus}}(\partial X; E, F)$ for which

$$B_\tau = e^{-i\tau/x} B e^{i\tau/x} : \mathcal{C}^\infty(X;E) \longrightarrow \mathcal{C}^\infty(X;F),\\ B_\tau \chi(x) F^*(u) \sim \sum_{j=0}^\infty x^j F^*(\hat{A}_j(\tau)u)\ \forall\ u \in \mathcal{C}^\infty(Y;E),\ \tau \in \mathbb{R}. \tag{A.9}$$

A choice of defining function, x, is involved here and, as noted above, the algebra does depend on this choice, although only through the trivialization of the normal bundle it determines. The algebra may be characterised by an appropriately uniform version of (A.9), although this is a rather cumbersome approach.

Instead the Schwartz kernels of its elements may be characterised directly on a cusp configuration space

$$X^2_{\text{cu}} = [X^2; \partial X \times \partial X; D_{\text{cu}}],\ \beta_{\text{cu}} : X^2_{\text{cu}} \longrightarrow X^2, \tag{A.10}$$

which is a blown-up version of the usual product X^2. Here $\partial X \times \partial X$ is the corner of X^2. If ∂X is not connected it should be replaced by the finitely many products between pairs of boundary hypersurfaces. Note that the description here diverges slightly from that in [6] where only the products of boundary hypersurfaces of X with themselves are considered and off-diagonal terms are not blown up. The submanifold $D_{\text{cu}} \subset \operatorname{ff}([X^2; \partial X \times \partial X])$ is determined by the cusp structure, in case ∂X is not connected there is one component corresponding to each pair of boundary faces. Thus, let x_H be an admissible defining function (corresponding to the chosen cusp structure) for a boundary face H. Then if x'_G is the same function in a second copy of X, for a possibly different boundary hypersurface, the function

$$s = \frac{x - x'}{x + x'} \in \mathcal{C}^\infty([X^2; \partial X \times \partial X],\ x = x_H,\ x' = x'_G \tag{A.11}$$

has the property that $s = \pm 1$ is the lift to $[X^2; \partial X \times \partial X]$ of the two local boundary faces, H as a boundary face in the first factor of X^2 and G as a boundary face in the second factor and, furthermore, $\rho = \frac{1}{2}(x + x')$ defines the new boundary hypersurface introduced in the blow up. In fact $[X^2; \partial X \times \partial X]$ is precisely the space obtained by appending s to $\mathcal{C}^\infty(X^2)$ (for each pair if boundary hypersurfaces). Then

$$D_{\text{cu}} = \{s = 0, \rho = 0\} \subset \operatorname{ff}([X^2; \partial X \times \partial X]) \tag{A.12}$$

is determined by (and determines) the cusp structure.

The configuration space X^2_{cu} has three type of boundary hypersurfaces. The 'old' boundary hypersurfaces, in two-to-one correspondence with the boundary faces of X, and three hypersurfaces (because the second blow-up disconnects the hypersurfaces in the first blow up), for each pair of boundary faces of X, corresponding to the two blow-ups in (A.11). We denote the last of these $\operatorname{ff}_{\text{cu}}$. The diagonal terms (corresponding to a product of a boundary hypersurface with itself) in this set are the only boundary hypersurfaces which meet the diagonal $\mathrm{Diag}_{\mathrm{cu}} \subset X^2_{\mathrm{cu}}$, defined as the closure of the inverse image of the interior of the diagonal of X^2, and they meet transversally. The part of the boundary other than $\mathrm{ff}_{\mathrm{cu}}$ will be denoted B_{nd} (nd for non-diagonal). Notice that the interior of the cusp front face is of the form

$$\operatorname{int}\mathrm{ff}_{\mathrm{cu}} \equiv \mathbb{R}_t \times \partial X \times \partial X \tag{A.13}$$

where the function $t = (x/x' - 1)/\rho$ gives a coordinate on the first factor. The kernels of the residual part of the cusp algebra are by definition those smooth densities on the interior of X^2 which are of the form

$$\Psi^{-\infty}_{\mathrm{cu}}(X) = \left\{A = A'\nu_{\mathrm{cu}}; A' \in \mathcal{C}^\infty(X^2_{\mathrm{cu}}),\ A' \equiv 0 \text{ at } B_{\mathrm{nd}}\right\}. \tag{A.14}$$

Here ν_{cu} is a smooth cusp density on the right fact of X, i.e. of the form ν/x^2 where ν is a smooth density in the usual sense.

