DIFFRACTION OF SINGULARITIES FOR THE WAVE EQUATION ON MANIFOLDS WITH CORNERS

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Abstract. We consider the fundamental solution to the wave equation on a manifold with corners. If the initial pole of the solution is appropriately situated, we show that the singularities which are diffracted by the corners (i.e., loosely speaking, are not propagated along limits of transversely reflected rays) are smoother than the main singularities of the solution. More generally, we show that subject to a hypothesis of nonfocusing, diffracted wavefronts of any solution to the wave equation are smoother than the incident singularities.

Contents

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1. Introduction

1.1. **The problem and its history.** Let X_0 be a manifold with corners, of dimension n, i.e., a manifold locally modeled on $(\mathbb{R}^+)^{f+1} \times \mathbb{R}^{n-f-1}$, endowed with an incomplete metric, smooth and non-degenerate up to the boundary. We consider the wave equation

$$\Box u \equiv D_t^2 u - \Delta u = 0 \text{ on } M_0 = \mathbb{R} \times X_0, \tag{1.1}$$

where $D_t = \imath^{-1}(\partial/\partial t)$ and Δ is the nonnegative Laplace-Beltrami operator; we will impose either Dirichlet or Neumann conditions at ∂X_0. As is well known, by the classic result of Duistermaat-Hörmander[1] (see [4]), that the wavefront set of a solution u propagates along null-bicharacteristics in the interior. However, the behavior of singularities striking the boundary and corners of M_0 is considerably subtler.

Indeed the propagation of singularities for the wave equation on manifolds with boundary is already a rather subtle problem owing the the difficulties posed by "glancing" bicharacteristics, those which are tangent to the boundary. Chazarain [1] showed that singularities striking the boundary transversely simply reflect according to the usual law of geometric optics (conservation of energy, tangential momentum, hence "angle of incidence equals angle of reflection") for the reflection of bicharacteristics. This result was extended in [23] and [24] by showing that, at glancing points, singularities may only propagate along certain generalized bicharacteristics. The continuation of these curves may fail to be unique at (non-analytic) points of infinite-order tangency as shown by Taylor [31]. Whether all of these branches of bicharacteristics can carry singularities is still not known.

As was shown initially in several special examples (namely those amenable to separation of variables) the interaction of wavefront set with a corner gives rise to new, *diffractive* phenomena, in which a single bicharacteristic carrying a singularity into a corner engenders singularities releasing from the corner along a whole family of bicharacteristics. For instance, a ray carrying a singularity transversely into a codimension-two corner will in general produce singularities on the entire cone of rays reflected in such a way as to conserve both energy and momentum tangent to the corner (see Figure 1) The first diffraction problem to be rigorously treated was that of the exterior of a wedge,[2] which was analyzed by Sommerfeld [30];

[1] This result, viewed in the context of hyperbolic equations, built on a considerable body of work prior to the introduction of the wavefront set; see especially [12], [15].

[2] This is not in fact a manifold with corners, but is quite closely related.

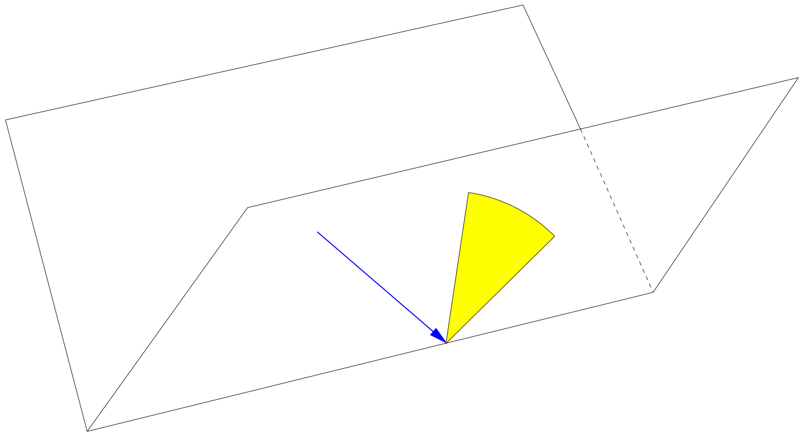

Figure 1. A ray carrying a singularity may strike a corner of codimension two and give rise to a whole family of diffracted singularities, conserving both energy and momentum along the corner.

subsequently, many related examples were analyzed by Friedlander [5], and more generally the case of exact cones was worked out explicitly by Cheeger-Taylor [2], [3] in terms of the functional calculus for the Laplace operator on the cross section of the cone. All of these explicit examples reveal that generically a diffracted wave arises from the interaction of wavefront set of the solution with singular strata of the boundary of the manifold; this has long been understood at a heuristic level, with the geometric theory of diffraction of by Keller [11] describing the classes of trajectories that ought to contribute to the asymptotics of the solution in various regimes.

Subsequent work has been focused primarily on characterizing the bicharacteristics on which singularities can propagate, and on describing the strength and form of the singularities that arise. The propagation of singularities on a wide class of manifolds including manifolds with boundary was first understood in the analytic case by Lebeau [13], [14]. In the $\mathcal{C}^\infty$ setting employed here, the special case of manifolds with conic singularities was studied by Melrose-Wunsch [26] and on *edge manifolds* (i.e., cone bundles) by Melrose-Vasy-Wunsch [25]. Vasy [34] obtained results analogous to Lebeau's in the case of manifolds with corners, and it is the results of this work that directly bear on the situation studied here.

While the foregoing results characterize which bicharacteristics may carry singularities for solutions to the wave equation, they ignore the question of the regularity of the diffracted front. In general, we expect a singularity measured microlocally with respect to Sobolev spaces an so in WF^s to propagate *strongly*, i.e. certain "geometric" bicharacteristics through that point are also in WF^s; in simple cases, these are known to be those which are locally approximable by bicharacteristics that miss the corners and reflect

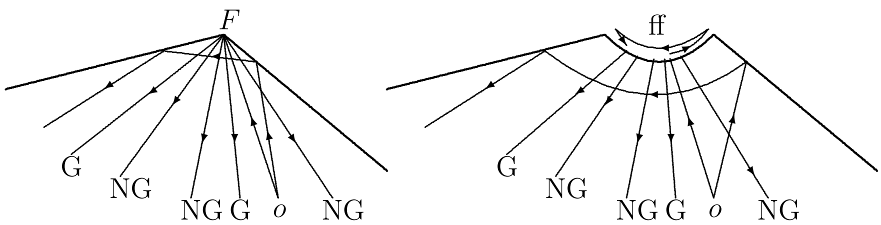

FIGURE 2. Geometric optic rays hitting a corner F, emanating from a point o. The rays labelled G are geometric at F, while those labelled NG are non-geometric at F. The leftmost geometric ray is a limit of rays like the unlabelled one shown on the figure that just miss F. The blown up version of the picture, i.e. where (r, θ) are introduced as coordinates at the origin, is shown on the right. The front face (i.e., the lift of the corner) is denoted ff. The reflecting line indicates the broken geodesic of length π induced on ff given by $r = 0$, $\theta \in [0, \theta_0]$. The total length of the three segments shown on ff is π; this can be thought of as the sum of three angles on the picture on the left: namely the angles between the incident ray and the right boundary (corresponding to the first segment), the right and left boundaries, finally the left boundary and the emanating reflected ray.

transversely off boundary hypersurfaces. More generally, we can define geometric bicharacteristics as follows: To begin, we *blow up* the corner, i.e. introduce polar coordinates around it; this serves to replace the corner with its inward-pointing normal bundle, which fibers over the corner with fiber given by one orthant of a sphere, $S^f \cap (\mathbb{R}^+)^{f+1}$. We will define *geometric* broken bicharacteristics passing through the corner as those that lift to the blown-up space to enter and leave the lift of the corner at points connected by *generalized broken geodesics* of length π with respect to the naturally defined metric on $S^f \cap (\mathbb{R}^+)^{f+1}$, undergoing specular reflection at its boundaries and corners.[3] (The fiber metric depends on the ambient metric and will not in general be the standard metric on the sphere). Bicharacteristics that enter and leave the corner at points in a fiber that are not at distance-π in this sense are referred to as "diffractive." See Figure 2.

It turns out that subject to certain hypotheses of *nonfocusing,* the singularities propagating along diffractive bicharacteristics emanating from the corner will be weaker than those on geometric bicharacteristics. In particular, the fundamental solution satisfies the nonfocusing condition, hence one consequence of our main theorem is as follows:

[3]The actual definition is considerably complicated by the existence of glancing rays, and is discussed in detail in §3.4.

Theorem 1.1. *Consider the fundamental solution u_o to the wave equation, with pole o sufficiently close to a corner, W, of codimension k with a correspondingly short geodesic from o to Y meeting each boundary face containing Y transversally. While u_o lies microlocally in* [4] $H^{-n/2+1-0}$*, it is less singular by $(k-1)/2$ derivatives along diffractive bicharacteristics emanating from W, that is it lies microlocally in $H^{(-n+k+1-0)/2}$.*

A more precise version of this result (with "sufficiently close..." eludicated) appears in Corollary 9.6.

A more general theorem on regularity of the diffracted wave subject to the nonfocusing condition is the central result of this paper. See §1.2 for a rough statement of the nonfocusing condition and §6 for technical details; the main theorem is stated and proved in §9.

There are a few related results known in special case. Gérard-Lebeau [7] explicitly analyzed the problem of an analytic conormal wave incident on an analytic corner in $\mathbb{R}^2$, obtaining a $1/2$-derivative improvement of the diffracted wavefront over the incident one. The first and third authors [26] obtained the results for manifolds with conic singularities, which the authors subsequently generalized to the case of edge manifolds [25].

1.2. **The hypothesis.** We now describe the nonfocusing hypothesis in more detail, in the context of the simplest geometric situation to which our results apply.

It is easily seen from the explicit form of the fundamental solution that it is not in general true that diffracted rays are more regular than incident singularities. For example, take Δ to be the Dirichlet Laplacian in a sectorial domain $\{r \in [0,\infty), \theta \in [0,\theta_0]\}$ in $\mathbb{R}^2$, and consider the solution

$$\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}}\phi(\theta)\delta(r-r_0), \tag{1.2}$$

where $\phi \in \mathcal{C}_c^\infty((0,\theta_0))$ is supported close to some value θ'. This solution is manifestly locally in $H^{1/2-0}$ by energy conservation. On the other hand one may see from the explicit form of the propagator in [2], [3] after convolution with $\phi(\theta)$ that a spherical wave of singularities emanates from the corner at time $t = r_0$, with regularity $H^{1/2-0}$, hence the same as the overall Sobolev regularity of the solution. The bicharacteristics along which singularities propagate are, for short time, just the lifts of the straight lines $r = r_0 \pm t$, $\theta \in \operatorname{supp}\phi$, hence travelling straight into or out of the vertex. Perturbing these slightly to make them miss the vertex, we see that in fact there are two "geometric" continuations[5] for each bicharacteristic, depending on whether we approximate it by geodesics passing to the left or to the right of the vertex (see Figure 2). Thus, the geometric continuations of the rays on which singularities strike the vertex are close to the two possible continuations of

[4]Here and henceforth we emply the notation $s-0$ to mean $s-\epsilon$ for all $\epsilon > 0$.

[5]What a geometric continuation of a bicharacteristic is in general will be elucidated in §3.4.

the single ray $\theta = \theta'$, hence do not include all points on the spherical wave. So we have an example in which there are "non-geometric" singularities of full strength.

The nonfocusing condition serves exactly to rule out such situations. The above example has the property that applying negative powers of $(\mathrm{Id} + D_\theta^2)$ does not regularize the short-time solution (or the initial data) as it is already $\mathcal{C}^\infty$ in the θ direction. In this simple example, the nonfocusing condition says precisely that the solution is regularized by negative powers of $(\mathrm{Id} + D_\theta^2)$, or, equivalently, that it can be written

$$(\mathrm{Id} + D_\theta^2)^N v,\ v \in H^s$$

for some s exceeding the overall Sobolev regularity. For instance, the fundamental solution

$$u = \frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}} \delta(r - r')\delta(\theta - \theta')$$

looks, after application of a sufficiently negative power of $(\mathrm{Id} + D_\theta^2)$, like a distribution of the form

$$\frac{\sin t\sqrt{\Delta}}{\sqrt{\Delta}} \delta(r - r') f(\theta)$$

with $f \in \mathcal{C}^M$, $M \gg 0$, hence we can write

$$u \in (\mathrm{Id} + D_\theta^2)^N H^{1/2-0},$$

for some $N \gg 0$, at least locally, away from the boundary. We also observe that the example (1.2) enjoys a property which is essential *dual* to the nonfocusing condition, to whit, fixed regularity under repeated application of D_θ. We refer to this property as "coisotropic regularity" (the terminology will be explained in §6) and it plays an essential role in our proof.

The nonfocusing condition and coisotropic regularity in a more general setting are subtler owing to their irreducibly microlocal nature: the operator D_θ has to be replaced by a family of operators characteristic along the flowout of the corner under consideration.

1.3. **Structure of the proof.** We now describe the logical structure of the proof, as it is somewhat involved. The heart of the argument is a series of propagation of singularities results, obtained by positive commutator methods. In order to be able to distinguish between "geometric" and "diffractive" bicharacteristics at a corner of M_0, we begin by performing a blowup of the corner, i.e. introducing polar coordinates centered at it, to obtain a new manifold with corners M. The commutants that we employ in our commutator argument *almost* lie in a calculus of pseudodifferential operators, the edge-b calculus, that behaves like Mazzeo's edge calculus [16] near the new boundary face introduced by the blowup (henceforth, the *edge*) and like Melrose's b calculus [19, 21, 22] at the remaining boundary faces. The complication, as in the previous work of Vasy [34] on propagation of singularities, is that in order to control certain error terms we in fact must employ a hybrid differential-pseudodifferential calculus, in which we keep track of certain terms involving differential operators normal to the boundary faces other than $\tilde{W}$.

Even this propagation result alone is insufficient to obtain our result, as it does not allow regularity of greater than a certain degree to propagate out of the edge, with the limitation in fact not exceeding the a priori regularity of the solution. What it does allow for, however, is the propagation of *coisotropic* regularity of arbitrarily high order, suitably microlocalized in the edge-b sense. This allows us to conclude that given a ray γ leaving the edge, if the solution enjoys coisotropic regularity along all rays incident upon the edge that are geometrically related to γ, then we may conclude coisotropic regularity along γ as well. (If some of these incident rays are glancing, i.e. tangent to the boundary, we require as our hypothesis actual *differentiability* globally at all incoming glancing rays rather than coisotropic regularity, which no longer makes sense; the version of the commutator argument that deals with these rays is the most technically difficult aspect of the argument.) In particular, then, *global* coisotropic regularity together with $\mathcal{C}^\infty$ regularity at glancing rays implies global coisotropic regularity leaving the edge away from glancing. We are then able to dualize this result to show that the nonfocusing condition propagates as well.

Consequently, we show that subject to the nonfocusing condition, in the model case of the sector considered above, if γ is an outgoing ray such that the solution is $\mathcal{C}^\infty$ along all incoming rays geometrically related to it,

$$u \in (\mathrm{Id} + D_\theta)^N H^s \text{ along } \gamma,$$

where $s = (-n + k + 1)/2 - 0$ for the fundamental solution near a general codimension-k corner on an n-manifold, i.e. is $1/2 - 0$ for our sector. On the other hand the microlocal propagation of coisotropic regularity shows that

$$D_\theta^k u \in H^{s_0} \text{ for all } k \text{ along } \gamma$$

where s_0 is the overall regularity of the solution ($-n/2 + 1 - 0$ for the fundamental solution). An interpolation argument then yields

$$u \in H^{s-0} \text{ along } \gamma,$$

proving the theorem.

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2. Geometry: metric and Laplacian

Let X_0 be a connected n-dimensional manifold with corners. We work locally, near a given point in the interior of a corner Y of codimension $f+1$.

Thus, we have local coordinates $x_1, \ldots, x_{f+1}, y_1, \ldots, y_{n-f-1}$ in which Y is given by $x_1 = \ldots = x_{f+1} = 0$. Suppose that g_0 is a smooth Riemannian metric on X_0, non-degenerate up to all boundary faces. We may always choose local coordinates in which it takes the form

$$g_0 = \sum a_{ij}\, dx_i\, dx_j + \sum b_{ij}\, dy_i\, dy_j + 2 \sum c_{ij}\, dx_i\, dy_j \tag{2.1}$$

with $c_{ij}|_Y = 0$. This can be arranged by changing the y variables to

$$y_j = y'_j + \sum x_k Y_{jk}(y')$$

while keeping the x_j unchanged. The the cross-terms then become

$$2 \sum c_{ij}\, dx_i\, dy'_j + 2 \sum b_{ij}\, Y_{jk}\, dy'_i\, dx_k,$$

which can be made to vanish by making the appropriate choice of the Y_{jk}, using the invertibility of $\{b_{ij}\}$.

Let $X = [X_0; Y]$, be the *real blowup* of Y in X_0 (see [21, 20]) and let $\tilde{Y}$ denote the front face of the blowup, which we also refer to as the *edge face*. Recall that the blow-up arises by identifying a neigborhood of Y in X_0 with the inward-pointing normal bundle N^+Y of Y in X_0 and blowing up the origin in the fibers of the normal bundle (i.e. introducing polar coordinates in the fibers). Since the normal bundle is trivialized by the defining functions of the boundary faces, a neighborhood of $\tilde{Y}$ in X is globally diffeomorphic to

$$[0, \infty) \times Y \times Z, \text{ where } Z = S^f \cap [0, \infty)^{f+1}.$$

We use coordinates $z_1, \ldots, z_f$ in Z; near a corner of Z of codimension k, these are divided into $z'_1, \ldots, z'_k \in [0, \infty)$ and $z''_{k+1}, \ldots, z''_f \in \mathbb{R}$. There is significant freedom in choosing the identification of a neighborhood of Y and the coordinates on the fibers of the normal bundle but the naturality of the smooth structure on the blown up manifold, $[X_0; Y]$, corresponds to the fact that these are smoothly related.

The metric g_0 identifies NY as a subbundle of $T_Y X_0$. This corresponds to coordinates (x_i, y_j) as above with the dy_j orthogonal to dx_i at Y. In the blow-up polar coordinates are introduced in the x_i but the y_j are left unchanged. It is convenient to think of these as polar coordinates induced by $\sum_{ij} a_{ij}\, dx_i\, dx_j$. In particular, we choose

$$x = \big(\sum_{ij} a_{ij}(0, y) x_i x_j \big)^{1/2}$$

as the 'polar variable' which is the defining function of the front face. With this choice, the metric takes the form

$$g = dx^2 + h(y, dy) + x^2 k(x, y, z, dz) + x k'(x, y, z, dx, dy, x\, dz). \tag{2.2}$$

More generally, one can simply consider the wider class of manifolds with corners with metrics of the form (2.2), we refer to these as 'edge metrics' for brevity. Note, however, that there are no results currently available in this wider setting that limit propagation of singularities to generalized broken

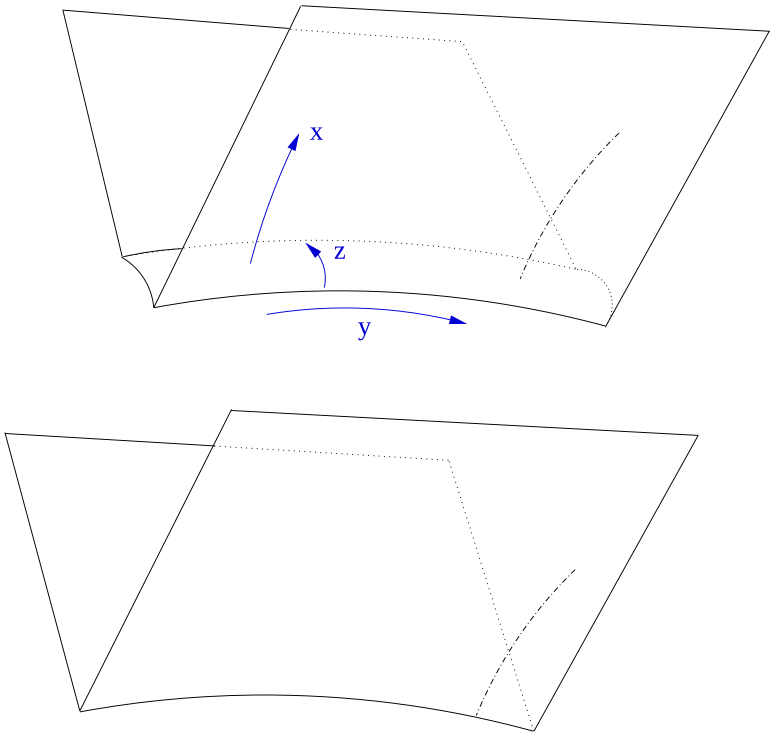

FIGURE 3. A manifold X_0 with corners of codimension two (below) and its blowup X (top). A geodesic hitting the codimension-two corner is shown, together with its lift to the blown-up space X, which then strikes the front "edge" face of the blowup.

bicharacteristics. Despite this, the results in §7 remain valid in this more general context.

Now, set

$$M = \mathbb{R}_t \times X,\ M_0 = \mathbb{R} \times X_0,\ \tilde{W} = \mathbb{R} \times \tilde{Y},$$

where $\tilde{W}$ now represents the space-time "edge."

Let $\mathrm{Diff}^*_{\mathrm{es}}(X)$ denote the filtered algebra of operators on $\mathcal{C}^\infty(X)$ generated by those vector fields that are tangent to the fibers of the front face $\tilde{Y}$ produce by the blow up; xD_x, xD_{y_j}, D_{z_j} form a local coordinte basis of these vector fields. See §3 and §5 for further explanation of this algebra and of our terminology. The same definition leads to the algebra of operators $\mathrm{Diff}^*_{\mathrm{es}}(M)$ on $\mathcal{C}^\infty(M)$ with local generating basis xD_t, xD_x, xD_{y_j}, D_{z_j}.

Lemma 2.1. *The Laplace operator $\Delta \in x^{-2}\,\mathrm{Diff}^2_{\mathrm{es}}(X)$ on X is of the form*

$$\Delta \in D_x^2 + \frac{f}{\imath x} D_x + \frac{1}{x^2}\Delta_Z + \Delta_Y + x^{-1}\,\mathrm{Diff}^2_{\mathrm{es}}(X)$$

where Δ_Z is the Laplace operator in Z with respect to the metric h (and hence depends parametrically on x and y) and Δ_Y is the Laplacian on Y with respect to the metric k.

In particular, $\Box = D_t^2 - \Delta_X \in x^{-2}\,\mathrm{Diff}^2_{\mathrm{es}}(M)$.

3. Bundles and bicharacteristics

In this section, we discuss several different geometric settings in which the propagation problem for $\Box$ on M_0 may be viewed. Somewhat loosely, each of these corresponds to a choice of a Lie algebra of vector fields with different boundary behavior; these then lead to distinct bundles of covectors, with the corresponding descriptors used as section headings here. The first, the "b"-bundle, can be considered either on M_0 or M. Indeed, the bundle of b-covectors on M_0 is the setting for the propagation results of [34]: these results are, as will be seen below, necessarily global in the corner, and do not distinguish between diffractive rays and geometric subset of them as defined below. In order to discuss the improvement in regularity which can occur for propagation along the geometric rays, two more bundles of covectors, lying over the blown-up space M, are introduced. These, the "edge-b" and "edge-smooth" bundles, keep track of local information in the fibers of the blowup of W in M_0, and allow us to distinguish the diffractive rays from geometric ones. The distinction between the edge-b and edge-smooth bundles comes only at the boundary of $\tilde{W}$, and the tension between the two presentations of the problem gives rise to reflection of singularities off boundary faces, uniformly up to the edge W.

In order to alleviate some of the notational burden on the reader, a table §3.5 is include in which the various bundles, their coordinates, their sections, and some of the maps among them are reviewed. The standard objects for a manifold with corners, Q, correspond to uniform smoothness up to all boundary faces, so $\mathcal{V}(Q)$ denotes the Lie algebra of smooth vector fields, TQ, the tangent bundle, of which $\mathcal{V}(Q)$ forms all the smooth sections, T^*Q is its dual, etc.

3.1. **b-Cotangent bundle.** Let $\mathcal{V}_{\mathrm{b}}(Q) \subset \mathcal{V}(Q)$ denote the Lie subalgebra of those smooth vector fields on the general manifold with corners Q, which are tangent to each boundary face. The local vector fields

$$x_1\partial_{x_1}, \dots, x_{f+1}\partial_{x_{f+1}}, \partial_{y_1}, \dots, \partial_{y_{n-f-1}};$$

form a basis over smooth functions, so $\mathcal{V}_{\mathrm{b}}(Q)$ is the space of $\mathcal{C}^\infty$-sections of a vector bundle, denoted

$$^{\mathrm{b}}TQ.$$

The dual bundle $^{\mathrm{b}}T^*Q$ therefore has sections spanned by

$$\frac{dx_1}{x_1}, \dots, \frac{dx_{f+1}}{x_{f+1}}, dy_1, \dots, dy_{n-f-1}.$$

The natural map

$$\pi_{\mathrm{s}\to\mathrm{b}} : T^*Q \longrightarrow {}^{\mathrm{b}}T^*Q \tag{3.1}$$

is the adjoint of the bundle map $\iota : {}^{\mathrm{b}}TQ \to TQ$ corresponding to the inclusion of $\mathcal{V}_{\mathrm{b}}(Q)$ in $\mathcal{V}(Q)$.

Canonical local coordinates on T^*M_0 correspond to decomposing a covector in terms of the basis as

$$\tau^{\mathrm{s}}\, dt + \sum_j \eta_j^{\mathrm{s}}\, dy_j + \sum_j \xi_j^{\mathrm{s}}\, dx_j,$$

and elements of ${}^{\mathrm{b}}T^*M_0$ may be written

$$\tau^{\mathrm{b}}\, dt + \sum_j \eta_j^{\mathrm{b}}\, dy_j + \sum_j \xi_j^{\mathrm{b}}\, \frac{dx_j}{x_j},$$

so defining canonical coordinates. The map (3.1) then takes the form

$$\pi_{\mathrm{s}\to\mathrm{b}}(x,y,t,\xi^{\mathrm{s}},\eta^{\mathrm{s}},\tau^{\mathrm{s}}) = (x,y,t,x\xi^{\mathrm{b}},\eta^{\mathrm{b}},\tau^{\mathrm{b}}) = (x,y,t,x\xi^{\mathrm{s}},\eta^{\mathrm{s}},\tau^{\mathrm{s}}),$$

with $x\xi^{\mathrm{s}} = (x_1\xi_1^{\mathrm{s}},\dots,x_{f+1}\xi_{f+1}^{\mathrm{s}})$.

The setting for the basic theorem on the propagation of singularities in [34] is ${}^{\mathrm{b}}T^*M_0$. In particular, generalized broken bicharacteristics, or GBB 's, are curves in ${}^{\mathrm{b}}T^*M_0$. In order to analyze the geometric improvement, spaces that will keep track of finer singularities are needed. Before introducing these, we first recall the setup for GBB's . Note that at W, $\pi_{\mathrm{s}\to\mathrm{b}}|_Y$ maps N^*W onto the zero section over W, and is injective on its complement, so we may make the identification

$$\pi_{\mathrm{s}\to\mathrm{b}}|_Y(N^*W) \cong T^*W.$$

We also recall that it is convenient to work on cosphere bundles. Since it is linear, $\pi_{\mathrm{s}\to\mathrm{b}}$ intertwines the $\mathbb{R}^+$-actions, but it does not induce a map on the corresponding cosphere bundles since it maps part of $T^*M_0 \setminus o$ into the zero section of ${}^{\mathrm{b}}T^*M_0$. However, on the characteristic set of $\Box$ this map is better behaved. Let

$$p_0 = \sigma_2(\Box) \in \mathcal{C}^\infty(T^*M_0 \setminus o)$$

be the standard principal symbol of $\Box \in \mathrm{Diff}^2(M_0)$; it is of the form

$$p_0 = (\tau^{\mathrm{s}})^2 - \Big(\sum A_{jk}\xi_j^{\mathrm{s}}\xi_k^{\mathrm{s}} + \sum B_{jk}\eta_j^{\mathrm{s}}\eta_k^{\mathrm{s}} + 2\sum C_{jk}\xi_j^{\mathrm{s}}\eta_k^{\mathrm{s}}\Big)$$

with $A_{jk}, B_{jk}, C_{jk} \in \mathcal{C}^\infty(M_0)$, $A_{jk} = A_{kj}$ and $B_{jk} = B_{kj}$, $C_{jk}|_{x=0} = 0$. Let

$${}^{\mathrm{s}}\Sigma_0 = p_0^{-1}(\{0\})/\mathbb{R}^+ \subset S^*M_0 \tag{3.2}$$

be the spherical image of the characteristic set of $\Box$. This has two connected components, ${}^{\mathrm{s}}\Sigma_{0,\pm}$, corresponding to $\tau^{\mathrm{s}} \gtrless 0$ since $\{\tau^{\mathrm{s}} = 0\} \cap {}^{\mathrm{s}}\Sigma_0 = \emptyset$. Now, $N^*W \subset \{\tau^{\mathrm{s}} = 0\}$, so $W \cap p^{-1}(0) = \emptyset$, meaning W is non-characteristic for $\Box$. Since N^*W is the null space of $\pi_{\mathrm{s}\to\mathrm{b}}$, there is an induced map on the sphere bundles $\widehat{\pi_{\mathrm{s}\to\mathrm{b}}} : {}^{\mathrm{s}}\Sigma_0 \longrightarrow {}^{\mathrm{b}}S^*M_0$; the range is denoted

$${}^{\mathrm{b}}\Sigma_0 = \widehat{\pi_{\mathrm{s}\to\mathrm{b}}}(p^{-1}(0))/\mathbb{R}^+ \subset {}^{\mathrm{b}}S^*M_0. \tag{3.3}$$

Again, ${}^{\mathrm{b}}\Sigma_0$ has two connected components corresponding to the sign of τ^{s} in ${}^{\mathrm{s}}\Sigma_0$ and hence the sign of τ^{b}. These will be denoted ${}^{\mathrm{b}}\Sigma_{0,\pm}$.

We use τ^{s}, resp. τ^{b}, to obtain functions homogeneous of degree zero on $T^*M_0 \setminus o$ inducing coordinates on S^*M_0 near ${}^{\mathrm{s}}\Sigma_0$:

$$x, y, t,\ \hat{\xi}^{\mathrm{s}} = \xi^{\mathrm{s}}/|\tau^{\mathrm{s}}|,\ \hat{\eta}^{\mathrm{s}} = \eta^{\mathrm{s}}/|\tau^{\mathrm{s}}|.$$

Note also that these coordinates are *global in the fibers of* $S^*M_0 \cap {}^{\mathrm{s}}\Sigma_{0,\pm} \to M_0$ for each choice of sign $\pm$.

$$\hat{\tau}^{\mathrm{s}} = \operatorname{sgn} \tau^{\mathrm{s}}$$

lifts to a constant function ± 1 on ${}^{\mathrm{s}}\Sigma_{0,\pm}$. There are similar coordinates on ${}^{\mathrm{b}}S^*M_0$ near ${}^{\mathrm{b}}\Sigma_0$.

In these coordinates,

(3.4)

$${}^{\mathrm{b}}\Sigma_0 \cap {}^{\mathrm{b}}S^*_W M_0 = \{(x, y, t, \hat{\xi}^{\mathrm{b}}, \hat{\eta}^{\mathrm{b}}) :\ x = 0,\ \hat{\xi}^{\mathrm{b}} = 0,\ \sum B_{jk}\hat{\eta}^{\mathrm{b}}_j\hat{\eta}^{\mathrm{b}}_k \leq 1\} \subset S^*W.$$

We also remark that with H_{p_0} denoting the Hamilton vector field of p_0,

$$\mathsf{H}_{\mathrm{s}} = |\tau^{\mathrm{s}}|^{-1}\mathsf{H}_{p_0}$$

is a homogeneous degree zero vector field near $p_0^{-1}(\{0\})$, thus can be regarded as a vector field on S^*M_0.

Now we define the b-hyperbolic and b-glancing sets by

$$\mathcal{G}_{W,\mathrm{b}} = \{q \in {}^{\mathrm{b}}S^*_W M_0 : |(\pi_{\mathrm{s}\to\mathrm{b}})^{-1}(q) \cap {}^{\mathrm{s}}\Sigma_0| = 1\} \tag{3.5}$$

and

$$\mathcal{H}_{W,\mathrm{b}} = \{q \in {}^{\mathrm{b}}S^*_W M_0 : |(\pi_{\mathrm{s}\to\mathrm{b}})^{-1}(q) \cap {}^{\mathrm{s}}\Sigma_0| \geq 2\}, \tag{3.6}$$

These are thus also subsets of S^*W. In local coordinates[6] they are given by

(3.7)

$$\mathcal{G}_{W,\mathrm{b}} = \{(x, y, t, \hat{\tau}^{\mathrm{b}}, \hat{\xi}^{\mathrm{b}}, \hat{\eta}^{\mathrm{b}}) :\ x = 0,\ \hat{\tau}^{\mathrm{b}} \in \{\pm 1\},\ \hat{\xi}^{\mathrm{b}} = 0,\ \sum B_{jk}\hat{\eta}^{\mathrm{b}}_j\hat{\eta}^{\mathrm{b}}_k = 1\}$$

$$\mathcal{H}_{W,\mathrm{b}} = \{(x, y, t, \hat{\tau}^{\mathrm{b}}, \hat{\xi}^{\mathrm{b}}, \hat{\eta}^{\mathrm{b}}) :\ x = 0,\ \hat{\tau}^{\mathrm{b}} \in \{\pm 1\},\ \hat{\xi}^{\mathrm{b}} = 0,\ \sum B_{jk}\hat{\eta}^{\mathrm{b}}_j\hat{\eta}^{\mathrm{b}}_k < 1\}.$$

Note that for $q \in {}^{\mathrm{b}}S^*_W M_0$, at the unique point q_0 in $(\pi_{\mathrm{s}\to\mathrm{b}})^{-1}(q) \cap {}^{\mathrm{s}}\Sigma_0$, we have $\hat{\xi}^{\mathrm{s}} = 0$, and correspondingly $\mathsf{H}_{\mathrm{s}}(q_0)$ is tangent to W, explaining the "glancing" terminology.

Now we discuss bicharacteristics.

Definition 3.1. A *generalized broken bicharacteristic*, or GBB , is a continuous map $\gamma : I \to {}^{\mathrm{b}}\Sigma_0$ such that for all $f \in \mathcal{C}^\infty({}^{\mathrm{b}}S^*M_0)$, real-valued,

$$\liminf_{s\to s_0} \frac{(f\circ\gamma)(s) - (f\circ\gamma)(s_0)}{s - s_0} \tag{3.8}$$

$$\geq \inf\{\mathsf{H}_{\mathrm{s}}(\pi^*_{\mathrm{s}\to\mathrm{b}} f)(q) : q \in \pi^{-1}_{\mathrm{s}\to\mathrm{b}}(\gamma(s_0)) \cap {}^{\mathrm{s}}\Sigma_0\}. \tag{3.9}$$

[6] The discrete variable $\hat{\tau}^{\mathrm{b}}$ is not, of course, part of the coordinate system, but serves to identify which of two components of the characteristic set we are in.

Remark 3.2. Replacing f by $-f$, we deduce that the inequality

$$\limsup_{s\to s_0}\frac{(f\circ\gamma)(s)-(f\circ\gamma)(s_0)}{s-s_0} \tag{3.10}$$

$$\leq \sup\{\mathsf{H}_{\mathrm{s}}(\pi^*_{\mathrm{s}\to\mathrm{b}}f)(q) : q\in \pi^{-1}_{\mathrm{s}\to\mathrm{b}}(\gamma(s_0))\cap{}^{\mathrm{s}}\Sigma_0\}. \tag{3.11}$$

also holds.

We recall an alternative description of GBB's , which was in fact Lebeau's definition [14]. (One could use this lemma as the defining property of GBB ; the equivalence of these two possible definitions is proved in [32, Lemma 7].)

Lemma 3.3. *(See* [32, Lemma 7]*.) Suppose γ is a GBB . Then*

(1) *If $\gamma(s_0)\in\mathcal{G}_{W,\mathrm{b}}$, let q_0 be the unique point in the preimage of $\gamma(s_0)$ under $\widehat{\pi_{\mathrm{s}\to\mathrm{b}}}=\pi_{\mathrm{s}\to\mathrm{b}}|_{{}^{\mathrm{s}}\Sigma_0}$. Then for all $f\in\mathcal{C}^\infty({}^{\mathrm{b}}S^*M_0)$ real valued, $f\circ\gamma$ is differentiable at s_0, and*
$$\frac{d(f\circ\gamma)}{ds}|_{s=s_0}=\mathsf{H}_{\mathrm{s}}\pi^*_{\mathrm{s}\to\mathrm{b}}f(q_0).$$
(2) *If $\gamma(s_0)\in\mathcal{H}_{W,\mathrm{b}}$, lying over a corner given in local coordinates by $x_j=0$, $j=1,\ldots,f+1$, there exists $\epsilon>0$ such that $x_j(\gamma(s))=0$ for $s\in(s_0-\epsilon,s_0+\epsilon)$ if and only if $s=s_0$. That is, γ does not meet the corner $\{x_1=\ldots=x_{f+1}=0\}$ in a punctured neighborhood of s_0.*

Remark 3.4. It also follows directly from the definition of GBB (by combining (3.8) and (3.10)) that, more generally, if the set

$$\{\mathsf{H}_{\mathrm{s}}(\pi^*_{\mathrm{s}\to\mathrm{b}}f)(q) : q\in \pi^{-1}_{\mathrm{s}\to\mathrm{b}}(\gamma(s_0))\cap{}^{\mathrm{s}}\Sigma_0\} \tag{3.12}$$

consists of a single value (for instance, if $\pi^{-1}_{\mathrm{s}\to\mathrm{b}}(\gamma(s_0))\cap{}^{\mathrm{s}}\Sigma_0$ is a single point), then $f\circ\gamma$ must be differentiable at s_0, with derivative given by this value. This is indeed how Lemma 3.3 is proved. The first part of the lemma follows because $\pi^{-1}_{\mathrm{s}\to\mathrm{b}}(\gamma(s_0))\cap{}^{\mathrm{s}}\Sigma_0$ is a single point, giving differentiability. On the other hand, the second half follows using $f=\sum\hat\xi^{\mathrm{b}}_j$, for which the single value in (3.12) is $-(1-\sum B_{ij}\hat\eta^{\mathrm{b}}_i\hat\eta^{\mathrm{b}}_j)<0$, for $\gamma(s_0)\in\mathcal{H}_{W,\mathrm{b}}$. Thus, f is locally strictly decreasing. Since $f(q')=0$ if $q'\in{}^{\mathrm{b}}S^*_WM_0\cap{}^{\mathrm{b}}\Sigma_0$, in particular at $\gamma(s_0)$, it is non-zero at $\gamma(s)$ for nearby but distinct values of s—so in particular for such s, $\gamma(s)\notin{}^{\mathrm{b}}S^*_WM_0\cap{}^{\mathrm{b}}\Sigma_0$, showing that γ leaves W instantaneously. In fact, this argument also demonstrates the following useful lemma.

Lemma 3.5. *Let U be a coordinate neighborhood around some $p\in W$, K a compact subset of U. Let $\epsilon_0>0$. Then there exists an $\delta>0$ with the following property. Suppose that γ is a GBB and $\gamma(s_0)\in{}^{\mathrm{b}}S^*_KM_0$. If $\sum_{j=1}^{f+1}\hat\xi^{\mathrm{b}}_j(\gamma(s_0))>0$ and $1-h(y(\gamma(s_0)),\hat\eta^{\mathrm{b}}(\gamma(s_0))>\epsilon_0$ then $\gamma|_{[s_0,s_0+\delta]}\cap{}^{\mathrm{b}}S^*_WM_0=\emptyset$, while if $\sum_{j=1}^{f+1}\hat\xi^{\mathrm{b}}_j(\gamma(s_0))<0$ and $1-h(y(\gamma(s_0)),\hat\eta^{\mathrm{b}}(\gamma(s_0))>\epsilon_0$ then $\gamma|_{[s_0-\delta,s_0]}\cap{}^{\mathrm{b}}S^*_WM_0=\emptyset$.*

Proof. Let $U_1 \subset U$ be open such that $K \subset U_1$, $\overline{U_1} \subset U$. GBB's are uniformly Lipschitz, i.e. with Lipschitz constant independent of the GBB , in compact sets (thus are equicontinuous in compact sets), so it follows that there is an $\delta_1 > 0$ such that $\gamma(s_0) \in {}^{\mathrm{b}}S^*_K M_0$ implies that $\gamma(s) \in {}^{\mathrm{b}}S^*_{U_1} M_0$ for $s \in [s_0 - \delta_1, s_0 + \delta_1]$. Now the uniform Lipschitz nature of the function $1 - h(y(\gamma(s)), \hat{\eta}^{\mathrm{b}}(\gamma(s)))$ shows that there exists $\delta_2 \in (0, \delta_1]$ such that for $|s - s_0| \leq \delta_2$, $1 - h(y(\gamma(s)), \hat{\eta}^{\mathrm{b}}(\gamma(s))) > \epsilon_0/2$. Now let $f = \sum \hat{\xi}^{\mathrm{b}}_j$. Then

$$\mathsf{H}_{\mathrm{s}}(\pi^*_{\mathrm{s}\to\mathrm{b}} f)|_{{}^{\mathrm{s}}\Sigma_0} = -\sum A_{ij}\hat{\xi}^{\mathrm{b}}_i\hat{\xi}^{\mathrm{b}}_j + \sum x_j F_{1j} = -(1 - \sum B_{ij}\hat{\zeta}^{\mathrm{b}}_i\hat{\zeta}^{\mathrm{b}}_j) + \sum x_j F_{2j},$$

with $F_{1j}, F_{2j} \in \mathcal{C}^\infty(S^* M_0)$, so there exist $\delta_3 > 0$ and $c > 0$ such that if $x_j < \delta_3$ for $j = 1, \ldots, f+1$, then $\mathsf{H}_{\mathrm{s}}(\pi^*_{\mathrm{s}\to\mathrm{b}} f)|_{{}^{\mathrm{s}}\Sigma_0} \leq -c$. Now if $x_j(\gamma(s_0)) > \delta_3/2$ for some j, the uniform Lipschitz character of $x_j \circ \gamma$ shows the existence of $\delta' > 0$ (independent of γ) such that $x_j(\gamma(s_0)) \neq 0$ for $|s - s_0| < \delta'$. On the other hand, if $x_j(\gamma(s_0) \leq \delta_3/2$ for all j, then the uniform Lipschitz character of $x_j \circ \gamma$ shows the existence of $\delta'' \in (0, \delta_2]$ such that $x_j(\gamma(s_0)) < \delta_3$ for $|s - s_0| < \delta''$, so $f(\gamma(s))$ is strictly decreasing on $[s_0 - \delta'', s_0 + \delta'']$. In particular, if $f(\gamma(s_0)) > 0$, then $f(\gamma(s)) > 0$ for $s \in [s_0 - \delta'', s_0]$, so $\gamma(s_0) \notin {}^{\mathrm{b}}S^*_W M_0$, and if $f(\gamma(s_0)) < 0$, then $f(\gamma(s)) < 0$ for $s \in [s_0, s_0 + \delta'']$, so $\gamma(s_0) \notin {}^{\mathrm{b}}S^*_W M_0$ again. This completes the proof of the lemma. □

We now recall the following statement, due to Lebeau.

Lemma 3.6. *(Lebeau,* [14, Proposition 1]*) If γ is a generalized broken bicharacteristic, $s_0 \in I$, $q_0 = \gamma(s_0)$, then there exist unique $\tilde{q}_+, \tilde{q}_- \in {}^{\mathrm{s}}\Sigma_0$ satisfying $\pi_{\mathrm{s}\to\mathrm{b}}(\tilde{q}_\pm) = q_0$ and having the property that if $f \in \mathcal{C}^\infty({}^{\mathrm{b}}S^* M_0)$ then $f \circ \gamma$ is differentiable both from the left and from the right at s_0 and*

$$\left(\frac{d}{ds}\right)(f \circ \gamma)|_{s_0\pm} = \mathsf{H}_{\mathrm{s}}\pi^*_{\mathrm{s}\to\mathrm{b}} f(\tilde{q}_\pm).$$

Definition 3.7. A generalized broken bicharacteristic segment γ, defined on $[0, s_0)$ or $(-s_0, 0]$, $\gamma(0) = q \in \mathcal{H}_{Y,b}$ is said to approach W *normally* as $s \to 0$ if for all j,

$$\lim_{s\to 0\pm} \frac{x_j(\gamma(s))}{s} \neq 0;$$

this limit always exists by [14, Proposition 1].

Remark 3.8. If γ approaches W normally then there is $s_1 > 0$ such that $\gamma(s) \in S^* M_0^\circ$ for $s \in (0, s_1)$ or $s \in (-s_1, 0)$ since $x_j(\gamma(0)) = 0$, and the one-sided derivative of $x_j \circ \gamma$ is non-zero.

While the actual derivatives depend on the choice of the defining functions x_j for the boundary hypersurfaces, the condition of normal incidence is independent of these choices.

3.2. **Edge-smooth cotangent bundle.** We now discuss another bundle, ultimately in order to discuss the refinement of GBB's that allows us to obtain a diffractive improvement. Let $\beta : M \to M_0$ be the blow-down map.

Let $\mathcal{V}_{\mathrm{es}}(M)$ denote the set of vector fields that are tangent to the fibers of $\beta|_{\tilde{W}} : \tilde{W} \to W$ (hence to $\tilde{W}$). This is a $\mathcal{C}^\infty(M)$-module, with sections locally spanned by

$$x\partial_x,\ x\partial_{y_j},\ \partial_{z'_j},\ \partial_{z''_j}.$$

(In fact, one can always use local coordinate charts without the z'' variables in this setting.) Under the blow-down map $\beta : M \to M_0$, elements of $\mathcal{V}(M_0)$ lift to vector fields of the form $x^{-1}V$, $V \in \mathcal{V}_{\mathrm{es}}(M)$, where x is a defining function of the front face, $\tilde{W}$. Conversely, $x^{-1}\mathcal{V}_{\mathrm{es}}(M)$ is spanned by the lift of elements of $\mathcal{V}(M_0)$ over $\mathcal{C}^\infty(M)$, i.e.

$$x^{-1}\mathcal{V}_{\mathrm{es}}(M) = \mathcal{C}^\infty(M) \otimes_{\mathcal{C}^\infty(M_0)} \beta^*\mathcal{V}(M). \tag{3.13}$$

Let ${}^{\mathrm{es}}TM$ denote the "edge-smooth" tangent bundle of M, defined as the bundle whose smooth sections are elements of $\mathcal{V}_{\mathrm{es}}(M)$; such a bundle exists by the above description of a local spanning set of sections. Let ${}^{\mathrm{es}}T^*M$ denote the dual bundle. Thus in the coordinates of §2, sections of ${}^{\mathrm{es}}T^*M$ are spanned by

$$\underline{\tau}\frac{dt}{x} + \underline{\xi}\frac{dx}{x} + \underline{\eta}\cdot\frac{dy}{x} + \underline{\zeta}'\cdot dz' + \underline{\zeta}''\cdot dz''. \tag{3.14}$$

By (3.13), taking into account that $dt^2 - g_0$ is a Lorentz metric on M_0, we deduce that its pull-back g to M is a Lorentzian metric on $x^{-1\,\mathrm{es}}T^*M$, i.e. that $x^{-2}g$ is a symmetric non-degenerate bilinear form on ${}^{\mathrm{es}}TM$ with signature $(+,-,\ldots,-)$. Correspondingly, the dual metric G has the property that x^2G is a Lorentzian metric on ${}^{\mathrm{es}}T^*M$. Note that G is the pull-back of $G_0 = \sigma_2(\Box) \in \mathcal{C}^\infty(T^*M_0 \setminus o)$. We thus conclude that $\sigma_2(\Box) \in \mathcal{C}^\infty(T^*M_0 \setminus o)$ lifts to an element of $x^{-2}\mathcal{C}^\infty({}^{\mathrm{es}}T^*M \setminus o)$; let

$$p = \sigma_{\mathrm{es},2}(x^2\Box) \in \mathcal{C}^\infty({}^{\mathrm{es}}T^*M \setminus o)$$

be such that $x^{-2}p$ is this lift, so

$$p|_{x=0} = \underline{\tau}^2 - (\underline{\xi}^2 + h(y,\underline{\eta}) + k(y,z,\underline{\zeta})).$$

Let ${}^{\mathrm{es}}\Sigma \subset {}^{\mathrm{es}}S^*M$ denote the characteristic set of $x^2\Box$, i.e., the set

$${}^{\mathrm{es}}\Sigma = p^{-1}(\{0\})/\mathbb{R}^+ = \{\sigma_{\mathrm{es},2}(x^2\Box) = 0\}/\mathbb{R}^+.$$

Thus, using the coordinates

$$x, y, t, z, \hat{\underline{\xi}} = \underline{\xi}/|\underline{\tau}|,\ \hat{\underline{\eta}} = \underline{\eta}/|\underline{\tau}|,\ \hat{\underline{\zeta}} = \underline{\zeta}/|\underline{\tau}|,\ \underline{\sigma} = |\underline{\tau}|^{-1}, \tag{3.15}$$

on ${}^{\mathrm{es}}T^*M$, valid where $\underline{\tau} \neq 0$, hence (outside the zero section) near where $p = 0$, and dropping $\underline{\sigma}$ to obtain coordinates on ${}^{\mathrm{es}}S^*M$,

$${}^{\mathrm{es}}\Sigma \cap {}^{\mathrm{es}}S^*_{\tilde{W}}M = \{(x = 0, t, y, z, \hat{\underline{\xi}}, \hat{\underline{\eta}}, \hat{\underline{\zeta}}) : \hat{\underline{\xi}}^2 + h(y,\hat{\underline{\eta}}) + k(y,z,\hat{\underline{\zeta}}) = 1\}. \tag{3.16}$$

The rescaled Hamilton vector field

$$\mathsf{H}_{\mathrm{es}} = |\underline{\tau}|^{-1}\mathsf{H}_p$$

is homogeneous of degree 0, and thus can be regarded as a vector field on ${}^{\mathrm{es}}S^*M$ which is tangent to ${}^{\mathrm{es}}\Sigma$. (Note that while H_{es} depends on the choice of x, and the particular homogeneous degree -1 function, $|\underline{\tau}|^{-1}$, used to re-homogenize H_p, these choices only change H_{es} by a positive factor, so its direction is independent of the choices—though our choices are in any case canonical.)

With the notation of [25, Section 7] (where it is explained slightly differently, as the underlying manifold is not a blow-up of another space), corresponding to the edge fibration

$$\beta : \tilde{W} \to W = Y \times \mathbb{R}_t,$$

there is a natural map

$$\varpi_{\mathrm{es}} : {}^{\mathrm{es}}T^*_{\tilde{W}}M \to T^*W.$$

In fact, in view of (3.13), the bundle $x^{-1\,\mathrm{es}}TM$ (whose sections are x^{-1} times smooth sections of ${}^{\mathrm{es}}TM$) can be identified with β^*TM_0, so one has a natural map $x^{-1\,\mathrm{es}}TM \to TM_0$. Dually, $x^{\mathrm{es}}T^*M$ can be identified with $\beta^*T^*M_0$, so one has a natural map $x^{\mathrm{es}}T^*M \to T^*M_0$. Multiplication by x maps ${}^{\mathrm{es}}T^*M$ to $x^{\mathrm{es}}T^*M$, and $\pi_{\mathrm{s}\to\mathrm{b}} : T^*M_0 \to {}^{\mathrm{b}}T^*M_0$ restricts to the quotient map $T^*_W M_0 \to T^*W = T^*M_0/N^*W$ over W, so ϖ_{es} is given by the composite map

$$\begin{aligned} {}^{\mathrm{es}}T^*_{\tilde{W}}M \ni \alpha \mapsto x\alpha \in x^{\mathrm{es}}T^*_{\tilde{W}}M \mapsto \beta_*(x\alpha) \in T^*_W M_0 \\ \mapsto [x\alpha] \in T^*W \subset {}^{\mathrm{b}}T^*M_0, \end{aligned}$$

which in local coordinates (3.14) is given by

$$\varpi_{\mathrm{es}}(x=0, y, t, z, \underline{\xi}, \underline{\eta}, \underline{\tau}, \underline{\zeta}) = (y, t, \underline{\eta}, \underline{\tau}).$$

The fibers can be identified with $\mathbb{R}_{\underline{\xi}} \times T^*Z$. In view of the $\mathbb{R}^+$-action on ${}^{\mathrm{es}}\Sigma$, this gives rise to a map $\varpi_{\mathrm{es}} : {}^{\mathrm{es}}\Sigma \to S^*W$, which is a fibration over $\mathcal{H}_{W,\mathrm{b}}$ (where $1 - h(y, \hat{\eta}^{\mathrm{b}}) > 0$) with fiber

$$\begin{aligned} \varpi_{\mathrm{es}}^{-1}(y, t, \hat{\eta}^{\mathrm{b}}) = \{(x=0, y, t, z, \hat{\underline{\xi}}, \hat{\underline{\eta}}, \hat{\underline{\zeta}}) : \hat{\underline{\eta}} = \hat{\eta}^{\mathrm{b}}, \\ |\hat{\underline{\xi}}|^2 + k(y, z, \hat{\underline{\zeta}}) = 1 - h(y, \hat{\eta}^{\mathrm{b}})\}; \end{aligned}$$

the fibers degenerate at $\mathcal{G}_{W,\mathrm{b}}$. Then H_{es} is tangent to the fibers of ϖ_{es}. In fact, as computed in [25, Proof of Lemma 2.3] (which is directly valid in our setting), using coordinates (3.15) on ${}^{\mathrm{es}}T^*M$,

(3.17)
$$\begin{aligned} -\frac{1}{2}\mathsf{H}_{\mathrm{es}} &= -\frac{1}{2}\underline{\sigma}\mathsf{H}_p \\ &= \hat{\underline{\xi}}(x\partial_x - \underline{\sigma}\partial_{\underline{\sigma}} - \hat{\underline{\zeta}} \cdot \partial_{\hat{\underline{\zeta}}}) + K^{ij}\hat{\underline{\zeta}}_i \partial_{z_j} + K^{ij}\hat{\underline{\zeta}}_i\hat{\underline{\zeta}}_j \partial_{\hat{\underline{\xi}}} - \frac{1}{2}\frac{\partial K^{ij}}{\partial z_k}\hat{\underline{\zeta}}_i\hat{\underline{\zeta}}_j \partial_{\hat{\underline{\zeta}}_k} + xH', \end{aligned}$$

with H' tangent to the boundary, hence as a vector field on ${}^{\mathrm{es}}S^*M$, restricted to ${}^{\mathrm{es}}S^*_{\tilde{W}}M$, H_{es} is given by

$$-\frac{1}{2}\mathsf{H}_{\mathrm{es}} = -\hat{\underline{\xi}}\hat{\underline{\zeta}}\partial_{\hat{\underline{\zeta}}} + K^{ij}\hat{\underline{\zeta}}_i \partial_{z_j} + K^{ij}\hat{\underline{\zeta}}_i\hat{\underline{\zeta}}_j \partial_{\hat{\underline{\xi}}} - \frac{1}{2}\frac{\partial K^{ij}}{\partial z_k}\hat{\underline{\zeta}}_i\hat{\underline{\zeta}}_j \partial_{\hat{\underline{\zeta}}_k}. \tag{3.18}$$

It is thus tangent to the fibers given by the constancy of $y, t, \hat{\underline{\eta}}$. Notice also that H_{es} is indeed tangent to the characteristic set, given by (3.16), and in ${}^{\mathrm{es}}S^*_{\tilde{W}}M$, it vanishes exactly at $\hat{\underline{\zeta}} = 0$. We let

$$\mathcal{R}_{\mathrm{es}} = \{q \in {}^{\mathrm{es}}\Sigma \cap {}^{\mathrm{es}}S^*_{\tilde{W}}M : \ \mathsf{H}_{\mathrm{es}}(q) = 0\} = \{(t, y, z, \hat{\underline{\xi}}, \hat{\underline{\eta}}, \hat{\underline{\zeta}}) \in {}^{\mathrm{es}}\Sigma : \ \hat{\underline{\zeta}} = 0\}$$

be the es-radial set.

