Boundaries, eta invariant and the determinant bundle

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Abstract. Cobordism invariance shows that the index, in K-theory, of a family of pseudodifferential operators on the boundary of a fibration vanishes if the symbol family extends to be elliptic across the whole fibration. For Dirac operators with spectral boundary condition, Dai and Freed [5] gave an explicit version of this at the level of the determinant bundle. Their result, that the eta invariant of the interior family trivializes the determinant bundle of the boundary family, is extended here to the wider context of pseudodifferential families of cusp type.

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Introduction

For a fibration of compact manifolds $M \longrightarrow B$ where the fibre is a compact manifold with boundary, the cobordism of the index can be interpreted as the vanishing of the (suspended) family index for the boundary

$$\mathrm{ind}_{\mathrm{AS}} : K^0_{\mathrm{c}}(T^*(\partial M/B)) \longrightarrow \mathrm{K}^1(B) \tag{1}$$

on the image of the restriction map

$$K_{\mathrm{c}}(T^*(M/B)) \longrightarrow K_{\mathrm{c}}(\mathbb{R} \times T^*(\partial M/B)) = K^0_{\mathrm{c}}(T^*(\partial M/B)). \tag{2}$$

This was realized analytically in [9] in terms of cusp pseudodifferential operators, namely that any elliptic family of cusp pseudodifferential operators can be perturbed by a family of order $-\infty$ to be invertible; this is described as the universal case in [1]. For the odd version of (1) and in the special case of Dirac operators, Dai and Freed in [5], showed that the τ (i.e. exponentiated η) invariant of a self-adjoint

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Dirac operator on an odd dimensional compact oriented manifold with boundary, with augmented Atiyah-Patodi-Singer boundary condition, defines an element of the inverse determinant line for the boundary Dirac operator. Here we give a full pseudodifferential version of this, showing that the τ invariant for a suspended (hence 'odd') family of elliptic cusp pseudodifferential operators, P, trivializes the determinant bundle for the indicial family $I(P)$

$$\tau = \exp(i\pi\eta(P)) : \mathrm{Det}(I(P) \longrightarrow \mathbb{C}^*, \tag{3}$$

which in this case is a doubly-suspended family of elliptic pseudodifferential operators; the relation to the Dirac case is discussed in detail.

This paper depends substantially on [10] where the determinant on $2n$ times suspended smoothing families is discussed. This determinant in the doubly suspended case is used to define the determinant bundle for any doubly-suspended elliptic family of pseudodifferential operators on a fibration (without boundary). As in the unsuspended case (see Bismut and Freed [4]), the first Chern class of the determinant bundle is the 2-form part of the Chern character of the index bundle. The realization of the eta invariant for singly suspended invertible families in [7] is extended here to the case of invertible families of suspended cusp operators. In the Dirac case this is shown to reduce to the eta invariant of Atiyah, Patodi and Singer for a self-adjoint Dirac operator with augmented APS boundary condition.

In the main body of the paper the consideration of a self-adjoint Dirac operator, $\eth$, is replaced by that of the suspended family, generalizing $\eth + it$, where t is the suspension variable. This effectively replaces the self-adjoint Fredholm operators, as a classifying space for odd K-theory, by the loop group of the small unitary group (see [2], p.81). One advantage of using suspended operators in this way is that the regularization techniques of [7] can be applied to define the eta invariant as an extension of the index. In order to discuss self-adjoint (cusp) pseudodifferential operators using this suspension approach, it is necessary to consider somewhat less regular (product-type) families, generalizing $A+it$, so we show how to extend the analysis to this larger setting.

As in [10], we introduce the various determinant bundles in a direct global form, as associated bundles to principal bundles (of invertible perturbations) instead of using the original spectral definition of Quillen [12]. In this way, the fact that the τ invariant gives a trivialization of the determinant bundle follows rather directly from the log-multiplicative property

$$\eta(A * B) = \eta(A) + \eta(B)$$

of the eta invariant.