This space of residual kernels is a module over $\mathcal{C}^\infty(X^2_{\mathrm{cu}})$ and can be thought of as the 'coefficient module'. This allows us to define related spaces of kernels directly. Most importantly, for any $m \in \mathbb{R}$ we set

$$\Psi^m_{\mathrm{cu}}(X) = I^m(X^2_{\mathrm{cu}}; \mathrm{Diag}_{\mathrm{cu}}) \otimes_{\mathcal{C}^\infty(X^2_{\mathrm{cu}})} \Psi^{-\infty}_{\mathrm{cu}}(X) \tag{A.15}$$

using the space of (classical) conormal distributions at the lifted diagonal. The expansion (A.9) corresponds precisely to the expansion in Taylor series of the kernels at $\mathrm{ff}_{\mathrm{cu}}$. In particular the leading term is determined by, and determines, the restriction of A' to $\mathrm{ff}_{\mathrm{cu}}$.

The leading term of the expansion in (A.9) is independent of choices, up to a positive scaling in the translation variable (constant on the factor ∂X). Thus, with only a choice of linear variable on an oriented one-dimensional vector space, we may identify the normal operator as a surjective algebra homomorphism

$$\Psi^m_{\mathrm{cu}}(X; E, F) \longrightarrow \Psi^m_{\mathrm{sus}}(\partial X; E, F),\ B \longmapsto A_0. \tag{A.16}$$

We call the associated family of pseudodifferential operators $I(B, \tau) = A_0(\tau) \in \Psi(\partial Y; E, F)$ the *indicial family*. Note that when ∂X is not connected the image space here consists of the 'matrices' labelled by two sets of the boundary components of ∂X, with entries which are suspended pseudodifferential operators between the components as discussed above. When the components are different the operators are necessarily smoothing, despite the formal presence of the order indicator m.

As noted above, the choice of a cusp structure on a compact manifold with boundary fixes a cusp tangent and cotangent bundle, isomorphic to the usual tangent and cotangent bundles, but not naturally so. Namely, if x is an admissible defining function for the boundary then the smooth vector fields, V on X satisfying

$$Vx \in x^2\mathcal{C}^\infty(X) \Longleftrightarrow V \in \mathcal{V}_{\mathrm{cu}}(X) \tag{A.17}$$

form a Lie algebra and $\mathcal{C}^\infty(X)$-module depending only on the underlying cusp structure and determining it. In local coordinates x, y_i, near a boundary point, $\mathcal{V}_{\mathrm{cu}}(X)$ is spanned over $\mathcal{C}^\infty(X)$ by $x^2\partial_x$, ∂_{y_i}. As a locally free module, this defines a smooth vector bundle over X (including over the boundary) such that

$$\mathcal{V}_{\mathrm{cu}}(X) = \mathcal{C}^\infty(X; {}^{\mathrm{cu}}TX). \tag{A.18}$$

The dual bundle, ${}^{\mathrm{cu}}T^*X$, is the natural carrier for the symbols of cusp pseudodifferential (or differential) operators. In particular, the standard symbol map over the interior of X extends by continuity to define (12) over any compact manifold with boundary. There is always a global quantisation map

$$q : S^m_{\mathrm{cl}}({}^{\mathrm{cu}}T^*X) \longrightarrow \Psi^m_{\mathrm{cu}}(X) \tag{A.19}$$