3.3. **Edge-b cotangent bundle.** Finally, we construct a bundle ${}^{\mathrm{eb}}T^*M$ over M that behaves like ${}^{\mathrm{b}}T^*M$ away from $\tilde{W}$, and behaves like ${}^{\mathrm{e}}T^*M$ near the interior of $\tilde{W}$. Before doing so, we remark that the pullback of ${}^{\mathrm{b}}T^*M_0$ to M is ${}^{\mathrm{b}}T^*M$, so $\beta : M \to M_0$ induces a map

$$\beta_\sharp : {}^{\mathrm{b}}T^*M \to {}^{\mathrm{b}}T^*M_0,$$

such that

$$\beta_\sharp|_{{}^{\mathrm{b}}T^*_wM} \to {}^{\mathrm{b}}T^*_{\beta(w)}M_0, \ w \in M,$$

is an isomorphism. It commutes with the $\mathbb{R}^+$-action, hence induces a map

$$\hat{\beta}_\sharp : {}^{\mathrm{b}}S^*M \to {}^{\mathrm{b}}S^*M_0,$$

such that

$$\hat{\beta}_\sharp|_{{}^{\mathrm{b}}S^*_wM} \to {}^{\mathrm{b}}S^*_{\beta(w)}M_0, \ w \in M,$$

is an isomorphism.

More precisely, ${}^{\mathrm{eb}}T^*M$ arises from the lift of vector fields on M_0 which are tangent to all faces of M_0 and vanish at W. (The set $\tilde{\mathcal{V}}$ of such vector fields is a $\mathcal{C}^\infty(M_0)$-module, but is *not* all sections of a vector bundle over M_0—unlike its analogue, $\mathcal{V}(M_0)$, in the construction of $\mathcal{V}_{\mathrm{es}}(M)$; locally $x_j\partial_{x_j}$, $j = 1, \ldots, f+1$, and $x_i\partial_{y_j}$, $i = 1, \ldots, f+1$, $j = 1, \ldots, n-f-1$, give a spanning list.)

Definition 3.9. Let $\mathcal{V}_{\mathrm{eb}}(M)$ consist of vector fields whose sections are tangent to all of ∂M and to the fibers of $\tilde{W}$.

This is again a $\mathcal{C}^\infty(M)$-module, and locally $x\partial_x$, $x\partial_{y_j}$, $z'_j\partial_{z'_j}$ and $\partial_{z''_j}$ give a spanning set; in fact

$$\mathcal{V}_{\mathrm{eb}}(M) = \mathcal{C}^\infty(M) \otimes_{\mathcal{C}^\infty(M_0)} \beta^*\tilde{\mathcal{V}}.$$

Thus, there is a vector bundle, called the "edge-b" tangent bundle of M, denoted ${}^{\mathrm{eb}}TM$, whose sections are exactly elements of $\mathcal{V}_{\mathrm{eb}}(M)$. Let ${}^{\mathrm{eb}}T^*M$ denote the dual bundle. Thus in the coordinates of §2, sections of ${}^{\mathrm{eb}}T^*M$ are spanned by

$$\tau\frac{dt}{x} + \xi\frac{dx}{x} + \eta \cdot \frac{dy}{x} + \sum \zeta'_i \frac{dz'_i}{z'_i} + \zeta'' \cdot dz''$$

In particular, we point out that the lift of $\sum x_j D_{x_j}$ from M_0 to M by β is xD_x, up to $x\mathcal{V}_{\mathrm{eb}}(M)$, hence considering their principal symbols gives

$$\sum_j \beta^*\xi^{\mathrm{b}}_j = \xi \text{ at } x = 0.$$

Dividing by $\beta^*|\tau^{\mathrm{b}}| = x^{-1}|\tau|$ yields

$$\sum_j \beta^* \hat{\xi}^{\mathrm{b}}_j = x\hat{\xi} + O(x^2),\ \hat{\xi} = \xi/|\tau|. \tag{3.19}$$

There exists a natural bundle map

$$\pi_{\mathrm{es}\to\mathrm{eb}} : {}^{\mathrm{es}}T^*M \to {}^{\mathrm{eb}}T^*M,$$

analogous to the bundle map $\pi_{\mathrm{s}\to\mathrm{b}} : T^*M_0 \to {}^{\mathrm{b}}T^*M_0$ of (3.1). In canonical coordinates, this maps

$$(\underline{\tau}, \underline{\xi}, \underline{\eta}, \underline{\zeta}', \underline{\zeta}'') \mapsto (\tau = \underline{\tau},\ \xi = \underline{\xi},\ \eta = \underline{\eta},\ \zeta'_i = \underline{\zeta}'_i z'_i,\ \zeta'' = \underline{\zeta}'').$$

This map commutes with the $\mathbb{R}^+$-action of dilations in the fibers, and maps $p^{-1}(\{0\}) \subset {}^{\mathrm{es}}T^*M \setminus o$ into the complement of the zero section of ${}^{\mathrm{eb}}T^*M$, so it gives rise to a map

$$\widehat{\pi_{\mathrm{es}\to\mathrm{eb}}} : {}^{\mathrm{es}}\Sigma \to {}^{\mathrm{eb}}S^*M.$$

Let

$${}^{\mathrm{eb}}\Sigma = \widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}({}^{\mathrm{es}}\Sigma) \subset {}^{\mathrm{eb}}S^*M.$$

In coordinates

$$x, y, t, z, \hat{\xi} = \xi/|\tau|,\ \hat{\eta} = \eta/|\tau|,\ \hat{\zeta} = \zeta/|\tau|,$$

on ${}^{\mathrm{eb}}S^*M$, and analogously defined coordinates on ${}^{\mathrm{es}}S^*M$,

$$\widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}(x, y, t, z, \underline{\hat{\xi}}, \underline{\hat{\eta}}, \underline{\hat{\zeta}}', \underline{\hat{\zeta}}'') = (x, y, t, z, \hat{\xi} = \underline{\hat{\xi}},\ \hat{\eta} = \underline{\hat{\eta}},\ \hat{\zeta}'_i = \underline{\hat{\zeta}}'_i z'_i,\ \hat{\zeta}'' = \underline{\hat{\zeta}}''),$$

so for $w \in \tilde{W}$, $z'_1(w) = \ldots = z'_p(w) = 0,\ z'_{p+1}(w) \neq 0, \ldots,\ z'_k(w) \neq 0$, with $p \geq 1$,

$$\begin{aligned} {}^{\mathrm{eb}}\Sigma \cap {}^{\mathrm{eb}}S^*_w M = \{(\hat{\xi}, \hat{\eta}, \hat{\zeta}) \in {}^{\mathrm{eb}}S^*_w M :\ & \hat{\zeta}'_1 = \ldots = \hat{\zeta}'_p = 0, \\ & 1 \geq \hat{\xi}^2 + h(y, \hat{\eta}) + k(y, z, (0, \ldots, 0, \hat{\zeta}'_{p+1}, \ldots, \hat{\zeta}'_k), \hat{\zeta}'')\}. \end{aligned}$$

We again also obtain a map $\varpi_{\mathrm{eb}} : {}^{\mathrm{eb}}\Sigma \cap {}^{\mathrm{eb}}S^*_{\tilde{W}}M \to S^*W$ analogously to ϖ_{es} which is a fibration over $\mathcal{H}_{W,\mathrm{b}}$; in local coordinates (on S^*W near the projection of ${}^{\mathrm{eb}}\Sigma$, $(y, t, \hat{\eta})$ are local coordinates, $\hat{\eta} = \eta/|\tau|$)

$$\varpi_{\mathrm{eb}}(0, y, t, z, \hat{\xi}, \hat{\eta}, \hat{\zeta}) = (y, t, \hat{\eta} = \underline{\hat{\eta}}). \tag{3.20}$$

More invariantly we can see this as follows. As discussed in [25, Section 7] in the setting where the fibers on $\tilde{W}$ have no boundaries, one considers the map

$$x\cdot : {}^{\mathrm{eb}}T^*M \to {}^{\mathrm{b}}T^*M$$

given by multiplication of the covectors by x away from $\tilde{W}$, which extends to a $\mathcal{C}^\infty$ map as indicated, namely

$$\begin{aligned} x\cdot : \tau\frac{dt}{x} + \xi\frac{dx}{x} + \eta \cdot \frac{dy}{x} + \sum \zeta'_i \frac{dz'_i}{z'_i} + \zeta'' \cdot dz'' \\ \mapsto \tau\, dt + x\xi\frac{dx}{x} + \eta\, dy + \sum x\zeta'_i \frac{dz'_i}{z'_i} + x\zeta'' \cdot dz''. \end{aligned}$$

Note that at $x = 0$, this gives

$$
\begin{aligned}
&x \cdot (\alpha) = \tau\, dt + \eta\, dy, \\
(3.21) \qquad &\alpha = \tau \frac{dt}{x} + \xi \frac{dx}{x} + \eta \cdot \frac{dy}{x} + \sum \zeta_i' \frac{dz_i'}{z_i'} + \zeta'' \cdot dz'' \in {}^{\mathrm{eb}}T^*_w M,\ w \in \tilde{W}.
\end{aligned}
$$

In particular, as the image under $(x\cdot) \circ \pi_{\mathrm{es}\to\mathrm{eb}}$ of $p^{-1}(\{0\}) \subset {}^{\mathrm{es}}T^*M \setminus o$ is disjoint from the zero section, and since multiplication by x commutes with the $\mathbb{R}^+$-action in the fibers, $\hat{\beta}_\sharp \circ (x\cdot)$ descends to a map

$$\varpi_{\mathrm{eb}} : {}^{\mathrm{eb}}\Sigma \to {}^{\mathrm{b}}\Sigma_0,$$

and away from $\tilde{W}$ it is given by the restriction of the natural identificantion of ${}^{\mathrm{eb}}S^*_{M\setminus\tilde{W}}M$ with ${}^{\mathrm{b}}S^*_{M_0\setminus W}M_0$, while at $\tilde{W}$, as (3.21) shows, is given by (3.20), where we consider $S^*W \subset {}^{\mathrm{b}}S^*M_0$, cf. (3.4).

We now introduce sets of covectors that are respectively elliptic, glancing, and hyperbolic with respect to the boundary faces of M_0 meeting at the corner W; these sets are thus of covectors over the boundary of $\tilde{W}$:

$$
\begin{aligned}
&\mathcal{E} = {}^{\mathrm{eb}}S^*_{\partial\tilde{W}}M \setminus {}^{\mathrm{eb}}\Sigma = \{q \in {}^{\mathrm{eb}}S^*_{\partial\tilde{W}}M\backslash o : (\widehat{\pi_{\mathrm{es}\to\mathrm{eb}}})^{-1}(q) = \emptyset\}, \\
&\mathcal{G} = \{q \in {}^{\mathrm{eb}}S^*_{\partial\tilde{W}}M\backslash o : |\widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}^{-1}(q)| = 1\}, \\
&\mathcal{H} = \{q \in {}^{\mathrm{eb}}S^*_{\partial\tilde{W}}M\backslash o : |\widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}^{-1}(q)| \geq 2\},
\end{aligned}
$$

so ${}^{\mathrm{eb}}\Sigma \cap {}^{\mathrm{eb}}S^*_{\partial\tilde{W}}M = \mathcal{G} \cup \mathcal{H}$.

In coordinates, note that, for instance, for

$$w \in \tilde{W},\ z_1'(w) = \ldots = z_p'(w) = 0,\ z_{p+1}'(w) \neq 0, \ldots,\ z_k'(w) \neq 0,$$

with $p \geq 1$,

(3.22)

$$
\begin{aligned}
\mathcal{E} \cap {}^{\mathrm{eb}}S^*_w M &= \{\exists j,\ 1 \leq j \leq p,\ \hat{\zeta}_j' \neq 0\} \\
&\qquad \cup \{1 < \hat{\xi}^2 + h(y, \hat{\eta}) + k(y, z, (0, \ldots, 0, \hat{\zeta}_{p+1}', \ldots, \hat{\zeta}_k'), \hat{\zeta}'')\}, \\
\mathcal{G} \cap {}^{\mathrm{eb}}S^*_w M &= \{\hat{\zeta}_1' = \ldots = \hat{\zeta}_p' = 0, \\
&\qquad 1 = \hat{\xi}^2 + h(y, \hat{\eta}) + k(y, z, (0, \ldots, 0, \hat{\zeta}_{p+1}', \ldots, \hat{\zeta}_k'), \hat{\zeta}'')\}, \\
\mathcal{H} \cap {}^{\mathrm{eb}}S^*_w M &= \{\hat{\zeta}_1' = \ldots = \hat{\zeta}_p' = 0, \\
&\qquad 1 > \hat{\xi}^2 + h(y, \hat{\eta}) + k(y, z, (0, \ldots, 0, \hat{\zeta}_{p+1}', \ldots, \hat{\zeta}_k'), \hat{\zeta}'')\}.
\end{aligned}
$$

Remark 3.10. The set $\mathcal{G}_{W,\mathrm{b}}$ defined in (3.5) represents rays that are glancing *with respect to the corner* W, i.e., are tangent to *all* boundary faces meeting at W, while $\mathcal{G}$ defined above describes the rays that are glancing with respect to *one or more* of the boundary faces meeting at W (see Figure 4). The sets $\mathcal{G}_{W,\mathrm{b}}$ and $\mathcal{H}_{W,\mathrm{b}}$ live in $S^*W \subset {}^{\mathrm{b}}S^*_W M_0$. This can be lifted to ${}^{\mathrm{b}}S^*M$ by β (since ${}^{\mathrm{b}}T^*M = \beta^{*\mathrm{b}}T^*M_0$), but in this picture $\mathcal{G}_{W,\mathrm{b}}$ and $\mathcal{H}_{W,\mathrm{b}}$ are *global* in the fibers of β, i.e., live over all of $\tilde{W}$, not merely over its boundary.

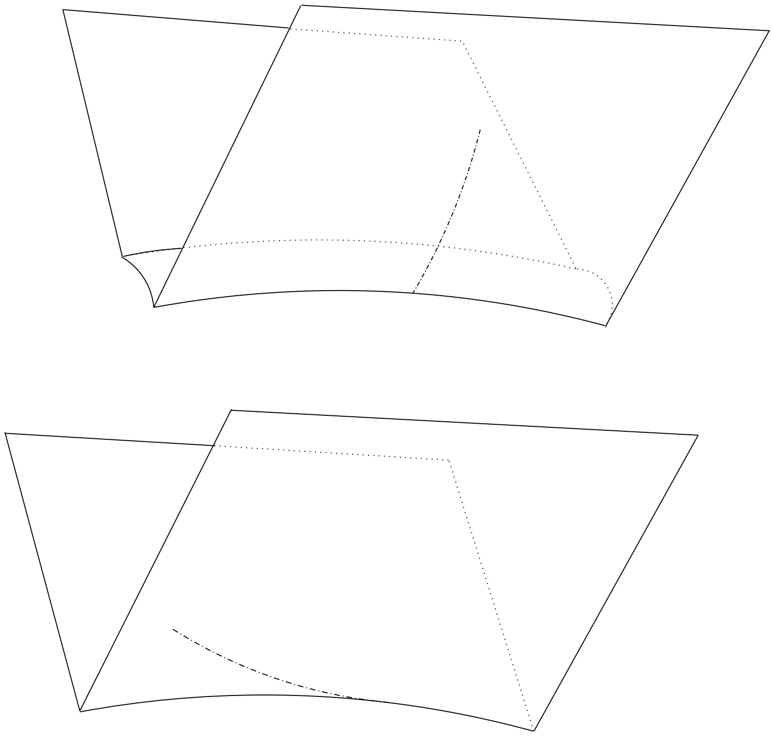

FIGURE 4. Glancing rays. The ray depicted at top, in M (projected down to X), terminates at a point in $\mathcal{G}$. The ray depicted at bottom, in M_0 (projected down to X_0), terminates at a point in $\mathcal{G}_{W,\mathrm{b}}$.

3.4. **Bicharacteristics.** We now turn to bicharacteristics in ${}^{\mathrm{eb}}\Sigma$, which will be the dynamical locus of the geometric improvement for the propagation result. Taking into account that H_{es} is tangent to the fibers of ϖ_{es}, one expects that over $\tilde{W}$, these bicharacteristics will lie in a single fiber of the related map ϖ_{eb}, i.e. $y, t, \hat\eta$ will be constant along these. The fibers of ϖ_{es} and ϖ_{eb} have a rather different character depending on whether they are over a point in $\mathcal{G}_{W,\mathrm{b}}$ or in $\mathcal{H}_{W,\mathrm{b}}$. Namely, over $\mathcal{G}_{W,\mathrm{b}}$ the fibers of ϖ_{es} resp. ϖ_{eb} are $\hat\xi^{\mathrm{b}} = 0, \hat\zeta^{\mathrm{b}} = 0$ resp. $\xi = 0, \zeta = 0$ i.e. they are the zero section. By contrast over $\alpha = (t, y, \hat\eta) \in \mathcal{H}_{W,\mathrm{b}}$, the fiber of ϖ_{es} is

$$\mathcal{H}_{\mathrm{es}\to\alpha,\mathrm{b}} = \{(t, y, z, \underline{\hat\xi}, \underline{\hat\eta} = \hat\eta, \underline{\hat\zeta}) \in {}^{\mathrm{es}}S^*M : \underline{\hat\xi}^2 + k(y, z, \underline{\hat\zeta}) = 1 - h(y, \underline{\hat\eta})\}$$

while that of ϖ_{eb} is

$$\begin{aligned}\mathcal{H}_{\mathrm{eb}\to\alpha,\mathrm{b}} = \{(t, y, z, \hat\xi, \hat\eta, \hat\zeta) \in {}^{\mathrm{eb}}S^*M :\ & \hat\zeta_1' = \ldots = \hat\zeta_p' = 0, \\ & \hat\xi^2 + k(y, z, (0, \ldots, 0, \hat\zeta_{p+1}', \ldots, \hat\zeta_k'), \hat\zeta'') \le 1 - h(y, \hat\eta)\}.\end{aligned}$$

The geometric improvement will take place over $\mathcal{H}_{W,\mathrm{b}}$, so from now on we concentrate on this set. Now, for $\alpha = (t, y, \hat{\eta}) \in \mathcal{H}_{W,\mathrm{b}}$

$$\mathcal{R}_{\mathrm{es}} \cap \mathcal{H}_{\mathrm{es}\to\alpha,\mathrm{b}} = \{(t, y, z, \hat{\underline{\xi}}, \hat{\underline{\eta}} = \hat{\eta}, \hat{\underline{\zeta}} = 0) \in {}^{\mathrm{es}}S^*M : \hat{\underline{\xi}}^2 = 1 - h(y, \hat{\underline{\eta}})\},$$

hence has two connected components which we denote by

$$\mathcal{R}_{\mathrm{es},\alpha,I/O} = \{(t, y, z, \hat{\underline{\xi}}, \hat{\underline{\eta}} = \hat{\eta}, \hat{\underline{\zeta}} = 0) \in {}^{\mathrm{es}}S^*M : \hat{\underline{\xi}} = \pm\,\mathrm{sgn}(\underline{\tau})\sqrt{1 - h(y, \hat{\underline{\eta}})}\},$$

with $\mathrm{sgn}(\underline{\tau})$ being the constant function ± 1 on the two connected components of ${}^{\mathrm{es}}\Sigma$.

Here the labels "I/O", stand for "incoming/outgoing." This is explained by

$$-\frac{1}{2}\mathsf{H}_{\mathrm{es}}t = -\hat{\underline{\tau}},\ -\frac{1}{2}\mathsf{H}_{\mathrm{es}}x = \hat{\underline{\xi}}x,$$

so in a neighborhood of $\mathcal{R}_{\mathrm{es},\alpha,I}$, $\mathsf{H}_{\mathrm{es}}t$ and $\mathsf{H}_{\mathrm{es}}x$ have the opposite signs, i.e. if t is increasing, x is decreasing along H_{es}, just as one would expect an 'incoming ray' to do; at outgoing points the reverse is the case.

We also let

$$\begin{aligned}\mathcal{R}_{\mathrm{eb},\alpha,I/O} &= \widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}(\mathcal{R}_{\mathrm{es},\alpha,I/O}) \\ &= \{(t, y, z, \hat{\xi}, \hat{\eta}, \hat{\zeta} = 0) \in {}^{\mathrm{eb}}S^*M : \hat{\xi} = \pm\,\mathrm{sgn}(\tau)\sqrt{1 - h(y, \hat{\eta})}\},\end{aligned}$$

and

$$\mathcal{R}_{\mathrm{eb},S,I/O} = \cup_{\alpha\in S}\mathcal{R}_{\mathrm{eb},\alpha,I/O}$$

for $S \subset \mathcal{H}_{W,\mathrm{b}}$.

Definition 3.11. An *edge generalized broken bicharacteristic*, or EGBB , is a continuous map $\gamma : I \to {}^{\mathrm{eb}}\Sigma$ such that for all $f \in \mathcal{C}^\infty({}^{\mathrm{eb}}S^*M)$, real-valued,

$$\begin{aligned}&\liminf_{s\to s_0}\frac{(f\circ\gamma)(s) - (f\circ\gamma)(s_0)}{s - s_0} \\ &\qquad \geq \inf\{\mathsf{H}_{\mathrm{es}}(\widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}^*f)(q) : q \in \widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}^{-1}(\gamma(s_0)) \cap {}^{\mathrm{es}}\Sigma\}.\end{aligned} \tag{3.23}$$

Lemma 3.12.
(1) *An* EGBB *outside* ${}^{\mathrm{eb}}S^*_{\tilde{W}}M$ *is a reparameterized* GBB *(under the natural identification of* ${}^{\mathrm{b}}S^*_{M_0\setminus W}M_0$ *with* ${}^{\mathrm{eb}}S^*_{M\setminus\tilde{W}}M$*), and conversely.*
(2) *If a point* q *on an* EGBB *lies in* ${}^{\mathrm{eb}}S^*_{\tilde{W}}M$*, then the whole* EGBB *lies in* ${}^{\mathrm{eb}}S^*_{\tilde{W}}M$*, in* $\varpi_{\mathrm{eb}}^{-1}(\varpi_{\mathrm{eb}}(q))$*, i.e. in the fiber of* ϖ_{eb} *through* q*.*
(3) *The only* EGBB *through a point in* $\mathcal{R}_{\mathrm{eb},\alpha,I/O}$ *is the constant curve.*
(4) *For* $\alpha \in \mathcal{H}_{W,\mathrm{b}}$*, an* EGBB *in* $\mathcal{H}_{\mathrm{eb}\to\alpha,\mathrm{b}}\setminus\mathcal{R}_{\mathrm{eb},\alpha,I/O}$ *projects to a reparameterized* GBB *in* ${}^{\mathrm{b}}T^*Z$*, hence to geodesic of length* π *in* $\tilde{W}$*.*

Proof.
(1) As H_{s} and H_{es} differ by an overall factor under the natural identification $\iota : S^*_{M_0\setminus W}M_0 \to {}^{\mathrm{es}}S^*_{M\setminus\tilde{W}}M$, namely

$$\iota_*\mathsf{H}_{\mathrm{s}} = |\tau^{\mathrm{s}}|^{-1}\,|\underline{\tau}|\,x^{-2}\mathsf{H}_{\mathrm{es}} = x^{-1}\mathsf{H}_{\mathrm{es}},$$

we obtain this immediately.

(2) The tangency of H_{es} to the fibers of ϖ_{es} means that if we set f equal to any of $\pm y_j, \pm t, \pm\hat{\eta}_j^{\mathrm{b}}$, $\mathsf{H}_{\mathrm{es}} f = 0$. By (3.23), then $(f \circ \gamma)'(s_0) = 0$ for all s_0, and for each of these choices. This ensures that γ remains in the fiber.

(3) H_{es} vanishes at the unique $q \in \widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}^{-1}(\gamma(s_0)) \cap {}^{\mathrm{es}}\Sigma$ if $\gamma(s_0) \in \mathcal{R}_{\mathrm{eb},\alpha,I/O}$. Moreover the function $\hat{\underline{\xi}} \circ \gamma$ is in C^1, as

$$\mathsf{H}_{\mathrm{es}}\hat{\underline{\xi}} = 2K(y,t,z,\hat{\underline{\zeta}}) = 2(1 - h(y,\hat{\underline{\eta}}) - \hat{\underline{\xi}}^2)$$

on ${}^{\mathrm{es}}\Sigma$. Thus, (3.23) entails that if $\hat{\underline{\xi}} = \pm\sqrt{1 - h(y,\hat{\underline{\eta}})}$ at some point on an EGBB , then it is constant.

(4) This follows from a reparameterization argument, as in [25], taking into account that H_{es} is tangent to the fibers of ${}^{\mathrm{es}}T^*_{\tilde{W}}M$, hence can be considered as a vector field on $\mathbb{R}_{\hat{\underline{\xi}}} \times T^*Z$. (In fact, a completely analogous argument takes place in [33, Section 6] in the setting of N-body scattering.) $\square$

Suppose now that $\gamma : [0,\delta_0] \to {}^{\mathrm{b}}\Sigma_0$ is a GBB with $\gamma(0) = \alpha \in \mathcal{H}_{W,\mathrm{b}}$. Thus, assuming $\delta_0 > 0$ is sufficiently small, by Lemma 3.5, $\gamma|_{(0,\delta_0]} \cap {}^{\mathrm{b}}S^*_W M_0 = \emptyset$. Since ${}^{\mathrm{b}}S^*_{M_0\setminus W}M_0$ is naturally diffeomorphic to ${}^{\mathrm{eb}}S^*_{M\setminus\tilde{W}}M$, we can lift $\gamma|_{(0,\delta_0]}$ to a curve $\tilde{\gamma} : (0,\delta_0] \to {}^{\mathrm{eb}}S^*M$ in a unique fashion. It is natural to ask whether this lifted curve extends continuously to 0, which is a question we now address.

The following is easily deduced from Lebeau, [14, Proposition 1] (stated here in Lemma 3.6) and its proof:

Lemma 3.13. *Suppose that $\alpha \in \mathcal{H}_{W,\mathrm{b}}$. There exists $\delta_0 > 0$ with the following property.*

Suppose $\gamma : [0,\delta_0] \to {}^{\mathrm{b}}S^*M_0$ is a GBB with $\gamma(0) = \alpha$. Let $\tilde{\gamma} : (0,\delta_0] \to {}^{\mathrm{eb}}S^*M$ be the unique lift of $\gamma|_{(0,\delta_0]}$ to ${}^{\mathrm{eb}}S^*M$. Then $\tilde{\gamma}$ (uniquely) extends to a continuous map $\tilde{\gamma} : [0,\delta_0] \to {}^{\mathrm{eb}}S^*M$, with $\tilde{\gamma}(0) \in \mathcal{R}_{\mathrm{eb},\alpha,O}$.

In addition, γ approaches W normally if and only if

$$\tilde{\gamma}(0) \notin {}^{\mathrm{eb}}S^*_{\partial\tilde{W}}M \cap \mathcal{R}_{\mathrm{eb},\alpha,O} = \mathcal{G} \cap \mathcal{R}_{\mathrm{eb},\alpha,O}.$$

The analogous results hold if $[0,\delta_0]$ is replaced by $[-\delta_0,0]$ and $\mathcal{R}_{\mathrm{eb},\alpha,O}$ is replaced by $\mathcal{R}_{\mathrm{eb},\alpha,I}$.

Remark 3.14. The proof in fact shows that δ_0 can be chosen independent of α as long as we fix some $K \subset \mathcal{H}_{W,\mathrm{b}} \subset {}^{\mathrm{b}}S^*_W M_0$ compact and require $\alpha \in K$.

Remark 3.15. The special case of a *normal* GBB segment γ, which lifts to a curve $\tilde{\gamma} : [0,\delta_0] \to {}^{\mathrm{eb}}S^*M$ starting at $\tilde{W}^\circ$, follows directly by the description of geodesic in edge metrics from [25], since normality implies that for sufficiently small $\delta_0 > 0$, $\gamma|_{(0,\delta_0]}$ has image disjoint from $x_j = 0$ for all j, i.e. the boundaries can be ignored, and one is simply in the setting of

[25]. This argument also shows that given $\alpha \in \mathcal{H}_{W,\mathrm{b}}$ and $p \in \mathcal{R}_{\mathrm{eb},\alpha,O} \setminus \mathcal{G}$, for sufficiently small $\delta_0 > 0$, there is a unique GBB $\gamma : [0,\delta_0] \to {}^{\mathrm{b}}S^*M_0$ with $\gamma(0) = \alpha$ such that the lift $\tilde{\gamma}$ of γ satisfies $\gamma(0) = p$.

Proof. Let $\alpha = (y_0, t_0, \hat{\eta}_0^{\mathrm{b}})$. First, by Lemma 3.5, $\gamma|_{(0,\delta_0]} \cap {}^{\mathrm{b}}S^*_W M_0 = \emptyset$ for $\delta_0 > 0$ sufficiently small, hence the lift $\tilde{\gamma}|_{(0,\delta_0]}$ exists and is unique. Lebeau proves in [14, Proof of Proposition 1] (with our notation) that

$$\lim_{s\to 0} \hat{\xi}(\tilde{\gamma}(s)) = \sqrt{1 - h(y_0, \hat{\eta}_0^{\mathrm{b}})} \;\text{ and }\; \frac{dx(\gamma(s))}{ds}|_{s=0} = 2\sqrt{1 - h(y_0, \hat{\eta}_0^{\mathrm{b}})} > 0.$$

This implies that

$$\sup\{|\hat{\underline{\zeta}}(q)| :\ q \in (\widehat{\pi_{\mathrm{s}\to\mathrm{b}}})^{-1}(\gamma(s))\} \to 0 \text{ as } s \to 0+,$$

since

$$\sum K_{ij}(y,z)\hat{\underline{\zeta}}_i\hat{\underline{\zeta}}_j = 1 - h(y,\underline{\hat{\eta}}) - \hat{\underline{\xi}}^2 + xG \le 1 - h(y,\underline{\hat{\eta}}) - \hat{\underline{\xi}}^2 + Cx$$

on ${}^{\mathrm{es}}\Sigma$, and $1 - h(y(\tilde{\gamma}(s)), \underline{\hat{\eta}}(\tilde{\gamma}(s))) - \hat{\underline{\xi}}(\tilde{\gamma}(s))^2 + Cx(\tilde{\gamma}(s)) \to 0$. It remains to show that the coordinates z_j have a limit as $s \to 0$. But by Lemma 3.6, $(dx_j \circ \gamma/ds)|_{s=0} = 2\hat{\xi}_j^{\mathrm{s}}(0)$ exists, and $\sum A_{ij}\hat{\xi}_i^{\mathrm{s}}(0)\hat{\xi}_j^{\mathrm{s}}(0) = 1 - h(y_0, \hat{\eta}_0^{\mathrm{b}}) > 0$. Thus, considering $z_j(\gamma(s)) = x_j(\gamma(s))/x(\gamma(s))$, L'Hôspital's rule shows that $\lim_{s\to 0+} z_j(\gamma(s)) = \hat{\xi}_j^{\mathrm{s}}(0)/\sqrt{1 - h(y_0, \hat{\eta}_0^{\mathrm{b}})}$ exists, finishing the proof of the first claim. The second claim follows at once from the last observation regarding $\lim_{s\to 0+} z_j(\gamma(s))$. ☐

We also need the following result, which is a refinement of Lemma 3.13, insofar as Lebeau's result only deals with a single GBB emanating from the corner W of M_0 : the following lemma extends Lemma 3.13 uniformly to GBB's starting close to but not at the corner. For simplicity of notation, we only state the results for the outgoing direction.

Lemma 3.16. *Suppose that $\alpha \in \mathcal{H}_{W,\mathrm{b}}$, $p \in \mathcal{R}_{\mathrm{eb},O,\alpha}$, $p_n \in {}^{\mathrm{eb}}S^*_{M\setminus\tilde{W}}M$, and $p_n \to p$ in ${}^{\mathrm{eb}}S^*M$. Suppose $\delta_0 > 0$ is sufficiently small (see following remark). Let $\gamma_n : [0,\delta_0] \to {}^{\mathrm{b}}S^*M_0$ be GBB such that $\gamma_n(0) = p_n$. For n sufficiently large, let $\tilde{\gamma}_n : [0,\delta_0] \to {}^{\mathrm{eb}}S^*M$ be the unique lift of γ_n to a map $[0,\delta_0] \to {}^{\mathrm{eb}}S^*M$. Then for N sufficiently large, $\{\tilde{\gamma}_n\}_{n\ge N}$ is equicontinuous.*

Remark 3.17. As $p_n \to p$, $\hat{\xi}(p_n) \to \hat{\xi}(p) > 0$, so there exists $N > 0$ such that

$$\sum_{j=1}^{f+1} \hat{\xi}_j^{\mathrm{b}}(p_n) = x(p_n)\hat{\xi}(p_n) + O(x(p_n)^2) > 0$$

for $n \ge N$; cf. (3.19). Thus, by Lemma 3.5, there exists $\delta_0 > 0$ such that $\gamma_n|_{[0,\delta_0]} \cap {}^{\mathrm{b}}S^*_W M_0 = \emptyset$ for $n \ge N$—this is the δ_0 in the statement of the lemma. Hence, for n sufficiently large, γ_n has a unique lift $\tilde{\gamma}_n$ to ${}^{\mathrm{eb}}S^*M$, since ${}^{\mathrm{eb}}S^*M$ and ${}^{\mathrm{b}}S^*M_0$ are naturally diffeomorphic away from $\tilde{W}$, resp. W as previously noted.

Proof. Note first that $\{\gamma_n\}_{n\in\mathbb{N}}$ is equicontinuous by Lebeau's result [14, Corollaire 2] (see also the proof of [14, Proposition 6])—indeed, this follows directly from our definition of GBB . This implies that $\{\tilde\gamma_n\}_{n\in\mathbb{N}}$ is equicontinuous at all $s_0\in(0,\delta_0]$, for given such a s_0, there exists $K_0\subset M_0$ compact disjoint from W such that $\gamma_n|_{[s_0,\delta_0]}$ has image in ${}^{\mathrm{b}}S^*_{K_0}M_0$, which is canonically diffeomorphic to ${}^{\mathrm{eb}}S^*_{\beta^{-1}(K_0)}M$. Thus, it remains to consider equicontinuity at 0.

For sufficiently large n, all γ_n have image in ${}^{\mathrm{b}}S^*_KM_0$ where K is compact and $K\subset O$ for a coordinate chart O on M_0. Thus, by the equicontinuity of γ_n, the coordinate functions

$$x_j\circ\gamma_n,\ t_j\circ\gamma_n,\ y_j\circ\gamma_n,\ \hat\xi^{\mathrm{b}}_j\circ\gamma_n,\ \hat\eta^{\mathrm{b}}_j\circ\gamma_n$$

are equicontinuous. We need to show that for the lifted curves, $\tilde\gamma_n$, the coordinate functions

$$x\circ\tilde\gamma_n,\ t_j\circ\tilde\gamma_n,\ y_j\circ\tilde\gamma_n,\ z_j\circ\tilde\gamma_n,\ \hat\xi\circ\tilde\gamma_n,\ \hat\eta\circ\tilde\gamma_n,\ \hat\zeta_j\circ\tilde\gamma_n$$

are equicontinuous at 0. By the above description, and $y_j\circ\tilde\gamma_n=y_j\circ\gamma_n$, $t_j\circ\tilde\gamma_n=t_j\circ\gamma_n$ and $\hat\eta_j\circ\tilde\gamma_n=\hat\eta^{\mathrm{b}}_j\circ\gamma_n$ are equicontinuous, as is $x\circ\tilde\gamma_n$ in view of $x=(\sum a_{ij}x_ix_j)^{1/2}$. Thus, it remains to consider $\hat\xi\circ\tilde\gamma_n$, $z_j\circ\tilde\gamma_n$ and $\hat\zeta_j\circ\tilde\gamma_n$.

Let $p=(0,y_0,z_0,\hat\xi_0,\hat\eta_0,0)$, and write $\mu=\hat\xi_0>0$. Thus,

$$\mu=\sqrt{1-h(y_0,\hat\eta_0)}.$$

Let $\epsilon_1>0$. One can show easily, as in the proof of Lebeau's [14, Proposition 1], that for all n sufficiently large (so that p_n sufficiently close to p) and $s_0>0$ sufficiently small,

$$s\in[0,s_0]\Longrightarrow\hat\xi\circ\tilde\gamma_n(s)\in[\mu-\epsilon_1,\mu+\epsilon_1]. \tag{3.24}$$

Indeed, $\mathsf{H}_{\mathrm{s}}\hat\xi=2x^{-1}\sum K_{ij}\underline{\hat\zeta}_i\underline{\hat\zeta}_j+F$ with F smooth, so $\mathsf{H}_{\mathrm{s}}\hat\xi\geq -C_0$ over the compact set K, hence

$$\hat\xi\circ\tilde\gamma_n(s)\geq\hat\xi(p_n)-C_0s. \tag{3.25}$$

On the other hand, on ${}^{\mathrm{es}}\Sigma$,

$$\underline{\hat\xi}^2=1-h(y,\underline{\hat\eta})-\sum K_{ij}(y,z)\underline{\hat\zeta}_i\underline{\hat\zeta}_j+xG\leq 1-h(y,\underline{\hat\eta})+C_1x,$$

hence on ${}^{\mathrm{eb}}\Sigma$,

$$\hat\xi^2\leq 1-h(y,\hat\eta)+C_1x.$$

Let

$$\Phi(x,y,\hat\eta^{\mathrm{b}})=\sqrt{1-h(y,\hat\eta^{\mathrm{b}})+C_1x};$$

this is thus a Lipschitz function on a neighborhood of α in ${}^{\mathrm{b}}S^*M_0$, hence there is $s_0'>0$ such that $\Phi\circ\gamma_n|_{[0,s_0']}$ is uniformly Lipschitz for n sufficiently large. Thus,

$$\begin{aligned}|\hat\xi(\tilde\gamma_n(s))|&\leq\Phi(\gamma_n(s))\leq\Phi(\alpha)+|\Phi(p_n)-\Phi(\alpha)|+|\Phi(\gamma_n(s))-\Phi(p_n)|\\&\leq\sqrt{1-h(y_0,\hat\eta_0)}+|\Phi(p_n)-\Phi(\alpha)|+C's.\end{aligned}$$

Thus, for sufficiently large n (so that p_n is close to p),

$$|\hat{\xi}(\tilde{\gamma}_n(s))| \leq \sqrt{1 - h(y_0, \hat{\eta}_0)} + \epsilon_1/2 + C's. \tag{3.26}$$

Combining (3.25) and (3.26) gives (3.24).

Now consider the function

$$\Theta = 1 - h(y, \hat{\eta}) - \hat{\xi}^2,$$

so $\pi^*_{\mathrm{s}\to\mathrm{b}}\Theta|_{{}^{\mathrm{s}}\Sigma_0 \cap S^* M_0} = \sum K_{ij}\underline{\hat{\zeta}}_i\underline{\hat{\zeta}}_j$. This satisfies

$$\mathsf{H}_{\mathrm{s}}\Theta = -2\hat{\xi}\mathsf{H}_{\mathrm{s}}\hat{\xi} + F_1 = -4x^{-1}\hat{\xi}\sum K_{ij}\underline{\hat{\zeta}}_i\underline{\hat{\zeta}}_j + F_2 = -4x^{-1}\hat{\xi}\Theta + F_3$$

with F_j smooth. Now,

$$x(p_n) + (\mu - \epsilon_1)s \leq x \circ \gamma_n(s) \leq x(p_n) + (\mu + \epsilon_1)s,$$

so

$$\frac{d}{ds}\Theta \circ \gamma_n + 4x^{-1}\hat{\xi}\Theta \leq C$$

implies that

$$\frac{d}{ds}\Theta_n + \frac{4(\mu - \epsilon_1)}{x(p_n) + (\mu + \epsilon_1)s}\Theta_n \leq C,$$

where we write $\Theta_n = \Theta \circ \gamma_n$. Multiplying through by

$$(x(p_n) + (\mu + \epsilon_1)s)^{4(\mu - \epsilon_1)/(\mu + \epsilon_1)}$$

gives

$$\begin{aligned}\frac{d}{ds}\big((x(p_n) + (\mu + \epsilon_1)s)^{4(\mu-\epsilon_1)/(\mu+\epsilon_1)}\Theta_n\big)& \\ \leq C(x(p_n) + (\mu + \epsilon_1)s)^{4(\mu-\epsilon_1)/(\mu+\epsilon_1)}&.\end{aligned} \tag{3.27}$$

Integration gives

(3.28)

$$\begin{aligned}&(x(p_n) + (\mu + \epsilon_1)s)^{4(\mu-\epsilon_1)/(\mu+\epsilon_1)}\Theta_n(s) - x(p_n)^{4(\mu-\epsilon_1)/(\mu+\epsilon_1)}\Theta_n(0) \\ &\quad\leq C'\big((x(p_n) + (\mu + \epsilon_1)s)^{1+4(\mu-\epsilon_1)/(\mu+\epsilon_1)} - x(p_n)^{1+4(\mu-\epsilon_1)/(\mu+\epsilon_1)}\big).\end{aligned}$$

Thus,

$$\begin{aligned}\Theta_n(s) \leq{}& \big(1 + (\mu + \epsilon_1)s/x(p_n)\big)^{-4(\mu-\epsilon_1)/(\mu+\epsilon_1)}\Theta_n(0) \\ &+ C'\big((x(p_n) + (\mu + \epsilon_1)s) - x(p_n)(1 + (\mu + \epsilon_1)s/x(p_n)\big)^{-4(\mu-\epsilon_1)/(\mu+\epsilon_1)}\big).\end{aligned} \tag{3.29}$$

Since

$$\big(1 + (\mu + \epsilon_1)s/x(p_n)\big)^{-4(\mu-\epsilon_1)/(\mu+\epsilon_1)} < 1,$$

this yields

$$\Theta_n(s) \leq \Theta_n(0) + C'(x(p_n) + (\mu + \epsilon_1)s) \tag{3.30}$$

On the other hand, as on ${}^{\mathrm{s}}\Sigma_0$, $\Theta = \sum K_{ij}\underline{\hat{\zeta}}_i\underline{\hat{\zeta}}_j + xF$, with F smooth, so $\Theta \geq -Cx$, we deduce that

$$\Theta_n(s) \geq -C(x(p_n) + (\mu + \epsilon_1)s).$$

Thus,

$$-C(x(p_n)+(\mu+\epsilon_1)s)\leq\Theta_n(s)\leq\Theta_n(0)+C'(x(p_n)+(\mu+\epsilon_1)s).$$

Suppose now that $\epsilon>0$ is given. As $p_n\to p$, there is an N such that for $n\geq N$, $Cx(p_n)+\Theta_n(0), C'x(p_n)\leq\epsilon/2$. Moreover, let $s_0>0$ such that $C(\mu+\epsilon_1)s_0, C'(\mu+\epsilon_1)s_0<\epsilon/2$. Then for $n\geq N$, $s\in[0,s_0]$, $-\epsilon\leq\Theta_n(s)-\Theta_n(0)\leq\epsilon$, giving the equicontinuity of Θ_n at 0 for $n\geq N$. In view of the definition of Θ_n and the already known equicontinuity of $y\circ\tilde\gamma_n$ and $\hat\eta\circ\tilde\gamma_n$, it follows that $(\xi\circ\tilde\gamma_n)^2$, hence $\xi\circ\tilde\gamma_n$ are equicontinuous. As on ${}^{\mathrm{es}}\Sigma$, $|\underline{\hat\zeta}|^2\leq C|\Theta|+C'x$, we also have $|\hat\zeta|^2\leq C|\Theta|+C'x$ there, so

$$|\hat\zeta\circ\tilde\gamma_n(s)-\hat\zeta(p_n)|\leq|\hat\zeta(p_n)|+|\hat\zeta\circ\tilde\gamma_n(s)|\leq|\hat\zeta(p_n)|+C|\Theta_n(s)|+C'x(\tilde\gamma_n(s)).$$

Given $\epsilon>0$, by the equicontinuity of Θ_n and $x\circ\tilde\gamma_n$, there is s_0 such that for $s\in[0,s_0]$, $C|\Theta_n(s)|+C'x(\tilde\gamma_n(s))<\epsilon/2$. As $\hat\zeta(p_n)\to 0$ due to $p_n\to p$, for n sufficiently large, $|\hat\zeta(p_n)|<\epsilon/2$, so for n sufficiently large and $s\in[0,s_0]$, $|\hat\zeta\circ\tilde\gamma_n(s)-\hat\zeta(p_n)|\leq\epsilon$, giving the equicontinuity of $\hat\zeta\circ\tilde\gamma_n$ at 0.

It remains to check the equicontinuity of $Z_n=z\circ\tilde\gamma_n$. But

$$\left|\frac{dZ_n}{ds}\right|\leq C\sup\{x(q)^{-1}|\underline{\hat\zeta}(q)|:\ q\in{}^{\mathrm{s}}\Sigma_0,\ \pi_{\mathrm{s}\to\mathrm{b}}(q)=\gamma_n(s)\},$$

and for such q, by (3.29),

$$\begin{aligned}x^{-2}|\underline{\hat\zeta}|^2&\leq Cx^{-2}(|\Theta|+x)\\&\leq C(x(p_n)+(\mu-\epsilon_1)s)^{-2}\big(1+(\mu+\epsilon_1)s/x(p_n)\big)^{-4(\mu-\epsilon_1)/(\mu+\epsilon_1)}\Theta_n(0)\\&\qquad+C(x(p_n)+(\mu+\epsilon_1)s),\end{aligned}$$

so

$$\begin{aligned}&x^{-1}|\underline{\hat\zeta}|\\&\leq C(x(p_n)+(\mu-\epsilon_1)s)^{-1}\big(1+(\mu+\epsilon_1)s/x(p_n)\big)^{-2(\mu-\epsilon_1)/(\mu+\epsilon_1)}\Theta_n(0)^{1/2}\\&\qquad+C\sqrt{x(p_n)+(\mu+\epsilon_1)s}\\&\leq Cx(p_n)^{-1}\big(1+(\mu-\epsilon_1)s/x(p_n)\big)^{-1-2(\mu-\epsilon_1)/(\mu+\epsilon_1)}\Theta_n(0)^{1/2}\\&\qquad+C\sqrt{x(p_n)+(\mu+\epsilon_1)s}.\end{aligned}$$

Thus, integrating the right hand side shows that

$$\begin{aligned}|Z_n(s)-Z_n(0)|&\leq C'\Theta_n(0)^{1/2}\Big(\big(1+(\mu-\epsilon_1)s/x(p_n)\big)^{-2(\mu-\epsilon_1)/(\mu+\epsilon_1)}-1\Big)\\&\qquad+C's\sqrt{x(p_n)+(\mu+\epsilon_1)s}\\&\leq C'\Theta_n(0)^{1/2}+C's\sqrt{x(p_n)+(\mu+\epsilon_1)s}.\end{aligned}$$

An argument as above gives the desired equicontinuity for n sufficiently large, completing the proof of the lemma. ◻

Corollary 3.18. *Suppose that $\alpha \in \mathcal{H}_{W,\mathrm{b}}$, $p \in \mathcal{R}_{\mathrm{eb},O,\alpha}$, $p_n \in {}^{\mathrm{eb}}S^*_{M\setminus\tilde{W}}M$, and $p_n \to p$ in ${}^{\mathrm{eb}}S^*M$. Let $\gamma_n : [0,\delta_0] \to {}^{\mathrm{b}}S^*M_0$ be GBB such that $\gamma_n(0) = p_n$. Then there is a GBB $\gamma : [0,\delta_0] \to {}^{\mathrm{b}}S^*M_0$ and γ_n has a subsequence, $\{\gamma_{n_k}\}$, such that $\gamma_{n_k} \to \gamma$ uniformly, the lift $\tilde{\gamma} : [0,\delta_0] \to {}^{\mathrm{eb}}S^*M$ of γ satisfies $\tilde{\gamma}(0) = p$, and the lift $\tilde{\gamma}_{n_k}$ of γ_{n_k} converges to $\tilde{\gamma}$ uniformly.*

Proof. As $p_n \to p$, it follows that there is a compact set $K_0 \subset M_0$ such that $\gamma_n(s) \in {}^{\mathrm{b}}S^*_{K_0}M_0$ for all n and all $s \in [0,\delta_0]$. Then by the compactness of the set of GBB's with image in ${}^{\mathrm{b}}S^*_K M_0$ in the topology of uniform convergence, [14, Proposition 6], γ_n has a subsequence, γ_{n_k}, uniformly converging to a GBB $\gamma : [0,\delta_0] \to {}^{\mathrm{b}}S^*M_0$. In particular, $\gamma(0) = \lim_k \gamma_{n_k}(0) = \lim_k \varpi_{\mathrm{eb}}(p_{n_k}) = \varpi_{\mathrm{eb}}(p) = \alpha$. By Lemma 3.13, γ lifts to a curve $\tilde{\gamma} : [0,\delta_0] \to {}^{\mathrm{eb}}S^*M$. We claim that $\tilde{\gamma}(0) = p$—once we show this, the corollary is proved.

Let $\tilde{\gamma}_n : [0,\delta_0] \to {}^{\mathrm{eb}}S^*M$ be the lift of γ_n. By Lemma 3.16, $\{\tilde{\gamma}_{n_k}\}_{k\in\mathbb{N}}$ is equicontinuous. Since for $\delta > 0$ $\gamma_{n_k}|_{[\delta,\delta_0]} \to \gamma$ uniformly, and these curves all have images in ${}^{\mathrm{b}}S^*_{K_1}M_0$ for some K_1 compact, disjoint from W, where ${}^{\mathrm{b}}S^*_{K_1}M_0$ and ${}^{\mathrm{eb}}S^*_{\beta^{-1}(K_1)}M$ are canonically diffeomorphic, we deduce that $\tilde{\gamma}_{n_k}|_{[\delta,\delta_0]} \to \tilde{\gamma}|_{[\delta,\delta_0]}$ uniformly; in particular $\{\tilde{\gamma}_{n_k}|_{[\delta,\delta_0]}\}$ is a Cauchy sequence in the uniform topology.

Let d be a metric on ${}^{\mathrm{eb}}S^*M$ giving rise to its topology. Given $\epsilon > 0$ let $\delta > 0$ be such that for $0 \le s \le \delta$ and for all n, one has $d(\tilde{\gamma}_n(s), \tilde{\gamma}_n(0)) = d(\tilde{\gamma}_n(s), p_n) < \epsilon/3$—this δ exists by equicontinuity. Next, let N be such that for $k,m \ge N$, $d(p_{n_k}, p_{n_m}) < \epsilon/3$ and for $k,m \ge N$, $\delta \le s \le \delta_0$, $d(\tilde{\gamma}_{n_k}(s), \tilde{\gamma}_{n_m}(s)) \le \epsilon/3$; such a choice of N exists by the uniform Cauchy statement above, and the convergence of $\{p_n\}$. Thus, for $k,m \ge N$ and $0 \le s \le \delta$,

$$d(\tilde{\gamma}_{n_k}(s), \tilde{\gamma}_{n_m}(s)) \le d(\tilde{\gamma}_{n_k}(s), p_{n_k}) + d(p_{n_k}, p_{n_m}) + d(p_{n_m}, \tilde{\gamma}_{n_m}(s)) \le \epsilon.$$

Since we already know the analogous claim for $\delta \le s \le \delta_0$, it follows that $\{\tilde{\gamma}_{n_k}\}$ is uniformly Cauchy, hence converges uniformly to a continuous map $\hat{\gamma} : [0,\delta_0] \to {}^{\mathrm{eb}}S^*M$. In particular, $\hat{\gamma}(0) = \lim_k \tilde{\gamma}_{n_k}(0) = \lim_k p_{n_k} = p$. But $\tilde{\gamma}_{n_k}|_{[\delta,\delta_0]} \to \tilde{\gamma}|_{[\delta,\delta_0]}$ uniformly for $\delta > 0$, so $\tilde{\gamma}|_{[\delta,\delta_0]} = \hat{\gamma}|_{[\delta,\delta_0]}$. The continuity of both $\tilde{\gamma}$ and $\hat{\gamma}$ now shows that $\tilde{\gamma} = \hat{\gamma}$, and in particular $\tilde{\gamma}(0) = p$ as claimed. $\square$

Now we are ready to introduce the bicharacteristics that turn out in general to carry full-strength, rather than weaker, diffracted, singularities.

Definition 3.19. A *geometric* GBB is a GBB $\gamma : (-s_0, s_0) \to {}^{\mathrm{b}}\Sigma_0$ with $q = \gamma(0) \in \mathcal{H}_{W,\mathrm{b}}$ such that there is an EGBB $\rho : \mathbb{R} \to {}^{\mathrm{eb}}T^*_{\tilde{W}}M$ with

$$\lim_{s\to-\infty} \rho(s) = \lim_{t\to 0^-} \tilde{\gamma}_-(t),$$
$$\lim_{s\to+\infty} \rho(s) = \lim_{t\to 0^+} \tilde{\gamma}_+(t),$$

with $\tilde{\gamma}_+$, resp. $\tilde{\gamma}_-$, denoting the lifts $\gamma|_{[0,\delta_0]}$, resp. $\gamma|_{[-\delta_0,0]}$, $\delta_0 > 0$ sufficiently small, to ${}^{\mathrm{eb}}S^*M$.

We say that two points $w, w' \in {}^{\mathrm{b}}\Sigma_0$ are *geometrically related* if they lie along a single geometric GBB .

Let T be a large parameter, fixed for the duration of the this paper.

Definition 3.20. For $p \in \mathcal{H}_{W,\mathrm{b}}$ the flowout of p, denoted $\mathcal{F}^W_{O,p}$, is the union of images $\gamma((0,T])$ of GBB's $\gamma : [0,T] \to {}^{\mathrm{b}}\Sigma_0$ with $\gamma(0) = p$.

For $p \in \mathcal{H}_{W,\mathrm{b}}$, the *regular part* of the flowout of p, denoted $\mathcal{F}^W_{O,p,\mathrm{reg}}$, is the union of images $\gamma((0,s_0))$ of normally approaching (or *regular*) GBB's $\gamma : [0,s_0) \to {}^{\mathrm{b}}\Sigma_0$ with $\gamma(0) = p$ and $\gamma(s) \in T^*M^\circ$ for $s \in (0,s_0)$.

The regular part of the flowout of a subset of $\mathcal{H}_{W,\mathrm{b}}$ is the union of the regular parts of the flowouts from the points in the set.

We let
$$\mathcal{F}^W_{O,p,\mathrm{sing}}$$
denote the union of images $\gamma(0,T]$ of *non*-normally-approaching GBB's γ, i.e. those GBB's γ with $\gamma(0) \in \mathcal{G} \cap \mathcal{R}_{\mathrm{eb}}$.

Incoming flowout and its regular part are defined correspondingly and denoted
$$\mathcal{F}^W_{I,p}, \mathcal{F}^W_{I,p,\mathrm{reg}}.$$
We let $\mathcal{F}^W_{I/O}$ denote the union of the flowouts of all $p \in \mathcal{H}_{W,\mathrm{b}}$.

We also need to define the flowout into or out of a single hyperbolic point $q \in \mathcal{R}_{\mathrm{eb},\alpha,I/O}$ (i.e. for $p \in \mathcal{H}_{W,\mathrm{b}}$ as above, we will consider the flow in/out to a single point in a fiber $q \in \mathcal{R}_{\mathrm{eb},p,I/O}$). By Remark 3.15, given such a q, there is a unique EGBB $\gamma(s)$ with $\lim_{s\to-\infty} \gamma(s) = q$.

Definition 3.21. For $q \in \mathcal{R}_{\mathrm{eb},I/O} \backslash {}^{\mathrm{eb}}S^*_{\partial\tilde{W}}M$, let $\mathcal{F}_{I/O,q}$ denote the image $\gamma((0,T])$ where γ is the unique EGBB with $\lim_{s\to\pm\infty} \gamma(s) = q$. Let $\mathcal{F}_{I/O,q,\mathrm{reg}}$ be defined as the union of $\gamma((0,s_0))$ with $\gamma(s) \in T^*M^\circ$ for all $s \in (0,s_0)$. Additionally, let $\mathcal{F}_{I/O}$ denote the union of all flowouts of $q \in \mathcal{R}_{\mathrm{eb},I/O} \backslash {}^{\mathrm{eb}}S^*_{\partial\tilde{W}}M$, and let $\mathcal{F} = \mathcal{F}_I \cup \mathcal{F}_O$.