The paper is organized as follows. In Section 1, we review the main properties of cusp operators. In Section 2, we consider a conceptually simpler situation which can be thought as an 'even' counterpart of our result. In Section 3, we present the determinant bundle as an associated bundle to a principal bundle; this definition is extended to family of $2n$-suspended elliptic operators in Section 4. This allows us in Section 5 to rederive a well-known consequence of the cobordism invariance of the index at the level of determinant bundles using the contractibility result of [9]. In Section 6, we introduce the notion of cusp suspended $*$-algebra, which is used in Section 7 to lift the determinant from the boundary. This lifted determinant is defined using the eta invariant for invertible suspended cusp operators introduced in Section 8. In Section 9, we prove the trivialization result and in Section 10 we relate it to the result of Dai and Freed [5] for Dirac operators. Finally, in Section 11, these results are extended to include the case of a self-adjoint family of elliptic cusp pseudodifferential operators. This involves the use of product-type suspended operators, which are discussed in the Appendix.

1. Cusp pseudodifferential operators

This section is intended to be a quick summary of the main properties of cusp pseudodifferential operators and ellipticity. We refer to [6], [8] and [9] for more details.

Let Z be a compact manifold with non-empty boundary ∂Z. Let $x \in \mathcal{C}^\infty(Z)$ be a defining function for the boundary, that is, $x \geq 0$ everywhere on Z,

$$\partial Z = \{z \in Z; x(z) = 0\}$$

and $dx(z) \neq 0$ for all $z \in \partial$Z. Such a choice of boundary defining function determines a cusp structure on the manifold Z, which is an identification of the normal bundle of the boundary ∂Z in Z with $\partial Z \times L$ for a 1-dimensional real vector space L. If E and F are complex vector bundles on Z, then $\Psi^m_{\mathrm{cu}}(Z; E, F)$ denotes the space of cusp pseudodifferential operators acting from $\mathcal{C}^\infty(Z; E)$ to $\mathcal{C}^\infty(Z; F)$ associated to the choice of cusp structure. Different choices lead to different algebras of cusp pseudodifferential operators, but all are isomorphic. We therefore generally ignore the particular choice of cusp structure.

A *cusp vector field* $V \in \mathcal{C}^\infty(Z, TZ)$ is a vector field such that $Vx \in x^2\mathcal{C}^\infty(Z)$ for any defining function consistent with the chosen cusp structure. We denote by $\mathcal{V}_{\mathrm{cu}}(Z)$ the Lie algebra of such vector fields. The *cusp tangent bundle* ${}^{\mathrm{cu}}TZ$ is the smooth vector bundle on Z such that $\mathcal{V}_{\mathrm{cu}} = \mathcal{C}^\infty(Z; {}^{\mathrm{cu}}TZ)$; it is isomorphic to TZ as a vector bundle, but not naturally so.

Let ${}^{\mathrm{cu}}S^*Z = ({}^{\mathrm{cu}}T^*Z \setminus 0)/\mathbb{R}^+$ be the *cusp cosphere bundle* and let R^m be the trivial complex line bundle on ${}^{\mathrm{cu}}S^*Z$ with sections given by functions over ${}^{\mathrm{cu}}T^*Z \setminus 0$ which are positively homogeneous of degree m.

Proposition 1.1. *For each $m \in \mathbb{Z}$, there is a* symbol map *giving a short exact sequence*

$$\Psi^{m-1}_{\mathrm{cu}}(Z; E, F) \longrightarrow \Psi^m_{\mathrm{cu}}(Z; E, F) \xrightarrow{\sigma_m} \mathcal{C}^\infty({}^{\mathrm{cu}}S^*Z; \hom(E, F) \otimes R^m). \tag{1.1}$$

Then $A \in \Psi^m_{\mathrm{cu}}(Z; E, F)$ is said to be *elliptic* if its symbol is invertible. In this context, ellipticity is not a sufficient condition for an operator of order 0 to be Fredholm on L^2.