(and similarly for operators between sections of vector bundles) which induces a 'full' symbol isomorphism

$$S^m_{\mathrm{cl}}({}^{\mathrm{cu}}T^*X)/S^{-\infty}({}^{\mathrm{cu}}T^*X) \simeq \Psi^m_{\mathrm{cu}}(X)/\Psi^{-\infty}_{\mathrm{cu}}(X). \tag{A.20}$$

The expansion (A.9) is not trivial even in the case of operators of symbolic order $-\infty$. In fact, it then gives a short exact sequence

$$\dot{\Psi}^{-\infty}(X;E,F) \longrightarrow \Psi^{-\infty}_{\mathrm{cu}}(X;E,F) \longrightarrow \Psi^{-\infty}_{\mathrm{sus}}(\partial X;E,F)[[x]]. \tag{A.21}$$

Here, the image consists of countably many copies of $\Psi^{-\infty}_{\mathrm{sus}}(\partial X;E,F)$ with an induced 'star' product (if $E=F$ or involving three bundles) which is only trivially multiplicative at the top level.

The elements of $\dot{\Psi}^{-\infty}(X;E)$ are of trace class but in general those of $\Psi^{-\infty}_{\mathrm{cu}}(X;E)$ are not, precisely because of the cusp density in (A.14), which is not integrable. For $z \in \mathbb{C}$ and $A \in \Psi^{-\infty}_{\mathrm{cu}}(X;E)$, $x^zAx^{-z} = B_z \in \Psi^{-\infty}_{\mathrm{cu}}(X;E)$. Moreover, for $\operatorname{Re} z > 1$, x^zA is then of trace class and the function

$$f_A(z) = \operatorname{Tr}(x^zA) = \int_{\mathrm{Diag}_{\mathrm{cu}}} x^zA|\mathrm{Diag}_{\mathrm{cu}} = \int_X x^zA'|\mathrm{Diag}_{\mathrm{cu}}\nu_{\mathrm{cu}},\ A \in \Psi^{-\infty}_{\mathrm{cu}}(X;E) \tag{A.22}$$

is holomorphic in $\operatorname{Re} z > 1$ with a meromorphic extension to the complex plane. Since the cusp density is of the form ν/x^2 in terms of a smooth density, the possible poles are at the points $1-\mathbb{N}_0$, $\mathbb{N}_0 = \{0,1,\dots,\}$ and are at most simple. The *boundary residue trace* is defined as the residue at $z=0$:

$$\operatorname{Tr}_{\mathrm{R},\partial}(A) = \lim_{z\longrightarrow 0} zf_A(z) : \Psi^{-\infty}_{\mathrm{cu}}(X;E) \longrightarrow \mathbb{C}. \tag{A.23}$$

It is independent of the choice of admissible defining function used to define it, is a trace and is given explicitly in terms of the second term in the expansion (A.9)

$$\begin{gathered}\operatorname{Tr}_{\mathrm{R},\partial}([A,B]) = 0\ \forall\ A,B \in \Psi^{-\infty}_{\mathrm{cu}}(X;E),\\ \operatorname{Tr}_{\mathrm{R},\partial}(A) = \frac{1}{2\pi}\int_{\mathbb{R}} \operatorname{Tr}(\hat{A}(\tau))d\tau.\end{gathered} \tag{A.24}$$

The regularized value of $f_A(z)$ at $z=0$ is the *regularized trace*

$$\overline{\operatorname{Tr}}(A) = \lim_{z\to 0}\left(f_A(z) - \frac{1}{z}\operatorname{Tr}_{\mathrm{R},\partial}(A)\right). \tag{A.25}$$

It does depend on the choice of admissible defining function x and is not a trace, although it reduces to the usual trace on the trace class elements

$$\overline{\operatorname{Tr}}(A) = \operatorname{Tr}(A)\ \forall\ A \in x^2\Psi^{-\infty}_{\mathrm{cu}}(X;E). \tag{A.26}$$

In this paper, we make substantial use of the *trace-defect formula*.