One needs some control over the intervals on which normally approaching GBB do not hit the boundary of M:

Lemma 3.22. *Suppose $K \subset \tilde{W}^\circ$ is compact, $\mathcal{K} \subset \mathcal{H}_{W,\mathrm{b}}$ is compact, $\epsilon_0 > 0$. Then there is $\delta_0 > 0$ such that if $\gamma : [0,\epsilon_0] \to {}^{\mathrm{b}}S^*M_0$ a GBB with lift $\tilde{\gamma}$, $\tilde{\gamma}(0) \in \mathcal{R}_{\mathrm{eb},\alpha,O} \cap {}^{\mathrm{eb}}S^*_K M$ for some $\alpha \in \mathcal{K}$, then $\tilde{\gamma}((0,\delta_0)) \cap {}^{\mathrm{eb}}S^*_{\partial M}M = \emptyset$.*

Remark 3.23. Another proof could be given that uses the description of the edge bicharacteristics in [25], since the GBB covered are normally incident.

Proof. First, by Lemma 3.5 there is a $\delta_0' > 0$ such that any GBB γ with $\gamma(0) \in \mathcal{K}$ satisfies $\gamma|_{(0,\delta_0']}$ disjoint from ${}^{\mathrm{b}}S^*_W M_0$.

Suppose now that there is no $\delta_0 > 0$ as claimed. Then there exist GBB's $\gamma_j : [0,\epsilon_0] \to {}^{\mathrm{b}}S^*M_0$, $p_j' \in {}^{\mathrm{eb}}S^*_K M \cap \mathcal{R}_{\mathrm{eb},\alpha_j,O}$, $\alpha_j \in \mathcal{K}$, and $\delta_j > 0$, $\delta_j \to 0$, such that $\gamma_j(\delta_j) \in {}^{\mathrm{b}}S^*_{\partial M_0} M_0$, and the lift $\tilde{\gamma}_j$ of γ_j satisfies $\tilde{\gamma}_j(0) = p_j'$. We may assume that $\delta_j < \epsilon_0/2$ and $\delta_j < \delta_0'$ for all j, hence $\gamma_j(\delta_j) \notin {}^{\mathrm{b}}S^*_W M_0$. By passing to a subsequence, using the compactness of $\mathcal{K}$ and of K, hence of ${}^{\mathrm{eb}}S^*_K M \cap \mathcal{R}_{\mathrm{eb},\mathcal{K},O}$, we may assume that $\{\alpha_j\}$ converges to some $\alpha \in \mathcal{K}$, and $\{p'_j\}$ converges to some $p \in {}^{\mathrm{eb}}S^*_K M \cap \mathcal{R}_{\mathrm{eb},\alpha,O}$. Using the continuity of $\tilde{\gamma}_j$ for each j, we may then choose some $0 < \epsilon_j < \delta_j$ such that $p_j = \tilde{\gamma}_j(\epsilon_j) \to p$ as well; note that $p_j \notin {}^{\mathrm{eb}}S^*_{\tilde{W}} M$. (We introduce ϵ_j to shift the argument of γ_j by ϵ_j, namely to ensure that $\gamma_j(.+\epsilon_j)$ at $s = 0$ is outside ${}^{\mathrm{b}}S^*_W M$, so Corollary 3.18 is applicable.) Thus, we can apply Corollary 3.18 to conclude that $\gamma_j(.+\epsilon_j) : [0, \epsilon_0/2] \to {}^{\mathrm{b}}S^* M_0$ has a subsequence γ_{n_j} such that $\gamma_{n_j}(.+\epsilon_{n_j})$ converges uniformly to a GBB γ, the lifts $\tilde{\gamma}_{n_j}(.+\epsilon_{n_j})$ also converge uniformly to the lift $\tilde{\gamma}$, and $\tilde{\gamma}(0) = p$. Thus, $\tilde{\gamma}_{n_j}((\delta_{n_j} - \epsilon_{n_j}) + \epsilon_{n_j}) \to \tilde{\gamma}(0) = p$ since $\delta_{n_j} - \epsilon_{n_j} \to 0$. As $\tilde{\gamma}_{n_j}((\delta_{n_j} - \epsilon_{n_j}) + \epsilon_{n_j}) \in {}^{\mathrm{eb}}S^*_{\partial M \setminus \tilde{W}} M$ and ${}^{\mathrm{eb}}S^*_{\partial M \setminus \tilde{W}^\circ} M$ is closed, it follows that $p \in {}^{\mathrm{eb}}S^*_{\partial M \setminus \tilde{W}^\circ} M$, contradicting $p \in {}^{\mathrm{eb}}S^*_K M$. This proves the lemma. $\square$

Corollary 3.24. *Suppose $U \subset \tilde{W}^\circ$ is open with $\bar{U} \subset \tilde{W}^\circ$ compact, $\mathcal{U} \subset \mathcal{H}_{W,\mathrm{b}}$ is open with $\bar{\mathcal{U}} \subset \mathcal{H}_{W,\mathrm{b}}$ compact. Then there is $\delta_0 > 0$ such that the set O of points $p \in {}^{\mathrm{eb}}S^* M$ for which there is a GBB γ with lift $\tilde{\gamma}$ such that $\tilde{\gamma}(0) \in {}^{\mathrm{eb}}S^*_U M \cap \mathcal{R}_{\mathrm{eb},\mathcal{U},O}$ and $\gamma(s) = p$ for some $s \in [0, \delta_0)$ is a $\mathcal{C}^\infty$ coisotropic submanifold of ${}^{\mathrm{eb}}S^* M$ transversal to ${}^{\mathrm{eb}}S^*_{\tilde{W}} M$.*

Proof. By Lemma 3.22, with $K = \bar{U}$, $\mathcal{K} = \bar{\mathcal{U}}$, there is a $\delta_0 > 0$ as in the lemma, hence the set O consists of points p for which the GBB γ only meet ∂M at $s = 0$, so (taking into account part (2) of Lemma 3.12 as well) O is a subset of the edge flow-out studied in [25] (e.g. by extending the edge metric g smoothly across the boundary hypersurfaces other than $\tilde{W}$). In particular, the properties of the flowout of such an open subset being $\mathcal{C}^\infty$, coisotropic[7] and transversal to ${}^{\mathrm{eb}}S^*_{\tilde{W}} M$ follow from Theorem 4.1 of [25]. $\square$

We now turn to properties of the singular flowout.

Lemma 3.25. *The singular flowout, $\mathcal{F}_{\mathrm{sing}}$, is closed in ${}^{\mathrm{eb}}S^*_{M \setminus \tilde{W}} M$.*

Proof. Suppose $p_n \in \mathcal{F}_{\mathrm{sing}}$, and let γ_n be such that the lift $\tilde{\gamma}_n$ of γ_n satisfies $\tilde{\gamma}_n(0) \in \mathcal{G} \cap \mathcal{R}_{\mathrm{eb}}$, and $\gamma_n(s_n) = p_n$, $s_n \in (0, T]$. Suppose that $p_n \to p \in {}^{\mathrm{eb}}S^*_{M \setminus \tilde{W}} M$. Then there exists a compact subset K of M such that $\gamma_n(s) \in {}^{\mathrm{b}}S^*_K M$ for all n and all $s \in [0, T]$. By passing to a subsequence we may assume that $s_n \to s$; as $p \notin {}^{\mathrm{eb}}S^*_{\tilde{W}} M$, $s \neq 0$. By passing to yet another subsequence we may also assume that $\gamma_n(0) \to q \in \mathcal{G} \cap \mathcal{R}_{\mathrm{eb}}$. Let $\epsilon_n > 0$,

[7] In [25], being coisotropic is considered as a property of submanifolds of a symplectic manifold, ${}^{\mathrm{e}}T^* M \setminus o$, M being an edge manifold. Conic submanifolds of ${}^{\mathrm{e}}T^* M \setminus o$ can be identified with submanifolds of ${}^{\mathrm{e}}S^* M$, and conversely, thus one can talk about submanifolds of ${}^{\mathrm{e}}S^* M$ being coisotropic. Alternatively, this notion could be defined using the *contact structure* of ${}^{\mathrm{e}}S^* M$, but for the sake of simplicity, and due to the role of symplectic structures in classical microlocal analysis, we did not follow this route in [25], necessitating making the connection via homogeneity here.

$\epsilon_n \to 0$, so $\gamma_n(\epsilon_n) \notin {}^{\mathrm{eb}}S^*_{\tilde{W}}M$ and $\gamma_n(\epsilon_n) \to q$. By Corollary 3.18 we conclude that $\gamma_n(.+\epsilon_n) : [0,T] \to {}^{\mathrm{b}}S^*M_0$ has a subsequence γ_{n_j} such that $\gamma_{n_j}(.+\epsilon_{n_j})$ converges uniformly to a GBB γ, the lifts $\tilde{\gamma}_{n_j}(.+\epsilon_{n_j})$ also converge uniformly to the lift $\tilde{\gamma}$, and $\tilde{\gamma}(0) = q$. In particular, as $\gamma_{n_j}((s_{n_j}-\epsilon_{n_j})+\epsilon_{n_j}) = \gamma_{n_j}(s_{n_j}) = p_{n_j} \to p$, and $s_{n_j} - \epsilon_{n_j} \to s$, $\gamma(s) = p$, so $p \in \mathcal{F}_{\mathrm{sing}}$ as claimed. □

Lemma 3.26. *Suppose $K \subset \tilde{W}^\circ$ is compact, $\mathcal{K} \subset \mathcal{H}_{W,\mathrm{b}}$ is compact. Then K has a neighborhood U in M and there is $\epsilon_0 > 0$ such that if $\gamma : [0,\epsilon_0] \to {}^{\mathrm{b}}S^*M_0$ is a GBB with lift $\tilde{\gamma}$, $\tilde{\gamma}(0) \in \mathcal{R}_{\mathrm{eb},O} \cap \mathcal{G}$, $\gamma(0) \in \mathcal{K}$ then $\tilde{\gamma}(s) \notin {}^{\mathrm{eb}}S^*_U M$ for $s \in (0,\epsilon_0]$.*

Proof. Let $\epsilon_0 > 0$ be such that any GBB γ with $\gamma(0) \in \mathcal{K}$ satisfies $\gamma|_{(0,\epsilon_0]}$ disjoint from ${}^{\mathrm{b}}S^*_W M_0$; such ϵ_0 exists by Lemma 3.5.

Now suppose that no U exists as stated. Then there exist GBB γ_n and $s_n \in (0,\epsilon_0]$ such that the lifts $\tilde{\gamma}_n$ of γ_n satisfies $\tilde{\gamma}_n(0) \in \mathcal{R}_{\mathrm{eb},O} \cap \mathcal{G}$, $\gamma_n(0) \in \mathcal{K}$ and $\pi(\tilde{\gamma}_n(s_n)) \to q$, $q \in K$, where $\pi : {}^{\mathrm{eb}}S^*M \to M$ is the bundle projection.

By the compactness of $\mathcal{K}$ and the compactness of $\cup_{\alpha \in \mathcal{K}} \mathcal{R}_{\mathrm{eb},O,\alpha} \cap \mathcal{G}$ we may pass to a subsequence (which we do not indicate in notation) such that $\gamma_n(0)$ converges to some $\alpha \in \mathcal{K}$ and $\tilde{\gamma}_n(0)$ converges to some $p \in \mathcal{R}_{\mathrm{eb},O} \cap \mathcal{G}$. We may further pass to a subsequence such that $s_n \to s_0 \in [0,\epsilon_0]$, and still further (taking into account the compactness of the fibers of ${}^{\mathrm{eb}}S^*M \to M$) that $\tilde{\gamma}_n(s_n) \to \tilde{p} \in {}^{\mathrm{eb}}S^*_K M$. Choose[8] $\epsilon_n \in (0,s_n)$ sufficiently small such that $\epsilon_n \to 0$ and $\gamma_n(\epsilon_n) \to p$. By Corollary 3.18 $\gamma_n(.+\epsilon_n)$ has a convergent subsequence γ_{n_k} such that $\gamma_{n_k}(.+\epsilon_{n_k})$ converge uniformly to a GBB γ and the lifts $\tilde{\gamma}_{n_k}(.+\epsilon_{n_k})$ converge uniformly to the lift $\tilde{\gamma}$ and $\tilde{\gamma}(0) = p$. Thus, $\tilde{\gamma}_{n_k}(s_{n_k}+\epsilon_{n_k}) \to \tilde{\gamma}(s_0)$, so $\tilde{\gamma}(s_0) = \tilde{p} \in {}^{\mathrm{eb}}S^*_K M$. But by the definition of ϵ_0, $\gamma(s_0) \notin {}^{\mathrm{b}}S^*_W M_0$ if $s_0 > 0$, while $s_0 = 0$ is impossible as $\tilde{\gamma}(0) = p \in {}^{\mathrm{eb}}S^*_{\partial\tilde{W}}M$, while $K \subset \tilde{W}^\circ$. This contradiction shows that the claimed U exists, proving the lemma. □

Corollary 3.27. *Suppose $K \subset \tilde{W}^\circ$ is compact, $\mathcal{K} \subset \mathcal{H}_{W,\mathrm{b}}$ is compact. Then K has a neighborhood U in M and there is $\epsilon_0 > 0$ such that if $o \in U \setminus \tilde{W}$ and γ is a GBB with $\gamma(0) \in {}^{\mathrm{b}}S^*_o M_0$ then for $s \in [-\epsilon_0, 0]$, $\gamma(s) \in \mathcal{K}$ implies γ is normally incident.*

In particular, if $q \in \tilde{W}^\circ$, $\alpha \in \mathcal{H}_{W,\mathrm{b}}$ and γ_0 is a GBB with $\gamma_0(0) = \alpha$ and lift $\tilde{\gamma}_0(0) \in {}^{\mathrm{eb}}S^*_q M$ then there is $\delta_0 > 0$ such that $s \in (0,\delta_0]$, $\gamma_0(s) \in {}^{\mathrm{b}}S^*_o M_0$ implies that every GBB γ with $\gamma(0) \in {}^{\mathrm{b}}S^*_o M_0$, $\gamma(s) = \alpha$, $s \in [-\epsilon_0, 0]$, is normally incident.

Proof. Let U and ϵ_0 be as in Lemma 3.26. If $o \in U$, γ is a GBB with $\gamma(0) \in {}^{\mathrm{b}}S^*_o M_0$, $s_0 \in [-\epsilon_0, 0]$, $\gamma(s_0) \in \mathcal{K}$ and γ is not normally incident, then the lift $\tilde{\gamma}$ of γ satisfies $\tilde{\gamma}(s_0) \in \mathcal{R}_{\mathrm{eb},O} \cap \mathcal{G}$ by Lemma 3.13. Thus, with $\gamma_0(s) = \gamma(s - s_0)$, so $\tilde{\gamma}_0(0) \in \mathcal{R}_{\mathrm{eb},O} \cap \mathcal{G}$, $\gamma_0(0) \in \mathcal{K}$, Lemma 3.26 shows that $\tilde{\gamma}_0(s) \notin {}^{\mathrm{eb}}S^*_U M$ for $s \in (0,\epsilon_0]$, contradicting $\tilde{\gamma}_0(-s_0) \in {}^{\mathrm{b}}S^*_o M_0$.

[8]Again, we do this so that Corollary 3.18 is applicable; cf. the proof of Lemma 3.22.

The second half follows by taking $\mathcal{K} = \{\alpha\}$, $K = \{q\}$. □

3.5. **A summary.** The following table summarizes a number of the most useful facts about the bundles that we have introduced above.

Manifold	M_0	M_0	M	M
Bundle	b	s	eb	es
Vector fields	$x_i\partial_{x_i}, \partial_{y_k}$	$\partial_{x_j}, \partial_{y_k}$	$x\partial_x, x\partial_y, z_i'\partial_{z_i'}, \partial_{z_j''}$	$x\partial_x, x\partial_y, \partial_{z_i}$
Dual coords	$\xi_j^{\mathrm{s}}, \eta_j^{\mathrm{s}}$	$\xi_j^{\mathrm{b}}, \eta_j^{\mathrm{b}}$	$\xi, \eta, \zeta', \zeta''$	$\underline{\xi}, \underline{\eta}, \underline{\zeta}$
Char. set	${}^{\mathrm{b}}\Sigma_0$	${}^{\mathrm{s}}\Sigma_0$	${}^{\mathrm{eb}}\Sigma$	${}^{\mathrm{es}}\Sigma$

(We have omitted time coordinates and their duals, as they behave just like y variables, and the notation follows suit.)

We also employ a number of maps among these structures, the most common being:

$$\pi_{\mathrm{s}\to\mathrm{b}} : T^*M_0 \to {}^{\mathrm{b}}T^*M_0,$$
$$\pi_{\mathrm{es}\to\mathrm{eb}} : {}^{\mathrm{es}}T^*M \to {}^{\mathrm{eb}}T^*M,$$
$$\varpi_{\mathrm{es}} : {}^{\mathrm{es}}T^*_{\tilde{W}}M \to T^*W,$$
$$\varpi_{\mathrm{eb}} : {}^{\mathrm{eb}}T^*_{\tilde{W}}M \to T^*W.$$

Recall that hats over maps indicate their restrictions to the relevant characteristic set.

4. Edge-b calculus

Recall from Definition 3.9 that $\mathcal{V}_{\mathrm{eb}}(M)$ is the space of smooth vector fields that are tangent to all of ∂M and tangent to the fibration of $\tilde{W} \subset \partial M$ given by blowdown. Thus, in local coordinates, $\mathcal{V}_{\mathrm{eb}}(M)$ is spanned over $\mathcal{C}^\infty(M)$ by the vector fields

$$x\partial_x,\ x\partial_t,\ x\partial_y,\ z_i'\partial_{z_i'}, \partial_{z''}$$

Definition 4.1. The space $\mathrm{Diff}^*_{\mathrm{eb}}(M)$ is the filtered algebra of operators over $\mathcal{C}^\infty(M)$ generated by $\mathcal{V}_{\mathrm{eb}}(M)$.

Recall also that $\mathcal{V}_{\mathrm{eb}}(M) = \mathcal{C}^\infty(M; {}^{\mathrm{eb}}TM)$, and ${}^{\mathrm{eb}}T^*M$ is the dual bundle of ${}^{\mathrm{eb}}TM$. In Appendix B the corresponding pseudodifferential operators are constructed.

Theorem 4.2. *There exists a pseudodifferential calculus* $\Psi^*_{\mathrm{eb}}(M)$ *microlocalizing* $\mathrm{Diff}^*_{\mathrm{eb}}(M)$.

The double space M^2_{eb} on which the kernels are defined is such that the quotient x/x' of the same boundary defining function on the left or right factor, lifts to be smooth except near the 'old' boundaries at which the kernels are required to vanish to infinite order. It follows that x/x' is a multiplier (and divider) on the space of kernels. This corresponds the action by conjugation of these defining fucntions, so it is possible to define a weighted version of the calculus. Set

$$\Psi^{m,l}_{\mathrm{eb}}(M) = x^{-l}\Psi^m_{\mathrm{eb}}(M).$$

Proposition 4.3. *$\Psi^{*,*}_{\mathrm{eb}}(M)$ is a bi-filtered calculus.*

Now, $\Psi^*_{\mathrm{eb}}(M)$ has all the properties (I–VII) of [25, Section 3], where $\mathcal{V}$ in [25, Section 3] is replaced by eb . Since the multiplier x/x' is identically equal to one on the lifted diagonal, the symbol is unaffected by this conjugation and hence the principal symbol map extends to

$$\sigma_{\mathrm{eb},m,l} : \Psi^{m,l}_{\mathrm{eb}}(M) \to x^{-l}S^m_{\mathrm{hom}}({}^{\mathrm{eb}}T^*M),$$

with the standard short exact sequence – see properties (III–IV). There are edge-b-Sobolev spaces, $H^s_{\mathrm{eb}}(M)$, defined via the elliptic elements of $\Psi^s_{\mathrm{eb}}(M)$, and on which the elements define bounded maps

$$A \in \Psi^m_{\mathrm{eb}}(M) \Longrightarrow A : H^s_{\mathrm{eb}}(M) \to H^{s-m}_{\mathrm{eb}}(M).$$

see property (VII).

The symbol of the commutator of $A \in \Psi^{m,l}_{\mathrm{eb}}(M)$ and $B \in \Psi^{m',l'}_{\mathrm{eb}}(M)$ is given by

$$\sigma_{\mathrm{eb},m+m'-1,l+l'}(\imath[A,B]) = \mathsf{H}_{\mathrm{eb},\sigma_{\mathrm{eb},m,l}(A)}(\sigma_{\mathrm{eb},m',l'}(B)).$$

In local coordinates the edge-b Hamilton vector field becomes

$$\begin{aligned}(4.1)\quad \mathsf{H}_{\mathrm{eb},f} = {}& \frac{\partial f}{\partial \xi}x\partial_x - \big(x\frac{\partial f}{\partial x} + \eta\cdot\frac{\partial f}{\partial \eta}\big)\partial_\xi + x\frac{\partial f}{\partial \eta}\partial_y + \big(\frac{\partial f}{\partial \xi}\eta - x\frac{\partial f}{\partial y}\big)\cdot\partial_\eta \\ &+ \sum\big(\frac{\partial f}{\partial \zeta'_j}z'_j\partial_{z'_j} - z'_j\frac{\partial f}{\partial z'_j}\partial_{\zeta'_j}\big) + \sum\big(\frac{\partial f}{\partial \zeta''_j}\partial_{z''_j} - \frac{\partial f}{\partial z''_j}\partial_{\zeta''_j}\big).\end{aligned}$$

In particular,

$$(4.2)\qquad x^{-k}\mathsf{H}_{\mathrm{eb},x^k a} = ka\partial_\xi + \mathsf{H}_{\mathrm{eb},a}.$$

In the space-time setting, where one of the y variables, t, is distinguished (and we still write y for the rest of the base variables), it is useful to rewrite this using the re-homogenized dual variables $\hat\eta = \eta/|\tau|$, $\hat\xi = \xi/|\tau|$, $\hat\zeta = \zeta/|\tau|$, $\sigma = |\tau|^{-1}$, valid near ${}^{\mathrm{eb}}\Sigma$, this becomes

$$\begin{aligned}(4.3)\quad \sigma^{-1}\mathsf{H}_{\mathrm{eb},f} = {}& \frac{\partial f}{\partial \hat\xi}x\partial_x - \big(x\frac{\partial f}{\partial x} - \sigma\frac{\partial f}{\partial \sigma} - \hat\zeta\cdot\frac{\partial f}{\partial \hat\zeta}\big)\partial_{\hat\xi} + x\frac{\partial f}{\partial \hat\eta}\partial_y \\ &- x\big(\sigma\frac{\partial f}{\partial \sigma} + \hat\eta\cdot\frac{\partial f}{\partial \hat\eta} + \hat\xi\frac{\partial f}{\partial \hat\xi} + \hat\zeta\cdot\frac{\partial f}{\partial \hat\zeta}\big)\partial_t - \frac{\partial f}{\partial \hat\xi}(\sigma\partial_\sigma + \hat\zeta\cdot\partial_{\hat\zeta}) \\ &+ x\frac{\partial f}{\partial t}\cdot(\sigma\partial_\sigma + \hat\eta\cdot\partial_{\hat\eta} + \hat\xi\partial_{\hat\xi} + \hat\zeta\cdot\partial_{\hat\zeta}) - x\frac{\partial f}{\partial y}\cdot\partial_{\hat\eta} \\ &+ \sum\big(\frac{\partial f}{\partial \hat\zeta'_j}z'_j\partial_{z'_j} - z'_j\frac{\partial f}{\partial z'_j}\partial_{\hat\zeta'_j}\big) + \sum\big(\frac{\partial f}{\partial \hat\zeta''_j}\partial_{z''_j} - \frac{\partial f}{\partial z''_j}\partial_{\hat\zeta''_j}\big).\end{aligned}$$

This is tangent to the fibers of $\varpi_{\mathrm{eb}} : {}^{\mathrm{eb}}\Sigma \cap {}^{\mathrm{eb}}S^*_{\tilde{W}}M \to S^*W$, in fact to its natural extension to a neighborhood of ${}^{\mathrm{eb}}\Sigma \cap {}^{\mathrm{eb}}S^*_{\tilde{W}}M$ in ${}^{\mathrm{eb}}S^*_{\tilde{W}}M$, so if $b \in \mathcal{C}^\infty({}^{\mathrm{eb}}S^*_{\tilde{W}}M)$ with $b|_{{}^{\mathrm{eb}}S^*_{\tilde{W}}M}$ constant along the fibers of this extension, then $\sigma^{\mu-1}\mathsf{H}_{\mathrm{eb},f}b \in x\mathcal{C}^\infty({}^{\mathrm{eb}}S^*M)$ for f homogeneous degree μ.

The fact that the operators are defined by kernels which are conormal means that there is an operator wave front set $\mathrm{WF}'_{\mathrm{eb}}$ for the eb-calculus, i.e. for $A \in \Psi^*_{\mathrm{eb}}(M)$, $\mathrm{WF}'_{\mathrm{eb}}(A) \subset {}^{\mathrm{eb}}S^*M$, with the properties (A)-(F) of [25, Section 3], so in particular algebraic operations are microlocal, see properties (A)-(B), and there are microlocal parametrices at points at which the principal symbol is elliptic, see property (E). These parametrices have error terms with which are smooth on the double space, but they are not compact.

As is the case for the b-calculus, $\Psi^*_{\mathrm{eb}}(M)$ has the additional property that the *radial vector fields* V_j for all boundary hypersurfaces $\{\rho_j = 0\}$ other than $\tilde{W}$, $[A, V_j] \in \rho_j\Psi^m_{\mathrm{eb}}(M)$ if $A \in \Psi^m_{\mathrm{eb}}(M)$, i.e. there is a gain of ρ_j over the *a priori* order. This follows from the analysis of the normal operators, see [25, Section 3]. In local coordinates a radial vector field for $z'_j = 0$ is given by $z'_j\,\partial_{z'_j}$; V_j being a radial vector field for $z'_j = 0$ means that $V_j - z'_j\,\partial_{z'_j} \in z_j\mathcal{V}_{\mathrm{eb}}(M)$. This latter requirement can easily be seen to be defined independently of choices of coordinate systems.

5. Differential-pseudodifferential operators

5.1. **The calculus.** We start by defining an algebra of operators which includes $\Box$. First, recall that $\mathcal{V}_{\mathrm{es}}(M)$ is the Lie algebra of vector fields that are tangent to the front face and to the fibers of the blow down map restricted to the front face, $\beta|_{\tilde{W}} : \tilde{W} \to W$ (but are not required to be tangent to other boundary faces). Thus, elements V of $\mathcal{V}_{\mathrm{es}}(M)$ define operators $V : \dot{\mathcal{C}}^\infty(M) \to \dot{\mathcal{C}}^\infty(M)$ and also $V : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$.

Definition 5.1. Let $\mathrm{Diff}_{\mathrm{es}}(M)$ be the filtered algebra of operators (acting either on $\dot{\mathcal{C}}^\infty(M)$ or $\mathcal{C}^\infty(M)$) over $\mathcal{C}^\infty(M)$ generated by $\mathcal{V}_{\mathrm{es}}(M)$.

We also let $\mathrm{Diff}^{k,l}_{\mathrm{es}}(M) = x^{-l}\,\mathrm{Diff}^k_{\mathrm{es}}(M)$; this is an algebra of operators acting on $\dot{\mathcal{C}}^\infty(M)$, and also on the space of functions classical conormal to $\tilde{W}$, $\cup_{s\in\mathbb{R}}x^{-s}\mathcal{C}^\infty(M)$.

Remark 5.2. Note that the possibility of the appearance of boundary terms requires care to be exercised with adjoints, as opposed to formal adjoints. See for instance Lemma 5.18.

We also remark that $\mathrm{Diff}^k_{\mathrm{es}}(M)$, hence $x^{-l}\,\mathrm{Diff}^k_{\mathrm{es}}(M)$, is closed under conjugation by x^{-r} where x is a defining function for W. This follows from the fact that $\mathrm{Diff}^1_{\mathrm{es}}(M)$ is so closed; the key property is that

$$x^r(x\partial_x)x^{-r} = (x\partial_x) - r \in \mathrm{Diff}^1_{\mathrm{es}}(M).$$

We will require, for commutator arguments that involve interaction of singularities with $\partial M\backslash\tilde{W}$, a calculus of mixed differential-pseudodifferential operators, mixing edge-b-pseudodifferential operators with these (more singular) edge-smooth differential operators.

Definition 5.3. Let

$$\mathrm{Diff}^k_{\mathrm{es}}\,\Psi^m_{\mathrm{eb}}(M)=\{\sum A_\alpha B_\beta : A_\alpha\in\mathrm{Diff}^k_{\mathrm{es}}(M),\ B_\beta\in\Psi^m_{\mathrm{eb}}(M)\}$$

Proposition 5.4. $\cup_{k,m}\mathrm{Diff}^k_{\mathrm{es}}\,\Psi^m_{\mathrm{eb}}(M)$ *is a filtered* $\mathcal{C}^\infty(M)$*-module, and an algebra under composition; it is commutative to top* eb*-order, i.e. for* $P\in\mathrm{Diff}^k_{\mathrm{es}}\,\Psi^m_{\mathrm{eb}}(M)$, $Q\in\mathrm{Diff}^{k'}_{\mathrm{es}}\,\Psi^{m'}_{\mathrm{eb}}(M)$,

$$[P,Q]\in\mathrm{Diff}^{k+k'}_{\mathrm{es}}\,\Psi^{m+m'-1}_{\mathrm{eb}}(M).$$

The key is the following lemma.

Lemma 5.5. *If* $A\in\Psi^m_{\mathrm{eb}}(M)$ *and* $Q\in\mathcal{V}_{\mathrm{es}}(M)$*, then*

$$[A,Q]=\sum Q_jA_j+B,\ [A,Q]=\sum A'_jQ'_j+B'_j \tag{5.1}$$

where $B,\ B'\in\Psi^m_{\mathrm{eb}}(M)$, $A_j,\ A'_j\in\Psi^{m-1}_{\mathrm{eb}}(M)$ *and* $Q_j,\ Q'_j\in\mathcal{V}_{\mathrm{es}}(M)$.

Proof. As both $\mathcal{V}_{\mathrm{es}}(M)$ and $\Psi^m_{\mathrm{eb}}(M)$ are $\mathcal{C}^\infty(M)$-modules, we can use a partition of unity, and it suffices to work locally and with a spanning set of vector fields. Since $xD_x,\ xD_{y_j},\ D_{z''_j}\in\mathcal{V}_{\mathrm{eb}}(M)$, the conclusion is automatic for Q chosen from among these vector fields since then $B=[A,Q]\in\Psi^m_{\mathrm{eb}}(M)$. Thus it only remains to consider the $Q=D_{z'_j}$ where z'_j is a defining function for one of the other boundary faces. Then for $\tilde Q=z'_jQ=z'_jD_{z'_j}\in\mathrm{Diff}^1_{\mathrm{eb}}(M)$, $[A,\tilde Q]\in\Psi^m_{\mathrm{eb}}(M))$. The normal operator at $z'_j=0$ satisfies $N_j([A,\tilde Q])=[N_j(A),N_j(\tilde Q)]$, and $N_j(\tilde Q)$ is scalar, and hence commutes with $N_j(A)$. Thus $N_j([A,\tilde Q])=0$, so $[A,\tilde Q]\in z'_j\Psi^m_{\mathrm{eb}}(M)$. Consequently,

$$-[A,Q]=[Q,A]=(z'_j)^{-1}[\tilde Q,A]+([(z'_j)^{-1},A]z'_j)(z'_j)^{-1}\tilde Q, \tag{5.2}$$

with the first term on the right hand side in $\Psi^m_{\mathrm{eb}}(M)$, the second of the form $\tilde AQ$, $\tilde A\in\Psi^{m-1}_{\mathrm{eb}}(M)$. This proves the first half of the lemma. The other part is similar. □

Proof of Proposition 5.4. The algebra properties follow immediately from the lemma. It only remains to verify the leading order commutativity.

As the bracket is a derivation in each argument, it suffices to consider P, Q lying in either $\mathcal{V}_{\mathrm{es}}(M)$ or $\Psi^*_{\mathrm{eb}}(M)$. If both operators are in $\Psi^*_{\mathrm{eb}}(M)$, the result follows from the symbol calculus. If $P,\ Q\in\mathcal{V}_{\mathrm{es}}(M)$, we have $[P,Q]=R\in\mathcal{V}_{\mathrm{es}}(M)$. We need to write R as a sum of elements of $\mathrm{Diff}^2_{\mathrm{es}}(M)$ times elements of $\Psi^{-1}_{\mathrm{eb}}(M)$. To this end, let Λ be an elliptic element of $\Psi^2_{\mathrm{eb}}(M)$ given by a sum of square of vector fields in $\mathcal{V}_{\mathrm{eb}}(M)$, e.g. in local coordinates

$$\Lambda=(x\partial_x)^2+(x\partial_t)^2+\sum(x\partial_y)^2+\sum(z'_i\partial_{z'_i})^2+\sum\partial^2_{z''_j}.$$

We write $\Lambda = \sum V_j^2$ for brevity. Let $\Upsilon \in \Psi_{\mathrm{eb}}^{-2}(M)$ be an elliptic parametrix for Λ. Then we may write

$$\mathrm{Id} = \sum V_j(V_j\Upsilon) + E,$$

with $E \in \Psi_{\mathrm{eb}}^{-\infty}(M)$. Now since $\mathcal{V}_{\mathrm{eb}}(M) \subset \mathcal{V}_{\mathrm{es}}(M)$, we certainly have $V_j \in \mathcal{V}_{\mathrm{es}}(M)$ for each j, hence $RV_j \in \mathrm{Diff}^2_{\mathrm{es}}(M)$. Moreover $V_j\Upsilon \in \Psi_{\mathrm{eb}}^{-1}(M)$. Thus,

$$R = \sum (RV_j)(V_j\Upsilon) + RE,$$

and we have shown that $R \in \mathrm{Diff}^2_{\mathrm{es}}\,\Psi_{\mathrm{eb}}^{-1}(M)$.

Finally, if $P \in \mathcal{V}_{\mathrm{es}}(M)$ and $Q \in \Psi_{\mathrm{eb}}^{m}(M)$ (or vice-versa) then using the lemma (and its notation) we may write

$$[P, Q] = \sum Q_j A_j + B.$$

Using the same method as above to write $B = \sum RV_j(V_j\Upsilon) + BE$ we find that $[P, Q] \in \mathrm{Diff}^1_{\mathrm{es}}\,\Psi_{\mathrm{eb}}^{m-1}(M)$. □

The above proof also yields the following useful consequence

Lemma 5.6. *For all m, $l \in \mathbb{R}$, and $k \in \mathbb{N}$,*

$$\mathrm{Diff}^m_{\mathrm{es}}\,\Psi^l_{\mathrm{eb}}(M) \subset \mathrm{Diff}^{m+k}_{\mathrm{es}}\,\Psi^{l-k}_{\mathrm{eb}}(M).$$

We note the following consequence of (5.2):

Lemma 5.7. *Let $A \in \Psi^m_{\mathrm{eb}}(M)$, $a = \sigma_{\mathrm{eb},m}(A)$. Then*

$$\imath[x^{-1}D_{z'_j}, A] = A_1 x^{-1} D_{z'_j} + x^{-1}A_0$$

where $A_0 \in \Psi^m_{\mathrm{eb}}(M)$, $A_1 \in \Psi^{m-1}_{\mathrm{eb}}(M)$,

$$\sigma_{\mathrm{eb},m}(A_0) = \frac{\partial a}{\partial z'_j}, \quad \sigma_{\mathrm{eb},m-1}(A_1) = \frac{\partial a}{\partial \zeta'_j} + \frac{\partial a}{\partial \xi}.$$

Note that this is exactly what one would expect from computation at the level of edge-b symbols: the Hamilton vector field of $\zeta'_i/(xz'_i)$ is $(\zeta'_i/(xz'_i))(\partial_{\zeta'_i} + \partial_\xi) + x^{-1}\partial_{z'_i}$.

Proof. This follows immediately from writing

$$[x^{-1}D_{z'_j}, A] = [x^{-1}, A]D_{z'_j} + x^{-1}[D_{z'_j}, A].$$

We then use (5.2) together with the following principal symbol calculations in $\Psi^*_{\mathrm{eb}}(M)$, see (4.1):

$$\imath\sigma_{\mathrm{eb},m}([Q', A]) = z'_j \partial_{z'_j} a,$$
$$\imath\sigma_{\mathrm{eb},m-1}([(z'_j)^{-1}, A]z'_j) = \partial_{\zeta'_j} a,$$
$$\imath\sigma_{\mathrm{eb},m-1}([x^{-1}, A]) = x^{-1}\partial_\xi a,$$

as well as $[x^l\Psi^k_{\mathrm{eb}}(M), x^{l'}\Psi^{k'}_{\mathrm{eb}}(M)] \subset x^{l+l'}\Psi^{k+k'-1}_{\mathrm{eb}}(M)$, which allows one to exchange factors after the previous steps without affecting the computed principal symbols. □

We now define the edge-smooth Sobolev spaces. It is with respect to these base spaces that we will measure regularity in proving propagation of edge-b wavefront set.

Definition 5.8. For $s \geq 0$ integer,

$$H_{\mathrm{es}}^{s,l-(f+1)/2}(M) = \{u \in x^l L_g^2(M) : \ A \in \mathrm{Diff}_{\mathrm{es}}^s(M) \Rightarrow Au \in x^l L_g^2(M)\}.$$

The norm in $H_{\mathrm{es}}^{s,l-(f+1)/2}(M)$, up to equivalence, is defined using any finite number of generators A_j for the finitely generated $\mathcal{C}^\infty(M)$-module $\mathrm{Diff}_{\mathrm{es}}^s(M)$ by

$$\|u\|_{H_{\mathrm{es}}^{s,l-(f+1)/2}(M)} = \left(\sum_j \|x^{-l} A_j u\|_{L_g^2(M)}^2\right)^{1/2}.$$

The space $H_{\mathrm{es},0}^{s,l-(f+1)/2}(M)$ is the closure of $\dot{\mathcal{C}}^\infty(M)$ in $H_{\mathrm{es}}^{s,l-(f+1)/2}(M)$.

Remark 5.9. The orders above are chosen so that setting $s = 0$, $l = 0$, we obtain $L_g^2(M) = H_{\mathrm{es}}^{0,-(f+1)/2}(M)$. Thus $x^{(f+1)/2} L_g^2(M) = L^2(M, x^{-(f+1)}\, dg)$ is the L^2-space corresponding to densities that are smooth up to all boundary hypersurfaces of M except $\tilde{W}$, and that are b-densities at the interior of $\tilde{W}$, meaning that $x(x^{-(f+1)}\, dg)$ is actually a smooth non-degenerate density on M. This convention keeps the weights consistent with [25].

Note also that the subspace $\mathcal{C}$ of $\mathcal{C}^\infty(M)$ given by

$$\mathcal{C} = x^\infty \mathcal{C}^\infty(M)$$

is dense in $H_{\mathrm{es}}^{s,l-(f+1)/2}(M)$ for all s and l; one could even require supports disjoint from $\tilde{W}$. Thus, the difference between $H_{\mathrm{es},0}^{s,l-(f+1)/2}(M)$ and $H_{\mathrm{es}}^{s,l-(f+1)/2}(M)$ corresponds to the behavior at the boundary hypersurfaces of M other than $\tilde{W}$, i.e. those arising from the boundary hypersurfaces of M_0, where the boundary conditions are imposed. Thus, this difference is similar to the difference between $H^s(\Omega)$ and $H_0^s(\Omega)$ for domains Ω with smooth boundary in a manifold.

The boundedness of $\Psi_{\mathrm{eb}}^0(M)$ on $H_{\mathrm{es}}^{1,1-(f+1)/2}(M)$ is an immediate consequence of the commutation property in Lemma 5.5.

Theorem 5.10. *$\Psi_{\mathrm{eb}}^0(M)$ is bounded on both $H_{\mathrm{es}}^{1,1-(f+1)/2}(M)$ and on the closed subspace $H_{\mathrm{es},0}^{1,1-(f+1)/2}(M)$.*

Remark 5.11. The more general case of $H_{\mathrm{es}}^{1,l-(f+1)/2}(M)$ with arbitrary l follows from the case of $l = 1$ using $x^{-l} A x^l \in \Psi_{\mathrm{eb}}^0(M)$ for $A \in \Psi_{\mathrm{eb}}^0(M)$.

In fact, reduction to $l = 0$ would make the proof below even more transparent.

The case of $H_{\mathrm{es}}^{s,l-(f+1)/2}(M)$ can be proved similarly, but we do not need this here.

Proof. As $\Psi^0_{\mathrm{eb}}(M) : \dot{\mathcal{C}}^\infty(M) \to \dot{\mathcal{C}}^\infty(M)$, the second statement follows from the first and the definition of $H^{1,1-(f+1)/2}_{\mathrm{es},0}(M)$.

As above, let $\mathcal{C}$ be the subspace of $\mathcal{C}^\infty(M)$ consisting of functions vanishing to infinite order at $\tilde{W}$, which is thus dense in $H^{1,1-(f+1)/2}_{\mathrm{es}}(M)$. Let $A \in \Psi^0_{\mathrm{eb}}(M)$. As $\Psi^0_{\mathrm{eb}}(M) : \mathcal{C} \to \mathcal{C}$, and A is bounded on $L^2_g(M)$, one merely needs to check that for $Q \in \mathrm{Diff}^1_{\mathrm{es}}(M)$ there exists $C > 0$ such that for $u \in \mathcal{C}$,

$$\|x^{-1}QAu\|_{L^2_g} \leq C\|u\|_{H^{1,1-(f+1)/2}_{\mathrm{es}}(M)}.$$

But

$$x^{-1}QAu = ([x^{-1}, A]x)(x^{-1}Qu) + x^{-1}[Q, A]u + A(x^{-1}Qu).$$

By Lemma 5.5, $[Q, A] = \sum A_jQ_j + B$, $B \in \Psi^0_{\mathrm{eb}}(M)$, $A_j \in \Psi^{-1}_{\mathrm{eb}}(M)$, $Q_j \in \mathrm{Diff}^1_{\mathrm{es}}(M)$, hence $x^{-1}[Q, A] = \sum(x^{-1}A_jx)(x^{-1}Q_j) + (x^{-1}Bx)x^{-1}$,

$$\begin{aligned} x^{-1}QAu = ([x^{-1}, A]x)(x^{-1}Qu) &+ \sum(x^{-1}A_jx)(x^{-1}Q_ju) \\ &+ (x^{-1}Bx)(x^{-1}u) + A(x^{-1}Qu), \end{aligned}$$

so the desired conclusion follows from

$$\|x^{-1}Qu\|_{L^2_g(M)}, \|x^{-1}Q_ju\|_{L^2_g(M)}, \|x^{-1}u\|_{L^2_g(M)} \leq C\|u\|_{H^{1,1-(f+1)/2}_{\mathrm{es}}(M)}, \ u \in \mathcal{C},$$

and additionally $[x^{-1}, A]x, x^{-1}A_jx \in \Psi^{-1}_{\mathrm{eb}}(M) \subset \Psi^0_{\mathrm{eb}}(M)$ (which are thus bounded on $L^2_g(M)$, just as $A, x^{-1}Bx \in \Psi^0_{\mathrm{eb}}(M)$ are). □

We can now define the eb-wave front set relative to a given Hilbert (or even Banach) space, which in practice will be either the Dirichlet form domain or a weighted edge-smooth Sobolev space serving as a stand-in for the Neumann form domain.

Definition 5.12. Let $\mathfrak{X}$ denote a Hilbert space on which operators in $\Psi^0_{\mathrm{eb}}(M)$ are bounded, with the operator norm of $\mathrm{Op}(a)$ depending on a fixed seminorm of a.

Let $q \in {}^{\mathrm{eb}}S^*M$, $u \in \mathfrak{X}$. For $m \geq 0, r \leq 0$, we say that $q \notin \mathrm{WF}^{m,r}_{\mathrm{eb},\mathfrak{X}}(u)$ if there exists $A \in \Psi^{m,r}_{\mathrm{eb}}(M)$ elliptic at q such that $Au \in \mathfrak{X}$. We also define $q \notin \mathrm{WF}^{\infty,r}_{\mathrm{eb},\mathfrak{X}}(u)$ if there exists $A \in \Psi^{0,r}_{\mathrm{eb}}(M)$ elliptic at q such that $PAu \in \mathfrak{X}$ for all $P \in \mathrm{Diff}^*_{\mathrm{eb}}(M)$.

There is an inclusion

$$\mathrm{WF}^{m,r}_{\mathrm{eb},\mathfrak{X}}\, u \subset \mathrm{WF}^{m',r'}_{\mathrm{eb},\mathfrak{X}}\, u$$

if

$$m \leq m', \ r \leq r'.$$

Remark 5.13. We could alter this definition to allow u a priori to lie in the larger space

$$\sum A_j(\mathfrak{X})$$

with $A_j \in \Psi_{\mathrm{eb}}^{\infty,0}(M)$; this would allow us to give a non-trivial definition of $\mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m,r} u$ even for $m < 0$.

The restriction to $r \leq 0$ is more serious: operators in $\Psi_{\mathrm{eb}}^{*,0}(M)$ would in general fail to be microlocal with respect to a putative $\mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m,r}(M)$ with $r > 0$.

Note also that if $\mathfrak{X}'$ is a closed subspace of $\mathfrak{X}$, with the induced norm, and if elements of $\Psi_{\mathrm{eb}}^{0}(M)$ restrict to (necessarily bounded) maps $\mathfrak{X}' \to \mathfrak{X}'$, then for $u \in \mathfrak{X}'$,

$$\mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m,r}(u) = \mathrm{WF}_{\mathrm{eb},\mathfrak{X}'}^{m,r}(u). \tag{5.3}$$

In particular, this holds with $\mathfrak{X} = H_{\mathrm{es}}^{k,l}(M)$ and $\mathfrak{X}' = H_{\mathrm{es},0}^{k,l}(M)$.

The eb-wave front set captures eb-regularity:

Lemma 5.14. *If $u \in \mathfrak{X}$, $r \leq 0$, $m \geq 0$ and $\mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m,r}(u) = \emptyset$, then $u \in H_{\mathrm{eb},\mathfrak{X},loc}^{m,r}(M)$, i.e. for all $A \in \Psi_{\mathrm{eb}}^{m,r}(M)$ with compactly supported kernel, $Au \in \mathfrak{X}$.*

Proof. This is a standard argument (see e.g. [34, Lemma 3.10]): For each $q \in {}^{\mathrm{eb}}S^*M$ there is $B_q \in \Psi_{\mathrm{eb}}^{m,r}(M)$ elliptic at q such that $B_q u \in \mathfrak{X}$. By compactness, ${}^{\mathrm{eb}}S^*M$ can be covered by $\bigcup_j \mathrm{ell}(B_{q_j})$ for finitely many points q_j. Now choose $Q \in \Psi_{\mathrm{eb}}^{-m,-r}(M)$ elliptic, and set $B = \sum Q B_{q_j}^* B_{q_j}$. Then B is elliptic and $Bu \in \mathfrak{X}$. As B has a parametrix $G \in \Psi_{\mathrm{eb}}^{-m,-r}(M)$ with $GB - \mathrm{Id} \in \Psi_{\mathrm{eb}}^{-\infty,0}(M)$,

$$Au = AG(Bu) + (A(\mathrm{Id} - GB))u, \text{ and } A(\mathrm{Id} - GB) \in \Psi_{\mathrm{eb}}^{-\infty,r}(M) \subset \Psi_{\mathrm{eb}}^{0,0}(M),$$

shows the claim. □

Pseudodifferential operators are microlocal, as follows by a standard argument:

Lemma 5.15. *(Microlocality) If $B \in \Psi_{\mathrm{eb}}^{s,l}(M)$ then for $r, r - l \leq 0$, $u \in \mathfrak{X}$,*

$$\mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m-s,r-l}(Bu) \subset \mathrm{WF}'(B) \cap \mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m,r}(u).$$

In particular, if $\mathrm{WF}'(B) \cap \mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m,r}(u) = \emptyset$ then $Bu \in H_{\mathrm{eb},\mathfrak{X},loc}^{m-s,r-l}(M)$.

Proof. We assume $m \geq s$ and $m \geq 0$ in accordance with the definition above; but the general case is treated easily by the preceeding remarks.

If $q \in {}^{\mathrm{eb}}S^*M$, $q \notin \mathrm{WF}'(B)$, let $A \in \Psi_{\mathrm{eb}}^{m-s,r-l}(M)$ be elliptic at q such that $\mathrm{WF}'(A) \cap \mathrm{WF}'(B) = \emptyset$. Thus $AB \in \Psi_{\mathrm{eb}}^{-\infty,r}(M) \subset \Psi_{\mathrm{eb}}^{0,0}(M)$, hence $ABu \in \mathfrak{X}$, so $q \notin \mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m-s,r-l}(Bu)$.

On the other hand, if $q \in {}^{\mathrm{eb}}S^*M$, $q \notin \mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m,r}(u)$, then there is $C \in \Psi_{\mathrm{eb}}^{m,r}(M)$ elliptic at q such that $Cu \in \mathfrak{X}$. Let G be a microlocal parametrix for C, so $G \in \Psi_{\mathrm{eb}}^{-m,-r}(M)$, and $q \notin \mathrm{WF}'(GC - \mathrm{Id})$. Let $A \in \Psi_{\mathrm{eb}}^{m-s,r-l}$ be elliptic at q and such that $\mathrm{WF}'(A) \cap \mathrm{WF}'(GC - \mathrm{Id}) = \emptyset$. Then

$$ABu = ABGCu + AB(\mathrm{Id} - GC)u,$$

and $AB(\mathrm{Id}-GC) \in \Psi_{\mathrm{eb}}^{-\infty,r}(M) \subset \Psi_{\mathrm{eb}}^{0,0}(M)$ since $\mathrm{WF}'(A)\cap\mathrm{WF}'(\mathrm{Id}-GC) = \emptyset$, so the second term on the right hand side is in $\mathfrak{X}$. On the other hand, $Cu \in \mathfrak{X}$ and $ABG \in \Psi_{\mathrm{eb}}^{0,0}(M)$, so $ABG(Cu) \in \mathfrak{X}$ as well, proving the wave front set containment.

The final claim follows immediately from this and Lemma 5.14. □

There is a quantitative version of the lemma as well. Since the proof is similar, cf. [34, Lemma 3.13], we omit it.

Lemma 5.16. *Suppose that $K \subset {}^{\mathrm{eb}}S^*M$ is compact, U is a neighborhood of K, $\tilde{K} \subset M$ compact.*

Let $Q \in \Psi_{\mathrm{eb}}^{s,l}(M)$ elliptic on K with $\mathrm{WF}'(Q) \subset U$ and the Schwartz kernel of Q supported in $\tilde{K} \times \tilde{K}$.

If $\mathcal{B}$ is a bounded family in $\Psi_{\mathrm{eb}}^{s,l}(M)$ with Schwartz kernel supported in $\tilde{K} \times \tilde{K}$ and with $\mathrm{WF}'(\mathcal{B}) \subset K$ then for $r, r-l \le 0$, there is $C > 0$ such that for $u \in \mathfrak{X}$ with $\mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m,r}(u) \cap U = \emptyset$,

$$\|Bu\|_{\mathfrak{X}} \le C(\|u\|_{\mathfrak{X}} + \|Qu\|_{\mathfrak{X}}).$$

5.2. **Dual spaces and adjoints.** We now discuss the dual spaces. *For simplicity of notation we suppress the loc and c subscripts for the local spaces and compact supports.* In principle this should only be done if M is compact, but, as this aspect of the material is standard, we feel that this would only distract from the new aspects. See for instance [34, Section 3] for a treatment where all the compact supports and local spaces are spelled out in full detail.

Recall now from Appendix A that if $\mathfrak{X}$ is a dense subspace of L^2_g, equipped with an inner product $\langle .,.\rangle_{\mathfrak{X}}$ in which it is a Hilbert space and the inclusion map ι into L^2_g is continuous, then there is a linear injective inclusion map $L^2_g \to \mathfrak{X}^*$ with dense range, namely

$$\iota^* = \iota^\dagger \circ j \circ c : \mathfrak{H} \to \mathfrak{X}^*$$

where $\iota^\dagger : (L^2_g(M))^* \to \mathfrak{X}^*$ is the standard adjoint map, $j : L^2_g(M) \to L^2_g(M)^*$ the standard conjugate-linear identification of a Hilbert space with its dual, and c is pointwise complex conjugation of functions. In particular, one has the chain of inclusions $\mathfrak{X} \subset L^2_g(M) \subset \mathfrak{X}^*$, and one considers $\mathfrak{X}^*$, together with these inclusions, the dual space of $\mathfrak{X}$ with respect to $L^2_g(M)$.

Definition 5.17. For $s \ge 0$, the dual space of $H_{\mathrm{es}}^{s,l}(M)$ with respect to the $L^2_g(M)$ inner product is denoted $\dot{H}_{\mathrm{es}}^{-s,-l-(f+1)}(M)$.

For $s \ge 0$, the dual space of the closed subspace

$$H_{\mathrm{es},0}^{s,l}(M) \equiv \dot{H}_{\mathrm{es}}^{s,l}(M)$$

is denoted $H_{\mathrm{es}}^{-s,-l-(f+1)}(M)$; this is a quotient space of $\dot{H}_{\mathrm{es}}^{-s,-l-(f+1)}(M)$. We denote the quotient map by

$$\rho : \dot{H}_{\mathrm{es}}^{-s,-l-(f+1)}(M) \to H_{\mathrm{es}}^{-s,-l-(f+1)}(M).$$

The standard characterization of these distribution spaces, by doubling across all boundary faces of M except $\tilde{W}$, is still valid—see [10, Appendix B.2] and [34, §3]. Note that for all s,l, elements of $\dot{H}_{\mathrm{es}}^{-s,-l-(f+1)}(M)$ are in particular continuous linear functionals on $\mathcal{C}$, which in turn is a dense subspace of $H_{\mathrm{es}}^{s,l}(M)$. In particular, they can be identified as elements of the dual $\mathcal{C}'$ of $\mathcal{C}$. Thus, were it not for the infinite order vanishing imposed at $\tilde{W}$ for elements of $\mathcal{C}$, these would be "supported distributions"—hence the notation with the dot. On the other hand, elements of $H_{\mathrm{es}}^{-s,-l-(f+1)}(M)$ are only continuous linear functionals on $\dot{\mathcal{C}}^\infty(M)$ (rather than on $\mathcal{C}$), though by the Hahn-Banach theorem can be extended to continuous linear functionals on $\mathcal{C}$ in a non-unique fashion.

If $P \in x^{-r}\operatorname{Diff}_{\mathrm{es}}^k(M)$, then it defines a continuous linear map

$$P : H_{\mathrm{es}}^{k,l}(M) \to H_{\mathrm{es}}^{0,l-r}(M).$$

Thus, its Banach space adjoint (with respect to the *sesquilinear* dual pairing) is a map

(5.4)
$$P^* : (H_{\mathrm{es}}^{0,l-r}(M))^* = \dot{H}_{\mathrm{es}}^{0,r-l-(f+1)}(M) \to (H_{\mathrm{es}}^{k,l}(M))^* = \dot{H}_{\mathrm{es}}^{-k,-l-(f+1)}(M),$$
$$\langle P^*u, v\rangle = \langle u, Pv\rangle,\ u \in \dot{H}_{\mathrm{es}}^{0,r-l-(f+1)}(M),\ v \in H_{\mathrm{es}}^{k,l}(M).$$

In principle, P^* depends on l and r. However, the density of $\mathcal{C}$ in these spaces shows that in fact it does not.

There is an important distinction here between considering P^* as stated, or as composed with the quotient map, $\rho \circ P^*$.

Lemma 5.18. *Suppose that $P \in x^{-r}\operatorname{Diff}_{\mathrm{es}}^k(M)$. Then there exists a unique $Q \in x^{-r}\operatorname{Diff}_{\mathrm{es}}^k(M)$ such that $\rho \circ P^* = Q$. However, in general, acting on $\mathcal{C}$, $P^* \neq Q$.*

If, on the other hand, $P \in x^{-r}\operatorname{Diff}_{\mathrm{eb}}^k(M)$, then there exists a unique $Q \in x^{-r}\operatorname{Diff}_{\mathrm{eb}}^k(M)$ such that $P^* = Q$.

Proof. For the first part we integrate by parts in $\langle u, Pv\rangle$ using $u, v \in \dot{\mathcal{C}}^\infty(M)$ (noting that $\dot{\mathcal{C}}^\infty(M)$ is dense in $H_{\mathrm{es},0}^{k,l}(M)$). Thus, one can localize. In local coordinates the density is $dg = Jx^f\,dx\,dy\,dz$, with $J \in \mathcal{C}^\infty(M)$, so for a vector field $V \in \mathcal{V}_{\mathrm{es}}(M)$, noting the lack of boundary terms due to the infinite order vanishing of u and v, one has (with the first equality being the definition of V^*)

$$\begin{aligned}\langle V^*u, v\rangle = \langle u, Vv\rangle &= \int u\,\overline{Vv}\,Jx^f\,dx\,dy\,dz\\ &= \int (J^{-1}x^{-f}V^\dagger(Jx^f u))\,\overline{v}\,Jx^f\,dx\,dy\,dz,\end{aligned}$$

where for $V = a(xD_x) + \sum b_j(xD_{y_j}) + \sum c_j D_{z_j}$, with $a, b_j, c_j \in \mathcal{C}^\infty(M)$,

$$V^\dagger = D_x x\overline{a} + \sum xD_{y_j}\overline{b_j} + \sum D_{z_j}\overline{c_j} \in \operatorname{Diff}_{\mathrm{es}}^1(M).$$

Conjugation of $V^\dagger$ by Jx^f still yields an operator in $\mathrm{Diff}^1_{\mathrm{es}}(M)$. This shows the existence (and uniqueness!) of the desired Q, namely

$$Q = J^{-1}x^{-f}V^\dagger Jx^f.$$

The density of $\dot{\mathcal{C}}^\infty(M)$ in $H^{0,r-l-(f+1)}_{\mathrm{es}}(M)$ now finishes the proof of the first claim when $P = V \in \mathcal{V}_{\mathrm{es}}(M)$, since this means that $\langle P^*u, v\rangle = \langle Qu, v\rangle$ for all $u \in H^{0,r-l-(f+1)}_{\mathrm{es}}(M)$, $v \in H^{k,l}_{\mathrm{es},0}(M)$. The general case follows by induction and adding weight factors (recalling Remark 5.2).

The same calculation works even if $u, v \in \mathcal{C}$ provided that $V \in \mathcal{V}_{\mathrm{eb}}(M)$: in this case D_{z_j} is replaced by vector fields tangent to all boundary faces, i.e. $D_{z''_j}$ and $z'_j D_{z'_j}$, for which there are no boundary terms—in the second case due to the vanishing factor z'_j. This proves the claim if $P \in x^{-r}\,\mathrm{Diff}^k_{\mathrm{eb}}(M)$.

Note, however, that this calculation breaks down if $u, v \in \mathcal{C}$ and $V \in \mathcal{V}_{\mathrm{es}}(M)$: the $D_{z'_j}$ terms gives rise to non-vanishing boundary terms in general, namely

$$\sum_j \int_{H_j} (-\imath)\overline{c_j}\, u\, \overline{v}\, Jx^f\, dx\, dy\, d\hat{z}_j = \sum_j \langle -\imath\overline{\sigma_{\mathrm{es},1}(V)(dx_j)}u, v\rangle_{H_j},$$

where H_j is the boundary hypersurface $z'_j = 0$, $d\hat{z}_j$ shows that dz'_j is dropped from the density, and on H_j one uses the density induced by the Riemannian density and dz'_j. This completes the proof of the lemma. □

We now define an extension of $\mathrm{Diff}_{\mathrm{es}}(M)$ as follows.

Definition 5.19. Let $x^{-r}\,\mathrm{Diff}^k_{\mathrm{es},\dagger}(M)$ denote the set of Banach space adjoints of elements of $x^{-r}\,\mathrm{Diff}^k_{\mathrm{es}}(M)$ in the sense of (5.4).

Also let $x^{-2r}\,\mathrm{Diff}^{2k}_{\mathrm{es},\sharp}(M)$ denote operators of the form

$$\sum_{j=1}^N Q_j P_j,\ P_j \in x^{-r}\,\mathrm{Diff}^k_{\mathrm{es}}(M),\ Q_j \in \mathrm{Diff}^k_{\mathrm{es},\dagger}(M).$$

For M non-compact, the sum is taken to be locally finite.

Thus, if $P \in x^{-2r}\,\mathrm{Diff}^{2k}_{\mathrm{es},\sharp}(M)$, P_j, Q_j as above, and $Q_j = R^*_j$, $R_j \in x^{-r}\,\mathrm{Diff}^k_{\mathrm{es}}(M)$, then

$$\langle Pu, v\rangle = \sum_{j=1}^N \langle P_j u, R_j v\rangle.$$

We are now ready to discuss Dirichlet and Neumann boundary conditions for $P \in x^{-2r}\,\mathrm{Diff}^{2k}_{\mathrm{es},\sharp}(M)$.

Definition 5.20. Suppose $P \in x^{-2r}\,\mathrm{Diff}^{2k}_{\mathrm{es},\sharp}(M)$. By the *Dirichlet operator* associated to P we mean the map

$$\rho \circ P : H^{k,l}_{\mathrm{es},0}(M) \to H^{-k,l-2r}_{\mathrm{es}}(M),$$

where $\rho : \dot{H}_{\mathrm{es}}^{-k,l-2r}(M) \to H_{\mathrm{es}}^{-k,l-2r}(M)$ is the quotient map. For $f \in H_{\mathrm{es}}^{-k,l-2r}(M)$ we say that $u \in H_{\mathrm{es},0}^{k,l}(M)$ solves *the Dirichlet problem* for $Pu = f$ if $\rho \circ Pu = f$. We also say in this case that $Pu = f$ with *Dirichlet boundary conditions.*

Similarly, for $f \in \dot{H}_{\mathrm{es}}^{-k,l-2r}(M)$ we say that $u \in H_{\mathrm{es}}^{k,l}(M)$ solves the *Neumann problem* for $Pu = f$ if $Pu = f$. We also say in this case that $Pu = f$ with *Neumann boundary conditions.* Correspondingly, for the sake of completeness, by the Neumann operator associated to P we mean P itself.

Remark 5.21. As noted in Lemma 5.18, when considering the action of $\mathrm{Diff}_{\mathrm{es}}(M)$ on $\dot{\mathcal{C}}^\infty(M)$, $\mathrm{Diff}_{\mathrm{es}}(M)$ is closed under adjoints (which thus map to $\mathcal{C}^{-\infty}(M)$, i.e. extendible distributions), so one can suppress the subscript $\sharp$ on $\mathrm{Diff}_{\mathrm{es},\sharp}(M)$. Thus, the subscript's main role is to keep the treatment of the Neumann problem clear—without such care, one would need to use quadratic forms throughout, as was done in [34].

We now turn to the action of $\Psi_{\mathrm{eb}}^{m,l}(M)$ on the dual spaces. Note that any $A \in \Psi_{\mathrm{eb}}^{m,l}(M)$ maps $\mathcal{C}$ to itself, and that $\Psi_{\mathrm{eb}}^{m,l}(M)$ is closed under formal adjoints, i.e. if $A \in \Psi_{\mathrm{eb}}^{m,l}(M)$ then there is a unique $A^* \in \Psi_{\mathrm{eb}}^{m,l}(M)$ such that $\langle Au, v\rangle = \langle u, A^* v\rangle$ for all $u, v \in \mathcal{C}$ —cf. $\mathrm{Diff}_{\mathrm{eb}}(M)$ in Lemma 5.18. We thus define $A : \mathcal{C}' \to \mathcal{C}'$ by $\langle Au, v\rangle = \langle u, A^* v\rangle$, $u \in \mathcal{C}'$, $v \in \mathcal{C}$. Since $\mathcal{C}$ is (even sequentially) dense in $\mathcal{C}'$ endowed with the weak-* topology, this definition is in fact the only reasonable one, and if $u \in \mathcal{C}$, the element of $\mathcal{C}'$ given by this is the linear functional induced by Au on $\mathcal{C}$.

Next, for subspaces of $\mathcal{C}'$ we have improved statements. In particular, most relevant here, dually to Theorem 5.10, any $A \in \Psi_{\mathrm{eb}}^{0}(M)$ is bounded on $\dot{H}_{\mathrm{es}}^{-1,l}(M)$ and on $H_{\mathrm{es}}^{-1,l}(M)$.

We now turn to an extension of $\mathrm{Diff}_{\mathrm{es}}\Psi_{\mathrm{eb}}(M)$. First, taking adjoints in Lemma 5.5, we deduce:

Lemma 5.22. *If $A \in \Psi_{\mathrm{eb}}^{m}(M)$ and $Q \in \mathrm{Diff}_{\mathrm{es},\dagger}^{1}(M)$, then $[A, Q] = \sum Q_j A_j + B$, $B \in \Psi_{\mathrm{eb}}^{m}(M)$, $A_j \in \Psi_{\mathrm{eb}}^{m-1}(M)$, $Q_j \in \mathrm{Diff}_{\mathrm{es},\dagger}^{1}(M)$.*

Similarly, $[A, Q] = \sum A'_j Q'_j + B'_j$, $B' \in \Psi_{\mathrm{eb}}^{m}(M)$, $A'_j \in \Psi_{\mathrm{eb}}^{m-1}(M)$, $Q'_j \in \mathrm{Diff}_{\mathrm{es},\dagger}^{1}(M)$.

Proof. The proof is an exercise in duality; we only spell out to emphasize our definitions. We have for $u \in \mathcal{C}'$, $v \in \mathcal{C}$,

$$\langle [A, Q]u, v\rangle = \langle (AQ - QA)u, v\rangle = \langle u, (Q^* A^* - A^* Q^*)v\rangle = \langle u, [Q^*, A^*]v\rangle$$

where $Q^* \in \mathrm{Diff}_{\mathrm{es}}^{1}(M)$, $A^* \in \Psi_{\mathrm{eb}}^{M}$. Thus, by Lemma 5.5 (applied with $\mathcal{V}_{\mathrm{es}}(M)$ replaced by $\mathrm{Diff}_{\mathrm{es}}^{1}(M)$), there exist $\tilde{A}_j \in \Psi_{\mathrm{eb}}^{m-1}(M)$, $\tilde{B} \in \Psi_{\mathrm{eb}}^{m}(M)$, $\tilde{Q}_j \in \mathrm{Diff}_{\mathrm{es}}^{1}(M)$ such that $[Q^*, A^*] = -[A^*, Q^*] = \sum \tilde{Q}_j \tilde{A}_j + \tilde{B}$. Thus,

$$\langle [A, Q]u, v\rangle = \langle u, (\sum \tilde{Q}_j \tilde{A}_j + \tilde{B})v\rangle = \langle (\tilde{A}_j^* \tilde{Q}_j^* + \tilde{B}^*)u, v\rangle,$$

with $\tilde{A}_j^* \in \Psi_{\mathrm{eb}}^{m-1}(M)$, $\tilde{B}^* \in \Psi_{\mathrm{eb}}^m(M)$, $\tilde{Q}_j^* \in \mathrm{Diff}^1_{\mathrm{es},\dagger}(M)$. This proves the second half of the lemma. The first half is proved similarly, using the second half of the statement of Lemma 5.5 rather than its first half. $\square$

In fact, the analogue of Lemma 5.7 also holds with $D_{z'_j}$ replaced by $D^*_{z'_j} \in \mathrm{Diff}^1_{\mathrm{es},\dagger}(M)$:

Lemma 5.23. *Let $A \in \Psi_{\mathrm{eb}}^{m,l}(M)$, $a = \sigma_{\mathrm{eb},m}(A)$. Then*

$$\imath[x^{-1}D^*_{z'_j}, A] = A_1 x^{-1} D^*_{z'_j} + x^{-1}A_0$$

where $A_0 \in \Psi_{\mathrm{eb}}^{m,l}(M)$, $A_1 \in \Psi_{\mathrm{eb}}^{m-1,l}(M)$,

$$\sigma_{\mathrm{eb},m}(A_0) = \frac{\partial a}{\partial z'_j}, \quad \sigma_{\mathrm{eb},m-1}(A_1) = \frac{\partial a}{\partial \zeta'_j} + \frac{\partial a}{\partial \xi}.$$

We thus make the following definition:

Definition 5.24. Let

$$\mathrm{Diff}^k_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^m(M) = \{\sum A_\alpha B_\beta : A_\alpha \in \mathrm{Diff}^k_{\mathrm{es},\sharp}(M),\ B_\beta \in \Psi_{\mathrm{eb}}^m(M)\}.$$

Using Proposition 5.4 and duality, as in the previous lemma, we deduce the following:

Proposition 5.25. $\mathrm{Diff}^k_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{m,l}(M)$ *is a $\Psi_{\mathrm{eb}}^0(M)$-bimodule, and*

$$\begin{aligned} &P \in \mathrm{Diff}^k_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{m,l}(M),\ A \in \Psi_{\mathrm{eb}}^{s,r}(M) \Rightarrow \\ &PA, AP \in \mathrm{Diff}^k_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{m+s,l+r}(M),\ [P,A] \in \mathrm{Diff}^k_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{m+s-1,l+r}(M). \end{aligned} \tag{5.5}$$

5.3. **Domains.** In this section, we discuss the relationship between Dirichlet and Neumann form domains of Δ and the scales of weighted Sobolev that we have introduced. First, we identify the Dirichlet quadratic form domain in terms of the edge-smooth Sobolev spaces.

The Friedrichs form domain of Δ with Dirichlet boundary conditions on X_0 is

$$H^1_0(X_0),$$

also denoted by $\dot{H}^1(X_0)$ (see [10, Appendix B.2]); we may also view this space as the completion of $\mathcal{C}_c^\infty(X_0)$ in the $H^1(X_0)$-norm,

$$\|u\|_{H^1(X_0)} = \|u\|_{L^2_{g_0}(X_0)} + \|du\|_{L^2_{g_0}(X_0;T^*X_0)}.$$

Equivalently in terms of "doubling" X_0 across all boundary hypersurfaces, $H^1_0(X_0)$ consists of H^1-functions on the "double" supported in X_0.

Lemma 5.26. *On $\dot{\mathcal{C}}^\infty(X) = \beta^*\dot{\mathcal{C}}^\infty(X_0)$, the norms*

$$\|u\|_{H^1(X_0)} = (\|u\|^2_{L^2_{g_0}} + \|du\|^2_{L^2_{g_0}})^{1/2}$$

and

$$\|u\|_{H^{1,1-(f+1)/2}_{\mathrm{es}}(X)}$$

are equivalent.

Proof. Multiplication by elements of $\mathcal{C}^\infty(X_0)$ is bounded with respect to both norms (with respect to $H^{1,1-(f+1)/2}_{\mathrm{es}}(X)$ even $\mathcal{C}^\infty(X)$ is bounded), so one can localize in X_0, or equivalently in X near a fiber $\beta^{-1}(p)$, $p \in W$, of $\tilde{W}$, and assume that u is supported in such a region.

Elements of $\mathcal{V}(X_0)$ lift under β to span $x^{-1}\mathcal{V}_{\mathrm{es}}(X)$ as a $\mathcal{C}^\infty(X)$-module by (3.13). In particular, merely since $\beta^*\mathcal{V}(X_0) \subset x^{-1}\mathcal{V}_{\mathrm{es}}(X)$, we obtain[9]

$$\|u\|_{H^1_0(X_0)} \lesssim \|u\|_{H^{1,1-(f+1)/2}_{\mathrm{es}}(X)}, \ u \in \dot{\mathcal{C}}^\infty(X). \tag{5.6}$$

We now prove the reverse inequality. By the spanning property, we have

$$\|x^{-1}Au\|_{L^2_g} \lesssim \|u\|_{H^1(X_0)} \tag{5.7}$$

for any $A \in \mathcal{V}_{\mathrm{es}}(X)$ as $\mathcal{C}^\infty(X)$ is bounded on $L^2_g(X) = L^2_{g_0}(X)$. As $\mathcal{V}_{\mathrm{es}}(X)$ together with the identity operator generates $\mathrm{Diff}^1_{\mathrm{es}}(X)$, we only need to prove

$$\|x^{-1}u\|_{L^2_g(X)} \lesssim \|u\|_{H^1(X_0)},$$

for $u \in \dot{\mathcal{C}}^\infty(X)$ supported near a fiber $\beta^{-1}(p)$, $p \in Y$, of $\tilde{Y}$. However, this follows easily from identifying a neighborhood of $\beta^{-1}(p)$ with $[0,\epsilon)_x \times O_y \times Z_z$, where $O \subset \mathbb{R}^{n-f-1}$, and using the Poincaré inequality in Z, namely that

$$\|u(x,y,.)\|_{L^2(Z)} \leq C\|(d_Z u)(x,y,.)\|_{L^2(Z)}, \quad u \in \dot{\mathcal{C}}^\infty(X_0).$$

Multiplying the square of both sides by x^{-2+f} and integrating in x, y, yields

$$\|x^{-1}u(x,y,.)\|_{L^2_g(X)} \leq C\|(x^{-1}d_Z u)(x,y,.)\|_{L^2_g(X)} \leq C'\|u\|_{H^1_0(X)}$$

by (5.7). $\square$

In view of the definition of $H^{1,1-(f+1)/2}_{\mathrm{es},0}(X)$ as the closure of $\dot{\mathcal{C}}^\infty(X)$ in $H^{1,1-(f+1)/2}_{\mathrm{es}}(X)$, we immediately deduce:

Proposition 5.27. *The Dirichlet form domain of Δ is given by*

$$H^1_0(X_0) = H^{1,1-(f+1)/2}_{\mathrm{es},0}(X) \tag{5.8}$$

in the strong sense that the natural (up to equivalence) Hilbert space norms on the two sides are equivalent. In particular, for $u \in H^1_0(X_0)$, we have

$$\|x^{-1}Qu\|_{L^2_g} \leq C\|u\|_{H^1(X_0)}$$

for all $Q \in \mathrm{Diff}^1_{\mathrm{es}}(X)$.

For Neumann boundary conditions the quadratic form domain is $H^1(X_0)$, whose lift is not quite so simple in terms of the edge-smooth spaces. However, we have the following lemma, which suffices for the edge-b propagation results below (with a slight loss).

[9] We use the notation that $a \lesssim b$ if there exists $C > 0$ such that $a \leq Cb$. Usually a and b depend on various quantities, e.g. on u here, and C is understood to be independent of these quantities.

Lemma 5.28. *We have* $H^{1,1-(f+1)/2}_{\mathrm{es}}(X) \subset H^1(X_0) \subset H^{1,-(f+1)/2}_{\mathrm{es}}(X)$, *with all inclusions being continuous.*

Proof. The first inclusion is an immediate consequence of (5.6) holding for $u \in \mathcal{C}$, $\mathcal{C}$ as in Remark 5.9 (thus dense in $H^{1,l-(f+1)/2}_{\mathrm{es}}(X)$ for all l), using again that elements of $\mathcal{V}(X_0)$ lift under β to span (and in particular lie in) $x^{-1}\mathcal{V}_{\mathrm{es}}(X)$ as a $\mathcal{C}^\infty(X)$-module by (3.13).

For the second inclusion, we need to prove that $\|Au\|_{L^2_g(X)} \leq C\|u\|_{H^1(X_0)}$ for $A \in \mathrm{Diff}^1_{\mathrm{es}}(X)$. As this is automatic for $A \in \mathcal{C}^\infty(X)$, we are reduced to considering $A \in \mathcal{V}_{\mathrm{es}}(X)$. But (5.7) still holds for $u \in \mathcal{C}$, so $\|Au\|_{L^2_g(X)} \leq C'\|x^{-1}Au\|_{L^2_g} \leq C\|u\|_{H^1(X_0)}$ for $A \in \mathcal{V}_{\mathrm{es}}(X)$. This finishes the proof of the lemma. $\square$

5.4. **The wave operator as an element of $x^{-2}\,\mathrm{Diff}^2_{\mathrm{es},\sharp}(M)$.** For $f \in \mathcal{C}$, in local coordinates,

$$df = (x\partial_x f)\,\frac{dx}{x} + \sum_j (x\partial_{y_j} f)\,\frac{dy_j}{x} + \sum_j (\partial_{z_j} f)\,dz_j.$$

Thus, the exterior derivative satisfies

$$d \in \mathrm{Diff}^1_{\mathrm{es}}(M; \underline{\mathbb{C}}, {}^{\mathrm{es}}T^*M),$$

with $\underline{\mathbb{C}}$ denoting the trivial bundle. As the dual Riemannian metric is of the form $x^{-2}G$, where G is a smooth fiber metric on ${}^{\mathrm{es}}T^*X$, and $\Delta = d^*d$, we deduce that $\Box \in x^{-2}\,\mathrm{Diff}^2_{\mathrm{es},\sharp}(M)$. However, we need a more precise description of $\Box$ for our commutator calculations.

So suppose now that U is a coordinate chart near a point q at $\partial\tilde{W}$ with coordinates (x, y, z', z'') centered at q, and recall from (2.2) that the Riemannian metric has the form

$$g = dx^2 + h(y, dy) + x^2 k(x, y, z, dz) + xk'(x, y, z, dx, dy, x\,dz). \tag{5.9}$$

By changing z'' if necessary (while keeping x, y, z' fixed), cf. the argument of § 2 leading to (5.10), we can arrange that the dual metric K of k have the form

(5.10)

$$K(0,y,z) = \sum_{i,j=1}^{k} k_{1,ij}(0,y,z)\underline{\zeta}'_i\underline{\zeta}'_j + \sum_{i=1}^{k}\sum_{j=k+1}^{f} k_{3,ij}(0,y,z)\underline{\zeta}'_i\underline{\zeta}''_j + \sum_{i,j=k+1}^{f} k_{2,ij}(0,y,z)\underline{\zeta}''_i\underline{\zeta}''_j, \qquad k_3|_C = 0.$$

We deduce the following lemma:

Lemma 5.29. *Let U be a coordinate chart near a point with $x = 0$ and $z' = 0$, and suppose that we have arranged that at*

$$C = \{x = 0,\ z' = 0\},$$

the vector spaces

$$\mathrm{sp}\{dz'_i,\ i=1,\dots,k\} \text{ and } \mathrm{sp}\{dz''_j,\ j=k+1,\dots,f\}$$

are orthogonal with respect to K*. With* $Q_i = x^{-1}D_{z'_i}$*, the wave operator satisfies*

$$\Box = \sum_{i,j} Q_i^* \kappa_{ij} Q_j + \sum_i (x^{-1}M_i Q_i + Q_i^* x^{-1} M'_i) + x^{-2}\tilde H \text{ on } U \tag{5.11}$$

with

$$\begin{gathered}\kappa_{ij}\in\mathcal{C}^\infty(M),\ M_i, M'_i\in\mathrm{Diff}^1_{\mathrm{eb}}(M),\ \tilde H\in\mathrm{Diff}^2_{\mathrm{eb}}(M)\\ \sigma_{\mathrm{eb},1}(M_i)=m_i=\sigma_{\mathrm{eb},1}(M'_i),\ \tilde h=\sigma_{\mathrm{eb},2}(\tilde H),\\ \kappa_{ij}|_{\tilde W}=-k_{1,ij}(y,z),\ m_i|_C=0,\ m_i|_{\tilde W}=-\frac12\sum_{j=k+1}^f k_{3,ij}\zeta''_j,\\ \tilde h|_{\tilde W}=\tau^2-\xi^2-h(y,\eta)-\sum_{i,j=k+1}^f k_{2,ij}(y,z)\zeta''_i\zeta''_j.\end{gathered} \tag{5.12}$$

We next note microlocal elliptic regularity.

Proposition 5.30. *Let* $u\in\mathfrak{X}\equiv H^{1,l}_{\mathrm{es}}(M)$*, and suppose that*

$$\Box u\in \dot H^{-1,l-2}_{\mathrm{es}}(M)=\mathfrak{Y}$$

with Dirichlet or Neumann boundary conditions. Then

$$\mathrm{WF}^{m,0}_{\mathrm{eb},\mathfrak{X}}(u)\subset {}^{\mathrm{eb}}\Sigma\cup\mathrm{WF}^{m,0}_{\mathrm{eb},\mathfrak{Y}}(\Box u).$$

In particular, if $\Box u=0$*, then* $\mathrm{WF}^{\infty,0}_{\mathrm{eb},\mathfrak{X}}(u)\subset {}^{\mathrm{eb}}\Sigma$.

Proof. The proof goes along the same lines as Proposition 4.6 of [34] and Theorem 8.11 of [25]; we thus provide a sketch. An essential ingredient the top-order commutativity of $x^l\,\mathrm{Diff}^2_{\mathrm{es},\sharp}\,\Psi^m_{\mathrm{eb}}(M)$, which allows us to treat all commutators as error terms. The key estimate is stated in Lemma 5.31 below.

Given the lemma, one proceeds by an inductive argument, showing that if $\mathrm{WF}^{s-1/2,0}_{\mathrm{eb},\mathfrak{X}}(u)\subset {}^{\mathrm{eb}}\Sigma\cup\mathrm{WF}^{s-1/2,0}_{\mathrm{eb},\mathfrak{Y}}(\Box u)$ (which is a priori known for $s=1/2$, starting our inductive argument) then $\mathrm{WF}^{s,0}_{\mathrm{eb},\mathfrak{X}}(u)\subset {}^{\mathrm{eb}}\Sigma\cup\mathrm{WF}^{s,0}_{\mathrm{eb},\mathfrak{Y}}(\Box u)$. In order to show this, one takes $A\in\Psi^{s,l+(f-1)/2}_{\mathrm{eb}}(M)$, with $\mathrm{WF}'(A)\cap{}^{\mathrm{eb}}\Sigma=\emptyset$, $\mathrm{WF}'(A)\cap\mathrm{WF}^{s,0}_{\mathrm{eb},\mathfrak{Y}}(\Box u)=\emptyset$. Λ_γ uniformly bounded in $\Psi^{0,0}_{\mathrm{eb}}(M)$, $\gamma\in(0,1]$, with $\Lambda_\gamma\in\Psi^{-1,0}_{\mathrm{eb}}(M)$ for all γ,

$$\sigma_{\mathrm{eb},0,0}(\Lambda_\gamma)=(1+\gamma(|\xi|^2+|\tau|^2+|\eta|^2+|\zeta|^2))^{-1},$$

so $A_\gamma=\Lambda_\gamma A$ is uniformly bounded in $\Psi^{m,l+(f-1)/2}_{\mathrm{eb}}(M)$ and $A_\gamma\to A$ in $\Psi^{m+\delta,l+(f-1)/2}_{\mathrm{eb}}(M)$ ($\delta>0$ fixed) as $\gamma\to0$. One then concludes by

Lemma 5.31 that for all $\epsilon \in (0,1]$,

$$|\langle \partial_t A_\gamma u, \partial_t A_\gamma u\rangle - \langle d_X A_\gamma u, d_X A_\gamma u\rangle| \leq \epsilon \|A_\gamma u\|^2_{H^{1,1-(f+1)/2}_{\mathrm{es}}(M)} + C\epsilon^{-1},$$

with C uniformly bounded, independent of γ. The microlocal ellipticity of the Dirichlet form, i.e.

$$\|A_\gamma u\|^2_{H^{1,1-(f+1)/2}_{\mathrm{es}}(X)} \leq C'|\langle \partial_t A_\gamma u, \partial_t A_\gamma u\rangle - \langle d_X A_\gamma u, d_X A_\gamma u\rangle|,$$

now gives that for sufficently small $\epsilon \in (0,1]$, $\epsilon\|A_\gamma u\|^2_{H^{1,1-(f+1)/2}_{\mathrm{es}}(M)}$ can be absorbed in $(C')^{-1}\|A_\gamma u\|^2_{H^{1,1-(f+1)/2}_{\mathrm{es}}(M)}$, and then one concludes that $\|A_\gamma u\|_{H^{1,1-(f+1)/2}_{\mathrm{es}}(M)}$ is uniformly bounded independent of γ. As $A_\gamma \to A$ strongly, one concludes by a standard argument $Au \in H^{1,1-(f+1)/2}_{\mathrm{es}}(M)$. Thus, $x^{l+(f-1)/2}Au \in H^{1,l}_{\mathrm{es}}(M)$, hence (as $\mathfrak{X} = H^{1,l}_{\mathrm{es}}(M)$)

$$\mathrm{ell}(A) \cap \mathrm{WF}^{s,0}_{\mathrm{eb},\mathfrak{X}}(u) = \emptyset,$$

completing the iterative step. □

As already mentioned, the key ingredient is the following lemma.

Lemma 5.31. *For Neumann boundary conditions, let $\mathfrak{X} = H^{1,l+1}_{\mathrm{es}}(M)$, $\mathfrak{Y} = \dot H^{-1,l-1}_{\mathrm{es}}(M)$; for Dirichlet boundary conditions let $\mathfrak{X} = H^{1,l+1}_{\mathrm{es},0}(M)$, $\mathfrak{Y} = H^{-1,l-1}_{\mathrm{es}}(M)$. Let $K \subset {}^{\mathrm{eb}}S^*M$ be compact, $U \subset {}^{\mathrm{eb}}S^*M$ open, $K \subset U$. Suppose that $\mathcal{A}$ is a bounded family of ps.d.o's in $\Psi^{m,l+(f+1)/2}_{\mathrm{eb}}(M)$ with $\mathrm{WF}'(\mathcal{A}) \subset K$, such that for $A \in \mathcal{A}$, $A \in \Psi^{m-1,l+(f+1)/2}_{\mathrm{eb}}(M)$ (but the bounds for A in $\Psi^{m-1,l+(f+1)/2}_{\mathrm{eb}}(M)$ are not necessarily uniform in A!). Then there exist $G \in \Psi^{m-1/2,0}_{\mathrm{eb}}(M)$, $\tilde G \in \Psi^{m,0}_{\mathrm{eb}}(M)$ with $\mathrm{WF}' G, \mathrm{WF}' \tilde G \subset U$ and $C_0 > 0$ such that for $\epsilon > 0$, $A \in \mathcal{A}$, one has in both the Neumann and the Dirichlet case, with the respective spaces $\mathfrak{X}$ and $\mathfrak{Y}$,*

$$\begin{aligned}
&u \in \mathfrak{X},\ \mathrm{WF}^{m-1/2,0}_{\mathrm{eb},\mathfrak{X}}(u) \cap U = \emptyset,\ \mathrm{WF}^{m,0}_{\mathrm{eb},\mathfrak{Y}}(\Box u) \cap U = \emptyset \Rightarrow\\
&|\langle \partial_t Au, \partial_t Au\rangle - \langle d_X Au, d_X Au\rangle|\\
&\qquad \leq \epsilon\|Au\|^2_{H^{1,1-(f+1)/2}_{\mathrm{es}}(M)} + C_0\big(\|u\|^2_{\mathfrak{X}} + \|Gu\|^2_{\mathfrak{X}}\\
&\qquad\qquad + \epsilon^{-1}\|\Box u\|^2_{\mathfrak{Y}} + \epsilon^{-1}\|\tilde G \Box u\|^2_{\mathfrak{Y}}\big).
\end{aligned} \tag{5.13}$$

Remark 5.32. Recall that $v \in H^{1,1-(f+1)/2}_{\mathrm{es}}(M)$ is equivalent to $d_X u \in L^2_g(M)$, $\partial_t u \in L^2_g(M)$ and $x^{-1}v \in L^2_g(M)$, so $\epsilon\|Au\|^2_{H^{1,1-(f+1)/2}_{\mathrm{es}}(M)}$ on the right hand side of (5.13) is comparable to the terms $\langle \partial_t Au, \partial_t Au\rangle$ and $\langle d_X Au, d_X Au\rangle$. However, if A is supported away from ${}^{\mathrm{eb}}\Sigma$, the Dirichlet form is microlocally elliptic, so this term can be absorbed into the left hand side, as was done in Proposition 5.30.

The hypotheses in (5.13) assure that the other terms on the right hand side are finite, independent of $A \in \mathcal{A}$.

Proof. Again, this follows the argument as Lemma 4.2 and 4.4 of [34] and Lemma 8.8 and 8.9 of [25], so we only sketch the proof. We sketch the Neumann argument; the Dirichlet case needs only simple changes. We have

$$\langle \partial_t u, \partial_t A^* A u\rangle - \langle d_X u, d_X A^* A u\rangle = \langle \Box u, A^* A u\rangle$$

for all $u \in \mathfrak{X}$ and $A \in \Psi_{\mathrm{eb}}^{m-1,l+(f-1)/2}(M)$ since $A^*Au \in H_{\mathrm{es}}^{1,-l-(f-1)}(M)$, which is mapped by $\Box$ into $\dot{H}_{\mathrm{es}}^{-1,-l-(f+1)}(M) = (H_{\mathrm{es}}^{1,l}(M))^*$. Modulo commutator terms, one can rewrite the left hand side as

$$\langle \partial_t Au, \partial_t Au\rangle - \langle d_X Au, d_X Au\rangle,$$

which is the left hand side of (5.13). The commutator terms can be estimated by the second and third terms (which do not depend on ϵ) on the right hand side of (5.13). The other terms on the right hand side arise by estimating (using that the dual of $H_{\mathrm{es}}^{1,l}(M)$ is $\dot{H}_{\mathrm{es}}^{-1,-l-(f+1)}(M)$)

$$\begin{aligned}
|\langle \Box u, A^* A u\rangle| &\leq \|A\Box u\|_{\dot{H}_{\mathrm{es}}^{-1,-1-(f+1)/2}(M)} \|Au\|_{H_{\mathrm{es}}^{1,1-(f+1)/2}(M)} \\
&\leq \epsilon^{-1}\|A\Box u\|^2_{\dot{H}_{\mathrm{es}}^{-1,-1-(f+1)/2}(M)} + \epsilon\|Au\|^2_{\dot{H}_{\mathrm{es}}^{-1,-1-(f+1)/2}(M)} \\
&= \epsilon^{-1}\|x^{-l-(f+1)/2}A\Box u\|^2_{\dot{H}_{\mathrm{es}}^{-1,l-1}(M)} + \epsilon\|Au\|^2_{\dot{H}_{\mathrm{es}}^{-1,-1-(f+1)/2}(M)},
\end{aligned}$$

and as $x^{-l-(f+1)/2}A$ is uniformly bounded in $\Psi_{\mathrm{eb}}^{m,0}(M)$, with wave front set in K, $\|x^{-l-(f+1)/2}A\Box u\|^2_{\mathfrak{Y}}$ can be estimated by a multiple of $\|\Box u\|^2_{\mathfrak{Y}} + \|\tilde{G}\Box u\|^2_{\mathfrak{Y}}$ in view of Lemma 5.16. This completes the proof. $\Box$

The following is analogous to Lemma 7.1 of [34] and Lemma 9.8 of [25] and states that near $\mathcal{G}$ the fiber derivatives $x^{-1}D_{z'_i}$ of microlocalized solutions Au to the wave equation can be controlled by a small multiple of the time derivative, modulo error terms (note that G is lower order than A by $1/2$). The theorem mentions a δ-neighborhood of a compact set $K \subset \mathcal{G}$ (for $\delta < 1$); by this we mean the set of points of distance $< \delta$ from K with respect to the distance induced by some Riemannian metric on ${}^{\mathrm{eb}}S^*M$. Note that if d and d' are the distance functions of two such Riemannian metrics, then there is a constant $c > 0$ such that every δ-neighborhood of K with respect to d is contained in a $c\delta$-neighborhood with respect to d', and vice versa, with c independent of δ (as $\delta \to 0$), so up to possibly rescaling, the choice of the Riemannian metric is not important. In particular, $\mathcal{G}$ is defined by $x = 0$, $z' = 0$, $1 - h(y,\hat{\eta}) - \hat{\xi}^2 - k(y,z,\hat{\zeta}' = 0,\hat{\zeta}'') = 0$, so the set given by

$$x < C'\delta,\ |z'| < C'\delta,\ |1 - h(y,\hat{\eta}) - \hat{\xi}^2 - k(y,z,\hat{\zeta}' = 0,\hat{\zeta}'')| < C'\delta,$$

is contained in a $C''\delta$-neighborhood of $\mathcal{G}$ for some $C'' > 0$, with C'' independent of δ (as long as C' is bounded).

Lemma 5.33. *For Dirichlet or Neumann boundary conditions let $\mathfrak{X}$ and $\mathfrak{Y}$ be as in Lemma 5.31.*

Let $K \Subset \mathcal{G}$. There exists $\delta_0 \in (0,1)$ and $C_0 > 0$ with the following property.

Let $0 < \delta < \delta_0$*, and* $\delta > 0$*, and let* U *be a* δ*-neighborhood of* K *in* ${}^{\mathrm{eb}}S^*M$*. Suppose* $\mathcal{A}$ *is a bounded family of ps.d.o's in* $\Psi_{\mathrm{eb}}^{m,l+(f+1)/2}(M)$ *with* $\mathrm{WF}'(\mathcal{A}) \subset U$*, such that for* $A \in \mathcal{A}$*,* $A \in \Psi_{\mathrm{eb}}^{m-1,l+(f+1)/2}(M)$*. Then there exist* $G \in \Psi_{\mathrm{eb}}^{m-1/2,0}(M)$*,* $\tilde{G} \in \Psi_{\mathrm{eb}}^{m,0}(M)$ *with* $\mathrm{WF}' G, \mathrm{WF}' \tilde{G} \subset U$ *and* $\tilde{C} = \tilde{C}(\delta) > 0$ *such that for* $A \in \mathcal{A}$*, one has in both the Neumann and the Dirichlet case, with the respective spaces* $\mathfrak{X}$ *and* $\mathfrak{Y}$*, that*

$$u \in \mathfrak{X},\ \mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m-1/2,0}(u) \cap U = \emptyset,\ \mathrm{WF}_{\mathrm{eb},\mathfrak{Y}}^{m,0}(\Box u) \cap U = \emptyset$$

implies

$$\begin{aligned}\sum \left\| x^{-1} D_{z_i'} A u \right\|^2 \\ \leq C_0 \delta \|D_t A u\|^2 + \tilde{C}\big(\|u\|_{\mathfrak{X}}^2 + \|Gu\|_{\mathfrak{X}}^2 + \|\Box u\|_{\mathfrak{Y}}^2 + \|\tilde{G}\Box u\|_{\mathfrak{Y}}^2\big).\end{aligned}$$

Proof. This is an analogue of Lemma 7.1 of [34] and Lemma 9.8 of [25], so we only indicate the main idea. By Lemma 5.31 one has control of the Dirichlet form in terms of the second through fifth terms on the right hand side, so it suffices to check that $\sum \left\| x^{-1} D_{z_i'} A u \right\|^2$ can be controlled by the Dirichlet form and $\delta \|D_t A u\|^2$. This uses that $K \Subset \mathcal{G}$, D_t is elliptic on ${}^{\mathrm{eb}}\Sigma$, and $\langle \tilde{H} A u, A u\rangle$ is small as $\mathrm{WF}'(A) \subset U$; see the aforementioned Lemma 7.1 of [34] and Lemma 9.8 of [25] for details. □

Corollary 5.34. *For Dirichlet or Neumann boundary conditions let* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *be as in Lemma 5.31. Let* $K \Subset \mathcal{G}$*,* $\delta > 0$*.*

Then there exists a neighborhood U *of* K *in* ${}^{\mathrm{eb}}S^*M$ *with the following property. Suppose that* $\mathcal{A}$ *is a bounded family of ps.d.o's in* $\Psi_{\mathrm{eb}}^{m,l+(f+1)/2}(M)$ *with* $\mathrm{WF}'(\mathcal{A}) \subset U$*, such that for* $A \in \mathcal{A}$*,* $A \in \Psi_{\mathrm{eb}}^{m-1,l+(f+1)/2}(M)$*. Then there exist* $G \in \Psi_{\mathrm{eb}}^{m-1/2,0}(M)$*,* $\tilde{G} \in \Psi_{\mathrm{eb}}^{m,0}(M)$ *with* $\mathrm{WF}' G, \mathrm{WF}' \tilde{G} \subset U$ *and* $\tilde{C} = \tilde{C}(\delta) > 0$ *such that for* $A \in \mathcal{A}$*, one has in both the Neumann and the Dirichlet case, with the respective spaces* $\mathfrak{X}$ *and* $\mathfrak{Y}$*, that*

$$u \in \mathfrak{X},\ \mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m-1/2,0}(u) \cap U = \emptyset,\ \mathrm{WF}_{\mathrm{eb},\mathfrak{Y}}^{m,0}(\Box u) \cap U = \emptyset$$

implies

$$\begin{aligned}\sum \left\| x^{-1} D_{z_i'} A u \right\|^2 \\ \leq \delta \|D_t A u\|^2 + C\big(\|u\|_{\mathfrak{X}}^2 + \|Gu\|_{\mathfrak{X}}^2 + \|\Box u\|_{\mathfrak{Y}}^2 + \|\tilde{G}\Box u\|_{\mathfrak{Y}}^2\big).\end{aligned}$$

Proof. Fix a Riemannian metric on ${}^{\mathrm{eb}}S^*M$. Let δ_0, C_0 be as in Lemma 5.33, and let $\delta' = \min(\delta_0/2, \delta/C_0)$. Applying Lemma 5.33 with δ' in place of δ gives the desired conclusion, if we let U be a δ'-neighborhood of K. □

Recall now that $C = \{x = 0,\ z' = 0\}$ denotes one boundary face of $\tilde{W}$ in local coordinates, and that as a vector field on ${}^{\mathrm{es}}T^*M$ tangent to $\tilde{W}$ (but not necessarily the other boundary faces), restricted to ${}^{\mathrm{es}}T^*_{\tilde W}M$, H_{es} is given by

$$-\frac{1}{2}\mathsf{H}_{\mathrm{es}} = \hat{\underline{\xi}}x\partial_x - \hat{\underline{\xi}}\underline{\sigma}\partial_{\underline{\sigma}} - \hat{\underline{\xi}}\hat{\underline{\zeta}}\partial_{\hat{\underline{\zeta}}} + K^{ij}\hat{\underline{\zeta}}_i\partial_{z_j} + K^{ij}\hat{\underline{\zeta}}_i\hat{\underline{\zeta}}_j\partial_{\hat{\underline{\xi}}} - \frac{1}{2}\frac{\partial K^{ij}}{\partial z_k}\hat{\underline{\zeta}}_i\hat{\underline{\zeta}}_j\partial_{\hat{\underline{\zeta}}_k};$$

see (3.17)-(3.18). We can expand the K^{ij} terms by breaking them up into z' and z'' components at C, using (5.10). This becomes particularly interesting at a point $q \in {}^{\mathrm{es}}\Sigma$ which is the unique point in the preimage of $p \in {}^{\mathrm{eb}}S^*_C M \cap \mathcal{G}$ under $\widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}$. At such points $\hat{\underline{\zeta}}' = 0$, so many terms vanish. One thus obtains

$$-\frac{1}{2}\mathsf{H}_{\mathrm{es}}(q) = \hat{\underline{\xi}}x\partial_x - \hat{\underline{\xi}}\underline{\sigma}\partial_{\underline{\sigma}} - \hat{\underline{\xi}}\hat{\underline{\zeta}}''\partial_{\hat{\underline{\zeta}}''} + k_{2,ij}\hat{\underline{\zeta}}''_i\partial_{z''_j} + k_{2,ij}\hat{\underline{\zeta}}''_i\hat{\underline{\zeta}}''_j\partial_{\hat{\underline{\xi}}} - \frac{1}{2}\frac{\partial k_{2,ij}}{\partial z''_k}\hat{\underline{\zeta}}''_i\hat{\underline{\zeta}}''_j\partial_{\hat{\underline{\zeta}}''_k}.$$

Pushing forward under $\widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}$, we obtain

$$\begin{aligned}(\widehat{\pi_{\mathrm{es}\to\mathrm{eb}}}_*\mathsf{H}_{\mathrm{es}})(p) = &- 2\hat{\xi}x\partial_x + 2\hat{\xi}\sigma\partial_\sigma + 2\hat{\xi}\hat{\zeta}''\partial_{\hat{\zeta}''} \\ &- 2k_{2,ij}\hat{\zeta}''_i\partial_{z''_j} - 2k_{2,ij}\hat{\zeta}''_i\hat{\underline{\zeta}}''_j\partial_{\hat{\xi}} + \frac{\partial k_{2,ij}}{\partial z''_k}\hat{\zeta}''_i\hat{\zeta}''_j\partial_{\hat{\zeta}''_k}.\end{aligned}$$

Below, this appears as the vector field $|\tau|V_0$, and will give the direction of propagation at glancing points in Theorem 7.7.

Lemma 5.35. *Let $Q_i = x^{-1}D_{z'_i}$, κ_{ij}, m_i, h be as in Lemma 5.29. For $A \in \Psi^{m,l}_{\mathrm{eb}}(M)$,*

$$\imath[\Box, A^*A] = \sum Q_i^* L_{ij} Q_j + \sum (x^{-1}L_i Q_i + Q_i^* x^{-1} L'_i) + x^{-2}L_0, \tag{5.14}$$

with

$$\begin{aligned}&L_{ij} \in \Psi^{2m-1,2l}_{\mathrm{eb}}(M),\ L_i, L'_i \in \Psi^{2m,2l}_{\mathrm{eb}}(M),\ L_0 \in \Psi^{2m+1,2l}_{\mathrm{eb}}(M),\\ &\sigma_{\mathrm{eb},2m-1}(L_{ij}) = 2aV_{ij}a,\ \textit{where } V_{ij} = \kappa_{ij}(\partial_{\zeta'_i} + \partial_{\zeta'_j} + 2\partial_\xi) + \mathsf{H}_{\mathrm{eb},\kappa_{ij}},\\ &\sigma_{\mathrm{eb},2m}(L_i) = \sigma_{\mathrm{eb},2m}(L'_i) = 2aV_i a,\ \textit{where}\end{aligned}$$

$$V_i = \sum_j \kappa_{ij}\partial_{z'_j} + \frac{1}{2}(m_i\partial_\xi + \mathsf{H}_{\mathrm{eb},m_i}) + \frac{1}{2}m_i(\partial_\xi + \partial_{\zeta'_i}), \tag{5.15}$$

$$\begin{aligned}&\sigma_{\mathrm{eb},2m+1}(L_0) = 2aV_0a,\ V_0 = 2\tilde{h}\partial_\xi + \mathsf{H}_{\mathrm{eb},\tilde{h}} + \sum_i m_i\partial_{z'_i},\\ &\mathrm{WF}'_{\mathrm{eb}}(L_{ij}), \mathrm{WF}'_{\mathrm{eb}}(L_i), \mathrm{WF}'_{\mathrm{eb}}(L'_i), \mathrm{WF}'_{\mathrm{eb}}(L_0) \subset \mathrm{WF}'_{\mathrm{eb}}(A).\end{aligned}$$

In particular, for $f \in \mathcal{C}^\infty({}^{\mathrm{eb}}S^*M)$ with $f|_{\tilde W} = \varpi^*_{\mathrm{eb}}\phi$ for some $\phi \in S^*W$,

$$V_{ij}f|_{\tilde W} = 0,\ V_i f|_{\tilde W} = 0,\ V_0 f|_{\tilde W} = 0. \tag{5.16}$$

Moreover, as smooth vector fields tangent to ${}^{\mathrm{eb}}T^*_{\tilde W}M$ *(but not necessarily tangent to the other boundaries),*

(5.17)
$$\begin{aligned}V_0|_C &= -2\xi\, x\partial_x - 2(\xi^2 + \sum_{ij} k_{2,ij}\zeta''_i\zeta''_j)\partial_\xi - 2\xi(\tau\,\partial_\tau + \eta\,\partial_\eta)\\ &\qquad - 2\sum_{i,j} k_{2,ij}\zeta''_i\partial_{z''_j} + \sum_{\ell,i,j}(\partial_{z''_\ell}k_{2,ij})\zeta''_i\zeta''_j\partial_{\zeta''_\ell},\\ V_{ij}|_C &= -k_{1,ij}(\partial_{\zeta'_i} + \partial_{\zeta'_j} + 2\partial_\xi) + \sum_\ell(\partial_{z''_\ell}k_{1,ij})\partial_{\zeta''_\ell},\ V_i|_C = -\sum_j k_{1,ij}\partial_{z'_j},\end{aligned}$$

and

(5.18)
$$\begin{aligned}&(|\tau|V_0\hat\xi)|_{\tilde W} = -2\sum_{ij} k_{2,ij}(0,y,z)\zeta''_i\zeta''_j\\ &(|\tau|V_i\hat\xi)|_{\tilde W} = -\sum_j k_{3,ij}(0,y,z)\zeta''_j,\ (|\tau|V_{ij}\hat\xi)|_{\tilde W} = -2k_{1,ij}(0,y,z),\\ &(|\tau|^{-s-1}x^{-r}V_0(|\tau|^s x^r))|_{\tilde W} = -2(r+s)\hat\xi,\ (|\tau|^{-s}x^{-r}V_i(|\tau|^s x^r))|_{\tilde W} = 0,\\ &(|\tau|^{-s+1}x^{-r}V_{ij}(|\tau|^s x^r))|_{\tilde W} = 0,\end{aligned}$$

Remark 5.36. This is the main commutator computation that we use in the next section. We stated explicitly the results we need. First, equation (5.16) shows that functions of the "slow variables" do not affect the commutator to leading order at $\tilde W$, hence they are negligible for all of our subsequent calculations.

Next, (5.17) gives the form of the commutator explicitly at C; this is what we need for hyperbolic or glancing propagation within $\tilde W$, i.e. at points of $\mathcal{H}$, resp. $\mathcal{G}$ away from radial points. These are sufficiently local that we only need the explicit calculation *at* C, rather than at all of $\tilde W$.

Finally, (5.18) contains the results we need at radial points in $\mathcal{G}$: there the construction is rather global in $\tilde W$, so it would be insufficient to state these results at C only. On the other hand, localization in $\hat\zeta$ is accomplished by localizing in $\hat\xi$, the "slow variables" and the characteristic set, so fewer features of V_{ij}, etc., are relevant.

Proof. By Lemma 5.29,

$$\begin{aligned}[\Box, A^*A] = &\sum_{i,j}([Q_i^*, A^*A]\kappa_{ij}Q_j + Q_i^*\kappa_{ij}[Q_j, A^*A] + Q_i^*[\kappa_{ij}, A^*A]Q_j)\\ &+ \sum_i(x^{-1}M_i[Q_i, A^*A] + [x^{-1}M_i, A^*A]Q_i\\ &\qquad + [Q_i^*, A^*A]x^{-1}M'_i + Q_i^*[x^{-1}M'_i, A^*A])\\ &+ [x^{-2}\tilde H, A^*A] \quad \text{on } U.\end{aligned}$$

The three terms on the first line of the right hand side are the only ones contributing to L_{ij}; in the case of the third term, via

$$\imath\sigma_{\mathrm{eb},2m-1}([\kappa_{ij}, A^*A]) = \mathsf{H}_{\mathrm{eb},\kappa_{ij}} a^2 = 2a\mathsf{H}_{\mathrm{eb},\kappa_{ij}} a,$$

while in the case of the first two terms by evaluating the commutators using Lemma 5.7 and taking only the A_1-terms, with the notation of the lemma. The A_0-terms of the first two commutators on the first line of the right hand side (with the notation of Lemma 5.7) contribute to L_i or L_i', as do the second and fourth terms on the second line and the A_1-term of the first and third terms on the second line. Finally, the expression on the third line, as well as the A_0-term of the first and third terms on the second line contribute to L_0. We also use (4.2) to remove the weight from $\mathsf{H}_{\mathrm{eb},x^{-1}m_i}$ and $\mathsf{H}_{\mathrm{eb},x^{-2}\tilde{h}}$, e.g. $x^2\mathsf{H}_{\mathrm{eb},x^{-2}\tilde{h}} = -2\tilde{h}\partial_\xi + \mathsf{H}_{\mathrm{eb},\tilde{h}}$.

The computation of the Hamilton vector fields at C then follows from Lemma 5.29 and (4.1) (recalling that t is one of the y-variables). □

6. Coisotropic regularity and non-focusing

In this section we recall from [25] the notion of coisotropic regularity and, dually, that of nonfocusing. We will be working microlocally near $\mathcal{F}_{\mathrm{reg}}$ and in particular, away from the difficulties of the glancing rays in $\mathcal{F}_{\mathrm{sing}}$. Consequently all the results in this section have proofs identical to those in [25, Section 4], where the fiber Z is without boundary.

Let $\mathcal{K}$ be a compact set in $\mathcal{R}^\circ_{\mathrm{eb}}$. By Lemma 3.25, there exists an open set $U \subset {}^{\mathrm{eb}}S^*M$ such that $\mathcal{K} \subset U$ and $\overline{U} \cap \mathcal{F} \subset \mathcal{F}_{\mathrm{reg}}$. Recall from Corollary 3.24 that in this case $\mathcal{F} \cap U$ is a *coisotropic* submanifold of ${}^{\mathrm{eb}}S^*M$—recall from Footnote 7 that a submanifold of ${}^{\mathrm{eb}}S^*M$ is coisotropic if the corresponding conic submanifold of ${}^{\mathrm{eb}}T^*M \setminus o$ is such.

In what follows, we let U be an arbitrary open subset of ${}^{\mathrm{eb}}S^*M$ satisfying $\overline{U} \cap \mathcal{F} \subset \mathcal{F}_{\mathrm{reg}}$, thus $U \cap \mathcal{F}$ is a $\mathcal{C}^\infty$ embedded coisotropic submanifold of ${}^{\mathrm{eb}}S^*M$; the foregoing remarks establish that such subsets are plentiful.

Definition 6.1. Given U as above, let $\mathcal{M}$ denote the module (over $\Psi^{0,0}_{\mathrm{eb}}(M)$) of operators $A \in \Psi^{1,0}_{\mathrm{eb}}(M)$ such that

- $\mathrm{WF}' A \subset U$,
- $\sigma_{\mathrm{eb},1}(A)|_{\mathcal{F}_{\mathrm{reg}}} = 0$.

Let $\mathcal{A}$ be the algebra generated by $\mathcal{M}$ with $\mathcal{A}^k = \mathcal{A} \cap \Psi^{k,0}_{\mathrm{eb}}(M)$.

Let $\mathfrak{X}$ be a Hilbert space on which $\Psi^{0,0}_{\mathrm{eb}}(M)$ acts, and let $K \subset U$. We say that[10] *u has coisotropic regularity of order k relative to $\mathfrak{X}$ in K if there exists $Q \in \Psi^{0,0}_{\mathrm{eb}}(M)$, elliptic on K, such that*

$$\mathcal{A}^k Q u \in \mathfrak{X}.$$

[10]Note that our choice of U containing K does not matter in the definition.

We say that u *satisfies the nonfocusing condition of degree k relative to $\mathfrak{X}$ on K if there exists $Q \in \Psi^{0,0}_{\mathrm{eb}}(M)$, elliptic on K, such that*

$$Qu \in \mathcal{A}^k(\mathfrak{X}).$$

We say that u is nonfocusing resp. coisotropic of degree k relative to $\mathfrak{X}$ on an arbitrary open subset S of $\mathcal{F}$ if for every open $O \subset S$ with closure disjoint from $\mathcal{F}_{\mathrm{sing}}$, it is nonfocusing resp. coisotropic on O of degree k with respect to $\mathfrak{X}$.

Remark 6.2.

(1) u is coisotropic on K if and only if u is coisotropic at every $p \in K$, i.e. on $\{p\}$ for every $p \in K$. This can be seen by a partition of unity and a microlocal elliptic parametrix construction, as usual.

(2) The conditions of coisotropic regularity and nonfocusing should be, loosely speaking, considered to be dual to one another; a precise statement to this effect appears in the proof of Theorem 9.4 below.

(3) Coisotropy and nonfocusing are only of interest on $\mathcal{F}_{\mathrm{reg}}$ itself: off of this set, to be coisotropic of order k with respect to $\mathfrak{X}$ means merely to be microlocally in $H^k_{\mathrm{eb},\mathfrak{X}}$ while to be nonfocusing means to be microlocally in $H^{-k}_{\mathrm{eb},\mathfrak{X}}$.

(4) Certainly, away from $\tilde{W}$, $\sigma(\Box)$ vanishes on $\mathcal{F}_{\mathrm{reg}}$, as the latter lies in the characteristic set Σ by definition. Splitting Σ into component according to the sign of τ, and letting $\Pi_\pm$ be pseudodifferential operators over M° microlocalizing near each of these components, we thus have

$$\Box\Pi_\pm = Q_\pm A_\pm + R$$

with $A_\pm$ in $\mathcal{M}$, $Q_\pm$ elliptic of order 1, and R smoothing.