Lemma 2. *The regularized trace of a commutator only depends on the indicial families (or the normal operators)*

$$\overline{\operatorname{Tr}}([A,B]) = \frac{1}{2\pi i}\int_{\mathbb{R}} \operatorname{Tr}\left(I(A,\tau)\frac{\partial}{\partial\tau}I(B,\tau)\right)d\tau,\ A,B \in \Psi^{-\infty}_{\mathrm{cu}}(X;E). \tag{A.27}$$

Proof. This is essentially the same as the corresponding formula in the b-calculus and can also be found in [7]. From the definition, for $\operatorname{Re} z >> 0$,

$$\operatorname{Tr}(x^z[A,B]) = \operatorname{Tr}([x^z,A]B) = \operatorname{Tr}(x^z A_z B),\ A_z = x^{-z}[x^z,A] \Longrightarrow \overline{\operatorname{Tr}}([A,B]) = -\operatorname{Tr}_{\mathrm{R},\partial}((D_{\log x}A)B). \tag{A.28}$$

Here, $A_z \in \Psi^{-\infty}_{\mathrm{cu}}(X;E)$ is entire in z and vanishes at $z=0$ and by definition

$$D_{\log x}A = -\frac{\partial}{\partial z}A_z\big|_{z=0}. \tag{A.29}$$

This is readily computed; the Schwartz kernel of A_z is $A(1-(x'/x)^z)$ so the z-derivative at $z=0$ has kernel $A\log(x/x')$. Notice that $s=0$ in (A.12) is $x/x'=1$, $x+x'=0$. Thus $\log(x/x')$ is a smooth function, away from $x=0$ or $x'=0$, which vanishes on D_{cu}. Lifted to X^2_{cu} it is therefore of the form ρt where t is given following (A.13). Thus in fact

$$D_{\log x}A \in x\Psi^{-\infty}_{\mathrm{cu}}(X;E),\ I(D_{\log x}A) = i\frac{d}{ds}I(A,s) \tag{A.30}$$

from which (A.27) follows. □

The naturality of the definition of the cusp algebra means that for a fibration with model fibre a compact manifold with boundary Z we may choose a smooth family of cusp structures, for instance by choosing a cusp structure on the total space, and so define the bundle of fibre-wise cusp operators $\Psi^m_{\mathrm{cu}}(M/B;E,F)$ for any bundles E and F over the total space and any m. Then the short exact sequence (12) on each fibre becomes

$$\Psi^{m-1}_{\mathrm{cu}}(M/B;E,F) \longrightarrow \Psi^m_{\mathrm{cu}}(M/B;E,F) \xrightarrow{\sigma_m} \mathcal{C}^\infty({}^{\mathrm{cu}}S^*(M/B);\hom(E,F)\otimes R^m). \tag{A.31}$$

Since R^m is trivial (canonically so if $m=0$) and ${}^{\mathrm{cu}}S^*(M/B)$ is isomorphic to the 'true' fibre cosphere bundle $S^*(M/B)$, the image space in (A.31) is an appropriate replacement for the corresponding space of symbols in the non-boundary case of Atiyah and Singer. In particular we may consider a family $P \in \Psi^m_{\mathrm{cu}}(M/B;E,F)$ to be 'elliptic' if $\sigma_m(P)$ has an inverse $s \in \mathcal{C}^\infty({}^{\mathrm{cu}}S^*(M/B);\hom(F,E)\otimes R^{-m})$; this is equivalent to pointwise invertibility of the symbol. Ellipticity, as in the boundaryless case, is equivalent to the existence of a parametrix $S \in \Psi^{-m}_{\mathrm{cu}}(M/B;F,E)$ such that $\operatorname{Id}-PS \in \Psi^{-\infty}_{\mathrm{cu}}(M/B;E)$ and $\operatorname{Id}-SP \in \Psi^{-\infty}_{\mathrm{cu}}(M/B;F)$. However, the important difference, due to the presence of the boundary, is that these errors are not necessarily compact as families of operators on L^2 and correspondingly, P need not be a family of Fredholm operators.

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