As a consequence of coisotropy, we have:

Lemma 6.3. *The module $\mathcal{M}$ is closed under commutators, and is finitely generated, i.e., there exist finitely many $A_i \in \Psi^0_{\mathrm{eb}}$ with $\sigma_{\mathrm{eb},1}(A_i)|_{\mathcal{F}_{\mathrm{reg}}} = 0$ such that*

$$\mathcal{M} = \{A \in \Psi^1_{\mathrm{eb}}(U) :\ \exists Q_i \in \Psi^0_{\mathrm{e}}(U),\ A = \sum_{i=0}^N Q_i A_i\}.$$

Moreover we may take A_N to have symbol $|\tau|^{-1}\sigma_{\mathrm{eb},2,0}(x^2\Box)$ and $A_0 = \mathrm{Id}$.

We thus also obtain

$$\mathcal{A}^k = \left\{ \sum_{|\alpha|\leq k} Q_\alpha \prod_{i=1}^N A_i^{\alpha_i},\ Q_\alpha \in \Psi^0_{\mathrm{eb}}(U) \right\} \tag{6.1}$$

where α runs over multiindices $\alpha : \{1,\dots,N\} \to \mathbb{N}_0$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$.

Corollary 6.4. *If u is coisotropic of order k on K relative to $\mathfrak{X}$ then there exists U open, $K \subset U$ such that for $\tilde{Q} \in \Psi_{\mathrm{eb}}^{0,0}(M)$, $\mathrm{WF}'(\tilde{Q}) \subset U$ implies $\tilde{Q}A^\alpha u \in \mathfrak{X}$ for $|\alpha| \leq k$.*

Conversely, suppose U is open and for $\tilde{Q} \in \Psi_{\mathrm{eb}}^{0,0}(M)$, $\mathrm{WF}'(\tilde{Q}) \subset U$ implies $\tilde{Q}A^\alpha u \in \mathfrak{X}$ for $|\alpha| \leq k$. Then for $K \subset U$, u is coisotropic of order k on K relative to $\mathfrak{X}$.

Proof. Suppose first that u is coisotropic of order k on K relative to $\mathfrak{X}$. By definition, there exists Q elliptic on K such that $\mathcal{A}^k Q u \subset \mathfrak{X}$. Let U be such that Q is elliptic on $\overline{U}$, $K \subset U$, and let $S \in \Psi_{\mathrm{eb}}^{0,0}(M)$ be a microlocal parametrix for Q, so $\mathrm{WF}'(R) \cap \overline{U} = \emptyset$ where $R = SQ - \mathrm{Id}$.

We prove the corollary by induction, with the case $k = 0$ being immediate as one can write $\tilde{Q}u = \tilde{Q}SQu + \tilde{Q}Ru$, $\tilde{Q}S \in \Psi_{\mathrm{eb}}^{0,0}(M)$ is bounded on $\mathfrak{X}$, $Qu \in \mathfrak{X}$, $\tilde{Q}R \in \Psi_{\mathrm{eb}}^{-\infty,0}(M)$ (for they have disjoint WF'), so $\tilde{Q}Ru \in \mathfrak{X}$.

Suppose now that $k \geq 1$, and the claim has been proved for $k-1$. By Lemma A.1, applied with $Q_n = Q$ (i.e. there is no need for the subscript n, or for uniformity),

$$A^\alpha Q = QA^\alpha + \sum_{|\beta| \leq |\alpha| - 1} C_\beta A^\beta.$$

Thus, for $|\alpha| = k$,

$$\tilde{Q}A^\alpha u = \tilde{Q}SQA^\alpha + \tilde{Q}RA^\alpha u$$

and

$$\tilde{Q}SQA^\alpha = \tilde{Q}SA^\alpha Q - \sum_{|\beta| \leq |\alpha| - 1} \tilde{Q}SC_\beta A^\beta$$

together with the induction hypothesis (due to which and to $\tilde{Q}SC_\beta \in \Psi_{\mathrm{eb}}^{0,0}(M)$ with $\mathrm{WF}'(\tilde{Q}SC_\beta) \subset U$, $\tilde{Q}SC_\beta A^\beta u \in \mathfrak{X}$) and $\tilde{Q}R \in \Psi_{\mathrm{eb}}^{-\infty,0}(M)$ imply $\tilde{Q}A^\alpha u \in \mathfrak{X}$, providing the inductive step.

The proof of the converse statement is similar. □

We now specialize to the case

$$\mathfrak{H} = L^2_g(I \times X_0)$$

where I is a compact interval. We additionally introduce another Hilbert space $\mathfrak{X} \subset \mathfrak{H}$, given by $H^1_0(I \times X_0)$ or $H^1(I \times X_0)$ with I an interval and the 0 denoting vanishing at $I \times \partial X_0$. Note that $\mathrm{Id} + \Delta : \mathfrak{X} \to \mathfrak{X}^*$ is an isometry.

Suppose K is compact. For $N \geq k + r$ we let $\mathfrak{Y}_K$ denote the subspace of $\mathfrak{H}$

$$\mathfrak{Y}_K = \{u \in \mathfrak{X}^* : \ \mathrm{WF}^N_{\mathrm{b},\mathfrak{H}^*}(u) \subset K, \ u \text{ is coisotropic of order } k \text{ w.r.t. } H^r_{\mathrm{b},\mathfrak{X}^*} \text{ on } K\}.$$

Let

$$\mathfrak{Z}_K = \{\phi \in \mathfrak{X} : u \text{ is coisotropic of order } k \text{ w.r.t. } H^r_{\mathrm{b},\mathfrak{X}} \text{ on } K\}.$$

Lemma 6.5. *Suppose that $K \subset O$, K compact, O open with compact closure, and $Q \in \Psi^0(M)$ such that* $\mathrm{WF}'(\mathrm{Id} - Q) \cap K = \emptyset$, $\mathrm{WF}'(Q) \subset O$. *Let*

$$\mathfrak{Y} = \{u \in \mathfrak{X}^* : \ T_N(\mathrm{Id} - Q)u \in \mathfrak{X}^*, \ |\alpha| \le k \Rightarrow T_r A^\alpha Q u \in \mathfrak{X}^*\} \tag{6.2}$$

and

$$\mathfrak{Z} = \{u \in \mathfrak{X} : \ |\alpha| \le k \Rightarrow T_r A^\alpha Q u \in \mathfrak{X}\} \tag{6.3}$$

Then

$$\mathfrak{Y}_K \subset \mathfrak{Y} \subset \mathfrak{Y}_{\overline{O}}.$$

and

$$\mathfrak{Z}_K \subset \mathfrak{Z} \subset \mathfrak{Z}_{\overline{O}}.$$

Proof. If $u \in \mathfrak{Y}_K$, then $\mathrm{WF}^N_{\mathrm{b},\mathfrak{H}^*}(u) \subset K$ implies that $T_N(\mathrm{Id} - Q)u \in \mathfrak{X}^*$. Moreover, since u is coisotropic on K, it is coisotropic on a neighborhood O' of K; we construct $Q' \in \Psi^0(M)$ with $\mathrm{WF}\, Q' \subset O'$, $\mathrm{WF}(\mathrm{Id} - Q') \cap K = \emptyset$. Then

$$T_r A^\alpha Q u = T_r A^\alpha Q' Q u + T_r A^\alpha (\mathrm{Id} - Q') Q u,$$

and the first term is in $\mathfrak{X}^*$ by coisotropy of u on O' while the latter is in $\mathfrak{X}^*$ by the wavefront condition on u.

On the other hand, if $u \in \mathfrak{Y}$, we have $T_N(\mathrm{Id} - Q)u \in \mathfrak{X}^*$ hence $\mathrm{WF}^N_{\mathrm{b},\mathfrak{H}^*}(u) \cap \mathrm{ell}(\mathrm{Id} - Q) = \emptyset$, so in particular, $\mathrm{WF}^N_{\mathrm{b},\mathfrak{H}^*}(u) \subset O^c$, since $\mathrm{Id} - Q$ must be elliptic on O^c. It remains, given $p \in \overline{O}$, to check coisotropic regularity at p. If $p \in \mathrm{ell}(\mathrm{Id} - Q)$, it again follows from the wavefront set condition, hence it suffices to consider $p \in \mathrm{ell}\, Q \supset [\mathrm{ell}(\mathrm{Id} - Q)]^c$; at such points coisotropic regularity follows from $T_r A^\alpha Q u \in \mathfrak{X}^*$.

The proof for $\mathfrak{Z}$ works analogously. □

Corollary 6.6. *Suppose $K = \cap_j O_j$, O_j open with compact closure, $\overline{O_{j+1}} \subset O_j$. Let $\mathfrak{Y}_j$, $\mathfrak{Z}_j$ be given by* (6.2), (6.3) *where Q_j satisfies* $\mathrm{WF}'(\mathrm{Id} - Q_j) \cap K = \emptyset$, $\mathrm{WF}'(Q_j) \subset O$. *Then $\mathfrak{Y}_K = \cap_j \mathfrak{Y}_j$, $\mathfrak{Z}_K = \cap_j \mathfrak{Z}_K$.*

In particular, $\mathfrak{Y}_K$ and $\mathfrak{Z}_K$ become Fréchet spaces when equipped with the $\mathfrak{Y}_j$, $\mathfrak{Z}_j$ norms.

Remark 6.7. It is easy to see that the Fréchet topology is independent of the choice of the particular O_j.

Proof. The fact that $\mathfrak{Y}_K \subset \cap_j \mathfrak{Y}_j$ follows from Lemma 6.5. For the reverse inequality, note that $u \in \cap \mathfrak{Y}_j \subset \cap \mathfrak{Y}_{\overline{O_j}}$ has $\mathrm{WF}_{\mathrm{b},\mathfrak{H}^*}(u) \subset \cap \overline{O_j} = K$. On the other hand, as $u \in \mathfrak{Y}_1$, $|\alpha| \le k \Rightarrow A^\alpha Q_1 u \in \mathfrak{X}$ and Q_1 is elliptic on K. Thus, $u \in \mathfrak{Y}_K$.

The same holds for $\mathfrak{Z}_K$. □

We now note the following functional-analytic facts:

Lemma 6.8. *Let Q be as above, and again let*

$$\mathfrak{Y} = \{u \in \mathfrak{X}^* : \ T_N(\mathrm{Id} - Q)u \in \mathfrak{X}^*, \ |\alpha| \leq k \Rightarrow T_r A^\alpha Q u \in \mathfrak{X}^*\}, \tag{6.4}$$

and

$$\mathfrak{Z} = \{u \in \mathfrak{X} : \ |\alpha| \leq k \Rightarrow T_r A^\alpha Q u \in \mathfrak{X}\}$$

Then the dual of $\mathfrak{Y}$ with respect to the space $\mathfrak{H}$ (see Appendix A) is

$$\mathfrak{Y}^* = \{u : \ u = v_0 + T_N(\mathrm{Id} - Q)v_1 + \sum_{|\alpha| \leq k} T_r A^\alpha Q v_\alpha, \ v_0, v_1, v_\alpha \in \mathfrak{X}\},$$

and the dual of $\mathfrak{Z}$ with respect to $\mathfrak{H}$ is

$$\mathfrak{Z}^* = \{u : \ u = v_0 + \sum_{|\alpha| \leq k} T_r A^\alpha Q v_\alpha, \ v_0, v_\alpha \in \mathfrak{X}^*\},$$

Proof. First consider the dual of $\mathfrak{Z}$ with respect to $\mathfrak{H}$. We apply the discussion of Appendix A leading to (A.7). More precisely, with the notation of the Appendix, we take $\mathfrak{H} = L^2_g(I \times X_0)$, and $\mathfrak{X} = H^1(I \times X_0)$, resp. $\mathfrak{X} = H^1_0(I \times X_0)$, as set out earlier. We also let $\mathcal{D} = \mathcal{C}^\infty(I \times X_0)$, resp. $\mathcal{D} = \dot{\mathcal{C}}^\infty(I \times X_0)$ (with the dot indicating infinite order vanishing at $I \times \partial X_0$). We define the operators B_k in (A.6) as follows: we take B_k, $k = 1, \ldots, N$, to be a collection of $\mathcal{C}^\infty$ vector fields on X_0 which span $\mathcal{V}(X)$ over $\mathcal{C}^\infty(X)$, $B_0 = \mathrm{Id}$, and define the $\mathfrak{X}$-norm on $\mathcal{D}$ by

$$\|u\|^2_{\mathfrak{X}} = \|B_0 u\|^2_{\mathfrak{H}} + \sum_{k=1}^N \|B_k u\|^2_{\mathfrak{H}}, \tag{6.5}$$

cf. (A.4); then $\mathfrak{X}$ is the completion of $\mathcal{D}$. Then we take the collection of A_j in defining the space $\mathfrak{Y}$ in Appendix A, with the norm (A.3), to be $T_r A_\alpha Q$, $|\alpha| \leq k$. Then our claim about $\mathfrak{Z}^*$ follows from (A.7) and (A.1), taking into account that the principal symbol of the conjugate a pseudo-differential operator by complex conjugation is the complex conjugate of the principal symbol of the original operator, so its vanishing on $\mathcal{F}_{\mathrm{reg}}$ is unaffected.

We now consider the dual of $\mathfrak{Y}$, with $\mathfrak{Y}$ given by (6.4). As $\mathrm{Id} + \Delta : \mathfrak{X} \to \mathfrak{X}^*$ is an isomorphism, the norm on $\mathfrak{X}^*$ is given by

$$\|u\|_{\mathfrak{X}^*} = \|(\mathrm{Id} + \Delta)^{-1} u\|_{\mathfrak{X}} = \sum_{k=0}^N \|B_k (\mathrm{Id} + \Delta)^{-1} u\|^2_{\mathfrak{H}},$$

with B_k as in (6.5), we are again in the setting leading to (A.7) with $\mathfrak{X}$ in the Appendix given by our $\mathfrak{X}^*$, the B_k in the Appendix given by $B_k(\mathrm{Id} + \Delta)^{-1}$, the space $\mathfrak{Y}$ in the Appendix being our space $\mathfrak{Y}$ in (6.4), and the A_j in the appendix given by $T_N(\mathrm{Id} - Q)$ and $T_r A^\alpha Q$, $|\alpha| \leq k$. Then our claim about $\mathfrak{Y}^*$ follows from (A.7) and (A.1). $\square$

Now let

$$\widetilde{\mathfrak{Y}}_K = \{u \in T_N(\mathfrak{X}) : \ u \text{ is nonfocusing of order } k \ \text{ w.r.t. } H^{-r}_{\mathrm{b},\mathfrak{X}} \text{ on } K\},$$

$$\widetilde{\mathfrak{Z}}_K = \{u \in \mathfrak{X}^* : \ u \text{ is nonfocusing of order } k \ \text{ w.r.t. } H^{-r}_{\mathrm{b},\mathfrak{X}^*} \text{ on } K\}.$$

Lemma 6.9. *Define* $\mathfrak{Y}$, $\mathfrak{Z}$ *as above. Then*

$$\widetilde{\mathfrak{Y}}_K \subset \mathfrak{Y}^* \subset \widetilde{\mathfrak{Y}}_{\overline{O}}.$$

and

$$\widetilde{\mathfrak{Z}}_K \subset \mathfrak{Z}^* \subset \widetilde{\mathfrak{Z}}_{\overline{O}}.$$

The proof follows that of Lemma 6.5 closely, using the characterization of $\mathfrak{Y}^*$ and $\mathfrak{Z}^*$ from Lemma 6.8.

We remark that away from $\tilde{W}$, we may always (locally) conjugate by an FIO to a convenient normal form: being coisotropic, locally $\mathcal{F}$ can be put in a model form $\zeta = 0$ by a symplectomorphism Φ in some canonical coordinates (y, z, η, ζ), see [10, Theorem 21.2.4] (for coisotropic submanifolds one has $k = n - l$, $\dim S = 2n$, in the theorem). We may moreover arrange the (z, ζ) coordinates (i.e. apply a further symplectomorphism) so that $\sigma(\Box) \circ \Phi = q\zeta_1$ for some symbol q elliptic in a small open set. We now quantize Φ to a FIO T, elliptic on some small neighborhood of a $w \in \mathcal{F}_{\mathrm{reg}}$, which can be arranged to have the following properties:

- $T\Box = QD_{z_1}T + R$ where $Q \in \Psi^1(M^\circ)$ is elliptic near $\Phi(w)$ and R is a smoothing operator.
- u has coisotropic regularity of order k (near w) with respect to H^s if and only if $D_z^\alpha Tu \in H^s$ whenever $|\alpha| < k$.
- u is nonfocusing of order k (near w) with respect to H^s if and only if $Tu \in \sum_{|\alpha| \leq k} D_z^\alpha H^s$.

Let $G \in \Psi^{-1}(M^\circ)$ be a parametrix for Q. As a consequence of the above observation, $\Box u = f$ implies that $D_{z_1}Tu - GTf \in \mathcal{C}^\infty$ microlocally near $\Phi(w)$, and if f is coisotropic of order k relative to H^{s-1}, then $D_z^\alpha GTf \in H^s$ for $|\alpha| \leq k$ (with an analogous statement for non-focusing) hence we have now sketched the proof of the following:

Proposition 6.10. *Suppose u is a distribution on M°, $\Box u = f$. If f is coisotropic of order k, resp. nonfocusing of order k, with respect to H^{s-1} then the coisotropic regularity of order k, resp. non-focusing regularity of order k, with respect to H^s, is invariant under the Hamilton flow over M°.*

In particular, for a solution to the wave equation, coisotropic regularity of order k with respect to H^s and nonfocusing of order k with respect to H^s are invariant under the Hamilton flow over M°.

(We remark that one could certainly give an alternative proof of this proposition by positive commutator arguments similar to, but much easier than, those used for propagation of edge regularity in the following section.)

Corollary 6.11. *Suppose that f is coisotropic, resp. non-focusing, of order k relative to H^{m-1}, supported in $t > T$. Let u be the unique solution of $\Box u = f$ with Dirichlet or Neumann boundary conditions, supported in $t > T$. Then u is coisotropic, resp. non-focusing, of order k relative to H^m at*

$p \in S^*M^\circ$ provided every[11] *GBB γ with $\gamma(0) = p$ has the property that there exists s_0 such that $t(\gamma(s_0)) < T$, and for $s \in [0, s_0]$ (or $s \in [s_0, 0]$, if $s_0 < 0$), $\gamma(s) \in S^*M^\circ$.*

The analogous statements hold if f is supported in $t < T$, and u is the unique solution of $\Box u = f$ supported in $t < T$, provided we replace $t(\gamma(s_0)) < T$ by $t(\gamma(s_0)) > T$.

Proof. This is an immediate consequence of Proposition 6.10, taking into account that u is coisotropic, resp. non-focusing, in $t < T$, by virtue of vanishing there. □

If $K \subset M^\circ$ is compact, then there is $\delta > 0$ such that if $p \in S^*_K M^\circ$ and γ is a GBB with $\gamma(0) = p$, then for $s \in (-\delta, \delta)$, $\gamma(s) \in M^\circ$. As s is equivalent to t as a parameter along GBB , we deduce the following result.

Corollary 6.12. *Suppose $K \subset M^\circ$ is compact. Suppose that f is coisotropic, resp. non-focusing, of order k relative to H^{m-1}, supported in $t > T$. Let u be the unique solution of $\Box u = f$ with Dirichlet or Neumann boundary conditions, supported in $t > T$. Then there exists $\delta_0 > 0$ such that u is coisotropic, resp. non-focusing, of order k relative to H^m at $p \in S^*_K M^\circ$ if $t(p) < T + \delta_0$.*

Of course, what happens to coisotropic regularity and nonfocusing when bicharacteristics reach $\tilde{W}$ is of considerable interest, and will be discussed below.

7. Edge propagation

This section contains a series of theorems that will enable us to track propagation of regularity into and back out of the edge. They are as follows:

- Theorem 7.1, which governs propagation of regularity into and out of the interior or $\tilde{W}$ as well as the microlocal propagation of coisotropic regularity there (i.e. iterated regularity under application of operators in $\mathcal{A}$).
- Theorem 7.3, which governs propagation of regularity into $\tilde{W}$ along glancing rays, tangent to one or more of the boundary faces meeting at $\mathbb{R} \times Y$ (in the blown-down picture).
- Theorem 7.6, which governs the propagation of edge regularity at non-radial hyperbolic points at the boundary of the edge $\tilde{W}$.
- Theorem 7.7, which governs the propagation of edge regularity at glancing points at the boundary of the edge $\tilde{W}$.

These theorems will then be assembled (together with the propagation over the interior of the edge, which we may simply quote from [25]) to yield the propagation of coisotropic regularity into and out of the edge in Theorem 8.1,

[11]The restriction of this GBB to $[0, s_0]$, if $s_0 \geq 0$, or $[s_0, 0]$ if $s_0 < 0$, is unique under these assumptions.

and this result is the key ingredient in proving the "geometric" improvement in regularity on the diffracted wave.

7.1. **Radial points in the interior of the edge.** The following theorem enables us to track edge wavefront set entering and leaving the edge at radial points over its interior.

Theorem 7.1 (Propagation at radial points in the interior of the edge). *(See* [25, Theorem 11.1]*) Let $u \in H^{1,l}_{\mathrm{es}}(M)$ solve $\Box u = 0$ with Dirichlet or Neumann boundary conditions.*

1. *Let $m > l + f/2$. Given $\alpha \in \mathcal{H}_{W,\mathrm{b}}$, and $p \in \mathcal{R}^{\circ}_{\mathrm{eb},\alpha,I}$, if $(\mathcal{F}_{I,p} \backslash \partial M) \cap \mathrm{WF}^m Au = \emptyset$, for all $A \in \mathcal{A}^k$ then $p \notin {\mathrm{WF}_{\mathrm{e}}}^{m,l'} Bu$ for all $l' < l$ and all $B \in \mathcal{A}^k$.*
2. *Let $m < l + f/2$. Given $\alpha \in \mathcal{H}_{W,\mathrm{b}}$, $p \in \mathcal{R}^{\circ}_{\mathrm{eb},\alpha,O}$, if a neighborhood U of p in ${}^{\mathrm{e}}S^*|_{\partial M} M$ is such that ${\mathrm{WF}_{\mathrm{e}}}^{m,l}(Au) \cap U \subset \partial \mathcal{F}_O$ for all $A \in \mathcal{A}^k$ then $p \notin {\mathrm{WF}_{\mathrm{e}}}^{m,l}(Bu)$ for all $B \in \mathcal{A}^k$.*

This theorem is literally the same theorem as [25, Theorem 11.1] as we are restricting our attention to the interior of the fibers, hence the presence of a fiber boundary in our setting is irrelevant. We thus refer the reader to [25] for the proof.

Remark 7.2. In fact, we could take $u \in H^{-\infty,l}_{\mathrm{es}}(M)$ here, but the restriction on regularity will be necessary in later theorems to maintain the boundary condition at the side faces $z_i' = 0$, and we prefer to keep a uniform hypothesis. Moreover, the boundary conditions are irrelevant; again, they are stated for the sake of uniformization.

7.2. **Propagation into radial points over the boundary of the edge.** We now turn to the question of propagation into the edge at glancing points, i.e. at points over the boundary of the fibers of M. Note that the hypotheses of this theorem are *global* in the boundary of the fiber: we do not attempt to distinguish different points in the fiber boundary.

Theorem 7.3 (Propagation into radial points over the boundary of the edge). *Let $u \in \mathfrak{X} \equiv H^{1,l}_{\mathrm{es}}(M)$ solve $\Box u = 0$ with Dirichlet or Neumann boundary conditions (see Definition 5.20).*

Let $m > l + f/2 - 1$ with $m \geq 0$, $l' \leq 0$. Suppose that $q \in \mathcal{H}_{W,\mathrm{b}}$ and there exists a neighborhood U of

$$\mathcal{R}_{\mathrm{eb},q,I} \cap {}^{\mathrm{eb}}S^*_{\partial \tilde{W}} M = \mathcal{R}_{\mathrm{eb},q,I} \cap \mathcal{G}$$

in ${}^{\mathrm{eb}}S^*M$ such that $\{x > 0\} \cap U \cap \mathrm{WF}^{m,*}_{\mathrm{eb},\mathfrak{X}} u = \emptyset$. Then ${}^{\mathrm{eb}}S^*_{\partial \tilde{W}} M \cap \mathrm{WF}^{m,l'}_{\mathrm{eb},\mathfrak{X}} u = \emptyset$ for all $l' < 0$.

Proof. Choose local coordinates on W, and let $q = (y_0, t_0, \hat{\tau}_0 \in \{\pm 1\}, \hat{\eta}_0) \in \mathcal{H}_{W,\mathrm{b}}$. Choose ξ_0 such that $\hat{\xi}_0^2 = 1 - h(y, \hat{\eta}_0)$ with $\operatorname{sgn} \hat{\xi}_0 = \operatorname{sgn} \tau_0$ (this is the incoming point).

One of the central issues in proving the theorem is to construct a symbol that is localized in the hypothesis region that is sufficiently close to being flow-invariant. To begin, we will need a localizer in the fiber variables. Fix any $K \Subset \tilde{W}^\circ$ and fix a small number $\epsilon_K > 0$. Let

$$\Upsilon : {}^{\mathrm{e}}S^*_K(M) \cap \{|\underline{\zeta}|/|\underline{\xi}| < \epsilon_K\} \to Z$$

be locally defined by

$$\Upsilon(q') = z(\exp_{z,\hat{\underline{\zeta}}} s_\infty \mathsf{H}_{\mathrm{es}}), \quad s_\infty = \frac{\operatorname{sgn} \hat{\underline{\xi}}}{\left|\hat{\underline{\zeta}}\right|_{K_q}} \arctan \frac{\left|\hat{\underline{\zeta}}\right|}{\hat{\underline{\xi}}}.$$

where $q' \in {}^{\mathrm{es}}S^*_K(M)$ has "edge-smooth" coordinates $(t, y, z, \tau, \xi, \eta, \zeta)$ (we are using the canonical identification of ${}^{\mathrm{e}}S^*M$ with ${}^{\mathrm{es}}S^*M$ away from $\partial M \backslash \tilde{W}$). This map is well-defined provided ϵ_K is chosen sufficiently small (so that the flow stays away from $\partial\tilde{W}$). The map simply takes a point over the boundary to its limit point in the fiber variables along the forward bicharacteristic flow, hence on $\tilde{W}$, we certainly have $\Upsilon_*(\mathsf{H}_{\mathrm{es}}) = 0$.

We now employ Υ to create a localizer away from $\partial\tilde{W}$. Fix

$$K'' \subset U' \subset K' \subset U \subset K \subset \tilde{W}^\circ$$

with K'', K', K compact and U', U open such that

(1) $\alpha \in {}^{\mathrm{es}}S^*_{K\backslash U}M$ and $\left|\hat{\underline{\zeta}}(\alpha)/\hat{\underline{\xi}}(\alpha)\right| < \epsilon_K$ imply $\Upsilon_K(\alpha) \in \tilde{W} \setminus K'$,

(2) $\alpha \in {}^{\mathrm{es}}S^*_{K''}M$ and $\left|\hat{\underline{\zeta}}(\alpha)/\hat{\underline{\xi}}(\alpha)\right| < \epsilon_K$ imply $\Upsilon_K(\alpha) \in U'$.

Now let $\chi \in \mathcal{C}^\infty(\tilde{W})$ be equal to 0 on $\tilde{W}\backslash K'$ and 1 on U'. For ϵ_K sufficiently small, $\chi \circ \Upsilon_K$ vanishes on ${}^{\mathrm{es}}S^*_{K\backslash U}M$, hence can be extended as 0 to ${}^{\mathrm{es}}S^*_{\tilde{W}\backslash U}M$ to define a $\mathcal{C}^\infty$ function. Thus, this extension of $\chi \circ \Upsilon_K$ is well defined and smooth on

$$\{\left|\hat{\underline{\zeta}}/\hat{\underline{\xi}}\right| < \epsilon_K,\ x < \epsilon_K\} \cup {}^{\mathrm{es}}S^*_{\tilde{W}\backslash U}M \subset {}^{\mathrm{es}}S^*M;$$

it equals 0 on the fibers over $\tilde{W}\backslash U$ and 1 on those over K''. But

$$\{\left|\hat{\zeta}/\hat{\xi}\right| < \epsilon'_K,\ x < \epsilon'_K\} \cup {}^{\mathrm{es}}S^*_{\tilde{W}\backslash U}M \subset \{\left|\hat{\underline{\zeta}}/\hat{\underline{\xi}}\right| < \epsilon_K,\ x < \epsilon_K\} \cup {}^{\mathrm{es}}S^*_{\tilde{W}\backslash U}M$$

for $\epsilon'_K > 0$ sufficiently small as on U, $\left|\hat{\underline{\zeta}}/\hat{\underline{\xi}}\right| \lesssim \left|\hat{\zeta}/\hat{\xi}\right|$, since $|z'_i|$ are all bounded away from 0 there. Due to the vanishing near $\partial\tilde{W}$, we can equivalently regard this extension of $\chi \circ \Upsilon_K$ as a $\mathcal{C}^\infty$ function ρ on the following subset of the edge-b cosphere bundle:

$$\{\left|\hat{\zeta}/\hat{\xi}\right| < \epsilon'_K,\ x < \epsilon'_K\} \cup {}^{\mathrm{eb}}S^*_{\tilde{W}\backslash U}M.$$

The resulting function satisfies

$$\mathsf{H}_{\mathrm{es}}(\rho) = O(x) \quad \text{on } \{\left|\hat{\zeta}/\hat{\xi}\right| < \epsilon'_K,\ x < \epsilon'_K\} \cup {}^{\mathrm{eb}}S^*_{\tilde{W}\backslash U}M. \tag{7.1}$$

and

$$\rho = 0 \text{ on } {}^{\mathrm{eb}}S^*_{\tilde{W}\setminus U}M. \tag{7.2}$$

It is convenient to extend ρ to all of ${}^{\mathrm{eb}}S^*M$ by defining it to be an arbitrary fixed positive constant, say 1, where it is not previously defined. Note that by (7.2), when we need to calculate derivatives of ρ in a commutator calculation, we may always assume that we are away from $\partial\tilde{W}$, hence use the edge-calculus Hamilton vector field result.

Now consider the function

$$\omega = |\hat{\eta} - \hat{\eta}_0|^2 + |y - y_0|^2 + \rho^2 + |t - t_0|^2.$$

(Note that keeping ω, $|\xi - \xi_0|$ and x sufficiently small on Σ_{eb} automatically means that $\hat{\zeta}$ is small as well.)

We now identify some appropriate neighborhoods in which to localize. First, choose $\epsilon_0, \epsilon_1 < 1$ such that

$$|\hat{\zeta}|^2 < \epsilon_0,\ x < \epsilon_1,\ \omega < \epsilon_1,\ |\xi - \xi_0|^2 < \epsilon_1 \Longrightarrow |\hat{\zeta}/\hat{\xi}| < \epsilon'_K/2.$$

Second, choose $\epsilon_2 < \epsilon_1$ such that

$$\Sigma_{\mathrm{eb}} \cap \{x < \epsilon_2\} \cap \{\omega < \epsilon_2\} \cap \{|\hat{\xi} - \hat{\xi}_0|^2 < \epsilon_2\} \subset \{|\hat{\zeta}|^2 < \frac{\epsilon_0}{2}\}.$$

Let

$$\tilde{K} = \{x \leq \epsilon_2\} \cap \{\omega \leq \epsilon_2\} \cap \{|\hat{\xi} - \hat{\xi}_0|^2 \leq \epsilon_2\} \cap \{|\hat{\zeta}|^2 \leq \frac{\epsilon_0}{2}\}.$$

Next, given $\delta > 0$, which will depend on $\tilde{K}$, let $U = U_\delta$ be as in Corollary 5.34. Finally, given any $\beta > 0$ (to be specified below) we will choose $\epsilon = \epsilon(\beta, \delta)$ so that

$$\epsilon, \epsilon(1+\beta) < \epsilon_2, \tag{7.3}$$

and so that

$$K_\epsilon = \{x \leq \epsilon,\ \omega \leq \epsilon(1+\beta),\ |\hat{\xi} - \hat{\xi}_0|^2 \leq \epsilon,\ |\hat{\zeta}|^2 \leq \epsilon_0\} \subset U = U_\delta.$$

(Note that $K_\epsilon \subset \tilde{K}$ by (7.3).)

Let $\phi \in \mathcal{C}_c^\infty([0,\epsilon))$, $\psi_0 \in \mathcal{C}_c^\infty([0,\epsilon_0))$, identically 1 on $[0, \epsilon_0/2]$, $\psi_1 \in \mathcal{C}_c^\infty([0,\epsilon))$, identically 1 on $[0,\epsilon/2]$, $\psi \in \mathcal{C}_c^\infty((-\infty,\epsilon))$, all non-increasing,

$$a = a_\epsilon = |\tau|^s x^{-r} \psi(\omega - \beta x)\psi_1(|\hat{\xi} - \hat{\xi}_0|^2)\phi(x)\psi_0(|\hat{\zeta}|^2)$$

Thus,

$$x \leq \epsilon,\ \omega \leq \epsilon(1+\beta),\ |\hat{\xi} - \hat{\xi}_0|^2 \leq \epsilon,\ |\hat{\zeta}|^2 \leq \epsilon_0 \text{ on } \operatorname{supp} a_\epsilon.$$

We usually suppress the ϵ-dependence of a below in our notation. Equation (7.3) ensures that $\epsilon(1+\beta) < 1$ on $\operatorname{supp} a$, so $\rho < 1$, and thus (7.1) holds. We have also arranged that

$$\left|\hat{\zeta}/\hat{\xi}\right| < \epsilon'_K/2 \text{ on } \operatorname{supp} a$$

and that $\psi_0(|\hat{\zeta}|^2) = 1$ on $\operatorname{supp}(\psi(\omega - \beta x)\phi(x)\psi_1(|\hat{\xi} - \hat{\xi}_0|^2)) \cap \Sigma_{\mathrm{eb}}$. This latter observation means that we need never consider derivatives falling on the ψ_0

term when computing the action of the Hamilton vector field on a. (The cutoff $\psi_0(\hat{\zeta})$ is therefore not necessary for correct localization of a, as that is achieved by the cutoffs in $\omega, \hat{\xi}$ and x if we restrict our attention to Σ_{eb}; rather this is necessary to make a a symbol, which it would not be if independent of ζ.)

We quantize a to $A \in \Psi_{\mathrm{eb}}^{s,r}(M)$, i.e. take any A with $\sigma_{\mathrm{eb},s}(A) = a$. By Lemma 5.35,

$$\imath[\Box, A^*A] = \sum Q_i^* L_{ij} Q_j + \sum (x^{-1} L_i Q_i + Q_i^* x^{-1} L_i') + x^{-2} L_0, \tag{7.4}$$

with

$$\begin{gathered} L_{ij} \in \Psi_{\mathrm{eb}}^{2s-1,2r}(M),\ L_i \in \Psi_{\mathrm{eb}}^{2s,2r}(M),\ L_0 \in \Psi_{\mathrm{eb}}^{2s+1,2r}(M), \\ \sigma_{\mathrm{eb},2s-1}(L_{ij}) = 2aV_{ij}a,\ \sigma_{\mathrm{eb},2s}(L_i) = 2aV_i a,\ \sigma_{\mathrm{eb},2s+1}(L_0) = 2aV_0 a, \\ \mathrm{WF}'_{\mathrm{eb}}(L_{ij}), \mathrm{WF}'_{\mathrm{eb}}(L_i), \mathrm{WF}'_{\mathrm{eb}}(L_i'), \mathrm{WF}'_{\mathrm{eb}}(L_0) \subset \mathrm{WF}'_{\mathrm{eb}}(A), \end{gathered} \tag{7.5}$$

with V_{ij}, V_i and V_0 smooth vector fields tangent on ${}^{\mathrm{eb}}T^*M$ tangent to $\tilde{W}$ and such that for $f \in \mathcal{C}^\infty({}^{\mathrm{eb}}S^*M)$ with $f|_{\tilde{W}} = \varpi_{\mathrm{eb}}^* \phi$ for some $\phi \in S^*W$,

$$V_{ij} f|_{\tilde{W}} = 0,\ V_i f|_{\tilde{W}} = 0,\ V_0 f|_{\tilde{W}} = 0. \tag{7.6}$$

In view of Corollary 5.34, we are led to regard the L_{ij} and L_i terms as negligible, *provided that* their principal symbol is bounded by a constant multiple of $\sigma_{\mathrm{eb},s+1}(L_0)$ times the appropriate power of $|\tau|$ (to arrange homogeneity of the same degree). Also by Lemma 5.35,

(7.7)

$$\begin{aligned} &(|\tau| V_0 \hat{\xi})|_{\tilde{W}} = -2 \sum_{ij} k_{2,ij}(0,y,z) \zeta_i'' \zeta_j'' \\ &(|\tau| V_i \hat{\xi})|_{\tilde{W}} = -\sum_j k_{3,ij}(0,y,z) \zeta_j'',\ (|\tau| V_{ij} \hat{\xi})|_{\tilde{W}} = -2k_{1,ij}(0,y,z), \\ &(|\tau|^{-s'-1} x^{-r'} V_0(|\tau|^{s'} x^{r'}))|_{\tilde{W}} = -2(r'+s')\hat{\xi},\ (|\tau|^{-s'} x^{-r'} V_i(|\tau|^{s'} x^{r'}))|_{\tilde{W}} = 0, \\ &(|\tau|^{-s'+1} x^{-r'} V_{ij}(|\tau|^{s'} x^{r'}))|_{\tilde{W}} = 0, \end{aligned}$$

In particular, with $s' = 0$, $r' = 1$, $|\tau|^{-1} V_0 x = -2\hat{\xi} x$, while $V_i x, V_{ij} x$ are $O(x^2)$.

In computing Va for various homogeneous degree $\mu - 1$ vector fields V on ${}^{\mathrm{eb}}T^*M$, we will employ the following arrangement of terms:

$$\begin{aligned} Va =& \psi(\omega - \beta x)\phi(x)\psi_0(|\hat{\zeta}|^2)\psi_1(|\hat{\xi} - \hat{\xi}_0|^2) V(|\tau|^s x^{-r}) \\ &+ |\tau|^s x^{-r} \psi_0(|\hat{\zeta}|^2)\phi(x)\psi_1(|\hat{\xi} - \hat{\xi}_0|^2)\psi'(\omega - \beta x)(V\omega - \beta V x) \\ &+ |\tau|^s x^{-r} \psi(\omega - \beta x)\psi_0(|\hat{\zeta}|^2)\psi_1(|\hat{\xi} - \hat{\xi}_0|^2) V(\phi(x)) \\ &+ |\tau|^s x^{-r} \psi(\omega - \beta x)\psi_0(|\hat{\zeta}|^2)\phi(x) V(\psi_1(|\hat{\xi} - \hat{\xi}_0|^2)) \\ &+ |\tau|^s x^{-r} \psi(\omega - \beta x)\phi(x)\psi_1(|\hat{\xi} - \hat{\xi}_0|^2) V(\psi_0(|\hat{\zeta}|^2)). \end{aligned}$$

As $|\tau|^{-1}V_0 x = -2\hat{\xi}x$ while $|\tau|^{-1}V_0\omega = xf$ for some $f \in \mathcal{C}^\infty({}^{\mathrm{eb}}S^*X)$, and $|\hat{\xi}|$ is bounded below on $\tilde{K}$ (which is a compact subset of ${}^{\mathrm{eb}}S^*M$), it follows that there exists $\beta > 0$ such that $-(\operatorname{sgn}\tau_0)|\tau|^{-1}(V_0\omega - \beta V_0 x) \geq x$ on $\tilde{K}$, and thus

$$-(\operatorname{sgn}\tau_0)|\tau|^{-1}(V_0\omega - \beta V_0 x) = xc_2^2 \tag{7.8}$$

for some smooth positive function c_2 defined on $\tilde{K}$, hence on a neighborhood of $\operatorname{supp} a$ in ${}^{\mathrm{eb}}S^*M$. Moreover,

$$V_0\psi_1(|\hat{\xi} - \hat{\xi}_0|^2) = -4(\hat{\xi} - \hat{\xi}_0)\psi_1'(|\hat{\xi} - \hat{\xi}_0|^2)\sum_{ij} k_{2,ij}(0, y, z)\zeta_i''\zeta_j''.$$

Similar computations hold for the V_i and V_{ij} terms, with result shown below in (7.9).

We start by discussing the terms in (7.4)-(7.5) in which the vector fields V_0, V_i, V_{ij} differentiate $\psi_1(|\hat{\xi} - \hat{\xi}_0|^2)$. These terms altogether have the form

$$\begin{gathered}\sum Q_i^* L_{ij,1} Q_j + \sum (x^{-1} L_{i,1} Q_i + Q_i^* x^{-1} L_{i,1}') + x^{-2} L_{0,1}, \text{ where}\\
\sigma_{\mathrm{eb},2s-1}(L_{ij,1})|_{\tilde{W}} = -(\operatorname{sgn}\tau_0)a_1 k_{1,ij}(0, y, z),\\
\sigma_{\mathrm{eb},2s}(L_{i,1})|_{\tilde{W}} = \sigma_{\mathrm{eb},2s}(L_{i,1}')|_{\tilde{W}} = -\frac{1}{2}(\operatorname{sgn}\tau_0)a_1\sum_j k_{3,ij}(0, y, z)\zeta_j'',\\
\sigma_{\mathrm{eb},2s+1}(L_{0,1})|_{\tilde{W}} = -(\operatorname{sgn}\tau_0)a_1\sum_{ij} k_{2,ij}(0, y, z)\zeta_i''\zeta_j'',\\
\text{with } a_1 = 8a(\operatorname{sgn}\tau_0)(\hat{\xi} - \hat{\xi}_0)|\tau|^s x^{-r}\psi(\omega - \beta x)\psi_0(|\zeta|^2)\phi(x)\psi_1'(|\hat{\xi} - \hat{\xi}_0|^2).\end{gathered}\tag{7.9}$$

On $\tilde{K} \cap \operatorname{supp}\psi_1'(|\hat{\xi} - \hat{\xi}_0|^2) \cap {}^{\mathrm{eb}}\Sigma$, $\hat{\xi} - \hat{\xi}_0$ has sign $-\operatorname{sgn}\hat{\xi}_0 = -\operatorname{sgn}\tau_0$, so $(\operatorname{sgn}\tau_0)(\hat{\xi} - \hat{\xi}_0) < 0$ there. Thus, noting that the right hand side on the last line is a square for x sufficiently small in view of $\psi_1' \leq 0$ when $(\operatorname{sgn}\tau_0)(\hat{\xi}_0 - \hat{\xi}) > 0$, it has the form

$$\begin{gathered}(-\operatorname{sgn}\tau_0)d_Z^* x^{-1} C_0^* C_0 x^{-1} d_Z + E_0 + E_0' + F_0,\\
C_0 \in \Psi_{\mathrm{eb}}^{s-1/2,r}(M),\ E_0, E_0' \in \operatorname{Diff}^2_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{2s-1,2r+2}(M)\\
\sigma_{\mathrm{eb},s-1/2}(C_0) = \Big(H((\operatorname{sgn}\tau_0)(\xi_0 - \xi))\tilde{\phi}(x)a_1\Big)^{1/2},\\
\mathrm{WF}'_{\mathrm{eb}}(E_0') \cap {}^{\mathrm{eb}}\Sigma = \emptyset,\ \mathrm{WF}'_{\mathrm{eb}}(E_0) \subset \{x > 0\} \cap \operatorname{supp} a,\\
F_0 \in \operatorname{Diff}^2_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{2s-2,2r+2}(M),\ \mathrm{WF}'_{\mathrm{eb}}(E_0'),\ \mathrm{WF}'_{\mathrm{eb}}(F_0) \subset \operatorname{supp} a,\end{gathered}\tag{7.10}$$

where H is the Heaviside step function (recall that $\psi_1' = 0$ near the origin, and $\psi' \leq 0$) and $\tilde{\phi} \in \mathcal{C}_c^\infty([0, \epsilon_2))$ is identically 1 near 0 and has sufficiently small support.

Next, the terms in (7.4)-(7.5) in which the vector fields V_0, V_i, V_{ij} differentiate $\psi(\omega-\beta x)$ have the form

$$\begin{aligned}
&\sum Q_i^* L_{ij,2} Q_j + \sum (x^{-1} L_{i,2} Q_i + Q_i^* x^{-1} L'_{i,2}) + x^{-2} L_{0,2}, \text{ where}\\
&\sigma_{\mathrm{eb},s-1}(L_{ij,2}) = a_2^2 f_{ij,2},\ \sigma_{\mathrm{eb},s}(L_{i,2}) = \sigma_{\mathrm{eb},s}(L'_{i,2}) = a_2^2 f_{i,2},\\
&\sigma_{\mathrm{eb},s+1}(L_{0,2}) = -(\operatorname{sgn}\tau_0) a_2^2 c_2^2,\ c_2 \text{ as in (7.8)}\\
&a_2^2 = -2xa|\tau|^{s+1} x^{-r} \psi'(\omega-\beta x)\psi_0(|\hat\zeta|^2)\phi(x)\psi_1(|\hat\xi-\hat\xi_0|^2),
\end{aligned}$$

with $f_{ij,2}$, $f_{i,2}$ smooth. Moreover, terms in (7.4)–(7.5) in which the vector fields V_0, V_i, V_{ij} differentiate $\psi_0(|\hat\zeta|^2)$ have wave front set disjoint from ${}^{\mathrm{eb}}\Sigma$ as already discussed, while the terms in which these vector fields differentiate $\phi(x)$ are supported in $\operatorname{supp} a \cap \{x>0\}$, where we will assume the absence of $\mathrm{WF}^{s-1,*}_{\mathrm{eb},\mathfrak{X}} u$ (the weight is indicated by an asterisk as we are away from $x=0$, so it is irrelevant).

Finally, the terms in (7.4)–(7.5) in which the vector fields V_0, V_i, V_{ij} differentiate $|\tau|^s x^{-r}$ have the form

$$\begin{aligned}
&\sum Q_i^* L_{ij,3} Q_j + \sum (x^{-1} L_{i,3} Q_i + Q_i^* x^{-1} L'_{i,3}) + x^{-2} L_{0,2}, \text{ where}\\
&\sigma_{\mathrm{eb},s-1}(L_{ij,3}) = a^2 x f_{ij,3},\ \sigma_{\mathrm{eb},s}(L_{i,3}) = \sigma_{\mathrm{eb},s}(L'_{i,3}) = a^2 x f_{i,3},\\
&\sigma_{\mathrm{eb},s+1}(L_{0,3}) = -a^2(\operatorname{sgn}\tau_0)|\tau| 2(s-r)c_3^2,
\end{aligned}$$

where $c_3^2|_{\tilde W} = 4(\operatorname{sgn}\tau_0)\hat\xi > 0$.

Finally, recall that terms with ψ_0 derivatives are supported in the elliptic set of $\Box$.

We are now ready to piece together the above information to compute the commutator $[\Box, A^*A]$. First we choose a family of operators convenient for adjusting orders: pick

$$T_\nu \in \Psi^{\nu,0}_{\mathrm{eb}}(M),\ \sigma_{\mathrm{eb},\nu}(T_\nu) = |\tau|^\nu \text{ near } \tilde K.$$

Thus, T_ν are simply weights, for $|\tau|^\nu$ is elliptic of order ν on a neighborhood of $\tilde K$.

Adding all the terms computed above, and rearranging them as needed, noting the top order commutativity in eb-order of $\mathrm{Diff}_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}(M)$, we finally deduce that

$$\begin{aligned}
&-\imath(\operatorname{sgn}\tau_0)[\Box, A^*A]\\
&= A_2^*\big(C_2^* x^{-2} C_2 + \sum_i (x^{-1} F_{i,2} Q_i + Q_i^* x^{-1} F'_{i,2}) + \sum_{ij} Q_i^* F_{ij,2} Q_j\big) A_2\\
&\quad + A^* T_{1/2}^*\big(C_3^* 2(s-r) x^{-2} C_3 + \sum_i (x^{-1} F_{i,3} Q_i + Q_i^* x^{-1} F'_{i,3})\\
&\qquad\qquad + \sum_{ij} Q_i^* F_{ij,3} Q_j\big) T_{1/2} A\\
&\quad + d_Z^* x^{-1} C_0^* C_0 x^{-1} d_Z + E + E' + R''
\end{aligned} \tag{7.11}$$

with

(1) $A_2 \in \Psi_{\text{eb}}^{s+1/2,r-1/2}(M)$, $\sigma_{\text{eb},s+1/2}(A_2) = a_2$, $\mathrm{WF}'_{\text{eb}}(A_2) \subset \operatorname{supp} a$
(2) $C_2, C_3 \in \Psi_{\text{eb}}^{0,0}(M)$; $F_{i,2}, F'_{i,2}, F_{i,3}, F'_{i,3} \in \Psi_{\text{eb}}^{-1,0}(M)$ and $F_{ij,2}, F_{ij,3} \in \Psi_{\text{eb}}^{-2,0}(M)$;
(3) On $\tilde{K}$, $\sigma_{\text{eb},0}(C_2) \neq 0$ and $\sigma_{\text{eb},0}(C_3) = (\operatorname{sgn} \tau_0)\hat{\xi} \neq 0$,
(4) $C_0 \in \Psi_{\text{eb}}^{s-1/2,r}(M)$, $\mathrm{WF}'_{\text{eb}}(C_0) \subset \operatorname{supp} a$,
(5) $E, E' \in \mathrm{Diff}^2_{\text{es},\sharp} \Psi_{\text{eb}}^{2s-1,2r+2}(M)$,
(6) $R'' \in \mathrm{Diff}^2_{\text{es},\sharp} \Psi_{\text{eb}}^{2s-2,2r+2}(M)$ (i.e. is lower order), $\mathrm{WF}'_{\text{eb}}(R'') \subset \operatorname{supp} a$,
(7) $\mathrm{WF}'_{\text{eb}} E \subset \{x > 0\} \cap \operatorname{supp} a$ (our hypothesis region),
(8) $\mathrm{WF}'_{\text{eb}} E' \cap {}^{\text{eb}}\Sigma = \emptyset$, $\mathrm{WF}'_{\text{eb}}(E') \subset \operatorname{supp} a$.

When we pair both sides of this equation (suitably regularized) with a solution to the wave equation the terms E, E' and R'' will be controlled respectively by the hypothesis on u in $x > 0$, microlocal elliptic regularity, and an inductive hypothesis in the iterative argument in which we improve the order by $1/2$ (or less) in each step. The remaining terms on the right hand side are either positive, or involve Q_i, and the latter terms are controlled by the former, by Corollary 5.34. Thus, save for the need to mollify to make sure that we can actually apply this commutator to u and pair it with u, and also be able to rewrite the commutator as the difference of products, this would give our positive commutator result, controlling $\|x^{-1}T_{1/2}C_3Au\|$.

We do, however, need to mollify. Let $\sigma > 0$ (typically we take $\sigma = 1/2$, always $\sigma \in (0, 1/2]$) $\Lambda_\gamma \in \Psi_{\text{eb}}^{-\sigma}(M)$ for $\gamma > 0$, such that $\{\Lambda_\gamma : \gamma \in (0, 1]\}$ is a bounded family in $\Psi^0_{\text{eb}}(M)$, and $\Lambda_\gamma \to \mathrm{Id}$ as $\gamma \downarrow 0$ in $\Psi^{\tilde{\epsilon}}_{\text{eb}}(M)$, for all $\tilde{\epsilon} > 0$. Let the principal symbol of Λ_γ, considered a bounded family in $\Psi^0_{\text{eb}}(M)$, be $(1+\gamma|\tau|^2)^{-\sigma/2}$ on a neighborhood of $\tilde{K}$. Let $A_\gamma = \Lambda_\gamma A$. We now have $A_\gamma \in \Psi_{\text{eb}}^{s-\sigma,r}(M)$ for $\gamma > 0$, and A_γ is uniformly bounded in $\Psi_{\text{eb}}^{s,r}(M)$, $A_\gamma \to A$ in $\Psi_{\text{eb}}^{s+\tilde{\epsilon},r}(M)$. Moreover,

$$\imath[\Box, A_\gamma^* A_\gamma] = \Lambda_\gamma^* \imath[\Box, A^*A]\Lambda_\gamma + A^*\imath[\Box, \Lambda_\gamma^*\Lambda_\gamma]A + \tilde{R}, \tag{7.12}$$

with $\tilde{R}$ uniformly bounded in $\mathrm{Diff}^2_{\text{es},\sharp} \Psi_{\text{eb}}^{2s-2,2r+2}(M)$ (hence lower order). Now, for a vector field V on ${}^{\text{eb}}T^*M$,

$$V(1+\gamma|\tau|^2)^{-\sigma/2} = -(\sigma/2)\gamma(1+\gamma|\tau|^2)^{-\sigma/2-1}V|\tau|^2.$$

Applying this, the general formula (7.4)–(7.5) with Λ_γ in place of A and (7.7) with $r' = 0$, $s' = 2$, we deduce that

$$\begin{aligned} &-(\operatorname{sgn}\tau_0)A^*\imath[\Box, \Lambda_\gamma^*\Lambda_\gamma]A \\ &\quad = A^*\Lambda_\gamma^* T_{1/2}^*\tilde{\Lambda}_\gamma^*\big(-2\sigma C_3^* x^{-2} C_3 + \sum_i (x^{-1}F_{i,4}Q_i + Q_i^* x^{-1} F'_{i,4}) \\ &\qquad\qquad + \sum_{ij} Q_i^* F_{ij,4} Q_j\big) T_{1/2}\tilde{\Lambda}_\gamma \Lambda_\gamma A + R'_\gamma, \end{aligned} \tag{7.13}$$

with $F_{ij,4} \in \Psi_{\mathrm{eb}}^{-2,0}(M)$, $F_{i,4}, F'_{i,4} \in \Psi_{\mathrm{eb}}^{-1,0}(M)$, $\tilde{\Lambda}_\gamma$ uniformly bounded in $\Psi_{\mathrm{eb}}^{0,0}(M)$ with principal symbol

$$\sigma_{\mathrm{eb}}(\tilde{\Lambda}_\gamma) = \big(\gamma|\tau|^2(1+\gamma|\tau|^2)^{-1}\big)^{1/2} \leq 1,$$

$C_3 \in \Psi_{\mathrm{eb}}^{0,0}(M)$ and $T_{1/2} \in \Psi_{\mathrm{eb}}^{1/2,0}(M)$ as in (7.11) and R'_γ uniformly bounded in $\mathrm{Diff}^2_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{2s-2,2r+2}(M)$, hence lower order. Note that this commutator has the *opposite* sign from (7.11), which limits our ability to regularize. However, as long as $\sigma' - \sigma > 0$, we can write

$$2\sigma' \,\mathrm{Id} - 2\sigma \tilde{\Lambda}_\gamma^* \tilde{\Lambda}_\gamma = B_\gamma^* B_\gamma$$

with B_γ uniformly bounded in $\Psi_{\mathrm{eb}}^{0,0}(M)$. Thus, if $s - r > \sigma$, taking σ' such that $\sigma < \sigma' < s - r$, we deduce that

$$\begin{aligned}
&-\imath(\operatorname{sgn}\tau_0)[\Box, A_\gamma^* A_\gamma] \\
&= A_{2,\gamma}^*\Big(C_2^* x^{-2} C_2 + \sum_i (x^{-1}F_{i,2}T_1 Q_i T_{-1} + T_{-1}Q_i^* T_1 x^{-1} F'_{i,2}) \\
&\qquad\qquad + \sum_{ij} T_{-1} Q_i^* Q_j F_{ij,2} T_1\Big) A_{2,\gamma} \\
&\quad + A_\gamma^* T_{1/2}^* \Big(C_3^* 2(s - r - \sigma') x^{-2} C_3 \\
&\qquad\qquad + \sum_i (x^{-1}F_{i,5}T_1 Q_i T_{-1} + T^{-1} Q_i^* T_1 x^{-1} F'_{i,5}) \\
&\qquad\qquad + \sum_{ij} T_{-1} Q_i^* Q_j F_{ij,5} T_1\Big) T_{1/2} A_\gamma \\
&\quad + A_\gamma^* T_{1/2}^* C_3^* B_\gamma^* B_\gamma C_3 T_{1/2} A_\gamma \\
&\quad + d_Z^* x^{-1} \Lambda_\gamma^* C_0^* C_0 \Lambda_\gamma x^{-1} d_Z + E_\gamma + E'_\gamma + R''_\gamma,
\end{aligned} \tag{7.14}$$

with the terms as in (7.11), in particular $F_{ij,5}, F'_{ij,5}$ as $F_{ij,3}$, etc., there, and $A_{2,\gamma} = A_2 \Lambda_\gamma$, etc. Here we rewrote the terms in (7.11) somewhat, inserting T_1 and T_{-1} in places (recall that $T_1 T_{-1}$ differs from Id by an element of $\Psi_{\mathrm{eb}}^{-1,0}(M)$ on $\tilde{K}$, and this difference can be absorbed in R''_γ) in order to be able to use Corollary 5.34 directly below. Applying both sides of (7.14) to u and pairing with u, we claim we may integrate by parts for any $\gamma > 0$ on the right hand side of the resulting expression to obtain

(7.15)
$$
\begin{aligned}
&-\imath(\operatorname{sgn}\tau_0)\langle[\Box, A_\gamma^* A_\gamma]u, u\rangle\\
&\;=\|x^{-1}C_2A_{2,\gamma}u\|^2 + 2(s-r-\sigma')\|x^{-1}C_3T_{1/2}A_\gamma u\|^2\\
&\quad+\sum_{ij}\langle Q_jF_{ij,2}T_1A_{2,\gamma}u, Q_iT_{-1}^*A_{2,\gamma}u\rangle\\
&\quad+\sum_i\big(\langle Q_iT_{-1}A_{2,\gamma}u, x^{-1}T_1^*F_{i,2}^*A_{2,\gamma}u\rangle + \langle x^{-1}T_1F'_{i,2}A_{2,\gamma}u, Q_iT_{-1}A_{2,\gamma}u\rangle\big)\\
&\quad+\sum_i\big(\langle Q_iT_{-1}T_{1/2}A_\gamma u, x^{-1}T_1^*F_{i,5}^*T_{1/2}A_\gamma u\rangle\\
&\qquad\qquad+\langle x^{-1}T_1F'_{i,5}T_{1/2}A_\gamma u, Q_iT_{-1}T_{1/2}A_\gamma u\rangle\big)\\
&\quad+\sum_{ij}\langle Q_jT_1F_{ij,5}T_{1/2}A_\gamma u, Q_iT_{-1}^*T_{1/2}A_\gamma u\rangle\\
&\quad+\|B_\gamma C_3T_{1/2}Au\|^2 + \|C_0\Lambda_\gamma x^{-1}d_Zu\|^2 + \langle(E_\gamma+E'_\gamma+R''_\gamma)u, u\rangle,
\end{aligned}
$$

and that we may similarly expand the left side by using

$$\langle[\Box, A_\gamma^*A_\gamma]u, u\rangle = \langle A_\gamma^*A_\gamma u, \Box u\rangle - \langle\Box u, A_\gamma^*A_\gamma u\rangle, \tag{7.16}$$

so that pairing with a solution to the wave equation yields identically zero.

We begin by justifying these two integrations by parts, after which we will read off the consequences. We start with the Dirichlet case. Note that the L^2_g-dual of $\mathfrak{Y} = H^{1,l}_{\mathrm{es},0}(M)$ is $H_{\mathrm{es}}^{-1,-l-(f+1)}(M)$ (where as usual the x^{f+1} factor derives from the difference between the metric density used in the pairing and the "edge-density" used to define the norm on $H^{\cdot,\cdot}_{\mathrm{es},0}(M)$). We have

$$x^{-2}\operatorname{Diff}^2_{\mathrm{es},\sharp}(M)\ni\Box:\mathfrak{Y}\to H_{\mathrm{es}}^{-1,l-2}(M) = x^{2l+(f+1)-2}(\mathfrak{Y})^*.$$

Here we suppressed the quotient map $\rho : \dot{H}_{\mathrm{es}}^{-1,l-2}(M)\to H_{\mathrm{es}}^{-1,l-2}(M)$, i.e. the stated mapping property is, strictly speaking, for $\rho\circ\Box$. Furthermore, the dual of $H^{s',r'}_{\mathrm{eb},\mathfrak{Y}}(M)$ is

$$\big(H^{s',r'}_{\mathrm{eb},\mathfrak{Y}}(M)\big)^* = H^{-s',-r'}_{\mathrm{eb},(\mathfrak{Y})^*}(M).$$

Equation (7.16) makes sense directly and naively for $\gamma>0$ if the products of $\Box$ with $A_\gamma^*A_\gamma\in\Psi^{2s-2\sigma,2r}_{\mathrm{eb}}(M)$ map $H^{s',r'}_{\mathrm{eb},\mathfrak{Y}}(M)$ to its dual, $H^{-s',-r'}_{\mathrm{eb},(\mathfrak{Y})^*}(M)$. We thus require

$$A_\gamma^*A_\gamma : H^{s',r'}_{\mathrm{eb},\mathfrak{Y}}(M)\to H^{-s',-r'-2l-(f+1)+2}_{\mathrm{eb},\mathfrak{Y}}(M)$$

which holds if

$$
\begin{aligned}
&s-\sigma\le s',\\
&\quad r\le r'+l+(f+1)/2-1.
\end{aligned}\tag{7.17}
$$

Following the same line of reasoning shows that if we are willing to settle for just (7.15), by contrast, we only require the milder hypotheses

$$\begin{aligned} s-\sigma &\leq s'+1/2, \\ r &\leq r'+l+(f+1)/2-1. \end{aligned} \tag{7.18}$$

In fact, we claim that (7.18) suffices for *both* (7.15) and (7.16), with the latter being obtained via the following subtler regularization.

This is best done by replacing u in the second slot of the pairing by a separate factor $\tilde{\Lambda}_{\gamma'}u$, where $\tilde{\Lambda}_\gamma$ is constructed just as Λ_γ, but with the greater degree of regularization $\sigma=1$. Thus we have a replaced the lost half of an edge derivative (on each factor) which obtains from assuming (7.18) instead of (7.17) and may again integrate by parts to obtain, for $\gamma,\gamma'>0$,

$$\begin{aligned} \langle [\Box, A_\gamma^* A_\gamma]u, \Lambda_{\gamma'}u\rangle &= \langle A_\gamma^* A_\gamma u, \Box\Lambda_{\gamma'}u\rangle - \langle \Box u, A_\gamma^* A_\gamma \Lambda_{\gamma'}u\rangle \\ &= \langle A_\gamma^* A_\gamma u, \Lambda_{\gamma'}\Box u\rangle + \langle A_\gamma^* A_\gamma u, [\Box, \Lambda_{\gamma'}]u\rangle - \langle \Box u, A_\gamma^* A_\gamma \Lambda_{\gamma'}u\rangle \end{aligned} \tag{7.19}$$

Now, $\Lambda_{\gamma'}\to \mathrm{Id}$ *strongly* (but not in norm) on $H^{s',r'}_{\mathrm{eb},\mathfrak{Y}}(M)$ and on $H^{s',r'}_{\mathrm{eb},\mathfrak{Y}^*}$ for all s',r'; this takes care of the first and third terms. Furthermore, $[\Box,\Lambda_{\gamma'}]\to 0$ *strongly* (but not in norm) as a map from $H^{s',r'}_{\mathrm{eb},\mathfrak{Y}}(M)$ to $H^{s'+1,r'}_{\mathrm{eb},H^{-1,l}_{\mathrm{es}}(M)}(M)$. Thus, letting $\gamma'\to 0$ shows (7.16) just under the assumption $s-\sigma\leq s'+1/2$, $r\leq r'-1+l+(f+1)/2$.

The Neumann case is completely analogous, except that then L^2_g-dual of $\mathfrak{X}=H^{1,l}_{\mathrm{es}}(M)$ is $\dot{H}^{-1,-l-(f+1)}_{\mathrm{es}}(M)$. We have

$$x^{-2}\operatorname{Diff}^2_{\mathrm{es},\sharp}(M)\ni\Box:\mathfrak{X}\to\dot{H}^{-1,l-2}_{\mathrm{es}}(M)=x^{2l+(f+1)-2}\mathfrak{X}^*.$$

Furthermore, the dual of $H^{s',r'}_{\mathrm{eb},\mathfrak{X}}(M)$ is

$$\big(H^{s',r'}_{\mathrm{eb},\mathfrak{X}}(M)\big)^* = H^{-s',-r'}_{\mathrm{eb},\mathfrak{X}^*}(M).$$

The rest of the argument proceeds unchanged.

Having justified our integrations by parts, we now show that we can absorb the Q_i-terms in (7.15) in the positive terms (uniformly as $\gamma\downarrow 0$) by using Corollary 5.34. Thus, given $\delta>0$, let U be as in Corollary 5.34; for sufficiently small $\epsilon>0$, $\operatorname{supp} a\subset U$. For instance, by Cauchy-Schwarz,

$$\begin{aligned} |\langle Q_j F_{ij,2}T_1 A_{2,\gamma}u, Q_i T^*_{-1}A_{2,\gamma}u\rangle| &\leq \|Q_i T^*_{-1}A_{2,\gamma}u\|^2 + \|Q_j F_{ij,2}T_1 A_{2,\gamma}u\|^2 \\ &\leq \delta\big(\|D_t T^*_{-1}A_{2,\gamma}u\|^2 + \|D_t F_{ij,2}T_1 A_{2,\gamma}u\|^2\big) \\ &\qquad + F\big(\|u\|^2_{H^{1,r+1/2-(f+1)/2}_{\mathrm{es}}(M)} + \|Gu\|^2_{H^{1,r+1/2-(f+1)/2}_{\mathrm{es}}(M)}\big), \end{aligned}$$

where $G\in\Psi^{s-1,0}_{\mathrm{eb}}(M)$. The the $\|Gu\|^2_{H^{1,r+1/2-(f+1)/2}_{\mathrm{es}}(M)}$ term can be estimated as $\langle R''_\gamma u,u\rangle$ since

$$\|Gu\|^2_{H^{1,r+1/2-(f+1)/2}_{\mathrm{es}}(M)} = \|x^{-r+1/2}d_M Gu\|^2_{L^2_g(M;\Lambda M)}, \tag{7.20}$$

and $(x^{-r+1/2}d_M G)^*(x^{-r+1/2}d_M G) \in \mathrm{Diff}^2_{\mathrm{es}}\,\Psi^{2s-2,2r+1}_{\mathrm{eb}}(M)$, hence in fact a little better than R''_γ, which has weight $2r+2$. Now, for $\digamma_0 > 0$ sufficiently large, depending on $\tilde K$ but *not* on $\epsilon > 0$ (as long as ϵ satisfies (7.3) and $\operatorname{supp} a \subset U$, i.e. $\epsilon > 0$ is sufficiently small), we have

$$\sigma_{\mathrm{eb},0,1}(D_t F_{ij,2} T_1) \le \digamma_0 \sigma_{\mathrm{eb},0,1}(x^{-1} C_2)$$

on a neighborhood of $\tilde K$. Thus,

$$\begin{aligned} &\|D_t F_{ij,2} T_1 A_{2,\gamma} u\|^2 \\ &\le 2\digamma_0 \|x^{-1} C_2 A_{2,\gamma} u\|^2 \\ &\qquad + \digamma'(\|x^{-r-1}u\|^2_{H^{1,r+1/2-(f+1)/2}_{\mathrm{es}}(M)} + \|G'u\|^2_{H^{1,r+1/2-(f+1)/2}_{\mathrm{es}}(M)}), \end{aligned}$$

with $G' \in \Psi^{s-1,0}_{\mathrm{eb}}(M)$ (so the last term behaves like (7.20)). Thus, if we choose $\delta > 0$ such that $8\digamma_0 n^2 \delta < 1$, the first term (for all i, j) can be absorbed in $\|x^{-1}C_2 A_{2,\gamma} u\|^2$, while the last two terms are estimated as $\langle R''_\gamma u, u\rangle$. Essentially identical arguments deal with all the other terms with Q_i and Q_j. In the case where Q_i is present on one side of the pairing only, we write, for instance,

$$\begin{aligned} &|\langle Q_i T_{-1} A_{2,\gamma} u, x^{-1} T_1^* F^*_{i,2} A_{2,\gamma} u\rangle| \\ &\le \delta^{-1/2}\|Q_i T_{-1} A_{2,\gamma} u\|^2 + \delta^{1/2}\|x^{-1} T_1^* F^*_{i,2} A_{2,\gamma} u\|^2. \end{aligned}$$

Using Corollary 5.34 on the first term, we have an estimate as above after possibly reducing $\delta > 0$.

Recall that uniform finiteness of $\left\|x^{-1} C_3 T_{1/2} A_\gamma u\right\|$ as $\gamma \downarrow 0$ will give absence of $\mathrm{WF}^{s-1/2,r-l-(f+1)/2}_{\mathrm{eb},\mathfrak{X}} u \cap \operatorname{ell} A$ (as always the contribution to the weight of $(f+1)/2$ comes from the metric weight while l comes from the weight in the definition of the base space, $\mathfrak{X}$). Similarly evaluating the other terms in the pairing, we take the extreme values of s', r' allowed by (7.18) to obtain

$$\begin{aligned} &\mathrm{WF}^{s-1/2-\sigma,r+1-l-(f+1)/2}_{\mathrm{eb},\mathfrak{X}} u \cap \mathrm{WF}' A = \emptyset, \\ &\text{and } \mathrm{WF}^{s-1/2,r+1-l-(f+1)/2}_{\mathrm{eb},\mathfrak{X}} u \cap \mathrm{WF}' A \cap \{x > 0\} = \emptyset, \\ &\text{and } s > r + \sigma,\ \sigma \in (0, 1/2] \\ &\Longrightarrow \mathrm{WF}^{s-1/2,r+1-l-(f+1)/2}_{\mathrm{eb},\mathfrak{X}} u \cap \operatorname{ell} A = \emptyset, \end{aligned} \tag{7.21}$$

or, relabeling,

$$\begin{aligned} \mathrm{WF}^{s,r}_{\mathrm{eb},\mathfrak{X}} u \cap \mathrm{WF}' A = \emptyset,\ &\mathrm{WF}^{s+\sigma,r}_{\mathrm{eb},\mathfrak{X}} u \cap \mathrm{WF}' A \cap \{x > 0\} = \emptyset, \\ &s > r + l + f/2 - 1, \\ &\Longrightarrow \mathrm{WF}^{s+\sigma,r}_{\mathrm{eb},\mathfrak{X}} u \cap \operatorname{ell} A = \emptyset. \end{aligned} \tag{7.22}$$

Recall here that $a = a_\epsilon$, and

$$0 < \epsilon < \epsilon' \Rightarrow \mathrm{WF}'(A_\epsilon) \cap {}^{\mathrm{eb}}\Sigma \subset \operatorname{ell} A_{\epsilon'} \cap {}^{\mathrm{eb}}\Sigma. \tag{7.23}$$

Finally, we show how to use (7.22) iteratively, together with an interpolation argument, to finish the proof of the theorem. A priori we have $u \in \mathfrak{X}$, i.e.

$$\mathrm{WF}^{0,0}_{\mathrm{eb},\mathfrak{X}}\, u = \emptyset,$$

If $0 > l + f/2 - 1$, we may iteratively apply (7.22) (shrinking $\epsilon > 0$ by an arbitratily small amount, using (7.23) to estimate the lower order error terms R''_γ) starting with $s = 0$ and always keeping $r = 0$, to obtain the conclusion of the theorem. (We choose $\sigma = 1/2$ at every stage in this process, until we are applying (7.22) with s such that $s + 1/2 > m$, at which point we finish the iteration by choosing $\sigma = m - s$ so as to retain our estimates on the wavefront set in the hypothesis region.)

However if $0 \leq l + f/2 - 1$, we may not apply (7.22) directly owing to the lack of positivity of the commutator, and we must employ an interpolation argument as follows. Applying (7.22) iteratively, this time with $r = r_0 < 0$ chosen sufficiently negative that we recover $0 > r_0 + l + f/2 - 1$, shows that we obtain

$$\mathrm{WF}^{m,r_0}_{\mathrm{eb},\mathfrak{X}}\, u \cap S = \emptyset, \tag{7.24}$$

with $S = \mathrm{WF}'\, A_\epsilon$ for some $\epsilon > 0$, A_ϵ constructed as above. Let

$$L = \sup\{r' : \mathrm{WF}^{m,r'}_{\mathrm{eb},\mathfrak{X}}\, u \cap S = \emptyset,\ r' \leq 0\}.$$

Note that the set on the right hand side is non-empty by (7.24). We aim to show that $L = 0$. To this end, note that if $L < 0$, then for any $r' < L$

$$\mathrm{WF}^{m,r'}_{\mathrm{eb},\mathfrak{X}}\, u \cap S = \mathrm{WF}^{0,0}_{\mathrm{eb},\mathfrak{X}} \cap S = \emptyset.$$

An interpolation then yields, for $\delta \in (0,1)$,

$$\mathrm{WF}^{m\delta,r'\delta}_{\mathrm{eb},\mathfrak{X}}\, u \cap S = \emptyset.$$

Note that for any $\delta \in (0,1)$ fixed, the compactness of S implies that for some $\epsilon' > \epsilon$,

$$\mathrm{WF}^{m\delta,r'\delta}_{\mathrm{eb},\mathfrak{X}}\, u \cap \mathrm{WF}'\, A_{\epsilon'} = \emptyset$$

still holds. If $\delta \in (0,1)$ in addition satisfies

$$m\delta > r'\delta + l + f/2 - 1 \tag{7.25}$$

then by iterating (7.22), shrinking ϵ' in each step (but keeping it larger than ϵ), we conclude that

$$\mathrm{WF}^{m,r'\delta}_{\mathrm{eb},\mathfrak{X}}\, u \cap S = \emptyset,$$

providing a contradiction with the definition of L if

$$r'\delta > L. \tag{7.26}$$

It remains to check whether $\delta \in (0,1)$ satisfying both (7.25) and (7.26) exists. This is evident from Figure 5, but a proof is as follows: we have $l + f/2 - 1 \geq 0$ by assumption (otherwise we are in the preceding case);

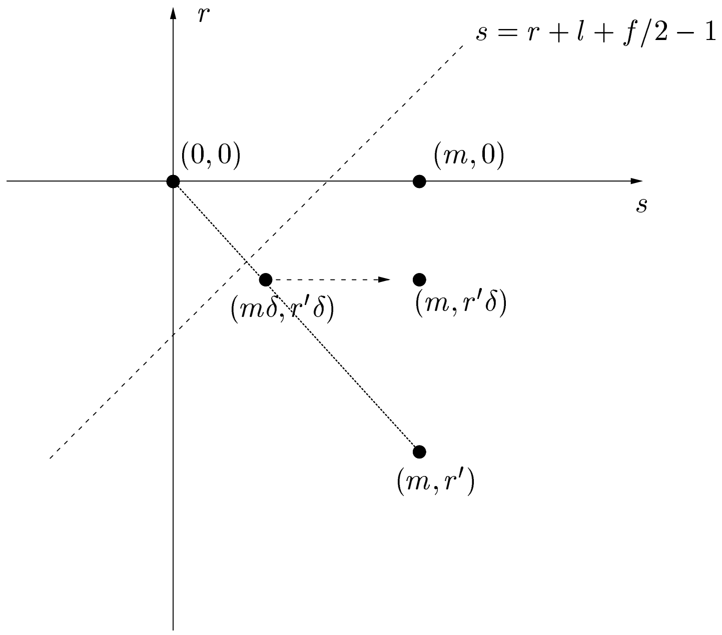

FIGURE 5. The interpolation argument. The figure shows the (s, r) plane, where we plot the values for which with have absence of $\mathrm{WF}^{m,r}_{\mathrm{eb},\mathfrak{X}}(u)$ (i.e. microlocal regularity of this order holds). We have a priori regularity of order $(0, 0)$ and wish to conclude regularity of order (m, l') with $l' < 0$. By (7.22) we may take a step to the right of length σ for any $\sigma \in (0, 1/2]$ provided that our starting regularity is below the line $s = r+l+f/2-1$ and that our endpoint is on $s \leq m$. If we know (m, r') regularity, we know regularity by interpolation on the whole line connecting this point to the origin; then starting on the interpolation line just below its intersection with $s = r+l+f/2-1$ allows us to achive $(m, r'\delta)$ regularity by moving to the right, thus improving the optimal weight for which we have our estimate.

moreover $m > 0$ (so that the theorem is not vacuous), and $l + f/2 - 1 < m$ by hypothesis. Thus, for any $r' < 0$,

$$0 \leq \frac{l + f/2 - 1}{m - r'} < \frac{m}{m - r'} < 1.$$

Setting

$$\delta_0(r') = \frac{l + f/2 - 1}{m - r'} \in [0, 1)$$

we see that (7.25) is an equality with $\delta = \delta_0(r)$ and that taking $\delta \in (\delta_0(r'), 1)$ yields (7.25). In particular, (7.25) is satisfied by $\delta = \delta(r', \lambda) = \delta_0(r')(1 + \lambda)$ for any $\lambda > 0$ sufficiently small. If $L < 0$, we have $\delta_0(L) < 1$, hence the function of r' and λ given by

$$r'\delta(r',\lambda) = r'\delta_0(r')(1+\lambda)$$

is strictly greater than L for $r' = L, \lambda = 0$. Hence increasing λ slightly and decreasing r' slightly preserves this relationship by continuity, and these choice of r' and δ yield $r'\delta > L$ as desired.

□

In order to verify the hypotheses of Theorem 7.3, which are stated at points over the edge, we will employ the following geometric result.

First note that if $q \in \mathcal{H}_{W,\mathrm{b}}$ then $\mathcal{R}_{\mathrm{eb},q,I}$ has a neighborhood O_1 in ${}^{\mathrm{eb}}S^*M$ and there is a $\delta_0 > 0$ such that any GBB γ with $\gamma(0) \in O_1 \cap \{x > 0\}$ satisfies $\gamma|_{[-\delta_0,0]} \cap {}^{\mathrm{b}}S^*_W M_0 = \emptyset$. Indeed, we simply need to take a coordinate neighborhood

$$\begin{aligned} O_1 = \{p \in {}^{\mathrm{eb}}S^*M : \hat\xi(p) < -\sqrt{1-h(q)}/2,\ x(p) < \epsilon_1,\ |y(p)-y(q)| < \epsilon_1, \\ |t(p)-t(q)| < \epsilon_1,\ |\hat\eta(p) - \hat\eta^{\mathrm{b}}(q)| < \epsilon_1\}, \end{aligned}$$

$\epsilon_1 > 0$ sufficiently small, since on its intersection with $\{x > 0\}$ (where ${}^{\mathrm{eb}}S^*M$ is naturally identified with ${}^{\mathrm{b}}S^*M_0$), $\sum \hat\xi^{\mathrm{b}}_j < 0$, hence Lemma 3.5 gives the desired δ_0 (cf. the argument of Remark 3.17). Thus, such GBB γ can be uniquely lifted to curves $\tilde\gamma$ in ${}^{\mathrm{eb}}S^*M$.

Lemma 7.4. *Suppose that $q \in \mathcal{H}_{W,\mathrm{b}}$. There exists $\epsilon_0 > 0$ with the following property.*

Suppose that $0 < \epsilon_1 \le \epsilon_0$, and U is a neighborhood of $\mathcal{F}^W_{I,q,\mathrm{sing}} \cap \{t = t(q) - \epsilon_1\}$. Then there is a neighborhood O of $\mathcal{R}_{\mathrm{eb},q,I} \cap \mathcal{G}$ in ${}^{\mathrm{eb}}S^*M$ such that for every (maximally extended) backward GBB γ with $\gamma(0) \in O \cap \{x > 0\}$ there is an $s_0 < 0$ such that $\gamma(s_0) \in U$, $\gamma(s) \notin {}^{\mathrm{b}}S^*_W M_0$ for $s \in [s_0, 0]$.

Proof. It follows from the discussion preceeding the statement of the lemma that there is a neighborhood O_1 of $\mathcal{R}_{\mathrm{eb},q,I}$ and $\delta_0 > 0$ such that every GBB γ defined on $[-\delta_0, 0]$, with $\gamma(0) \in O_1 \cap \{x > 0\}$ satisfies $\gamma(s) \notin {}^{\mathrm{b}}S^*_W M_0$ for $s \in [-\delta_0, 0]$. As $t(\gamma(s)) - t(\gamma(0)) = 2\hat\tau^{\mathrm{b}}(\gamma(0))s$, this implies that there is an $\epsilon_0 > 0$ such that for $t(\gamma(s)) \in [t(q) - \epsilon_0, t(q)]$, $\gamma(s) \notin {}^{\mathrm{b}}S^*_W M_0$.

Suppose now for the sake of contradiction that there is no neighborhood O of $\mathcal{R}_{\mathrm{eb},q,I} \cap \mathcal{G}$ in ${}^{\mathrm{eb}}S^*M$ such that for every (maximally extended) backward GBB γ with $\gamma(0) \in O \cap \{x > 0\}$, there exists $s_0 < 0$ with $\gamma(s_0) \in U$. As $\mathcal{R}_{\mathrm{eb},q,I} \cap \mathcal{G}$ is compact, we conclude that there is a sequence of points $p_n \in O_1 \subset {}^{\mathrm{eb}}S^*M$ with $x(p_n) > 0$ (so p_n can be regarded as a point in ${}^{\mathrm{b}}S^*M_0$) and GBB γ_n such that

- $\gamma_n(0) = p_n$,
- the image of γ_n disjoint from U,
- $p_n \to p \in \mathcal{R}_{\mathrm{eb},q,I} \cap \mathcal{G}$.

By Corollary 3.18, $\{\gamma_n\}$ has a subsequence $\{\gamma_{n_k}\}$ converging uniformly to a GBB γ such that the lift $\tilde\gamma$ of γ to ${}^{\mathrm{eb}}S^*M$ satisfies $\tilde\gamma(0) = p$. Thus, by

Lemma 3.13, γ is not normally incident, so the image of γ is in $\mathcal{F}^W_{I,q,\mathrm{sing}}$, and thus intersects $\mathcal{F}^W_{I,q,\mathrm{sing}} \cap \{t = \epsilon_1\}$. As $\gamma_{n_k} \to \gamma$ uniformly, for large enough k, γ_{n_k} intersects U, providing a contradiction. Thus, there exists O such that for every (maximally extended) backward GBB γ with $\gamma(0) \in O \cap \{x > 0\}$, there exists $s_0 < 0$ with $\gamma(s_0) \in U$. We may assume that $O \subset O_1$ by replacing O by $O \cap O_1$ if needed.

To finish the proof, we note that, provided $\epsilon_0 > 0$ is sufficiently small, if $\gamma(0) \in O \subset O_1$, $t(\gamma(s)) \in [t(q) - \epsilon_0, t(q)]$ implies $\gamma(s) \notin {}^{\mathrm{b}}S^*_W M_0$. $\square$

Theorem 7.3 and this lemma immediately give the following Corollary.

Corollary 7.5. *Let $u \in \mathfrak{X} \equiv H^{1,l}_{\mathrm{es}}(M)$ solve $\Box u = 0$ with Dirichlet or Neumann boundary conditions.*

Let $m > l + f/2 - 1$ with $m \geq 0$, $l' \leq 0$. Suppose that $q \in \mathcal{H}_{W,\mathrm{b}}$ and $\mathcal{F}^W_{I,q,\mathrm{sing}} \cap \mathrm{WF}^{m,*}_{\mathrm{eb},\mathfrak{X}} u = \emptyset$. Then ${}^{\mathrm{eb}}S^*_{\partial\tilde{W}} M \cap \mathrm{WF}^{m,l'}_{\mathrm{eb},\mathfrak{X}} u = \emptyset$ for all $l' < 0$.

Proof. Let $\epsilon_0 > 0$ be as in Lemma 7.4. As $\mathrm{WF}^{m,*}_{\mathrm{eb},\mathfrak{X}} u$ is closed, $\mathcal{F}^W_{I,q,\mathrm{sing}} \cap \{t = t(q) - \epsilon_0/2\}$ has a neighborhood U disjoint from $\mathrm{WF}^{m,*}_{\mathrm{eb},\mathfrak{X}} u$. By Lemma 7.4, $\mathcal{R}_{\mathrm{eb},q,I} \cap \mathcal{G}$ has a neighborhood O such that every backward GBB γ with $\gamma(0) \in O \cap \{x > 0\}$ intersects U and is disjoint from ${}^{\mathrm{b}}S^*_W M_0$. By the propagation of singularities, [34], $\mathrm{WF}^{m,*}_{\mathrm{eb},\mathfrak{X}}(u) \cap O \cap \{x > 0\} = \mathrm{WF}^{m,*}_{\mathrm{b},\mathfrak{X}}(u) \cap O \cap \{x > 0\} = \emptyset$. Note that this uses the fact that every backward GBB γ with $\gamma(0) \in O \cap \{x > 0\}$ intersects U and is disjoint from ${}^{\mathrm{b}}S^*_W M_0$, for we do not assume that u lies in a b-derivative of $\mathfrak{X}$ as we allow arbitrary weights at $\tilde{W}$. Thus, by Theorem 7.3, ${}^{\mathrm{eb}}S^*_{\partial\tilde{W}} M \cap \mathrm{WF}^{m,l'}_{\mathrm{eb},\mathfrak{X}} u = \emptyset$ for all $l' < 0$. $\square$

7.3. **Propagation at hyperbolic points within the edge.** Now we consider propagation within ${}^{\mathrm{eb}}S^*_{\tilde{W}} M$, away from the radial points. The propagation away from $\partial\tilde{W}$ is given by the results in [25]: on the edge cosphere bundle over $\tilde{W}^\circ$, we find that $\mathrm{WF}_{\mathrm{eb}}\, u = \mathrm{WF}_{\mathrm{eb},\mathfrak{X}}\, u$ (with, say, $\mathfrak{X} = H^{1,l}_{\mathrm{es}}(M)$, for Dirichlet boundary conditions, $\mathfrak{X} = H^{1,l}_{\mathrm{es},0}(M)$ for Neumann boundary conditions—though this is irrelevant since we are working away from $\partial\tilde{W}$) given by is a union of integral curves of $\mathsf{H}_{\mathrm{es}}|_{{}^{\mathrm{es}}S^*_{\tilde{W}} M}$, given by (3.18), i.e.

$$\frac{1}{2}\mathsf{H}_{\mathrm{es}} = -\hat{\underline{\xi}}\hat{\underline{\zeta}}\partial_{\hat{\underline{\zeta}}} + K^{ij}\hat{\underline{\zeta}}_i\partial_{z_j} + K^{ij}\hat{\underline{\zeta}}_i\hat{\underline{\zeta}}_j\partial_{\hat{\underline{\xi}}} - \frac{1}{2}\frac{\partial K^{ij}}{\partial z_k}\hat{\underline{\zeta}}_i\hat{\underline{\zeta}}_j\partial_{\hat{\underline{\zeta}}_k},$$

where, as beforehand, hats denote variables divided by $|\underline{\tau}|$, hence coordinates in the edge-smooth cosphere bundle (which over $\tilde{W}^\circ$ is canonically identified with the edge cotangent bundle). This leaves open only the question of how bicharacteristics reaching $z' = 0$ interact with those leaving $z' = 0$, i.e. the problem of reflection/diffraction from the boundary faces and corners of Z. Since the propagation over the interior of $\tilde{W}$ can be considered as a special case of propagation at $\mathcal{G} \setminus \mathcal{R}_{\mathrm{eb}}$, see Theorem 7.7 (with no z' variables, i.e. with $k = 0$ in the notation of this theorem), we do not state the interior propagation result of [25] here explicitly.

Let us thus begin by considering a *hyperbolic* point $q \in \mathcal{H}$ given by

$$x = 0,\ t = t_0,\ y = y_0,\ z' = 0,\ z'' = z_0'',\ \hat{\xi} = \hat{\xi}_0,\ \hat{\eta} = \hat{\eta}_0,\ \hat{\zeta}' = 0,\ \hat{\zeta}_0'', \tag{7.27}$$

in edge-b canonical coordinates. Thus, in addition to $\zeta' = 0$, we have

$$1 > \hat{\xi}^2 + h(y_0, \hat{\eta}_0) + k(y_0, z' = 0, z_0'', \hat{\zeta}' = 0, \hat{\zeta}_0'').$$

In the special case that z' is a variable in $\mathbb{R}^1$, i.e. if q lies on a codimension-one boundary face of eb, then two points in ${}^{\mathrm{eb}}S^*_{\tilde{W}}(M)$ lie above q and two edge bicharacteristics in ${}^{\mathrm{eb}}S^*_{\tilde{W}^\circ}(M)$ contain q in their closures; we denote them $\gamma_\pm$ with the $\pm$ given by $\mathrm{sgn}(\zeta^{s'} \cdot z')$; we will take $\gamma_\pm$ to be only the segments of these bicharacteristics in $|z'| < \epsilon \ll 1$ in order not to enter into global considerations. Our sign convention is such that $\gamma_\pm$ tends toward q under the forward resp. backward bicharacteristic flow. What we will show in this case is that if $u \in \mathfrak{X} \equiv H^{1,l}_{\mathrm{es}}(M)$ and $\Box u = 0$ with Dirichlet or Neumann boundary conditions then

$$\gamma_- \cap \mathrm{WF}^{m,0}_{\mathrm{eb},\mathfrak{X}}\, u = \emptyset \Longrightarrow \gamma_+ \cap \mathrm{WF}^{m,0}_{\mathrm{eb},\mathfrak{X}}\, u = \emptyset \quad \text{for any } m.$$

More generally, we have the following result, which via standard geometric arguments (see [24]) implies the propagation along EGBB's through p :

Theorem 7.6. *For Neumann boundary conditions, let $\mathfrak{X} = H^{1,l}_{\mathrm{es}}(M)$, $\mathfrak{Y} = \dot{H}^{-1,l-2}_{\mathrm{es}}(M)$; for Dirichlet boundary conditions let $\mathfrak{X} = H^{1,l}_{\mathrm{es},0}(M)$, $\mathfrak{Y} = H^{-1,l-2}_{\mathrm{es}}(M)$.*

Let $u \in \mathfrak{X}$ solve $\Box u = f$, $f \in \mathfrak{Y}$. Let $p \in \mathcal{H}$ be given by (7.27). Let U be an open neighborhood of p in ${}^{\mathrm{eb}}S^*_{\tilde{W}}(M)$ let $m \in \mathbb{R}$, $l' \leq 0$, and suppose that $\mathrm{WF}^{m+1,l'}_{\mathrm{eb},\mathfrak{Y}}(f) \cap U = \emptyset$. Then

$$U \cap \{\sum \hat{\zeta}_i' > 0\} \cap \mathrm{WF}^{m,l'}_{\mathrm{eb},\mathfrak{X}}(u) = \emptyset \Longrightarrow p \notin \mathrm{WF}^{m,l'}_{\mathrm{eb},\mathfrak{X}}(u).$$

Thus, the hypothesis region of the theorem, in which we make a wavefront assumption lies within the points with at least one z_i' non-zero, i.e. away from ${}^{\mathrm{eb}}S^*_C M$, where $C = \{x = 0,\ z' = 0\}$, and with momenta directed *toward* the boundary $z' = 0$.

Proof. As usual, one needs to prove that if in addition to the hypotheses above $p \notin \mathrm{WF}^{m-1/2,l'}_{\mathrm{eb},\mathfrak{X}}(u)$ then $p \notin \mathrm{WF}^{m,l'}_{\mathrm{eb},\mathfrak{X}}(u)$, with a slightly more controlled (but standard) version if $m = \infty$. So we assume $p \notin \mathrm{WF}^{m-1/2,l'}_{\mathrm{eb},\mathfrak{X}}(u)$ from now on.

For a constant β to be determined later, let

$$\phi = \sum \hat{\zeta}_i' + \beta\omega \tag{7.28}$$

where

$$\omega = \left|z'\right|^2 + \left|z'' - z_0''\right|^2 + \left|\hat{\zeta}'' - \hat{\zeta}_0''\right|^2 + |y - y_0|^2 + |t - t_0|^2 + x^2 + |\hat{\eta} - \hat{\eta}_0| + \left|\hat{\xi} - \hat{\xi}_0\right|^2$$

Then for β sufficiently small, we have

$$|\tau|^{-1}\mathsf{H}_p\phi > 0.$$

Now let $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$ with support in $[0,\infty)$ and $\chi_0(s) = \exp(-1/s)$ for $s > 0$. Thus, $\chi_0'(s) = s^{-2}\chi_0(s)$. Take $\chi_1 \in \mathcal{C}^\infty(\mathbb{R})$ to have support in $[0,\infty)$, to be equal to 1 on $[1,\infty)$ and to have $\chi_1' \geq 0$ with $\chi_1' \in \mathcal{C}_c^\infty((0,1))$. Finally, let $\chi_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ be supported in $[-2c_1, 2c_1]$ and be identically equal to 1 on $[-c_1, c_1]$. Pick $\delta < 1$. Set

$$a = |\tau|^s x^{-r}\chi_0(\mathsf{M}(1-\phi/\delta))\chi_1(\sum \hat{\zeta}_j'/\delta + 1)\chi_2(|\hat{\zeta}'|^2).$$

Note that on the support of a, we have

$$\sum \zeta_j' > -\delta, \tag{7.29}$$

hence we also obtain

$$0 \leq \omega < 2\frac{\delta}{\beta}. \tag{7.30}$$

Thus, by keeping δ and δ/β both small, we can keep the support of a within any desired neighborhood of $\zeta' = 0$, $\omega = 0$.

We now quantize a to $A \in \Psi_{\mathrm{eb}}^{s,r}(M)$. We claim that

$$\begin{aligned} \imath[\Box, A^*A] \\ &= B^*(\sum D_{z_i'}^* C_{ij} D_{z_j'} + R_0 + \sum(R_i D_{z_i'} + D_{z_i'}^* R_i') + \sum D_{z_i'} R_{ij} D_{z_j'})B \\ &\qquad + A^*WA + R'' + E + E' \end{aligned} \tag{7.31}$$

where

(1) $B \in \Psi_{\mathrm{eb}}^{s+1/2,r+1}(M)$ has symbol

$$\mathsf{M}^{1/2}\tau^{s+1/2}x^{-r-1}\delta^{-1/2}\chi_1\chi_2\sqrt{\chi_0\chi_0'}$$

(2) $C_{ij} \in \Psi_{\mathrm{eb}}^{-2,0}(M)$, and the symbol-valued quadratic form $\sigma(C_{ij})$ is strictly positive definite on a neighborhood of $\mathrm{WF}_{\mathrm{eb}}' B$,

(3) R, R_i, R_i', R_{ij} are in $\Psi_{\mathrm{eb}}^{0,0}(M)$, $\Psi_{\mathrm{eb}}^{-1,0}(M)$, $\Psi_{\mathrm{eb}}^{-1,0}(M)$, and $\Psi_{\mathrm{eb}}^{-2,0}(M)$ respectively and have (unweighted) symbols bounded by multiples of $\sqrt{\delta}(\sqrt{\beta} + 1/\sqrt{\beta})$.

(4) $R'' \in \mathrm{Diff}_{\mathrm{es},\sharp}^2 \Psi_{\mathrm{eb}}^{-2,0}(M)$, $E, E' \in \mathrm{Diff}_{\mathrm{es},\sharp}^2 \Psi_{\mathrm{eb}}^{-1,0}(M)$,

(5) E is microsupported where we have assumed regularity,

(6) $W \in \mathrm{Diff}_{\mathrm{es},\sharp}^2 \Psi_{\mathrm{eb}}^{-1,2}(M)$,

(7) E' is supported off the characteristic set.

These terms arise as follows. Applying Lemma 5.35, we have (with $Q_i = x^{-1}D_{z_i'}$)

$$\imath[\Box, A^*A] = \sum Q_i^* L_{ij} Q_j + \sum(x^{-1}L_iQ_i + Q_i^* x^{-1}L_i') + x^{-2}L_0, \tag{7.32}$$

with

$$
\begin{aligned}
&L_{ij} \in \Psi_{\mathrm{eb}}^{2s-1,2r}(M),\ L_i, L_i' \in \Psi_{\mathrm{eb}}^{2s,2r}(M),\ L_0 \in \Psi_{\mathrm{eb}}^{2s+1,2r}(M),\\
&\sigma_{\mathrm{eb},2m-1}(L_{ij}) = 2aV_{ij}a,\ V_{ij} = \kappa_{ij}(\partial_{\zeta_i'} + \partial_{\zeta_j'} + 2\partial_\xi) + \mathsf{H}_{\mathrm{eb},\kappa_{ij}},\\
&\sigma_{\mathrm{eb},2m}(L_i) = \sigma_{\mathrm{eb},2m}(L_i') = 2aV_i a,\\
&V_i = \sum_j \kappa_{ij}\partial_{z_j'} + \frac{1}{2}(m_i\partial_\xi + \mathsf{H}_{\mathrm{eb},m_i}) + \frac{1}{2}m_i(\partial_\xi + \partial_{\zeta_i'}),\\
&\sigma_{\mathrm{eb},2m+1}(L_0) = 2aV_0 a,\ V_0 = 2\tilde{h}\partial_\xi + \mathsf{H}_{\mathrm{eb},\tilde{h}} + \sum_i m_i \partial_{z_i'},\\
&\mathrm{WF}_{\mathrm{eb}}'(L_{ij}), \mathrm{WF}_{\mathrm{eb}}'(L_i), \mathrm{WF}_{\mathrm{eb}}'(L_i'), \mathrm{WF}_{\mathrm{eb}}'(L_0) \subset \mathrm{WF}_{\mathrm{eb}}'(A).
\end{aligned}
\tag{7.33}
$$

with

(7.34)

$$
\begin{aligned}
V_0|_C = &-2\xi\, x\partial_x - 2(\xi^2 + \sum_{ij} k_{2,ij}\zeta_i''\zeta_j'')\partial_\xi - 2\xi(\tau\,\partial_\tau + \eta\,\partial_\eta)\\
&- 2\sum_{i,j} k_{2,ij}\zeta_i''\partial_{z_j''} + \sum_{\ell,i,j}(\partial_{z_\ell''}k_{2,ij})\zeta_i''\zeta_j''\partial_{\zeta_\ell''},\\
V_{ij}|_C = &-k_{1,ij}(\partial_{\zeta_i'} + \partial_{\zeta_j'} + 2\partial_\xi) + \sum_\ell (\partial_{z_\ell''}k_{1,ij})\partial_{\zeta_\ell''},\ V_i|_C = -\sum_j k_{1,ij}\partial_{z_j'},
\end{aligned}
$$

First we evaluate the terms in L_{ij} coming from terms in which V_{ij} hits $\chi_0(\mathsf{M}(1-\phi/\delta))$. The main contribution will be from the derivatives falling on $\hat{\zeta}'$, with the rest controlled by shrinking β; in particular,

$$V_{ij}(\phi/\delta) = -2k_{1,ij} + r_{ij}$$

with

$$|r_{ij}| \le \mathrm{const}(\beta\sqrt{\omega} + \sqrt{\omega});$$

on the support of a, this is in turn controlled by a multiple of

$$\beta\sqrt{\delta/\beta} + \sqrt{\delta/\beta}.$$

Thus, from these two terms, we obtain corresponding terms in $\imath[\Box, A^*A]$ of the forms

$$B^*\big(\sum D_{z_i'}^* C_{ij} D_{z_j'}\big)B$$

and

$$B^*\big(\sum D_{z_i'}^* R_{ij} D_{z_j'}\big)B$$

respectively.

Similarly, terms with V_i and V_0 hitting χ_0 go into the R_i and R_0 terms in (7.31) respectively.

The terms arising from

$$V_\bullet\big(\chi_1(\sum \hat{\zeta}_j'/\delta + 1)\big)$$

are supported on the hypothesis region, $\{\sum \hat{\zeta}_j' < 0\}$, hence give commutator terms of the form $x^{-2}E$ above.

The terms arising from

$$V_\bullet\big(\chi_2(|\hat{\zeta}'|^2)\big)$$

lie off of the characteristic set, hence give commutator terms of the form $x^{-2}E'$ above.

The term arising from differentiating $|\tau|^s x^{-r}$ gives the commutator term A^*WA.

As we are interested in edge-b wavefront set, the term $D^*_{z'_i}C_{ij}D_{z'_j}$ is slightly inconvenient, but we note that owing to strict positivity of C_{ij} we may replace it by a multiple of $\Delta_{z'}/x^2$ plus another positive term. Rewriting $\Delta_{z'}/x^2 = (\Delta_{z'}/x^2 + \Box) - \Box$, and noting that the first of these terms is in $x^{-2}\operatorname{Diff}^2_{\mathrm{eb}}(M)$ and elliptic on the hyperbolic set, we see that we in fact have

$$\begin{aligned}(7.35)\quad \imath[A^*A,\Box] = R'''\Box + B^*(C^*C + \sum D^*_{z'_i}\tilde{C}_{ij}D_{z'_j} + R_0 \\ + \sum D^*_{z'_i}R'_i + R_iD_{z'_i} + A^*WA + \sum D^*_{z'_i}R_{ij}D_{z'_j})B + R'' + E + E'\end{aligned}$$

where $\tilde{C}_{ij} \in \Psi^{0,2}_{\mathrm{eb}}(M)$ is a positive matrix of operators (this is a priori true only at the symbolic level, but we may absorb lower-order terms in R_{ij}).

Following [34], we find that for any $\digamma > 0$,

$$(7.36)\quad |\langle R_0w, w\rangle| \leq C(\sqrt{\delta})(\sqrt{\beta} + 1/\sqrt{\beta})\|w\|^2 + \digamma^{-1}\left\|R'_0w\right\|^2 + \digamma\|w\|^2,$$

where $R'_0 \in \Psi^{-1}_{\mathrm{eb}}$ has the same microsupport as R_0, and is *one order lower*. Here we have employed L^2 boundedness of $\Psi^{0,0}_{\mathrm{eb}}(M)$, or more specifically, the square-root argument used to prove it (cf. [34] for details, specifically the treatment following (6.18)). By the same token, we can estimate

$$\begin{aligned}(7.37)\quad \Big\langle R_iD_{z'_i}w, w\Big\rangle \leq C(\sqrt{\delta})(\sqrt{\beta} + 1/\sqrt{\beta})(\left\|T_{-1}D_{z'_i}w\right\|^2 + \|w\|^2) \\ + 2\digamma\|w\|^2 + \digamma^{-1}\left\|\tilde{R}_iD_{z'_i}w\right\|^2,\end{aligned}$$

where $\tilde{R}_i \in \Psi^{-2,0}_{\mathrm{eb}}(M)$ has the same microsupport as R_i, and is one order lower. We also compute

(7.38)

$$\begin{aligned}\left|\Big\langle R_{ij}D_{z'_i}w, D_{z'_j}w\Big\rangle\right| \leq C(\sqrt{\delta})(\sqrt{\beta} + 1/\sqrt{\beta})(\left\|T_{-1}D_{z'_i}w\right\|^2 + \left\|T_{-1}D_{z'_j}w\right\|^2) \\ + 2\digamma(\left\|T_{-1}D_{z'_j}w\right\|^2 + \left\|T_{-1}D_{z'_j}w\right\|^2) + 2\digamma\|w\|^2 \\ + \digamma^{-1}\left\|\tilde{R}_{ij}D_{z'_i}w\right\|^2 + \digamma^{-1}\left\|\tilde{R}'_{ij}D_{z'_j}w\right\|^2,\end{aligned}$$

where $\tilde{R}_{ij}, \tilde{R}'_{ij} \in \Psi^{-3}_{\mathrm{eb}}(M)$ have the same microsupport as R_{ij}, and are one order lower. Although the argument is identical to that in [34], [25], we reproduce the derivation of (7.38) for the convenience of the reader; (7.36) and (7.37) follow by similar (easier) arguments. To begin, we note that

$T_1^* R_{ij} \in \Psi_{\mathrm{eb}}^{-1}(M)$ has symbol bounded by $C(\sqrt{\delta})(\sqrt{\beta}+1/\sqrt{\beta})$, hence by the Hörmander square-root argument

$$\|T_1^* R_{ij} u\|^2 \leq C(\sqrt{\delta})(\sqrt{\beta}+1/\sqrt{\beta})\|u\|^2 + \left\|\tilde{R}_{ij} u\right\|^2 \tag{7.39}$$

with $\tilde{R}_{ij}$ as described above. Now write $D_{z'_j} w = T_1 T_{-1} D_{z'_j} w - F D_{z'_j} w$; this permits us to expand

$$\left\langle R_{ij} D_{z'_i} w, D_{z'_j} w\right\rangle = \left\langle T_1^* R_{ij} D_{z'_i} w, T_{-1} D_{z'_j} w\right\rangle - \left\langle R_{ij} D_{z'_i} w, F D_{z'_j} w\right\rangle.$$

The first term on the right may be controlled, using (7.39) and Cauchy-Schwarz, by the RHS of (7.38); the second term may also be so estimated by again applying Cauchy-Schwarz and absorbing $\left\|F D_{z'_j} w\right\|^2$ into a term $\left\|\tilde{R}'_{ij} D_{z'_j} w\right\|^2$ with appropriately enlarged $\tilde{R}'_{ij}$.

Now we turn to making our commutator argument. Let u be a solution to

$$\Box u = f$$

with Dirichlet or Neumann boundary conditions. Choose $\Lambda_\gamma \in \Psi_{\mathrm{eb}}^{-1}(M)$ converging to the identity as $\gamma \downarrow 0$ as in §7.2. Note that by making $\Lambda_\gamma \in \Psi_{\mathrm{eb}}^{-1}(M)$, we are combining the roles of the regularizer $\Lambda_\gamma \in \Psi_{\mathrm{eb}}^{-\sigma}(M)$ in §7.2, required for obtaining an improvement over the a priori assumptions, and the regularizer $\tilde{\Lambda}_{\gamma'}$ used to justify the pairing argument, see (7.19). Let $A_\gamma = \Lambda_\gamma A$ with A constructed as above. As before, we have

$$\imath[\Box, A_\gamma^* A_\gamma] = \Lambda_\gamma^* \imath[\Box, A^* A]\Lambda_\gamma + A^* \imath[\Box, \Lambda_\gamma^* \Lambda_\gamma] A + \tilde{R}, \tag{7.40}$$

with $\tilde{R}$ uniformly bounded in $\mathrm{Diff}^2_{\mathrm{es},\sharp} \Psi_{\mathrm{eb}}^{2s-2,2r+2}(M)$ (hence lower order), and where

$$[\Box, \Lambda_\gamma^* \Lambda_\gamma] \in \mathrm{Diff}^2_{\mathrm{es},\sharp} \Psi_{\mathrm{eb}}^{-1,2}(M)$$

is uniformly bounded, and in fact

$$[\Box, \Lambda_\gamma^* \Lambda_\gamma] = \Lambda_\gamma^* \tilde{W}_\gamma \Lambda_\gamma,$$

with $\tilde{W}_\gamma$ uniformly bounded in $\mathrm{Diff}^2_{\mathrm{es},\sharp} \Psi_{\mathrm{eb}}^{-1,2}(M)$, cf. (7.13).

Now we pair $A_\gamma^* A_\gamma$ with u. Letting $B_\gamma = B\Lambda_\gamma$, provided integrations by parts can be justified, we have

$$\begin{aligned}
&\langle A_\gamma \Box u, A_\gamma u\rangle - \langle A_\gamma, A_\gamma \Box u\rangle \\
&\quad = \langle \imath[A_\gamma^* A_\gamma, \Box] u, u\rangle = \|C B_\gamma u\|^2 + \sum \left\langle \tilde{C}_{ij} D_{z'_j} B_\gamma u, D_{z'_i} B_\gamma u\right\rangle \\
&\quad + \langle R_0 B_\gamma u, B_\gamma u\rangle + \sum \left\langle D_{z'_i} B_\gamma u, R_i B_\gamma u\right\rangle + \left\langle R'_i B_\gamma u, D_{z'_i} B_\gamma u\right\rangle \\
&\quad + \sum \left\langle D_{z'_j} R_{ij} B_\gamma u, D_{z'_i} B_\gamma u\right\rangle + \langle W_\gamma A_\gamma u, A_\gamma u\rangle + \langle (R''_\gamma + E_\gamma + E'_\gamma) u, u\rangle
\end{aligned} \tag{7.41}$$

where W_γ is uniformly bounded in $\mathrm{Diff}^2_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{-1,2}(M)$ and comprises both the W term from above and the term containing $[\Box, \Lambda_\gamma^*\Lambda_\gamma]$. The integrations by parts may by justified, for any $\gamma > 0$, if

$$\mathrm{WF}^{s',l'}_{\mathrm{eb},\mathfrak{X}}\, u \cap \mathrm{WF}'\, A = \emptyset \tag{7.42}$$

whenever

$$s-1 \leq s', \quad r \leq l' + l + \frac{(f+1)}{2} - 1$$

since then the products of $\Box$ with $A_\gamma^* A_\gamma$ map $H^{s',l'}_{\mathrm{eb},\mathfrak{X}}(M)$ to its dual (as required in the Neumann setting), as well as mapping $H^{s',l'}_{\mathrm{eb},\mathfrak{Y}}(M)$, $\mathfrak{Y} = H^{1,l}_{\mathrm{es},0}(M)$, to its dual (as required in the Dirichlet setting). We take $s' = m - 1/2$, hence $s = m+1/2$, and $r = l'+l+\frac{(f+1)}{2}-1$ here, and note that it suffices to have the microlocal assumptions (7.42) rather than global assumptions in view the microlocality of $\Psi^{(}_{\mathrm{eb}}M)$, see Lemma 5.15 and Lemma 5.16.

We now examine the terms on the RHS. The first two are positive. To the third, we apply (7.36), with $w = B_\gamma u$: if δ, δ/β, and $\digamma$ are sufficiently small, we may absorb the first and third terms on the RHS of (7.36) in $\|CB_\gamma u\|$, while the lower-order second is uniformly bounded by our wavefront assumptions. Likewise, applying (7.37) and (7.38), we may choose $\digamma, \delta, \beta$ so as to absorb terms involving $\digamma$ and $(\sqrt{\delta})(\sqrt{\beta} + 1/\sqrt{\beta})$ in $\left\|x^{-1}CB_\gamma u\right\|^2$; the $\digamma^{-1}$ terms, as they are lower order, remain bounded. Moreover, as $\chi_0'(s) = s^{-2}e^{-1/s}$ for $s > 0$ and vanishes for $s \leq 0$,

$$\mathsf{M}^2(1-\phi/\delta)^2\chi_0'(\mathsf{M}(1-\phi/\delta)) = \chi_0(\mathsf{M}(1-\phi/\delta)).$$

Thus,

$$\begin{aligned}
&|\tau|^{1/2}x^{-1}a\\
&= |\tau|^{s+1/2}x^{-r-1}\mathsf{M}(1-\phi/\delta)\sqrt{\chi_0(\mathsf{M}(1-\phi/\delta))\chi_0'(\mathsf{M}(1-\phi/\delta))}\\
&\qquad\qquad\qquad \chi_1(\textstyle\sum \hat{\zeta}_j'/\delta + 1)\chi_2(|\hat{\zeta}'|^2)\\
&= \mathsf{M}^{1/2}\delta^{1/2}(1-\phi/\delta)b \leq 2\mathsf{M}^{1/2}\delta^{1/2}b
\end{aligned}$$

as $\phi \geq \sum \hat{\zeta}_i' \geq -\delta$ on $\operatorname{supp} a$. We deduce that $\|T_{1/2}x^{-1}A_\gamma u\|$ can be estimated by $4\mathsf{M}^{1/2}\delta^{1/2}\|CB_\gamma u\|$ plus lower order terms, and hence, for M chosen sufficently small, we may absorb the W_γ term in $\|CB_\gamma u\|^2$.

Finally we consider the LHS of (7.41). We have

(7.43)
$$\begin{aligned}
|\langle A_\gamma \Box u, A_\gamma u\rangle| &\leq \|(T_{-1/2})^* x A_\gamma \Box u\|\,\|T_{1/2}x^{-1}A_\gamma u\| + |\langle xA_\gamma\Box u, x^{-1}FA_\gamma u\|\\
&\leq \digamma_1^{-1}\|(T_{-1/2})^* xA_\gamma\Box u\|^2 + \digamma_1\|T_{1/2}x^{-1}A_\gamma u\|^2 + |\langle xA_\gamma \Box u, x^{-1}FA_\gamma u\|
\end{aligned}$$

with $F \in \Psi_{\mathrm{eb}}^{-1,0}(X)$, hence $x^{-1}FA_\gamma$ uniformly bounded in $\Psi_{\mathrm{eb}}^{s-1,r+1}(M)$, xA_γ uniformly bounded in $\Psi_{\mathrm{eb}}^{s,r-1}(M)$, so as $s = m+1/2$, and $r = l'+l+\frac{(f+1)}{2}-1$, the last term is uniformly bounded by the a priori assumptions. Similarly, $\|(T_{-1/2})^* x A_\gamma \Box u\|$ is uniformly bounded, as $(T^{-1/2})^* x A_\gamma$ is uniformly bounded in $\Psi_{\mathrm{eb}}^{s-1/2,r-1}(M)$, while $\|T_{1/2} x^{-1} A_\gamma u\|^2$ can be absorbed in $\|CB_\gamma u\|^2$ (for $\digamma_1$ sufficiently small) as discussed above.

The net result is that
$$\left\|x^{-1} CBu_\gamma\right\|^2$$
remains uniformly bounded as $\gamma \downarrow 0$. Noting that $CB_\gamma \to CB$ strongly (cf. the proof of Lemma A.3), $CB \in \Psi_{\mathrm{eb}}^{s+1/2,r+1}(M)$ is elliptic at q, $s = m + 1/2$, and $r = l' + l + \frac{(f+1)}{2} - 1$, $\mathfrak{X} = H_{\mathrm{es}}^{1,l}(M)$, we can complete the proof in the standard manner. □

7.4. **Propagation at glancing points within the edge.** Let $q \in \mathcal{G} \backslash \mathcal{R}_{\mathrm{eb}}$ be given by
$$x = 0,\ t = t_0,\ y = y_0,\ z' = 0,\ z'' = z_0'',\ \hat{\xi} = \hat{\xi}_0,\ \hat{\eta} = \hat{\eta}_0,\ \hat{\zeta}' = 0,\ \hat{\zeta}'' = \hat{\zeta}_0''.$$
As $q \in \mathcal{G}$,
$$\hat{\xi}_0^2 + h(y_0, \hat{\eta}_0) + k(y, z, \hat{\zeta}' = 0, \hat{\zeta}_0'') = 1.$$
As $q \notin \mathcal{R}_{\mathrm{eb}}$,
$$\hat{\xi}_0^2 + h(y_0, \hat{\eta}_0) < 1,$$
so $\hat{\zeta}_0'' \neq 0$, and $h(y_0, \hat{\eta}_0) < 1$, so $\varpi_{\mathrm{eb}}(q) \in \mathcal{H}_{W,\mathrm{b}}$. We will let Π locally denote the coordinate projection onto the variables
$$(\hat{\xi}, z'', \hat{\zeta}'');$$
Let W be a homogeneous vector field equal to V_0 (from (5.17)) at q, and extended in local coordinates to a constant vector field in $(\hat{\xi}, z'', \hat{\zeta}'')$.

Theorem 7.7. *For Neumann boundary conditions, let* $\mathfrak{X} = H_{\mathrm{es}}^{1,l}(M)$, $\mathfrak{Y} = \dot{H}_{\mathrm{es}}^{-1,l-2}(M)$*; for Dirichlet boundary conditions let* $\mathfrak{X} = H_{\mathrm{es},0}^{1,l}(M)$, $\mathfrak{Y} = H_{\mathrm{es}}^{-1,l-2}(M)$.

Let $u \in \mathfrak{X}$ *solve* $\Box u = f$, $f \in \mathfrak{Y}$. *Let* $q \in \mathcal{G} \backslash \mathcal{R}_{\mathrm{eb}}$ *be as above, and suppose that* $m \in \mathbb{R}$, $l' \leq 0$, *and* $q \notin \mathrm{WF}_{\mathrm{eb},\mathfrak{Y}}^{m+1,l'}(f)$.

There exists $\delta_0 > 0$ *and* $C_0 > 0$ *such that for all* $\delta \in (0, \delta_0)$ *and* $\beta \in (C_0\delta, 1)$,

(7.44)
$$\big(q' \in \Sigma_{\mathrm{eb}},\ |\Pi(q') - \Pi(q) - \delta \mathsf{W}| < \delta\beta \ \text{ and } |z'(q')| < \delta\beta \Longrightarrow q' \notin \mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m,l'} u\big) \\ \Longrightarrow q \notin \mathrm{WF}_{\mathrm{eb},\mathfrak{X}}^{m,l'} u$$

Proof. Below we will choose $\delta_0 > 0$ sufficiently small so that $\mathrm{WF}_{\mathrm{eb},\mathfrak{Y}}^{m+1,l'}(f)$ is disjoint from a δ_0-neighborhood of q (see the discussion before Lemma 5.33).

Let k be the codimension of the face over which q lies. Let ρ_{2n-2k} be the degree-zero homogeneous function with
$$\rho_{2n-2k}|_{\tilde{W}} = \tau^{-2} \tilde{h}|_{\tilde{W}},$$

with $\tilde{h}$ as in Lemma 5.29; note that $\rho_{2n-2k}(q) = 0$ by (3.22) and $d\rho_{2n-2k}(q) \neq 0$ since at least one of the $d\hat{\zeta}''_{k+1}(q), \ldots, d\hat{\zeta}''_f(q)$ components of $d\rho_{2n-2k}(q)$ is non-zero, in view of the quadratic nature of $\tau^2(1-\rho_{2n-2k})$ in the fibers of the cotangent bundle and (3.22) and $\hat{\zeta}''_0 \neq 0$ as observed above. We remark that, with V_0 as in (5.17),

$$V_0\rho_{2n-2k}|_{\tilde{W}} = 0.$$

Note that ${}^{\mathrm{eb}}S^*M$ has dimension $2(n+1)-1 = 2n+1$, thus, with $C = \{x = 0, z' = 0\}$, $\mathcal{G} \cap {}^{\mathrm{eb}}S^*_C M$ has dimension $2n+1-2k-2 = 2n-2k-1$ in view of (3.22). We proceed by remarking that by (5.16),

$$(\mathsf{W}x)|_{\tilde{W}} = 0,\ (\mathsf{W}y_j)|_{\tilde{W}} = 0,\ (\mathsf{W}t)|_{\tilde{W}} = 0,\ (\mathsf{W}\hat{\eta}_j)|_{\tilde{W}} = 0,$$

so t, y_j, $\hat{\eta}_j$ give $1 + 2(n-f-1) = 2(n-f)-1$ homogeneous degree zero functions on ${}^{\mathrm{eb}}T^*M$ (or equivalently $\mathcal{C}^\infty$ functions on ${}^{\mathrm{eb}}S^*M$) whose restrictions to $\mathcal{G} \cap {}^{\mathrm{eb}}S^*_C M$ have linearly independent differentials at q. We let $\rho_2, \ldots, \rho_{2n-2f}$ be given by these functions, and let $\rho_1 = x$. We next remark that, in the notation of (5.10),

$$\mathsf{W}\hat{\xi}(q) = -2k_2(x = 0, y_0, z' = 0, z''_0), \hat{\zeta}''_0) < 0$$

as $\hat{\zeta}_0 \neq 0$, hence $\mathsf{W}(q) \neq 0$. Further, we let ρ_j, $j = 2n-2f+1, \ldots, 2n-2k-1$, be degree-zero homogeneous functions on ${}^{\mathrm{eb}}T^*M$ (or equivalently $\mathcal{C}^\infty$ functions on ${}^{\mathrm{eb}}S^*M$) such that $d(\rho_2|_{\mathcal{G}}), \ldots, d(\rho_{2n-2k-1}|_{\mathcal{G}})$ have linearly independent differentials at q, and such that

$$\mathsf{W}\rho_j(q) = 0.$$

Such functions ρ_j exist as $\mathcal{G} \cap {}^{\mathrm{eb}}S^*_C M$ has dimension $2n+1-2k-2 = 2n-2k-1$, so the $2n-2f-1$ functions $\rho_2, \ldots, \rho_{2n-2f}$ can be complemented by some functions $\rho_{2n-2f+1}, \ldots, \rho_{2n-2k-1}$ to obtain $2n-2k-1$ functions whose pullbacks to $\mathcal{G} \cap {}^{\mathrm{eb}}S^*_C M$ have linearly independent differentials and which are annihilated by W at q, for the space of such one-forms is $2n-2k-2$ dimensional. Thus, by dimensional considerations (using $\mathsf{W}(q) \neq 0$), $\{d\rho_j|_{\mathcal{G}}(q) : \ j = 2, \ldots, 2n-2k-1\}$ spans the space of one-forms on $\mathcal{G}$ annihilated by $\mathsf{W}(q)$, and $d\rho_2, \ldots, d\rho_{2n-2k-1}$ together with $d(\hat{\xi}|_{\mathcal{G}} - \hat{\xi}_0)$ span $T^*\mathcal{G}$. Let

$$\omega_0 = \sum_{j=1}^{2n-2k-1} \rho_j^2;$$

then keeping in mind that $|x| \leq \omega_0^{1/2}$, and with V_0, V_i, V_{ij} as in (5.17),

$$\begin{aligned}
&|\tau|^{-1}|V_0\omega_0| \lesssim \sqrt{\omega_0}(\sqrt{\omega_0} + |\hat{\xi} - \hat{\xi}_0| + |z'|),\\
&|V_i\omega_0| \lesssim \sqrt{\omega_0}\, x \leq \omega_0, \qquad |\tau|\,|V_{ij}\omega_0| \lesssim \sqrt{\omega_0},
\end{aligned}$$

by (5.17) and (5.15). Note also that

$$\begin{aligned}
&\big||\tau|^{-1}|V_0|z'|^2\big| \lesssim |z'|(|z'| + x)\\
&\big|V_i|z'|^2\big| \lesssim |z'|, \qquad |\tau|\big|V_{ij}|z'|^2\big| \lesssim |z'|^2.
\end{aligned}$$

Let $\omega = \omega_0 + |z'|^2$. Then

(7.45)
$$|\tau|^{-1}|V_0\omega| \lesssim \sqrt{\omega}(\sqrt{\omega} + |\hat{\xi} - \hat{\xi}_0|), \qquad |V_i\omega| \lesssim \sqrt{\omega}, \qquad |\tau|\,|V_{ij}\omega| \lesssim \sqrt{\omega}.$$

Let
$$\phi = \hat{\xi} - \hat{\xi}_0 + \frac{1}{\beta^2\delta}\omega$$

By (5.18),

$$\begin{aligned} &\big||\tau|^{-1} V_0\hat{\xi} + 2\sum_{ij} k_{2,ij}\hat{\zeta}_i''\hat{\zeta}_j''\big| \lesssim x \leq \omega^{1/2}, \\ &|V_i\hat{\xi}| \lesssim (|z'| + x) \lesssim \omega^{1/2}, \qquad |\tau|\,|V_{ij}\hat{\xi}| \lesssim 1. \end{aligned} \tag{7.46}$$

In particular, as $\hat{\zeta}_0'' \neq 0$,
$$|\tau|^{-1} V_0\hat{\xi} \leq -c_0 + C_1'\omega^{1/2},$$
for some $c_0 > 0$, $C_1' > 0$.

Set
$$a = |\tau|^s x^{-r}\chi_0(\mathsf{M}(2 - \phi/\delta))\chi_1((\hat{\xi} - \hat{\xi}_0 + \delta)/\beta\delta + 1)\chi_2(|\hat{\zeta}'|^2).$$

We always assume for this argument that $\beta < 1$, so on $\operatorname{supp} a$ we have
$$\phi \leq 2\delta \text{ and } \hat{\xi} - \hat{\xi}_0 \geq -\beta\delta - \delta \geq -2\delta.$$

Since $\omega \geq 0$, the first of these inequalities implies that $\hat{\xi} - \hat{\xi}_0 \leq 2\delta$, so on $\operatorname{supp} a$
$$|\hat{\xi} - \hat{\xi}_0| \leq 2\delta. \tag{7.47}$$

Hence,
$$\omega \leq \beta^2\delta(2\delta - (\hat{\xi} - \hat{\xi}_0)) \leq 4\delta^2\beta^2. \tag{7.48}$$

Moreover, on $\operatorname{supp} d\chi_1$,
$$\hat{\xi} - \hat{\xi}_0 \in [-\delta - \beta\delta, -\delta],\ \omega^{1/2} \leq 2\beta\delta, \tag{7.49}$$
so this region lies in the hypothesis region of (7.44) after β and δ are both replaced by appropriate constant multiples.

We now quantize a to $A \in \Psi^{s,r}_{\mathrm{eb}}(M)$. By Lemma 5.35,
$$\imath[\Box, A^*A] = \sum Q_i^* L_{ij} Q_j + \sum(x^{-1}L_iQ_i + Q_i^*x^{-1}L_i') + x^{-2}L_0, \tag{7.50}$$
with

$$\begin{aligned} &L_{ij} \in \Psi^{2s-1,2r}_{\mathrm{eb}}(M),\ L_i, L_i' \in \Psi^{2s,2r}_{\mathrm{eb}}(M),\ L_0 \in \Psi^{2s+1,2r}_{\mathrm{eb}}(M), \\ &\sigma_{\mathrm{eb},2s-1}(L_{ij}) = 2aV_{ij}a, \\ &\sigma_{\mathrm{eb},2s}(L_i) = \sigma_{\mathrm{eb},2s}(L_i') = 2aV_ia,\ \sigma_{\mathrm{eb},2s+1}(L_0) = 2aV_0a, \\ &\mathrm{WF}'_{\mathrm{eb}}(L_{ij}), \mathrm{WF}'_{\mathrm{eb}}(L_i), \mathrm{WF}'_{\mathrm{eb}}(L_i'), \mathrm{WF}'_{\mathrm{eb}}(L_0) \subset \mathrm{WF}'_{\mathrm{eb}}(A), \end{aligned} \tag{7.51}$$

with V_{ij}, V_i and V_0 as above, given by (5.17). Thus, we obtain

$$\begin{aligned}(7.52)\quad & \imath[\Box, A^*A] \\ &= B^*(C^*C + \sum D_{z_i'}^* C_{ij} D_{z_j'} + \sum (R_i D_{z_i'} + D_{z_i'}^* R_i') + \sum D_{z_i'}^* R_{ij} D_{z_j'})B \\ &\qquad + A^*WA + R'' + E + E'\end{aligned}$$

where

(1) $B \in \Psi_{\mathrm{eb}}^{s+1/2,r+1}(M)$ has symbol
$$\mathsf{M}^{1/2}|\tau|^{s+1/2}x^{-r-1}\delta^{-1/2}\chi_1\chi_2\sqrt{\chi_0\chi_0'}$$
(2) $C \in \Psi_{\mathrm{eb}}^{0,0}(M)$, has strictly positive symbol on a neighborhood of $\mathrm{WF}'_{\mathrm{eb}} B$, given by $(-V_0\phi)^{1/2}$ near $\mathrm{WF}'_{\mathrm{eb}} B$,
(3) $C_{ij} \in \Psi_{\mathrm{eb}}^{-2,0}(M)$, (C_{ij}) positive semidefinite,
(4) R_i, R_i', R_{ij} are in $\Psi_{\mathrm{eb}}^{-1,0}(M)$, $\Psi_{\mathrm{eb}}^{-1,0}(M)$, and $\Psi_{\mathrm{eb}}^{-2,0}(M)$ respectively and have (unweighted) symbols r_i, r_i', r_{ij} with

$$(7.53)\qquad |\tau|\,|r_i|, |\tau|\,|r_i'|, |\tau|^2\,|r_{ij}| \lesssim 1/\beta,$$

(5) $W \in \mathrm{Diff}^2_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{-1,-2}(M)$,
(6) $R'' \in \mathrm{Diff}^2_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{-2,0}(M)$, $E, E' \in \mathrm{Diff}^2_{\mathrm{es},\sharp}\Psi_{\mathrm{eb}}^{-1,0}(M)$,
(7) E is microsupported where we have assumed regularity,
(8) E' is supported off the characteristic set.

These terms arise as follows. By (7.45), (7.46), (7.47), and (7.49),

$$\begin{aligned}V_0\phi &= V_0(\hat\xi - \hat\xi_0) + \frac{1}{\beta^2\delta}V_0\omega \\ &\leq -c_0 + C_1'\omega^{1/2} + \frac{1}{\beta^2\delta}C_1''\omega^{1/2}(\omega^{1/2} + |\hat\xi - \hat\xi_0|) \\ &\leq -c_0 + 2(C_1' + C_1'')(\delta + \frac{\delta}{\beta}) \leq -c_0/4 < 0.\end{aligned}$$

provided that $\delta < \frac{2}{16(C_1'+C_1'')}$, $\frac{\beta}{\delta} > \frac{16(C_1'+C_1'')}{2}$, i.e. that δ is small, but β/δ is not too small—roughly, β can go to 0 at most as a multiple of δ (with an appropriate constant) as $\delta \to 0$. Recall also that $\beta < 1$, so there is an upper bound as well for β, but this is of no significance as we let $\delta \to 0$. Thus, we define C to have principal symbol equal to the product of $(-V_0\phi)^{1/2}$ times a cutoff function identically 1 in a neighborhood of $\operatorname{supp} a$, but with sufficiently small support so that $-V_0\phi > 0$ on it. Thus, the L_0-term of (7.50) gives rise to the C^*C term of (7.52), as well as contributing to the E and E' terms (where χ_1 and χ_2 are differentiated), W (where the weight $|\tau|^s x^{-r}$ is differentiated) and the lower order term R''.

Similarly, the L_i, L_i' and L_{ij} terms in which V_i or V_{ij} differentiates χ_1 or χ_2 contribute to the E and E' terms, while those in which they differentiate the weight contributes to the W term, so it remains to consider when V_i and V_{ij} differentiate χ_0. As we keep $\beta < 1$,

$$|V_i\phi| \le |V_i\hat{\xi}| + |V_i\omega| \lesssim 1 + (\beta^2\delta)^{-1}\omega^{1/2} \lesssim 1 + \beta^{-1} \lesssim \beta^{-1},\ |V_{ij}\phi| \lesssim \beta^{-1},$$

which thus proves the estimates on the terms arise this way, namely R_i, R_i', R_{ij}, above.

We now employ Lemma 5.33 to estimate the $D_{z_i'}$ terms as in the proof of Theorem 7.3. Note that we are using the finer result, Lemma 5.33, rather than its corollary here (unlike in Theorem 7.3), to obtain better control over the constant in front of the $D_{z_i'}$ terms as we shrink δ and β. The important fact is that $\mathcal{G} \cap {}^{\mathrm{eb}}S^*_C M$ is defined by $\rho_{2n-2k} = 0$, $x = 0$, $z' = 0$, and

$$\rho_{2n-2k}, x, |z'| \lesssim \omega^{1/2} \le 2\delta\beta$$

on the wave front set of $C_{ij}, R_i, R_i', R_{ij}$. Thus, we can apply Lemma 5.33 for a $C_1\beta\delta$-neighborhood of a compact subset of $\mathcal{G}$. Noting that $xD_tT_{-1} \in \Psi^{0,0}_{\mathrm{eb}}(M)$, we conclude that, with $B_\gamma = B\Lambda_\gamma$, and for Neumann boundary conditions,

$$\begin{aligned} &\sum \left\| D_{z_i'} T_{-1} B_\gamma u \right\|^2 \le C_0 C_1 \beta\delta \|B_\gamma u\|^2 + \tilde{C}\mathcal{Q}(u, Gu, \tilde{G}u) \\ &\mathcal{Q}(u, Gu, \tilde{G}u) = \|u\|^2_{H^{1,r-(f-1)/2}_{\mathrm{es}}(M)} + \|Gu\|^2_{H^{1,r-(f-1)/2}_{\mathrm{es}}(M)} \\ &\qquad\qquad + \|\Box u\|^2_{\dot{H}^{-1,r-(f+3)/2}_{\mathrm{es}}} + \|\tilde{G}\Box u\|^2_{\dot{H}^{-1,r-(f+3)/2}_{\mathrm{es}}}, \end{aligned} \tag{7.54}$$

where $G \in \Psi^{s,0}_{\mathrm{eb}}(M)$, $\tilde{G} \in \Psi^{s+1/2,0}_{\mathrm{eb}}(M)$ (independent of γ) with wave front set in a neighborhood of $\operatorname{supp} a$. For Dirichlet conditions we simply replace

$$H^{1,r-(f-1)/2}_{\mathrm{es}}(M) \text{ by } H^{1,r-(f-1)/2}_{\mathrm{es},0}(M)$$

and

$$\dot{H}^{-1,r-(f+3)/2}_{\mathrm{es}}(M) \text{ by } H^{-1,r-(f+3)/2}_{\mathrm{es}}(M).$$

Note that by (7.53) we have for all $w \in L^2$,

$$\|T_1^* R_i^* w\| \le C_2\beta^{-1}\|w\| + \|\tilde{R}_i w\|,\ \tilde{R}_i \in \Psi^{-1,0}_{\mathrm{eb}}(M),$$

with $\tilde{R}_i$ having the same microsupport as R_i. But

$$\begin{aligned} &|\langle R_i D_{z_i'} B_\gamma u, B_\gamma u\rangle| \\ &\le |\langle R_i T_1 D_{z_i'} T_{-1} B_\gamma u, B_\gamma u\rangle| + |\langle R_i [T_1, D_{z_i'}] T_{-1} B_\gamma u, B_\gamma u\rangle| \\ &\le \|D_{z_i'} T_{-1} B_\gamma u\|\, \|T_1^* R_i^* B_\gamma u\| + |\langle \hat{R}_i B_\gamma u, B_\gamma u\rangle|,\ \hat{R}_i \in \mathrm{Diff}^1_{\mathrm{es}} \Psi^{-2,0}_{\mathrm{eb}}(M), \end{aligned}$$

and $|\langle \hat{R}_i B_\gamma u, B_\gamma u\rangle|$ can be estimated by the inductive hypothesis, while

$$\begin{aligned} &\|D_{z_i'} T_{-1} B_\gamma u\|\, \|T_1^* R_i^* B_\gamma u\| \\ &\le (C_0 C_1 \beta\delta)^{1/2} C_2 \beta^{-1} \|B_\gamma u\|^2 + \tilde{C}^{1/2} \mathcal{Q}(u, Gu, \tilde{G}u)^{1/2} \|B_\gamma u\| \\ &\le C_2 (C_0 C_1 \delta/\beta)^{1/2} \|B_\gamma u\|^2 + \digamma^{-1} \tilde{C}\mathcal{Q}(u, Gu, \tilde{G}u) + \digamma \|B_\gamma u\|^2. \end{aligned}$$

As $F > 0$ is freely chosen, the main point is that if δ/β is sufficiently small, the first term can be absorbed into $\|CB_\gamma u\|^2$, for the principal symbol of C is bounded below by $(c_0/4)^{1/2}$ on $\operatorname{supp} a$. Since the R_i' term is analogous, and the R_{ij} term satisfies better estimates (for one uses (7.54) directly, rather than its square root, as for R_i), the proof can be finished as in Theorem 7.6. $\square$

Finally, applying arguments that go back to [24] we may put together Theorems 7.6 and 7.7 to obtain propagation of edge-b wavefront set along EGBB's over the edge face:

Theorem 7.8. *For Neumann boundary conditions, let $\mathfrak{X} = H^{1,l}_{\mathrm{es}}(M)$, $\mathfrak{Y} = \dot{H}^{-1,l-2}_{\mathrm{es}}(M)$; for Dirichlet boundary conditions let $\mathfrak{X} = H^{1,l}_{\mathrm{es},0}(M)$, $\mathfrak{Y} = H^{-1,l-2}_{\mathrm{es}}(M)$.*

Let $u \in \mathfrak{X}$ solve $\Box u = f$, $f \in \mathfrak{Y}$. Then for all $s \in \mathbb{R} \cup \{\infty\}$, $l' \leq 0$,

$$\big((\mathrm{WF}^{s,l'}_{\mathrm{eb},\mathfrak{X}}(u) \setminus \mathrm{WF}^{s+1,l'}_{\mathrm{eb},\mathfrak{Y}}(f)) \cap {}^{\mathrm{eb}}S^*_{\tilde{W}}M\big) \subset {}^{\mathrm{eb}}\Sigma$$

is a union of maximally extended EGBB's in ${}^{\mathrm{eb}}\Sigma \setminus \mathrm{WF}^{s+1,l'}_{\mathrm{eb},\mathfrak{Y}}(f)$.

8. Propagation of fiber-global coisotropic regularity

We now state a microlocal result on the propagation of coisotropy. The result says that coisotropic regularity propagates along EGBB's provided that we also have infinite order regularity along all rays arriving at radial points in $\mathcal{G}$.

Theorem 8.1 (Microlocal propagation of coisotropy). *Suppose that $u \in H^{1,l}_{\mathrm{es}}(M)$, $\Box u = 0$, with Dirichlet or Neumann boundary conditions (see Definition 5.20), $p \in \mathcal{H}_{W,\mathrm{b}}$. Suppose also that*

(i) *$q \in (\mathcal{H}_{\mathrm{eb}\to p,\mathrm{b}} \cap \mathcal{R}_{\mathrm{eb},O}) \setminus {}^{\mathrm{eb}}S^*_{\partial\tilde{W}}M$,*
(ii) *u has coisotropic regularity of order $k \in \mathbb{N}$ relative to H^m on the coisotropic $\mathcal{F}^W_{I,\mathrm{reg}}$ in an open set containing all points in $\mathcal{F}^W_{I,p,\mathrm{reg}} \cap \{0 < x < \delta\}$ that are geometrically related to $\mathcal{F}_{O,q}$.*
(iii) *$\mathrm{WF}_{\mathrm{b}}(u) \cap \mathcal{F}^W_{p,I,\mathrm{sing}} = \emptyset$.*

Then u has coisotropic regularity of order k relative to $H^{m'}$ for all

$$m' < \min(m, l + f/2)$$

on $\mathcal{F}^W_{O,\mathrm{reg}}$, in a neighborhood of $\mathcal{F}_{O,q,\mathrm{reg}}$.

Proof. The second numbered assumption and propagation of WF_{e} through incoming radial points, Theorem 7.1 part (1), implies that along EGBB's in the backward flow of q which pass through $\mathcal{R}_{\mathrm{eb},I} \setminus \mathcal{G}$ there is no $\mathrm{WF}^{m,\tilde{l}}_{\mathrm{eb}}$, with

$$\tilde{l} = \min(l, m - f/2 - 0).$$

In view of Theorem 7.3, the third assumption gives the same along EGBB's in the backward flow of q which pass through $\mathcal{R}_{\mathrm{eb},I} \cap \mathcal{G}$. Thus, near q, but on the EGBB's in the backward flow of q, there is no $\mathrm{WF}_{\mathrm{eb}}^{m,\tilde{l}}$ at all. Propagation of singularities through q (Theorem 7.1 part (2)) then gives no $\mathrm{WF}_{\mathrm{e}}{}^{\tilde{m},\tilde{l}}$, $\tilde{m} = \min(m, \tilde{l} + f/2 - 0)$, on the flow-out. Substituting in $\tilde{l}$, we see that $\tilde{m} = \min(m, l + f/2 - 0)$, giving no $\mathrm{WF}_{\mathrm{eb}}^{m',\tilde{l}}$. Thus in $x > 0$, near the flow out, there is no $\mathrm{WF}^{m'}$, which gives the case $k = 0$.

We now turn to the general case, $k \neq 0$. To begin, note that assumption ii and Theorem 7.1 imply that in fact we have coisotropic regularity of order k relative to $H_{\mathrm{eb}}^{m,\tilde{l}}$ at all $q' \in \mathcal{R}_{\mathrm{eb},p,I}$ that are connected to q by an EGBB . This in turn yields absence of $\mathrm{WF}_{\mathrm{eb}}^{m+k,\tilde{l}}$ in a *neighborhood* of each such q in ${}^{\mathrm{eb}}S^*_{\tilde{W}}M$, as the operators in $\mathcal{A}$ are all characteristic only at the radial points over $\tilde{W}$. By Theorem 7.8 followed by the second part of Theorem 7.1, we then achieve coisotropic regularity of order k relative to $H_e^{m',\tilde{l}}$ at q, hence in a neighborhood as well. $\square$

Corollary 8.2. *Suppose that $u \in H^{1,l}_{\mathrm{es}}(M)$, $\Box u = 0$, with Dirichlet or Neumann boundary conditions, $p \in \mathcal{H}_{W,\mathrm{b}}$, $k \in \mathbb{N}$. Suppose also that*

(i) *u has coisotropic regularity of order k relative to H^m on the coisotropic $\mathcal{F}^W_{I,\mathrm{reg}}$ in a neighborhood of $\mathcal{F}_{I,p,\mathrm{reg}}$,*
(ii) $\mathrm{WF}_{\mathrm{b}}(u) \cap \mathcal{F}^W_{p,I,\mathrm{sing}} = \emptyset$.

Then u has coisotropic regularity of order k relative to $H^{m'}$ for all

$$m' < \min(m, l + f/2)$$

on $\mathcal{F}^W_{O,\mathrm{reg}}$ in a neighborhood of $\mathcal{F}^W_{O,p,\mathrm{reg}}$.

Finally, we prove that the regularity with respect to which coisotropic regularity is gained in the above results is not, in fact, dependent on the weight l :

Corollary 8.3. *Suppose that $u \in H^1(M_0)$, $\Box u = 0$, with Dirichlet or Neumann boundary conditions, $p \in \mathcal{H}_{W,\mathrm{b}}$, $k \in \mathbb{N}$, $\epsilon > 0$. There exists k' (depending on k and ϵ) such that*

(i) *u has coisotropic regularity of order k' relative to H^s on the coisotropic $\mathcal{F}^W_{I,\mathrm{reg}}$ in a neighborhood of $\mathcal{F}^W_{I,p,\mathrm{reg}}$,*
(ii) $\mathrm{WF}_{\mathrm{b}}(u) \cap \mathcal{F}^W_{p,I,\mathrm{sing}} = \emptyset$.

implies that u has coisotropic regularity of order k relative to $H^{s-\epsilon}$ on $\mathcal{F}^W_{O,\mathrm{reg}}$ in a neighborhood of $\mathcal{F}^W_{O,p,\mathrm{reg}}$.

Proof. Consider Dirichlet boundary conditions first. Then

$$u \in H^{1,1-(f+1)/2}_{\mathrm{es},0}(M).$$

Thus, by Corollary 8.2, u has coisotropic regularity of order k' relative to[12] $H^{m-\epsilon}$, $m < \min(s, 1/2)$ on $\mathcal{F}_{O,\mathrm{reg}}$ near $\mathcal{F}_{O,p,\mathrm{reg}}$, strictly away from ∂M.

[12]An improved version of the argument, using the time derivative to shift the domain, gives coisotropy of order k' relative to $H^{s-1/2-\epsilon}$.

On the other hand, by the propagation of singularities, [34, Corollary 8.4], u is in H^s along $\mathcal{F}^W_{O,p}$. Hence the theorem follows by the interpolation result of the following lemma, Lemma 8.4.

Consider Neumann boundary conditions next. Then $u \in H^{1,-(f+1)/2}_{\mathrm{es}}(M)$, so by Corollary 8.2, u has coisotropic regularity of order k' relative to $H^{m-\epsilon}$, $m < \min(s, -1/2)$ on $\mathcal{F}^W_{O,\mathrm{reg}}$ near $\mathcal{F}^W_{O,p,\mathrm{reg}}$, strictly away from ∂M.

Proceeding now as in the Dirichlet case, using [34, Corollary 8.4], we complete the proof. □

Lemma 8.4. *Suppose that u is in H^s microlocally near some point q away from ∂M, and it is coisotropic of order N relative to H^m near q with $s > m$. Then for $\epsilon > 0$ and $k < \frac{\epsilon N}{s-m}$, u is coisotropic of order k relative to $H^{s-\epsilon}$ near q.*

In particular, if u is in H^s microlocally near some point q away from ∂M and u is coisotropic (of order ∞) relative to H^m near q with $s > m$, then u is coisotropic relative to $H^{s-\epsilon}$ for all $\epsilon > 0$.

Proof. If $Q \in \Psi^0(M)$ and $\mathrm{WF}'(Q)$ lies sufficiently close to q, then the hypotheses are globally satisfied by $u' = Qu$. Moreover, being coisotropic, locally $\mathcal{F}$ can be put in a model form $\zeta = 0$ by a symplectomorphism Φ in some canonical coordinates (y, z, η, ζ), by [10, Theorem 21.2.4] (for coisotropic submanifolds one has $k = n - l$, $\dim S = 2n$, in the theorem). Further reducing $\mathrm{WF}'(Q)$ if needed, and using an elliptic 0th order Fourier integral operator F with canonical relation given by Φ to consider the induced problem for $v = Fu' = FQu$, we may thus assume that $v \in H^s$, and $D_z^\alpha v \in H^m$ for all α, i.e. $\langle D_z\rangle^N v \in H^m$. Considering the Fourier transform $\hat{v}$ of v, we then have $\langle \eta, \zeta\rangle^s \hat{v} \in L^2$, $\langle \eta, \zeta\rangle^m \langle \zeta\rangle^N \hat{v} \in L^2$. But this implies $\langle \eta, \zeta\rangle^{m\theta + s(1-\theta)} \langle \zeta\rangle^{N\theta} \hat{v} \in L^2$ for all $\theta \in [0, 1]$ by interpolation (indeed, in this case by Hölder's inequality). In particular, taking $\theta = \frac{\epsilon}{s-m}$, $\langle \eta, \zeta\rangle^{s-\epsilon} \langle \zeta\rangle^k \hat{v} \in L^2$ if $k < \frac{N\epsilon}{s-m}$, and the lemma follows. □

9. Geometric theorem

The final essential ingredient in the proof of the geometric theorem is the dualization of the coisotropic propagation result, Corollary 8.3. We now prove such a result.

Theorem 9.1. *Let $u \in H^{-\infty}_{\mathrm{b},H^1_{loc}}(M_0)$ satisfy the wave equation with Dirichlet or Neumann boundary conditions. Let $p \in \mathcal{H}_{W,\mathrm{b}}$, and $w \in \mathcal{F}^W_{O,p,\mathrm{reg}}$.*

Suppose $k \in \mathbb{N}$ and $\epsilon > 0$. Then there is $k' \in \mathbb{N}$ (depending on k and ϵ) such that if $\mathrm{WF}_{\mathrm{b}}(u) \cap \mathcal{F}^W_{I,p,\mathrm{sing}} = \emptyset$ and u is non-focusing of order k relative to H^s on a neighborhood of $\mathcal{F}^W_{I,p,\mathrm{reg}}$ in $\mathcal{F}^W_{I,\mathrm{reg}}$ then u is non-focusing of order k' relative to $H^{s-\epsilon}$ at w.

Proof. We assume $s \leq 0$ to simplify notation; we return to the general case at the end of the argument.

Let $T = t(p)$, and choose $T_0 < T < T_1$ sufficiently close to T. Let χ be smooth step function such that $\chi \equiv 1$ on a neighborhood of $[T, \infty]$ and $\chi \equiv 0$ on a neighborhood of $(-\infty, T_0]$. We find that

$$v \equiv \chi u$$

satisfies $\Box v = f$ with $f = [\Box, \chi]u$, and v vanishes on a neighborhood of $(-\infty, T_0] \times X$. Thus, we write

$$\Box_+^{-1} f = v.$$

By propagation of singularities, [34], only singularities of f on $\mathcal{F}_{I,p}^W$ affect regularity at w, i.e. if $\mathcal{F}_{I,p}^W \cap \mathrm{WF}^{s-1}_{\mathrm{b},\dot{H}^{-1}(M_0)}(f) = \emptyset$ then $w \notin \mathrm{WF}^s_{\mathrm{b},H^1(M_0)}(u)$, hence in particular w is non-focusing of order 0 relative to H^{s+1}. Thus,

$$\begin{aligned} Q_0 \in \Psi_{\mathrm{b}}^0(M_0),\ \mathrm{WF}'(\mathrm{Id} - Q_0) \cap \mathcal{F}_{I,p}^W \cap {}^{\mathrm{b}}S^*_{\operatorname{supp} d\chi} M_0 = \emptyset \\ \Rightarrow w \notin \mathrm{WF}_{\mathrm{b},H^1(M_0)}(\Box_+^{-1}((\mathrm{Id} - Q_0) f)), \end{aligned}$$

so it suffices to analyze $\Box_+^{-1}(Q_0 f)$. We choose $\mathrm{WF}'(Q_0)$ sufficiently small such that

$$\begin{aligned} q \in \mathrm{WF}'(Q_0) \Rightarrow \text{ either } q \notin \mathrm{WF}_{\mathrm{b}}(f) \text{ or} \\ f \text{ is non-focusing of order } k \text{ at } q \text{ relative to } H^{s-1}; \end{aligned}$$

this is possible by our hypotheses. We may thus replace f by $f_0 = Q_0 f$, assume that f_0 is the sum of a distribution that is non-focusing of order k relative to H^{s-1} and is supported in M_0° plus an element of $H^\infty_{\mathrm{b},H^1(M_0)}$, and show that $\Box_+^{-1} f_0$ is non-focusing at w of order k' (for some k' to be determined) relative to $H^{s-\epsilon}$.

Let

$$T_0' < T_0 < T_1 < T_1'.$$

We regard $[T_0, T_1]$ as the time interval for analysis, but we enlarge it to $[T_0', T_1']$ in order to be able to apply some b-ps.d.o's with symbol elliptic for $t \in [T_0, T_1]$ to elements of our function spaces. (The ends of the interval would be slightly troublesome.) We define a Hilbert space $\mathfrak{X}$ to be $\mathfrak{X} = H^1([T_0', T_1'] \times X_0)$ in the case of Neumann conditions, or $\mathfrak{X} = H^1_0([T_0', T_1'] \times X_0)$ in the case of Dirichlet conditions, where 0 indicates vanishing enforced at $[T_0', T_1'] \times \partial X_0$ (but not at the endpoints of the time interval). Let $\mathfrak{X}^*$ be the L^2_g-dual of $\mathfrak{X}$.

We further let

(i)
$$T_0 < \tilde{T}_0 < t_0' < t_0 < T < t_1 < t_1' < T_1$$
such that $\operatorname{supp} d\chi \subset (t_0, T)$.

(ii) $\chi_0 \in \mathcal{C}^\infty(\mathbb{R})$ such that $\operatorname{supp}(1-\chi_0) \subset (\tilde{T}_0, +\infty)$, $\operatorname{supp} \chi_0 \subset (-\infty, t_0')$.

(iii) $\mathcal{U}_0$ be an open set with $\overline{\mathcal{U}_0} \subset \{t \in (t_0', T)\}$, $\overline{\mathcal{U}_0} \cap \mathcal{F}^W_{I,\mathrm{sing}} = \emptyset$ and $\mathrm{WF}_{\mathrm{b},\mathfrak{X}^*}(f_0) \subset \mathcal{U}_0$.

(iv) $\mathcal{U}_1$ be a neighborhood of w with $\overline{\mathcal{U}_1} \subset \{t \in (T, t_1')\}$ and $\overline{\mathcal{U}_1} \cap \mathcal{F}^W_{O,\mathrm{sing}} = \emptyset$.

(v) $B_0, B_1 \in \Psi^0_{\mathrm{b}}(M)$ with

$$\mathrm{WF}'(B_j) \subset \mathcal{U}_j,\ w \notin \mathrm{WF}'(\mathrm{Id} - B_1),\ \mathrm{WF}'(\mathrm{Id} - B_0) \cap \mathrm{WF}_{\mathrm{b},\mathfrak{X}^*}(f_0) = \emptyset,$$

and with Schwartz kernel supported in $(t_0', t_1')^2 \times X^2$.

(vi) A_i, $i = 1, \dots, N$, denote first-order pseudodifferential operators, generating $\mathcal{M}$ as defined in §6, but now locally over a neighborhood of $\overline{\mathcal{U}_0} \cup \overline{\mathcal{U}_1}$ in M°, and with kernels compactly supported in M°.

(vii) $T_\nu \in \Psi^\nu_{\mathrm{b}}(M)$ with elliptic principal symbol on $[T_0, T_1] \times X_0$ with Schwartz kernel supported in $(T_0', T_1')^2 \times (X_0)^2$. Thus, T_ν can be applied to elements of $\mathfrak{X}$ and $\mathfrak{X}^*$.

Now suppose that we are given r and $\epsilon > 0$. Then, with k as in the statement of the Theorem, Corollary 8.3 gives a $k' = k'(r, \epsilon, k)$ using $s = r$ in the notation of that corollary. We let $\mathfrak{Y}$ be a space of microlocally coisotropic functions on $\mathrm{ell}(B_1)$ relative to $\mathfrak{X}^*$ which are in addition extremely well-behaved elsewhere (they will be finite-order conormal to the boundary) in $\{t \geq t_0'\}$, but are merely in $H^1_{\mathrm{b},\mathfrak{X}^*}$ for t near T_0: Let $N > r > 1 + \epsilon$ and set

$$\mathfrak{Y} = \{\psi \in \mathfrak{X}^* :$$
$$\|T_N(\mathrm{Id} - B_1 - \chi_0)\psi\|^2_{\mathfrak{X}^*} + \|T_1 \chi_0 \psi\|^2_{\mathfrak{X}^*} + \sum_{|\alpha| \leq k'} \|T_r A^\alpha B_1 \psi\|^2_{\mathfrak{X}^*} < \infty\}.$$

Thus,

$$\psi \in \mathfrak{Y} \Rightarrow \mathrm{WF}^N_{\mathrm{b},\mathfrak{X}^*}(\psi) \subset \mathrm{WF}'(B_1) \cup {}^{\mathrm{b}}S^*_{\mathrm{supp}\,\chi_0} M_0,$$

and

$$\|\psi\|_{H^1_{\mathrm{b},\mathfrak{X}^*}} \lesssim \|\psi\|_{\mathfrak{Y}} \tag{9.1}$$

for ψ supported in $[T_0, T_1] \times X_0$ (where the T_s are elliptic).

Also, let $\mathfrak{Z}$ be the space of microlocally coisotropic functions on $\mathrm{ell}(B_0)$ relative to $\mathfrak{X}$ (and just in $\mathfrak{X}$ elsewhere):

$$\mathfrak{Z} = \{\phi \in \mathfrak{X} : \sum_{|\alpha| \leq k} \|T_{r-1-\epsilon} A^\alpha B_0 \phi\|^2_{\mathfrak{X}} < \infty\}.$$

Note that as discussed in Section 6 (in particular, Lemma 6.8)

$$\mathfrak{Z}^* = \mathfrak{X}^* + \sum_{|\alpha| \leq k} T_{r-1-\epsilon} A^\alpha B_0 \mathfrak{X}^*,$$

so by our assumption on f_0, $f_0 \in \mathfrak{Z}^*$, provided $-(r - 1 - \epsilon) - 1 \leq s - 1$, i.e. provided[13] $r \geq -s + 1 + \epsilon$. Moreover, if $v_0 \in \mathfrak{Y}^*$, then

$$v_0 \in \mathfrak{X} + T_N(\mathrm{Id} - B_1 - \chi_0)\mathfrak{X} + T_1 \chi_0 \mathfrak{X} + \sum_{|\alpha| \leq k'} T_r A^\alpha B_1 \mathfrak{X}. \tag{9.2}$$

[13]Note that for such r, $r \geq 1 + \epsilon$ as required above, since $s \leq 0$.

In particular, as $w \notin \mathrm{WF}'(\mathrm{Id} - B_1 - \chi_0) \cup {}^{\mathrm{b}}S^*_{\operatorname{supp}\chi_0} M_0$, v_0 is non-focusing at w of order k' relative to H^{-r+1}, hence relative to $H^{s-\epsilon}$, if we actually choose $r = -s + 1 + \epsilon$.

For $I \subset [T_0, T_1]$, let $\dot{\mathcal{D}}_I$ denote the subspace of $H^{\infty}_{\mathrm{b},\mathfrak{X}}$ consisting of functions supported in $I \times X$, $\dot{\mathcal{E}}_I$ denote the subspace $H^{\infty}_{\mathrm{b},\mathfrak{X}^*}$ consisting of functions supported in $I \times X$, so $\Box : \dot{\mathcal{D}}_I \to \dot{\mathcal{E}}_I$ is continuous, $\dot{\mathcal{D}}_I \subset \mathfrak{Z}$, $\dot{\mathcal{E}}_I \subset \mathfrak{Y}$ are dense with continuous inclusions. Also let $\dot{\mathcal{D}} = \dot{\mathcal{D}}_I$, $\dot{\mathcal{E}} = \dot{\mathcal{E}}_I$ for $I = (T_0, T_1)$. Finally we also let $\mathfrak{Z}_I$ be space of restrictions of elements of $\mathfrak{Z}$ to I, and analogously for $\mathfrak{Y}_I$. Then, as we will prove in Lemma 9.2[14], Corollary 8.3 implies that

$$\phi \in \dot{\mathcal{D}} \Rightarrow \|\phi\|_{\mathfrak{Z}} \leq C\|\Box\phi\|_{\mathfrak{Y}}, \tag{9.3}$$

where vanishing for t near T_1 is used. In fact, we prove a somewhat more precise statement:[15]

Lemma 9.2. *For $\tau \in [T_0, T_1)$, $\tau' > \tau$,*

$$\phi \in \dot{\mathcal{D}} \Rightarrow \|\phi\|_{\mathfrak{Z}_{[\tau',T_1]}} \leq C\|\Box\phi\|_{\mathfrak{Y}_{[\tau,T_1]}}. \tag{9.4}$$

Proof of Lemma: Recall first that by standard energy estimates (taking into account the vanishing of ϕ near T_1)

$$\phi \in \dot{\mathcal{D}} \Rightarrow \|\phi\|_{\mathfrak{X}_{[\tau,T_1]}} \lesssim \|\Box\phi\|_{\mathfrak{X}^*_{[\tau,T_1]}} + \|D_t\Box\phi\|_{\mathfrak{X}^*_{[\tau,T_1]}} \lesssim \|\Box\phi\|_{\mathfrak{Y}_{[\tau,T_1]}}. \tag{9.5}$$

Thus, we only need to prove that for $|\alpha| \leq k'$,

$$\|T_{r-1-\epsilon}A^\alpha B_0\phi\|_{\mathfrak{X}_{[\tau',T_1]}} \lesssim \|\Box\phi\|_{\mathfrak{Y}_{[\tau,T_1]}}.$$

If $\Box\phi$ is supported away from ∂M, then this follows from Corollary 6.11 and Corollary 8.3. In general, let $Q \in \Psi^0(M)$ be such that $\mathrm{WF}'(B_1) \cap \mathrm{WF}'(\mathrm{Id} - Q) = \emptyset$, and Q has compactly supported Schwartz kernel in $(M^\circ)^2$. Then $Q\Box\phi$ has support away from ∂M, so

$$\|T_{r-1-\epsilon}A^\alpha B_0\Box_-^{-1}(Q\Box\phi)\|_{\mathfrak{X}_{[\tau',T_1]}} \lesssim \|Q\Box\phi\|_{\mathfrak{Y}_{[\tau,T_1]}} \lesssim \|\Box\phi\|_{\mathfrak{Y}_{[\tau,T_1]}}, \tag{9.6}$$

where $\Box_-^{-1}$ denotes the backward solution of the inhomogeneous wave equation. On the other hand,

$$\|(\mathrm{Id} - Q)\Box\phi\|_{H^N_{b,\mathfrak{X}^*}(M_0)} \lesssim \|\Box\phi\|_{\mathfrak{Y}_{[\tau,T_1]}},$$

so by propagation of b-regularity [34],

$$\|\Box_-^{-1}((\mathrm{Id} - Q)\Box\phi)\|_{H^{N-1}_{b,\mathfrak{X}}(M_0)} \lesssim \|\Box\phi\|_{\mathfrak{Y}_{[\tau,T_1]}},$$

[14]The only reason for Corollary 8.3 combined with Corollary 6.11 not yielding the result immediately is that Corollary 8.3 is stated for the *homogeneous* wave equation. This suffices for our purposes as we only require inhomogeneities that are very regular near the boundary, hence the propagation result of [34] is adequate.

[15]Notice that ϕ is merely supported in (T_0, T_1) here; not in (τ, T_1), which would be (9.3) on $[\tau, T_1]$, except for the loss of going from τ' to τ.

hence the much weaker statement

$$\|T_{r-1-\epsilon}A^\alpha B_0\Box_-^{-1}((\mathrm{Id}-Q)\Box\phi)\|_{\mathfrak{X}_{[\tau',T_1]}} \lesssim \|\Box\phi\|_{\mathfrak{Y}_{[\tau,T_1]}}, \tag{9.7}$$

also holds. Combining (9.6) and (9.7) proves (9.4). *This concludes the proof of Lemma 9.2,* and hence of (9.3) as well.

In particular, recalling that $\dot{\mathcal{D}} = \dot{\mathcal{D}}_{(T_0,T_1)}$, (9.3) shows that for $\psi \in \mathrm{Ran}_{\dot{\mathcal{D}}}\Box$ there is a unique $\phi \in \dot{\mathcal{D}}$ such that $\psi = \Box\phi$; we denote this by $\phi = \Box^{-1}\psi$. Thus,

$$\|\Box^{-1}\psi\|_{\mathfrak{Z}} \leq C\|\psi\|_{\mathfrak{Y}},\ \psi \in \mathrm{Ran}_{\dot{\mathcal{D}}}\Box.$$

Now consider the linear functional on $\mathrm{Ran}_{\dot{\mathcal{D}}}\Box$ given by

$$\psi \mapsto \langle f_0, \Box^{-1}\psi\rangle,\ \psi \in \mathrm{Ran}_{\dot{\mathcal{D}}}\Box,$$

which satisfies

$$|\langle f_0, \Box^{-1}\psi\rangle| \leq \|f_0\|_{\mathfrak{Z}^*}\|\Box^{-1}\psi\|_{\mathfrak{Z}} \leq C\|f_0\|_{\mathfrak{Z}^*}\|\psi\|_{\mathfrak{Y}},\ \psi \in \mathrm{Ran}_{\dot{\mathcal{D}}}\Box.$$

This has a unique extension to a continuous linear functional ℓ on $\overline{\mathrm{Ran}_{\dot{\mathcal{D}}}\Box}$, the closure of $\mathrm{Ran}_{\dot{\mathcal{D}}}\Box$ in $\mathfrak{Y}$.

If we used the Hahn-Banach theorem at this point to extend the linear functional ℓ further to a linear functional v_0 on all of $\mathfrak{Y}$, we would obtain a solution of the wave equation $\Box v_0 = f_0$ on (T_0,T_1), as $\langle\Box v_0,\phi\rangle = \langle v_0,\Box\phi\rangle = \langle f_0,\phi\rangle$ for $\phi \in \dot{\mathcal{D}}$, which is indeed non-focusing at w, but we need not just *any* solution, but the forward solution, $\Box_+^{-1}f_0$. So we proceed by extending the linear functional ℓ to a continuous linear functional L on

$$\overline{\mathrm{Ran}_{\dot{\mathcal{D}}_{(T_0,t_1')}}\Box} + \overline{\dot{\mathcal{E}}_{(T_0,t_0')}}, \tag{9.8}$$

first, in such a manner that the extension is ℓ on the first summand and vanishes on the second summand. If we actually have such an extension, then we can further extend it to all of $\mathfrak{Y}$, then vanishing on the first summand shows that it solves the wave equation on (T_0,t_1'), while vanishing on the second summand shows that it vanishes on (T_0,t_0'), so its restriction as a distribution on (T_0,t_0') is indeed $\Box_+^{-1}f_0$. In order to obtain such an extension we show:

Lemma 9.3.

(i) *ℓ vanishes on the intersection of the two summands, so L is well-defined as a (not necessarily continuous) linear map,*

(ii) *The subspace (9.8) of $\mathfrak{Y}$ is closed, and given an element $\psi+\rho$ in the sum, there is a representation*[16] *$\tilde\psi+\tilde\rho$ of $\psi+\rho$ as a sum of elements of the two summands such that one can estimate the $\mathfrak{Y}$-norm of $\tilde\psi$ and $\tilde\rho$ in terms of $\psi+\rho$.*

[16]Since the intersection of the summands is non-trivial, this can only be true for some representation, not all representations!

PROOF OF LEMMA: We start with the statement regarding intersection of the summands in (9.8). Thus, we claim that if

$$\operatorname{supp} f_0 \subset [t_0, t_1] \times X,$$

then

$$\psi \in \overline{\operatorname{Ran}_{\dot{\mathcal{D}}_{[T_0,t_1']}} \Box} \text{ and } \operatorname{supp}\psi \subset (T_0, t_0') \Rightarrow \ell(\psi) = 0. \tag{9.9}$$

To see this let $\psi_j \to \psi$ in $\mathfrak{Y}$, $\psi_j = \Box\phi_j$, $\phi_j \in \dot{\mathcal{D}}_{[T_0,t_1']}$. Then $\{\psi_j\}$ is Cauchy in $\mathfrak{Y}$, hence $\{\phi_j\}$ is Cauchy in $\mathfrak{Z}$ by (9.3), hence converges to some $\phi \in \mathfrak{Z}$. By the support condition on ϕ_j, $\operatorname{supp}\phi \subset [T_0, t_1']$. As $\Box\phi_j \to \Box\phi$ in $\mathfrak{X}^*$ (for $\Box : \mathfrak{X} \to \mathfrak{X}^*$ is continuous), and $\Box\phi_j = \psi_j \to \psi$ in $\mathfrak{Y}$, hence in $\mathfrak{X}^*$, we deduce that $\psi = \Box\phi$, i.e. $\Box\phi$ is supported in (T_0, t_0'). Thus, $\psi_j|_{(t_0',T_1)} \to 0$ in the $\mathfrak{Y}$ topology hence $\phi_j|_{[t_0,T_1]} \to 0$ in the $\mathfrak{Z}$ topology using (9.4) with $\tau = t_0'$, $\tau' = t_0$, so, by the support condition on f,

$$|\langle f_0, \phi_j\rangle| \leq \|f_0\|_{\mathfrak{Z}^*_{[t_0,t_1]}} \|\phi_j\|_{\mathfrak{Z}_{[t_0,t_1]}} \to 0,$$

so we deduce that $\langle f_0, \phi\rangle = 0$ as claimed.

Next we turn to the closedness of the sum in (9.8). First, we claim that if $\psi \in \operatorname{Ran}_{\dot{\mathcal{D}}_{(T_0,t_1')}} \Box$, $\rho \in \dot{\mathcal{E}}_{(T_0,t_0')}$ then there exist $\tilde{\psi} \in \operatorname{Ran}_{\dot{\mathcal{D}}_{(T_0,t_1')}} \Box$, $\tilde{\rho} \in \dot{\mathcal{E}}_{(T_0,\tilde{T}_0)}$ such that

$$\psi + \rho = \tilde{\psi} + \tilde{\rho} \text{ and } \|\tilde{\psi}\|_{\mathfrak{Y}} \lesssim \|\psi + \rho\|_{\mathfrak{Y}}.$$

Indeed, let $\chi_+ \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\operatorname{supp}\chi_+ \subset (T_0, +\infty) \text{ and } \operatorname{supp}(1 - \chi_+) \subset (-\infty, \tilde{T}_0).$$

Let $\Box_-^{-1}(\psi + \rho)$ denote the backward solution of the inhomogeneous wave equation; i.e. the unique solution of $\Box\tilde{u} = \psi + \rho$ which vanishes on (t_1, T_1). Then let

$$\tilde{\psi} = \Box(\chi_+ \Box_-^{-1}(\psi + \rho)) \in \operatorname{Ran}_{\dot{\mathcal{D}}_{(T_0,t_1')}} \Box,$$

so

$$\tilde{\rho} \equiv \psi + \rho - \tilde{\psi} = (1 - \chi_+)(\psi + \rho) - [\Box, \chi_+]\Box_-^{-1}(\psi + \rho) \in \dot{\mathcal{E}}_{(T_0,\tilde{T}_0)}.$$

Moreover,

$$\tilde{\psi} = \chi_+(\psi + \rho) + [\Box, \chi_+]\Box_-^{-1}(\psi + \rho) \tag{9.10}$$

satisfies

$$\|\tilde{\psi}\|_{\mathfrak{Y}} \lesssim \|\psi + \rho\|_{\mathfrak{Y}},$$

as follows by inspecting the two terms on the right hand side of (9.10): for the first this is clear, for the second this follows from $\|\psi+\rho\|_{H^1_{\mathrm{b},\mathfrak{X}^*}} \lesssim \|\psi+\rho\|_{\mathfrak{Y}}$, see (9.1), hence one has a bound in $\mathfrak{X}$ for $\Box_-^{-1}(\psi + \rho)$ by (9.5), and then $\operatorname{supp}[\Box, \chi_+] \subset \operatorname{supp} d\chi_+ \subset (T_0, \tilde{T}_0)$ gives the desired bound in $\mathfrak{Y}$. *This concludes the proof of Lemma 9.3.*

Thus, if $\psi_j \in \operatorname{Ran}_{\dot{\mathcal{D}}_{(T_0,t_1')}} \Box$, $\rho_j \in \dot{\mathcal{E}}_{(T_0,\tilde{T}_0)}$ and $\psi_j + \rho_j$ converges to some $\nu \in \mathfrak{Y}$ then defining $\tilde{\psi}_j$ and $\tilde{\rho}_j$ as above, we deduce that due to the Cauchy property of $\{\tilde{\psi}_j+\tilde{\rho}_j\}$, $\{\tilde{\psi}_j\}$ is Cauchy in $\mathfrak{Y}$, hence so is $\{\tilde{\rho}_j\}$, thus by the completeness of $\mathfrak{Y}$ they converge to elements in $\psi \in \overline{\mathrm{Ran}_{\dot{\mathcal{D}}_{(T_0,t_1')}}\Box}$, resp. $\rho \in \overline{\dot{\mathcal{E}}_{(T_0,\tilde{T}_0)}}$ with $\psi+\rho=\nu$. This shows that $\overline{\mathrm{Ran}_{\dot{\mathcal{D}}_{(T_0,t_1')}}\Box}+\overline{\dot{\mathcal{E}}_{(T_0,\tilde{T}_0)}}$ is closed, and indeed gives an estimate[17] that if $\nu \in \overline{\mathrm{Ran}_{\dot{\mathcal{D}}_{(T_0,t_1')}}\Box}+\overline{\dot{\mathcal{E}}_{(T_0,\tilde{T}_0)}}$ then there exists $\psi \in \overline{\mathrm{Ran}_{\dot{\mathcal{D}}_{(T_0,t_1')}}\Box}$ and $\rho \in \overline{\dot{\mathcal{E}}_{(T_0,\tilde{T}_0)}}$ such that $\psi+\rho=\nu$ and

$$\|\psi\|_{\mathfrak{Y}}+\|\rho\|_{\mathfrak{Y}} \lesssim \|\nu\|_{\mathfrak{Y}}. \tag{9.11}$$

As mentioned earlier, this construction allows us to define a unique continuous linear functional L on

$$\overline{\mathrm{Ran}_{\dot{\mathcal{D}}_{(T_0,t_1')}}\Box}+\overline{\dot{\mathcal{E}}_{(T_0,t_0')}},$$

in such a way that it is ℓ on the first summand and it vanishes on the second summand: uniqueness is automatic, existence (without continuity) follows from (9.9), as the two functionals agree on the intersection of the two spaces, while continuity follows from (9.11). Then we extend L by the Hahn-Banach theorem to a linear functional v_0 on $\mathfrak{Y}$.

Then $v_0 \in \mathfrak{Y}^*$ solves $\Box v_0 = f_0$ on (T_0,t_1'), since for $\phi \in \dot{\mathcal{D}}_{(T_0,t_1')}$

$$\langle \Box v_0,\phi\rangle = \langle v_0,\Box\phi\rangle = \langle f_0,\phi\rangle,$$

and v vanishes on (T_0,t_0), for it vanishes on $\dot{\mathcal{D}}_{(T_0,t_0)}$, i.e. on test functions supported there, so it is the restriction of the forward solution of the wave equation to (T_0,T_1). We have thus shown that if $f_0 \in \mathfrak{Z}^*$ is supported in $[t_0,t_1]$, which holds if f_0 satisfies the support condition, is microlocally non-focusing on $\mathcal{U}_0$, and is conormal to the boundary elsewhere, then the forward solution of $\Box v_0 = f_0$ is in $\mathfrak{Y}^*$ (cf. Lemma 6.9), hence by (9.2) it is in particular microlocally non-focusing of order k' relative to $H^{s-\epsilon}$ at w. This completes the proof of the theorem if $s \leq 0$.

If $s>0$, one could use a similar argument relative to slightly different spaces: the only reason for the restriction is that elements of $\mathfrak{Y}$ lie in $\mathfrak{X}^*$ and a larger space (which would thus have a smaller dual relative to L^2) would be required to adapt the argument. However, it is easy to reduce the general case to $s \leq 0$: replacing u by $\tilde{u}=(1+D_t^2)^N u$, $N>s/2$, $\tilde{u}$ is non-focusing of order k relative to H^{s-2N} on a neighborhood of $\mathcal{F}^W_{I,p,\mathrm{reg}}$ and solves the wave equation, hence it is non-focusing of order k' relative to $H^{s-2N-\epsilon}$ at w by the already established $s \leq 0$ case of this theorem, and then the microlocal ellipticity of $(1+D_t^2)^N$ near the characteristic set (recall that w is over the interior of M_0) shows that u itself is non-focusing of order k' relative to $H^{s-\epsilon}$ at w, as claimed. $\Box$

[17]This estimate follows from the open mapping theorem, given that the sum is closed, but the direct argument yields it anyway.

As a consequence of the proposition of nonfocusing, we are now able to prove our main theorem:

Theorem 9.4. *Let $u \in H^1_{loc}(M_0)$ satisfy the wave equation with Dirichlet or Neumann boundary conditions. Let $p \in \mathcal{H}_{W,\mathrm{b}}$, and $w \in \mathcal{F}^W_{O,p,\mathrm{reg}}$.*
Assume

(i) *u satisfies the nonfocusing condition relative to H^s on an open neighborhood of $\mathcal{F}^W_{I,p,\mathrm{reg}}$ in $\mathcal{F}^W_{I,\mathrm{reg}}$,*
(ii) $\mathrm{WF}^s\, u \cap \{w' \in \mathcal{F}^W_{I,p,\mathrm{reg}} : w', w$ *are geometrically related*$\} = \emptyset$,
(iii) $\mathrm{WF}_{\mathrm{b}}(u) \cap \mathcal{F}^W_{p,I,\mathrm{sing}} = \emptyset$.

Then

$$w \notin \mathrm{WF}^{s-0}(u).$$

Proof. By using a microlocal partition of unity (cf. the argument at the beginning of Theorem 9.1), we may arrange that (ii) is strengthened to

$$\mathrm{WF}^{\infty}\, u \cap \{w' \in \mathcal{F}^W_{I,p,\mathrm{reg}} : w', w \text{ are geometrically related}\} = \emptyset, \tag{9.12}$$

for if a microlocal piece $\tilde{u}$ is in H^s near $\mathcal{F}^W_{I,p}$, then it is also such at w.

Let $r < s$. On the one hand, by conditions (i) and (iii), u satisfies the non-focusing condition (of some, possibly large, order k') relative to H^r at w due to Theorem 9.1. On the other hand, by Theorem 8.1, (9.12) and condition (iii), u is microlocally coisotropic at w, i.e. there exists $S \in \mathbb{R}$ such that[18] microlocally near w

$$A^{\alpha} u \in H^S \quad \forall \alpha. \tag{9.13}$$

We can interpolate non-focusing and (9.13) as discussed in [25] to conclude that microlocally near w, $u \in H^{r-0}$. Since $r < s$ is arbitrary, this proves the result. □

Corollary 9.5. *Let u be a solution to $\Box u = 0$ with Dirichlet or Neumann boundary conditions, and let $p \in \mathcal{H}_{W,\mathrm{b}}$. Suppose that for some $\epsilon_0 > 0$, in a neighborhood of $\mathcal{F}^W_{I,p}(\epsilon_0)$ in ${}^{\mathrm{b}}S^*_{M_0\setminus W} M_0$, u is a Lagrangian distribution of order s with respect to $\mathcal{L} \subset T^* M_0^\circ$, a conic Lagrangian such that $\mathcal{L} \cap \mathcal{F}^W_{I,p,\mathrm{sing}} = \emptyset$ and the intersection of $\mathcal{L}$ and $\mathcal{F}^W_{I,\mathrm{reg}}$ is transverse at $\mathcal{F}^W_{I,p,\mathrm{reg}}$.*
Then if $w \in \mathcal{F}^W_{O,\mathrm{reg}}$ is not geometrically related to any point in $\mathcal{L}$,

$$w \notin \mathrm{WF}^{-s-n/4+(k-1)/2-0}\, u = \emptyset$$

where k is the codimension of W.

The a priori regularity of such a solution is $H^{-s-n/4-0}$ so this represents a gain in regularity along the diffracted wave of $(k-1)/2 - 0$ derivatives.

Corollary 9.5 follows immediately from Section 14 of [25] and Theorem 9.4.

[18]The particular choice of S is dependent on the background regularity of the solution, which in turn can be low, depending on the order of nonfocusing relative to H^s.

Corollary 9.6. *Let $\gamma : (-\epsilon_0, 0] \to {}^{\mathrm{b}}\Sigma_0$ be a GBB normally incident at W, $\gamma(0) = \alpha \in \mathcal{H}_{W,\mathrm{b}}$, and let $\overline{\gamma}$ be its projection to M_0°. Given $o \in \overline{\gamma}((-\epsilon_0, 0))$, let u_o be the forward fundamental solution of $\Box$, i.e. $u_o = \Box_+^{-1}\delta_o$.*

There exists $\epsilon > 0$ such that if $o \in \overline{\gamma}((-\epsilon, 0))$ then for all $w \in \mathcal{F}^W_{O,\alpha,\mathrm{reg}}$, such that w is not geometrically related to point in ${}^{\mathrm{b}}S^*_o M_0 \cap {}^{\mathrm{b}}\Sigma_0$,

$$w \notin \mathrm{WF}^{(-n+k+1)/2-0}\, u_o$$

where k is the codimension of W.

Note that this represents a gain of $(k-1)/2-0$ derivatives relative to the overall regularity of the fundamental solution, which lies in $H^{-n/2+1-0}$.

Proof. The hypotheses on the location of o ensure that, with $\mathcal{L}$ denoting the flowout of ${}^{\mathrm{b}}S^*_o M_0 \cap {}^{\mathrm{b}}\Sigma_0$, $\mathcal{L}$ is disjoint from $\mathcal{F}^W_{I,\alpha,\mathrm{sing}}$ in view of Corollary 3.27. Thus, the microlocal setting is the same as that of [25], hence the hypotheses of Corollary 9.5 are satisfied. □

Appendix A. Some functional analysis

We often encounter the following setup. Suppose that $\mathfrak{H}$, $\mathfrak{Y}$ are Banach, resp. locally convex, spaces, and

$$\iota : \mathfrak{Y} \to \mathfrak{H}$$

is a continuous injection with dense range (so one can think of $\mathfrak{Y}$ as a subspace of $\mathfrak{H}$ with a stronger topology). Let $\mathfrak{H}'$, $\mathfrak{Y}'$ denote the spaces of linear functionals on $\mathfrak{H}$, $\mathfrak{Y}$ with the norm, resp. the $\mathfrak{H}$-weak and $\mathfrak{Y}$-weak topologies (which is the weak-* topology in the Banach space setting). Then the adjoint of ι is the map

$$\iota^\dagger : \mathfrak{H}' \to \mathfrak{Y}', \qquad \iota^\dagger \ell(v) = \ell(\iota v),\ \ell \in \mathfrak{H}',$$

and $\iota^\dagger$ is continuous in the respective topologies. The injectivity of ι implies that $\iota^\dagger$ has dense range, while the fact that ι has dense range implies that $\iota^\dagger$ is injective. Thus, one can think of $\mathfrak{H}'$ as a subspace of $\mathfrak{Y}'$, with a stronger topology.

If $\mathfrak{H}$ is a Hilbert space with inner product $\langle .,. \rangle_{\mathfrak{H}}$ $\mathbb{C}$-linear in the first argument, there is a canonical (conjugate-linear) isomorphism $j_{\mathfrak{H}} : \mathfrak{H} \to \mathfrak{H}'$ given by $j_{\mathfrak{H}}(u)(v) = \langle v, u \rangle_{\mathfrak{H}}$. Suppose also that there is a canonical conjugate linear isomorphism

$$c_{\mathfrak{H}} : \mathfrak{H} \to \mathfrak{H},\ c_{\mathfrak{H}}^2 = \mathrm{Id},\ \langle u, c_{\mathfrak{H}} v \rangle = \langle v, c_{\mathfrak{H}} u \rangle;$$

if $\mathfrak{H}$ is a function space, this is usually given by pointwise complex conjugation. Thus,

$$T_{\mathfrak{H}} = j_{\mathfrak{H}} \circ c_{\mathfrak{H}} : \mathfrak{H} \to \mathfrak{H}'$$

is a linear isomorphism. Thus, if $A : \mathfrak{Y} \to \mathfrak{H}$ is continuous linear, then $A^\dagger : \mathfrak{H}' \to \mathfrak{Y}'$ continuous linear, and

$$A^\flat = A^\dagger \circ j_{\mathfrak{H}} \circ c_{\mathfrak{H}} : \mathfrak{H} \to \mathfrak{Y}'$$

is continuous and linear. In particular, letting A be our continuous injection,

$$\iota^\flat = \iota^\dagger \circ j_{\mathfrak{H}} \circ c_{\mathfrak{H}} : \mathfrak{H} \to \mathfrak{Y}'$$

is linear, injective with dense range, so $\mathfrak{H}$ can be considered a subspace of $\mathfrak{Y}'$ (with a stronger topology). In particular,

$$\iota^\flat \circ \iota : \mathfrak{Y} \to \mathfrak{Y}'$$

is also injective with dense range. One considers the triple $(\mathfrak{Y}', \mathfrak{H}, \iota)$ the $\mathfrak{H}$-dual of $\mathfrak{Y}$; we will denote this either simply by $\mathfrak{Y}'$,or by $\mathfrak{Y}^*$ if we want to emphasize the inclusion of $\mathfrak{Y}$ into $\mathfrak{Y}'$ via $\mathfrak{H}$, in what follows. Note that if $\mathfrak{Y}$ is also a Hilbert space with a canonical conjugate linear isomorphism[19] $c_{\mathfrak{Y}}$,

$$\iota \circ c_{\mathfrak{Y}} = c_{\mathfrak{H}} \circ \iota,\ c_{\mathfrak{Y}}^2 = \mathrm{Id},$$

then we have the canonical linear isomorphism $T_{\mathfrak{Y}} = j_{\mathfrak{Y}} \circ c_{\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{Y}'$, and it is important to keep in mind that $T_{\mathfrak{Y}}$ is (usually) different from $\iota^\flat\iota = \iota^\dagger \circ T_{\mathfrak{H}} \circ \iota$:

$$\begin{aligned} &T_{\mathfrak{Y}}(u)(v) = \langle v, c_{\mathfrak{Y}} u\rangle_{\mathfrak{Y}}, \\ &\iota^\flat\iota(u)(v) = \langle \iota v, (c_{\mathfrak{H}} \circ \iota)u\rangle_{\mathfrak{H}} = \langle \iota v, (\iota \circ c_{\mathfrak{Y}})u\rangle_{\mathfrak{H}}, \end{aligned}$$

for $u, v \in \mathfrak{Y}$. A simple example, when X a compact manifold with a smooth non-vanishing density ν is obtained by $\mathfrak{Y} = \dot{\mathcal{C}}^\infty(X)$ (a Fréchet space) and $\mathfrak{H} = L^2_\nu(X)$ with respect to the density ν, with $\iota : \mathfrak{Y} \to \mathfrak{X}$ the inclusion. Then $\iota^\flat\iota : \dot{\mathcal{C}}^\infty(X) \to \mathcal{C}^{-\infty}(X)$ is the standard inclusion of Schwartz functions in tempered distributions: $\iota^\flat\iota f(\phi) = \int f\phi\,\nu$.

In fact, we shall always consider a setting with $\mathcal{D}$ a dense subspace of $\mathfrak{H}$, with a locally convex topology, with respect to which the inclusion map is continuous (i.e. which is stronger than the subspace topology), so using the linear isomorphism $j_{\mathfrak{H}} \circ c_{\mathfrak{H}} : \mathfrak{H} \to \mathfrak{H}'$, we have continuous inclusions, with dense ranges,

$$\mathcal{D} \subset \mathfrak{H} \equiv \mathfrak{H}' \to \mathcal{D}'.$$

Suppose now that $A : \mathcal{D} \to \mathcal{D}$, hence $A^\dagger : \mathcal{D}' \to \mathcal{D}'$, and suppose that $A^\dagger$ maps $\mathcal{D}$, i.e. more precisely the range of $\iota^\flat\iota$ (with $\iota : \mathcal{D} \to \mathfrak{H}$ the inclusion), to itself, and let

$$\tilde{A} = (\iota^\flat\iota)^{-1} A^\dagger (\iota^\flat\iota) : \mathcal{D} \to \mathcal{D}.$$

Then for $f, \phi \in \mathcal{D}$

$$\begin{aligned} \langle \iota\phi, \iota\tilde{A}f\rangle_{\mathfrak{H}} &= (\iota^\flat\iota c_{\mathcal{D}}\tilde{A}f)(\phi) = (c_{\mathcal{D}'}\iota^\flat\iota\tilde{A}f)(\phi) = (c_{\mathcal{D}'}A^\dagger\iota^\flat\iota f)(\phi) \\ &= \overline{(A^\dagger\iota^\flat\iota f)(c_{\mathcal{D}}\phi)} = \overline{(\iota^\flat\iota f)(Ac_{\mathcal{D}}\phi)} = \overline{\langle \iota c_{\mathcal{D}} f, \iota A c_{\mathcal{D}}\phi\rangle} \\ &= \langle \iota c_{\mathcal{D}} A c_{\mathcal{D}}\phi, \iota f\rangle, \end{aligned} \tag{A.1}$$

so $\tilde{A}$ is the formal adjoint of $c_{\mathcal{D}}Ac_{\mathcal{D}}$ with respect to the $\mathfrak{H}$ inner product.

Given a Hilbert space $\mathfrak{H}$ as above, hence an inclusion of $\mathcal{D}$ into $\mathcal{D}'$, we shall also have to consider subspaces $\mathfrak{X}$ of $\mathcal{D}'$ with a locally convex topology,

[19] Again, pointwise complex conjugation on function spaces is a good example.

which contain the image of $\mathcal{D}$ in $\mathcal{D}'$ (under the $\mathfrak{H}$-induced inclusion map), and such that the inclusion maps

$$\mathcal{D} \hookrightarrow \mathfrak{X} \hookrightarrow \mathcal{D}'$$

are continuous, with dense range, hence one has the corresponding sequence of adjoint maps, which are continuous, with dense range, when all the duals are equipped with the weak topologies. As $(\mathcal{D}')' = \mathcal{D}$, one obtains

$$\mathcal{D} \hookrightarrow \mathfrak{X}' \hookrightarrow \mathcal{D}'.$$

If further

$$\mathcal{D} \hookrightarrow \mathfrak{X} \hookrightarrow \mathfrak{H} \hookrightarrow \mathcal{D}'$$

continuous, with dense ranges, then

$$\mathcal{D} \hookrightarrow \mathfrak{H}' \equiv \mathfrak{H} \hookrightarrow \mathfrak{X}' \hookrightarrow \mathcal{D}',$$

and similarly if one had the reverse inclusion between $\mathfrak{X}$ and $\mathfrak{H}$.

One way that subspaces such as $\mathfrak{Y}$ arise is by considering a a finite number of continuous linear maps $A_j : \mathcal{D} \to \mathcal{D}$, such that there exist continuous extensions $A_j : \mathcal{D}' \to \mathcal{D}'$ (which are then unique by the density of $\mathcal{D}$ in $\mathcal{D}'$), hence $A_j : \mathfrak{X} \to \mathcal{D}'$, $j = 1, \dots, k$. Then, in what essentially amounts to constructing a "joint maximal domain" for the A_j, and writing $\iota_{\mathfrak{X}\mathcal{D}'} : \mathfrak{X} \to \mathcal{D}'$ for the inclusion, let

$$\mathfrak{Y} = \{u \in \mathfrak{X} : \ \forall j, A_j u \in \operatorname{Ran} \iota_{\mathfrak{X}\mathcal{D}'}\} \tag{A.2}$$

with

$$\|u\|_{\mathfrak{Y}}^2 = \|u\|_{\mathfrak{X}}^2 + \sum \|\iota_{\mathfrak{X}\mathcal{D}'}^{-1} A_j u\|_{\mathfrak{X}}^2, \tag{A.3}$$

where the injectivity of $\iota_{\mathfrak{X}\mathcal{D}'}$ was used. If $\{u_n\}$ is Cauchy in $\mathfrak{Y}$, then it is such in $\mathfrak{X}$, so converges to some $u \in \mathfrak{X}$, and thus $A_j u_n \to A_j u \in \mathcal{D}'$. Moreover, if $\{u_n\}$ is Cauchy in $\mathfrak{Y}$ then $\iota_{\mathfrak{X}\mathcal{D}'}^{-1} A_j u_n$ is Cauchy in $\mathfrak{X}$ so converges to some $v_j \in \mathfrak{X}$, hence $A_j u_n \to \iota_{\mathfrak{X}\mathcal{D}'} v_j$ in $\mathfrak{X}$. Thus, $A_j u = \iota_{\mathfrak{X}\mathcal{D}'} v_j$, so $A_j u \in \operatorname{Ran} \iota_{\mathfrak{X}\mathcal{D}'}$, and $A_j u_n \to A_j u$ in $\mathfrak{X}$, proving that $\mathfrak{Y}$ is a Hilbert space. We will simply write A_j for $A_j|_{\mathfrak{Y}} : \mathfrak{Y} \to \mathfrak{X}$. Note that $\mathcal{D} \subset \mathfrak{Y}$, so $\mathfrak{Y}$ is dense in $\mathfrak{X}$. However, $\mathcal{D}$ is *not* necessarily dense in $\mathfrak{Y}$.

An example is given by $\mathfrak{X} = L^2_g(X)$ on a compact manifold with or without boundary, g a Riemannian metric, A_j be a finite set of $\mathcal{C}^\infty$ vector fields which span all vector fields over $\mathcal{C}^\infty(X)$, $\mathcal{D}$ either $\mathcal{C}^\infty(X)$ or $\dot{\mathcal{C}}^\infty(X)$; then $\mathfrak{Y} = H^1_g(X)$. If $\mathcal{D} = \mathcal{C}^\infty(X)$, then $\mathcal{D}$ is dense in $\mathfrak{Y}$, but if $\mathcal{D} = \dot{\mathcal{C}}^\infty(X)$, then this is not in general the case: it fails if the boundary of X is non-empty. Other examples are given coisotropic distributions, where the A_j are products of first order ps.d.o's characteristic on a coisotropic manifold; see a general discussion below for spaces given by such ps.d.o's.

Another way a subspace like $\mathfrak{Y}$ might arise from continuous linear maps $A_j : \mathcal{D} \to \mathcal{D}$ is the following. In a "joint minimal domain" construction, one can define

$$\|u\|_{\tilde{\mathfrak{Y}}}^2 = \|u\|_{\mathfrak{X}}^2 + \sum \|A_j u\|_{\mathfrak{X}}^2, \tag{A.4}$$

as above, and let $\tilde{\mathfrak{Y}}$ be the completion of $\mathcal{D}$ with respect to this norm, so $\tilde{\mathfrak{Y}}$ is a Hilbert space. Moreover, the inclusion map $\iota_{\mathcal{D}\mathfrak{X}} : \mathcal{D} \to \mathfrak{X}$ as well as A_j extend to continuous linear maps

$$\widetilde{\iota_{\mathcal{D}\mathfrak{X}}} = \iota_{\tilde{\mathfrak{Y}}\mathfrak{X}}, \widetilde{A_j} : \tilde{\mathfrak{Y}} \to \mathfrak{X},$$

and $\iota_{\tilde{\mathfrak{Y}}\mathfrak{X}}$ has dense range (for $\mathcal{D}$ canonically injects into the completion). In addition, with $\mathfrak{Y}$ as above, the inclusion map $\iota_{\mathcal{D}\mathfrak{Y}} : \mathcal{D} \to \mathfrak{Y}$ extends continuously to a map

$$\widetilde{\iota_{\tilde{\mathfrak{Y}}\mathfrak{Y}}} : \tilde{\mathfrak{Y}} \to \mathfrak{Y}$$

which is an isometry, and is in particular injective. This in particular shows that the inclusion map from $\tilde{\mathfrak{Y}}$ to $\mathfrak{X}$ is also injective, with a dense range. For X a manifold with boundary and $\mathcal{D} = \dot{\mathcal{C}}^\infty(X)$, A_j vector fields as above, one has $\mathfrak{Y} = H^1_0(X)$; with $\mathcal{D} = \mathcal{C}^\infty(X)$, one has $\mathfrak{Y} = H^1(X)$.

Note that the closure of $\mathcal{D}$ in $\mathfrak{Y}$ is $\tilde{\mathfrak{Y}}$, so $\mathcal{D}$ is dense in $\mathfrak{Y}$ if and only if $\mathfrak{Y} = \tilde{\mathfrak{Y}}$ (i.e. $\widetilde{\iota_{\tilde{\mathfrak{Y}}\mathfrak{Y}}}$ is surjective). *From this point on we* assume *that* $\mathfrak{Y} = \tilde{\mathfrak{Y}}$. This is true, for instance, if one is given $B_1, \ldots, B_r \in \Psi^{1,0}_{\mathrm{eb}}(M)$, and $A_1, \ldots, A_k$ are up to s-fold products of these, as shown below in Lemma A.3. Thus, $\mathfrak{Y}' \subset \mathcal{D}'$ (i.e. the inclusion map is injective).

Using the inclusion map $\iota_{\mathfrak{Y}\mathfrak{X}}$ we can now identify the dual of $\mathfrak{Y}$ with respect to $\mathfrak{H}$. We start with the case $\mathfrak{H} = \mathfrak{X}$. By the Riesz lemma, $\mathfrak{Y}' = T_{\mathfrak{Y}}\mathfrak{Y}$ and $T_{\mathfrak{Y}}$ is unitary, where

$$T_{\mathfrak{Y}}v(u) = \langle u, c_{\mathfrak{Y}}v\rangle_{\mathfrak{Y}}.$$

But

$$\begin{aligned} T_{\mathfrak{Y}}(v)(u) = \langle u, c_{\mathfrak{Y}}v\rangle_{\mathfrak{Y}} &= \langle \iota_{\mathfrak{Y}\mathfrak{H}}u, \iota_{\mathfrak{Y}\mathfrak{H}}c_{\mathfrak{Y}}v\rangle_{\mathfrak{H}} + \sum_j \langle A_j u, A_j c_{\mathfrak{Y}}v\rangle_{\mathfrak{H}} \\ &= j_{\mathfrak{H}}\iota_{\mathfrak{Y}\mathfrak{H}}c_{\mathfrak{Y}}v(\iota_{\mathfrak{Y}\mathfrak{H}}u) + \sum_j j_{\mathfrak{H}}A_j c_{\mathfrak{Y}}v(A_j u). \end{aligned}$$

Thus,

$$T_{\mathfrak{Y}}(v)(u) = \big((\iota^\flat_{\mathfrak{Y}\mathfrak{H}}\iota_{\mathfrak{Y}\mathfrak{H}} + \sum A^\flat_j c_{\mathfrak{H}}A_j c_{\mathfrak{Y}})v\big)(u).$$

We conclude that

$$T_{\mathfrak{Y}}v = (\iota^\flat_{\mathfrak{Y}\mathfrak{H}}\iota_{\mathfrak{Y}\mathfrak{H}} + \sum A^\flat_j c_{\mathfrak{H}}A_j c_{\mathfrak{Y}})v,$$

and

$$\mathfrak{Y}^* = (\iota^\flat_{\mathfrak{Y}\mathfrak{H}}\iota_{\mathfrak{Y}\mathfrak{H}} + \sum A^\flat_j c_{\mathfrak{H}}A_j c_{\mathfrak{Y}})\mathfrak{Y}. \tag{A.5}$$

This also shows that

$$\mathfrak{Y}^* = \operatorname{Ran}\iota^\dagger_{\mathfrak{Y}\mathfrak{H}} + \sum \operatorname{Ran} A^\dagger_j,$$

for $\supset$ follows from the definition of $\iota^\dagger$, etc., while $\subset$ follows from (A.5). We recall here that by (A.1), $A^\dagger_j$ is the formal adjoint of $c_{\mathcal{D}}A_j c_{\mathcal{D}}$.

More generally, we do not need to assume $\mathfrak{X} = \mathfrak{H}$; rather assume that

$$\langle u, v\rangle_{\mathfrak{X}} = \sum_k \langle B_k u, B_k v\rangle_{\mathfrak{H}}, \tag{A.6}$$

where $B_k : \mathfrak{X} \to \mathfrak{H}$ are continuous linear maps (and there is no assumption on the relationship in the sense of inclusions between $\mathfrak{X}$ and $\mathfrak{H}$). Then

$$\begin{aligned} T_{\mathfrak{Y}}(v)(u) = \langle u, c_{\mathfrak{Y}} v\rangle_{\mathfrak{Y}} &= \langle \iota_{\mathfrak{Y}\mathfrak{X}} u, \iota_{\mathfrak{Y}\mathfrak{X}} c_{\mathfrak{Y}} v\rangle_{\mathfrak{X}} + \sum_j \langle A_j u, A_j c_{\mathfrak{Y}} v\rangle_{\mathfrak{X}} \\ &= \sum_k \left(\langle B_k \iota_{\mathfrak{Y}\mathfrak{X}} u, B_k \iota_{\mathfrak{Y}\mathfrak{X}} c_{\mathfrak{Y}} v\rangle_{\mathfrak{H}} + \sum_j \langle B_k A_j u, B_k A_j c_{\mathfrak{Y}} v\rangle_{\mathfrak{H}} \right) \\ &= \sum_k \left(j_{\mathfrak{H}} B_k \iota_{\mathfrak{Y}\mathfrak{X}} c_{\mathfrak{Y}} v (B_k \iota_{\mathfrak{Y}\mathfrak{X}} u) + \sum_j j_{\mathfrak{H}} B_k A_j c_{\mathfrak{Y}} v (B_k A_j u) \right) \\ &= \left(\sum_k \left((B_k \iota_{\mathfrak{Y}\mathfrak{X}})^\flat c_{\mathfrak{H}} B_k \iota_{\mathfrak{Y}\mathfrak{X}} + \sum_j (B_k A_j)^\flat c_{\mathfrak{H}} B_k A_j \right) c_{\mathfrak{Y}} v \right)(u), \end{aligned}$$

so we conclude as above, using $(B_k A_j)^\flat = A^\dagger B_k^\flat$, etc., that

$$\mathfrak{Y}^* = \operatorname{Ran} \iota_{\mathfrak{Y}\mathfrak{X}}^\dagger + \sum_j A_j^\dagger; \tag{A.7}$$

note that the same computation as in (A.1) with factors of ι omitted shows that $A_j^\dagger$ is the formal adjoint of $c_{\mathcal{D}} A_j c_{\mathcal{D}}$.

We now prove the density lemma mentioned above. We start by commuting bounded families of operators through products of first order ps.d.o's.

Lemma A.1. *Let $r \geq 1$. For $s \in \mathbb{N}$, let J_s be the set of maps $\{j : \{1, \ldots, s\} \to \{1, \ldots, r\}$.*

Suppose that $A_1, \ldots, A_r \in \Psi^{1,0}_{\mathrm{eb}}(M)$, and for $s \in \mathbb{N}$ and $j \in J_s$, let

$$A_j = A_{j_1} \ldots A_{j_s}.$$

Then for $k \in \mathbb{N}$, $j \in J_k$ and $\{Q_n\}$ a uniformly bounded family in $\Psi^{0,0}_{\mathrm{eb}}(M)$,

$$A_j Q_n = Q_n A_j + \sum_{s \leq k-1} \sum_{i \in J_s} C_{in} A_i,$$

with $\{C_{in} :\ n \in \mathbb{N}\}$ uniformly bounded in $\Psi^{0,0}_{\mathrm{eb}}(M)$, and the uniform bounds are microlocal (so in particular $\mathrm{WF}'(\{C_{in} :\ n \in \mathbb{N}\}) \subset \mathrm{WF}'(\{Q_n :\ n \in \mathbb{N}\})$).

Moreover, for $\epsilon > 0$, if $Q_n \to \mathrm{Id}$ in $\Psi^{\epsilon,0}_{\mathrm{eb}}(M)$ then $C_{in} \to 0$ in $\Psi^{\epsilon,0}_{\mathrm{eb}}(M)$ as $n \to \infty$.

Remark A.2. We do not need the microlocality of the uniform bounds below, but it is useful elsewhere.

Proof. We proceed by induction, with $k=0$ being clear.

Suppose $k \geq 1$, and the statement has been proved with k replaced by $k-1$. Then for $j \in J_k$,

$$A_j Q_n = Q_n A_j + [A_{j_1}, Q_n] A_{j_2} \dots A_{j_k} + \dots + A_{j_1} \dots A_{j_{k-1}} [A_{j_k}, Q_n].$$

Note that $[A_{j_m}, Q_n] \in \Psi^{0,0}_{\mathrm{eb}}(M)$ uniformly, and in a microlocal sense (and $[A_{j_m}, Q_n] \to 0$ in $\Psi^{\epsilon,0}_{\mathrm{eb}}(M)$ if $Q_n \to \mathrm{Id}$ in $\Psi^{0,0}_{\mathrm{eb}}(M)$). Thus, the first two terms are of the stated form. For the others, there are $l \leq k-1$ factors in front of the commutator, which is bounded in $\Psi^{0,0}_{\mathrm{eb}}(M)$ (and converges to 0 in $\Psi^{\epsilon,0}_{\mathrm{eb}}(M)$ if $Q_n \to \mathrm{Id}$ in $\Psi^{\epsilon,0}_{\mathrm{eb}}(M)$), so by the inductive hypothesis

$$A_{j_1} \dots A_{j_l} [A_{j_{l+1}}, Q_n]$$

can be rewritten as $\sum_{s \leq l} \sum_{i \in J_s} C_{s,in} A_i$, hence

$$A_{j_1} \dots A_{j_l} [A_{j_{l+1}}, Q_n] A_{j_{l+2}} \dots A_{j_k}$$

is rewritten as $C_{s,in} A_{i_1} \dots A_{i_s} A_{j_{l+2}} \dots A_{j_k}$ with $s + (k-(l+1)) \leq l + k - (l+1) = k-1$ factors of the A's, hence is of the stated form. $\square$

Lemma A.3. *Suppose that $B_1, \dots, B_r \in \Psi^{1,0}_{\mathrm{eb}}(M)$, and let $\mathfrak{X}$ be a Hilbert space on which $\Psi^{0,0}_{\mathrm{eb}}(M)$ acts, with operator norm on $\mathfrak{X}$ bounded by a fixed $\Psi^{0,0}_{\mathrm{eb}}(M)$-seminorm, and suppose that $\mathcal{D} \subset \mathfrak{X}$ a dense subspace with a locally convex topology, and with all $Q \in \Psi^{-\infty,0}_{\mathrm{eb}}(M)$, $Q : \mathfrak{X} \to \mathcal{D}$ continuous, while for all $Q \in \Psi^{m,0}_{\mathrm{eb}}(M)$, $Q : \mathcal{D} \to \mathfrak{X}$ is continuous, with bound given by a fixed $\Psi^{m,0}_{\mathrm{eb}}(M)$ seminorm and a fixed seminorm on $\mathcal{D}$.*

For $k \in \mathbb{N}$, let

$$\mathfrak{Y} = \{u \in \mathfrak{X} : \ \forall s \leq k, \ \forall j \in J_s, A_j u \in \mathfrak{X}\}$$

with

$$\|u\|^2_{\mathfrak{Y}} = \|u\|^2_{\mathfrak{X}} + \sum_{s \leq k} \sum_{j \in J_s} \|A_j u\|^2_{\mathfrak{X}},$$

Then $\mathcal{D}$ is dense in $\mathfrak{Y}$.

Proof. We start by observing that if Q_n is a uniformly bounded family in $\Psi^{0,0}_{\mathrm{eb}}(M)$, $Q \in \Psi^{0,0}_{\mathrm{eb}}(M)$ and $Q_n \to Q$ in $\Psi^{\epsilon,0}_{\mathrm{eb}}(M)$ for $\epsilon > 0$, then $Q_n \to Q$ strongly on $\mathfrak{X}$. Indeed, Q_n is uniformly bounded on $\mathfrak{X}$ by the assumptions of the lemma, so it suffices to prove that for a dense subset of $\mathfrak{X}$, which we take to be $\mathcal{D}$, $u \in \mathcal{D}$ implies $Q_n u \to Qu$ in $\mathfrak{X}$. But this is immediate, for $Q_n \to Q$ in $\Psi^{\epsilon,0}_{\mathrm{eb}}(M)$, hence as a map $\mathcal{D} \to \mathfrak{X}$, by the assumptions of the lemma.

Now let $\Lambda_n \in \Psi^{-\infty,0}_{\mathrm{eb}}(M)$ uniformly bounded in $\Psi^{0,0}_{\mathrm{eb}}(M)$ and $\Lambda_n \to \mathrm{Id}$ in $\Psi^{\epsilon,0}_{\mathrm{eb}}(M)$ for $\epsilon > 0$, so $\Lambda_n \to \mathrm{Id}$ strongly on $\mathfrak{X}$. We claim that for $s \leq k$,

$$u \in \mathfrak{Y}, \ j \in J_s \Rightarrow A_j \Lambda_n u \to A_j u \text{ as } n \to \infty \text{ in } \mathfrak{X}. \tag{A.8}$$

Since $\Lambda_n u \in \mathcal{D}$, this will prove the lemma. Note that $\Lambda_n u \to u$ in $\mathfrak{X}$.

By Lemma A.1, for $j \in J_s$,

$$A_j \Lambda_n = \Lambda_n A_j + \sum_{l \leq s-1} \sum_{i \in J_l} C_{in} A_i,$$

with $\{C_{in} : n \in \mathbb{N}\}$ uniformly bounded in $\Psi^{0,0}_{\mathrm{eb}}(M)$, $C_{in} \to 0$ in $\Psi^{\epsilon,0}_{\mathrm{eb}}(M)$ for $\epsilon > 0$. Correspondingly, $\{C_{in} : n \in \mathbb{N}\}$ is uniformly bounded as operators on $\mathfrak{X}$, and $C_{in} \to 0$ strongly on $\mathfrak{X}$. Since $A_j u \in \mathfrak{X}$ and $A_i u \in \mathfrak{X}$ for $i \in J_l$, $l \leq s$, we deduce that $A_j \Lambda_n u \to A_j u$ in $\mathfrak{X}$, completing the proof. □

Appendix B. The edge-b calculus

Let M be, for this section, a general compact manifold with corners and let W be one of its boundary hypersurfaces. In the body of the paper above, M is obtained by the blow up of a boundary face Y of a manifold with corners M_0 and W is the front face of the blow up, i.e. the preimage of Y under the blow-down map. In fact the discussion is mostly local in the interior of Y and hence we could assume that Y has locally maximal codimension, so that it, and hence W has no boundary. We shall not make this assumption here and include the setting obtained by blow-up without actually restricting the discussion to it. Instead, we shall suppose that W is equipped with a fibration

$$\begin{tikzcd} Z \arrow[r, dash] & W \arrow[d, "\phi"] \\ & Y. \end{tikzcd} \tag{B.1}$$

Since the manifolds here may have corners, this is to be a fibration in that sense, so the typical fiber, Z, is required to be a compact manifold with corners, the base, Y, is a manifold with corners and ϕ is supposed to be locally trivial in the sense that each $p \in Y$ has a neighborhood U over which there is a diffeomorphism giving a commutative diagram

$$\begin{tikzcd} \phi^{-1}(U) \arrow[rr, "\simeq"] \arrow[dr, "\phi"'] & & U \times Z \arrow[dl, "\pi_U"] \\ & U. & \end{tikzcd} \tag{B.2}$$

Mainly for notational reasons we will also assume that Y is connected.

Let $\mathcal{V}_{\mathrm{eb}}(M) \subset \mathcal{V}_{\mathrm{b}}(M)$ be the Lie subalgebra of all those smooth vector fields on M which are tangent to all boundary (hypersur-)faces and in addition are tangent to the fibers of ϕ on W. The calculus of edge-b pseudodifferential operators will be constructed in this setting, it is determined by M and ϕ and microlocalizes $\mathcal{V}_{\mathrm{eb}}(M)$.

In case the ϕ has a single fiber, i.e. $Z = W$, corresponding to the case that W is not blown up at all, the Lie algebra $\mathcal{V}_{\mathrm{eb}}(M)$ reduces to $\mathcal{V}_{\mathrm{b}}(M)$ and the desired microlocalization is just the algebra of b-pseudodifferential operators on M as a manifold with corners. The construction of this algebra is discussed in [19], [22] and of course W is in no way singled out amongst the boundary hypersurfaces. The pseudodifferential operators are described in terms of their Schwartz kernels, which are the conormal distributions with respect to the resolved diagonal in a blown-up version of M^2, with the additional constraint of vanishing rapidly at boundary faces which do not meet the lifted diagonal. The resolved double space in this case is

$$M^2_{\mathrm{b}} = [M^2; \mathcal{B}],\ \mathcal{B} = \{B \times B; B \in \mathcal{M}_1(M)\}. \tag{B.3}$$

Here, in generality, $\mathcal{M}_p(M)$ is the collection of all *connected* boundary faces of codimension p of the manifold with corners M. It is of crucial importance that the lift to M^2_{b} of the diagonal, is a p-submanifold – the lift in this case is the closure (in M^2_{b}) of the inverse image of the interior of the diagonal:

$$\mathrm{Diag}_{\mathrm{b}} = \mathrm{cl}(\mathrm{Diag}(M) \cap \mathrm{int}(M^2)). \tag{B.4}$$

Then the operators on functions correspond to the kernels

$$\Psi^m(M) = \left\{A \in I^m(M^2_{\mathrm{b}}; \beta^*\Omega_R); A \equiv 0 \text{ at } \partial M^2_{\mathrm{b}} \setminus \mathrm{ff}(\beta)\right\} \tag{B.5}$$

where $\mathrm{ff}(\beta)$ is the collection of boundary faces produced by the blow-ups defining the combined blow-down map $\beta : M^2_{\mathrm{b}} \longrightarrow M^2$.

The composition properties of these operators, including the fact that the 'small calculus' is an algebra, can be obtained geometrically from the corresponding triple space

$$\begin{aligned} M^3_{\mathrm{b}} &= [M^3; \mathcal{B}^3, \mathcal{B}^2],\ \mathcal{B}^3 = \{B^3; B \in \mathcal{M}_1(M)\}, \\ \mathcal{B}^2 &= \{M \times B \times B,\ B \times M \times B,\ B \times B \times M; B \in \mathcal{M}_1(M)\}. \end{aligned} \tag{B.6}$$

There is considerable freedom in the order of blow-ups here and this is sufficient to show that the three projections, π_O, from M^3 to M^2 lift to 'stretched projections' $\pi_{O,\mathrm{b}} : M^3_{\mathrm{b}} \longrightarrow M^2_{\mathrm{b}}$, $O = S, C, F$, corresponding to the left, outer and right two factors respectively; these maps are b-fibrations and factor through a product of M^2_{b} and M in each case.

As already noted, it is crucial for the definition (B.5) that the lifted diagonal $\mathrm{Diag}_{\mathrm{b}}$ be a p-submanifold, meaning that it meets the boundary locally as a product. This also turns out to be essential in the *construction* of M^2_{eb} below.

There is another extreme case in which the microlocalization of the Lie algebra $\mathcal{V}_{\mathrm{eb}}$ is well-established, namely when W is the only boundary hypersurface, so M is a manifold with boundary; this is the case of an 'edge' alone, with no other boundaries. The construction of a geometric resolution in this case can be found in [16] and [17]. It is quite parallel to, and of course includes as a special case, the b-algebra on a manifold with boundary. In the general edge case when the fibration ϕ is non-trivial (but W itself has no boundary) the centre blown up in (B.3), which would be W^2, is replaced by the fiber diagonal

$$\begin{aligned}\mathrm{Diag}_\phi &= \{(p,p') \in W^2; \phi(p) = \phi(p')\} = (\phi\times\phi)^{-1}(\mathrm{Diag}(F))\\ M^2_\phi &= [M^2; \mathrm{Diag}_\phi].\end{aligned} \tag{B.7}$$

Similarly, the triple space is obtained by blow-up of the triple fiber product

(B.8)

$$\mathrm{Diag}^3_\phi = \{(p,p',p'') \in W^3; \phi(p) = \phi(p') = \phi(p'')\} = (\phi\times\phi\times\phi)^{-1}(\mathrm{Diag}^3(F))$$

and then the three partial fiber diagonals, the inverse images, Diag^O_ϕ, $O = S, C, F$ of Diag_ϕ under the three projections $\pi_O : M^3 \longrightarrow M^2$:

$$M^3_\phi = [M^3; \mathrm{Diag}^3_\phi; \mathrm{Diag}^S_\phi, \mathrm{Diag}^C_\phi, \mathrm{Diag}^F_\phi]. \tag{B.9}$$

Again the three projections lift to b-fibrations

$$M^3_\phi \overset{\pi_{F,\phi}}{\underset{\pi_{C,\phi}}{\overset{\xrightarrow{\pi_{S,\phi}}}{\rightrightarrows}}} M^2_\phi. \tag{B.10}$$

The microlocalization of $\mathcal{V}_{\mathrm{eb}}$ is accomplished here by the combination of these two constructions. The diagonal, even for a manifold with boundary, is not a p-submanifold – does not meet the boundary faces in a product manner – and as already noted this is remedied by the b-resolution. Since the fiber diagonal in W^2 is the inverse image of the diagonal in Y it too is not a p-submanifold in case Y has boundary, but then the partial b-resolution of M resolves it to a p-submanifold which can then be blown up.

More explicitly, the boundary hypersurfaces of M, other than W itself, fall into three classes according to their behavior relative to ϕ. Namely there may be some disjoint from W; these are relatively unimportant in the discussion below. Otherwise the intersection of W and such a boundary hypersurface, B, is a boundary hypersurface $B \cap W$ of W. The remaining two cases correspond to this being the preimage under ϕ of a boundary hypersurface of Y or, if this is not the case, then $B \cap W$ is the union of boundary hypersurfaces of the fibers of ϕ, corresponding to a fixed boundary hypersurface of Z. In brief, the boundary hypersurfaces $B \in \mathcal{M}_1(M) \setminus \{W\}$ which meet W correspond either to the boundary hypersurfaces of Y or of Z. Let $\mathcal{B}'$, $\mathcal{B}(Z)$ and $\mathcal{B}(Y) \subset \mathcal{M}_1(M)$ denote the three disjoints subsets into which $\mathcal{M}_1(M) \setminus \{W\}$ is so divided.

To define the double space on which the kernels are conormal distributions with respect to the lifted diagonal, just as in both special cases discussed above, we make one blow up for each of the boundary hypersurfaces. For those other than W, this is the same as for the b-double space for M, which is to say the corners, $B \times B$, are to be blown up for all $B \in \mathcal{M}_1(M) \setminus \{W\}$. Since these submanifolds are mutually transversal boundary faces within M^2 they may be blown up in any order with the same final result. For W we wish to blow up the fiber diagonal, Diag_ϕ, in (B.7). This is certainly a manifold with corners, since it is the fiber product of W with itself as a bundle over Y, given by ϕ. However, as noted, it is not embedded as a p-submanifold if Y has non-trivial boundary. If x_i and y_j are respectively boundary defining functions and interior coordinates near some boundary point of Y, and x', y', x'', y'' are their local lifts to W^2 under the two copies of ϕ, then $\mathrm{Diag}_\phi \subset W^2$ is the 'diagonal' $x' = x''$, $y' = y''$. Near a boundary point of Y this is not a p-submanifold.

Lemma B.1. *The fiber diagonal* Diag_ϕ *lifts to a p-submanifold of* $[M^2; \mathcal{B}(Y)]$.

We will still denote the lifted submanifold as Diag_ϕ.

Proof. Since $\mathrm{Diag}_\phi \subset W^2$ and this is the smallest boundary face of M^2 with this property, under the blow up of other boundary faces of M^2, Diag_ϕ lifts to the subset (always a submanifold in fact) of the lift of W^2 under the blow up of the intersection of W^2 with the boundary face which is the center of the blow up. That is, to track the behavior of Diag_ϕ we need simply blow up the intersections of the elements of $\mathcal{B}(Y)$ with W^2, inside W^2. This corresponds to exactly the 'boundary resolution' of Y^2 to Y^2_{b} as discussed briefly above. So the diagonal in Y lifts to be a p-submanifold. Since ϕ is a fibration over Y, it follows easily from the local description that Diag_ϕ lifts to a p-submanifold of the blow up, $[W^2; \mathcal{B}(Y) \cap W^2]$ and hence to a p-submanifold of $[M^2; \mathcal{B}(Y)]$ as claimed. $\square$

Thus blowing up the elements of $\mathcal{B}(Y)$ in M^2 resolves Diag_ϕ to a p-submanifold, by resolving the diagonal in Y^2. The defining functions of the elements of $\mathcal{B}(Z)$ restrict to defining functions of boundary faces of the lift of Diag_ϕ so all the remaining boundary faces, in $\mathcal{B}' \cup \mathcal{B}(Z)$ are transversal to this lift. Such transversality is preserved under blow up of boundary faces, so we may define the eb-double space in several equivalent ways as regards the order of the blow-ups and in particular:-

$$\begin{aligned} M^2_{\mathrm{eb}} =& [M^2; \mathcal{B}^2, \mathrm{Diag}_\phi] \\ \equiv& [M^2; \mathcal{B}(Y)^2, \mathrm{Diag}_\phi, \mathcal{B}(Z)^2, (\mathcal{B}')^2], \end{aligned} \tag{B.11}$$

where the 'squares' mean the set of self-products of the elements and the ordering within the boundary faces is immaterial.

The fibration ϕ restricts to a fibration, ϕ_B, of $B \cap W$ for each $B \in \mathcal{B}(Z)$, over the same base Y. For each $B \in \mathcal{B}(Y)$ instead ϕ restricts to $B \cap W$ to a fibration, again denoted ϕ_B, over $Y(B)$, the corresponding boundary hypersurface of Y. Thus considering $B \in \mathcal{B}(Z)$ or $B \in \mathcal{B}(Y)$ as manifolds with corners on their own, each inherits a fibration structure as initially given on $W \subset M$ on the intersection $B \cap W \in \mathcal{M}_1(B)$. For the elements of $\mathcal{B}'$ there is a corresponding trivial structure with no W.

Lemma B.2. *The diagonal in* M^2 *lifts to a p-submanifold of* M^2_{eb} *and the 'front faces' of* M^2_{eb}*, those boundary hypersurfaces produced by blow up, are of the form* $B^2_{\mathrm{eb}} \times [0,1]$*, corresponding to each* $B \in \widetilde{\mathcal{B}} = \mathcal{M}_1(M) \setminus \{W\}$ *with* *its induced fibration structure, and that corresponding to* Diag_ϕ *is the pull-back of the bundle* $[W^2; (\mathcal{B}(Z) \cap (W))^2]$, *defined by blowing up the diagonal corners of the fibers, to a (closed) quarter ball bundle over* Y.

Proof. These statements are all local and follow by elementary computations in local coordinates. □

Thus, the definition of the 'small' calculus of edge-b pseudodifferential operators is directly analogous to (and of course extends in generality) (B.5):

$$\Psi^m_{\mathrm{eb}}(M) = \left\{A \in I^m(M^2_{\mathrm{eb}}; \beta^*\Omega_R); A \equiv 0 \text{ at } \partial M^2_{\mathrm{b}} \setminus \mathrm{ff}(\beta)\right\} \tag{B.12}$$

where the particular fibration ϕ is not made explicit in the notation. The fact that these kernels define operators on $\dot{\mathcal{C}}^\infty(M)$ and $\mathcal{C}^\infty(M)$ reduces to the fact that push-forward off the right factor of M, which is to say under the left projection, gives a continuous map

$$(\pi_{L,\phi})_* : \Psi^m_{\mathrm{eb}}(M) \longrightarrow \mathcal{C}^\infty(M). \tag{B.13}$$

The principal symbol map is well-defined at the level of conormal distributions, taking values in the smooth homogeneous fiber-densities of the non-zero part of the conormal bundle to the submanifold in question. In this case $N^*\mathrm{Diag}_{\mathrm{eb}} = {}^{\mathrm{eb}}T^*M$ is a natural identification and the density factors cancel as in the standard case so

$$\sigma_m : \Psi^m_{\mathrm{eb}}(M) \longrightarrow \mathcal{C}^\infty({}^{\mathrm{eb}}S^*M; N_m) \tag{B.14}$$

where N_m is the bundle of functions which are homogeneous of degree $-m$.

The structure of the front faces leads directly to the 'symbolic' structure of the (small) algebra of pseudodifferential operators. Namely, there are homomorphisms to model operator algebras corresponding to each boundary face of M. For those other than W the model is a parameterized ('suspended') family of edge-b (or for those boundary faces not meeting W plain b-) operators corresponding to the fibrations of boundary hypersurfaces of W and for W the model is

The corresponding triple space can be defined by essentially the same modifications to the construction of M^3_{b} as correspond to obtaining M^2_{eb} in place of M^2_{b}.

Lemma B.3. *Under the blow-down map for the partial triple b-product*

$$\beta^3_{\tilde{\mathcal{B}}} : M^3_{\tilde{\mathcal{B}}} = [M^3; \mathcal{B}(Y)^3; \mathcal{B}(Y)^2; \mathcal{B}(Z)^3; (\mathcal{B}')^3; \mathcal{B}(Z)^2, (\mathcal{B}')^2] \longrightarrow M^3 \tag{B.15}$$

the triple fiber diagonal and the three partial fiber diagonals

$$\begin{aligned} \mathrm{Diag}^3_\phi &= \{(p, p', p'') \in W^3; \phi(p) = \phi(p') = \phi(p'')\}, \\ \mathrm{Diag}^O_\phi &= (\pi_O)^{-1}(\mathrm{Diag}_\phi),\ O = S, C, F, \end{aligned} \tag{B.16}$$

all lift to p-submanifolds.

Proof. This reduces to the same argument as above, namely that the triple and three partial diagonals in Y^3 are resolved to p-submanifolds in Y^3_{b} and the effect of the first two sets of blow-ups in (B.15) on Y^3 is to replace it by Y^3_{b} and hence to resolve the submanifolds in (B.16). Under the subsequent blow-ups of boundary faces any p-submanifold lifts to a p-submanifold. $\square$

Thus we may define the edge-b triple space to be

$$M^3_{\mathrm{eb}} = [M^3_{\widetilde{\mathcal{B}}}; \mathrm{Diag}^3_\phi; \mathrm{Diag}^S_\phi; \mathrm{Diag}^C_\phi; \mathrm{Diag}^F_\phi]. \tag{B.17}$$

Proposition B.4. *The three partial diagonals lift to b-submanifolds intersecting in the lifted triple diagonal and the three projections lift to b-fibrations*

$$M^3_{\mathrm{eb}} \; \begin{matrix} \xrightarrow{\pi_{F,\mathrm{eb}}} \\ \xrightarrow{\pi_{S,\mathrm{eb}}} \\ \xrightarrow[\pi_{C,\mathrm{eb}}]{} \end{matrix} \; M^2_\phi. \tag{B.18}$$

where $\pi_{O,\mathrm{eb}}$ is transversal to the other two lifted diagonals.

Proof. The existence of the stretched projections as smooth maps follows from the possibility of commutation of blow-ups. For the sake of definiteness, concentrate on π_F, the projection onto the right two factors.

After the blow up of the triple fiber diagonal in (B.17), the three partial fiber diagonals are disjoint so the other two can be blown up last. When it is to be blown up, the triple fiber diagonal is a submanifold of Diag^F_ϕ so the order can be exchanged, showing that there is a composite blown-down map

$$M^3_{\mathrm{eb}} \longrightarrow [M^3_{\widetilde{\mathcal{B}}}; \mathrm{Diag}^F_\phi]. \tag{B.19}$$

The manifold with corners $M^3_{\widetilde{\mathcal{B}}}$ is the b-resolved triple product where the boundary hypersurface W is ignored. The commutation arguments showing the existence of a composite blow-down map $M^3_{\mathrm{b}} \longrightarrow M \times M^2_{\mathrm{b}}$ carry over directly to give an alternative construction

$$M^3_{\widetilde{\mathcal{B}}} = [M \times M^2_{\widetilde{\mathcal{B}}}; \mathcal{F}] \tag{B.20}$$

where $\mathcal{F}$ consists of those boundary faces in (B.15) which involve a defining function on the first factor of M^3 – so all the triple products and the double products with boundary hypersurface in the first factor. These are all transversal to Diag^F_ϕ, realized as a p-submanifold of $M \times M^2_{\widetilde{\mathcal{B}}}$ so can be commuted past it in the blow up, giving the map $\pi_{F,\phi}$ in (B.18). That it is a b-submersion follows from its definition as a composite of blow-downs of boundary faces, together with the corresponding fact for the edge case. That it is a b-fibration follows from the fact that the image of a boundary hypsersurface is either a boundary hypersurface or the whole manifold since this is true locally in the interior of boundary hypersurfaces. $\square$

These facts together show that the small calculus of edge-b pseudodifferential operators, as defined in (B.12) is a filtered algebra. It also follows directly that the symbol (B.14) is multiplicative as in the standard case. The extension to operators on sections of bundles is essentially notational.

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