More generally, one can consider the space of full symbols of order m

$$\mathcal{S}^m_{\mathrm{cu}}(Z;E,F)=\rho^{-m}\mathcal{C}^\infty(\overline{{}^{\mathrm{cu}}T^*Z};\hom(E,F)),$$

where ρ is a defining function for the boundary (at infinity) in the radial compactification of ${}^{\mathrm{cu}}T^*Z$. After choosing appropriate metrics and connections, one can define a quantization map following standard constructions

$$q:\mathcal{S}^m_{\mathrm{cu}}(Z;E,F)\longrightarrow\Psi^m_{\mathrm{cu}}(Z;E,F) \tag{1.2}$$

which induces an isomorphism of vector spaces

$$\mathcal{S}^m_{\mathrm{cu}}(Z;E,F)/\mathcal{S}^{-\infty}_{\mathrm{cu}}(Z;E,F)\cong\Psi^m_{\mathrm{cu}}(Z;E,F)/\Psi^{-\infty}_{\mathrm{cu}}(Z;E,F). \tag{1.3}$$

If Y is a compact manifold without boundary and E is a complex vector bundle over Y, there is a naturally defined algebra of suspended pseudodifferential operators, which is denoted here $\Psi^*_{\mathrm{sus}}(Y;E)$. For a detailed discussion of this algebra (and the associated modules of operators between bundles) see [7]. An element $A\in\Psi^m_{\mathrm{sus}}(Y;E)$ is a one-parameter family of pseudodifferential operators in $\Psi^m(Y;E)$ in which the parameter enters symbolically. A suspended pseudodifferential operator is associated to each cusp pseudodifferential operator by 'freezing coefficients at the boundary.' Given $A\in\Psi^m_{\mathrm{cu}}(Z;E,F)$, for each $u\in\mathcal{C}^\infty(Z;E)$, $Au\big|_{\partial Z}\in\mathcal{C}^\infty(\partial Z;F)$ depends only on $u\big|_{\partial Z}\in\mathcal{C}^\infty(\partial Z;E)$. The resulting operator $A_\partial:\mathcal{C}^\infty(\partial Z;E)\longrightarrow\mathcal{C}^\infty(\partial Z;F)$ is an element of $\Psi^m(\partial Z;E,F)$. More generally, if $\tau\in\mathbb{R}$ then

$$\begin{gathered}\Psi^m_{\mathrm{cu}}(Z;E,F)\ni A\longmapsto e^{i\frac{\tau}{x}}Ae^{-i\frac{\tau}{x}}\in\Psi^m_{\mathrm{cu}}(Z;E,F)\text{ and}\\ I(A,\tau)=(e^{i\frac{\tau}{x}}Ae^{-i\frac{\tau}{x}})_\partial\in\Psi^m_{\mathrm{sus}}(\partial Z;E,F)\end{gathered} \tag{1.4}$$

is the *indicial family* of A.

Proposition 1.2. *The* indicial homomorphism *gives a short exact sequence,*

$$x\Psi^m_{\mathrm{cu}}(Z;E,F)\longrightarrow\Psi^m_{\mathrm{cu}}(Z;E,F)\xrightarrow{I}\Psi^m_{\mathrm{sus}}(\partial Z;E,F).$$

There is a power series expansion for operators $A\in\Psi^m_{\mathrm{cu}}(Z;E,F)$ at the boundary of which $I(A)$ is the first term. Namely, if x is a boundary defining function consistent with the chosen cusp structure there is a choice of product decomposition near the boundary consistent with x and a choice of identifications of E and F with their restrictions to the boundary. Given such a choice the 'asymptotically translation-invariant' elements of $\Psi^m_{\mathrm{cu}}(Z;E,F)$ are well-defined by

$$[x^2D_x,A]\in x^\infty\Psi^m_{\mathrm{cu}}(Z;E,F) \tag{1.5}$$

where D_x acts through the product decomposition. In fact

$$\left\{A\in\Psi^m_{\mathrm{cu}}(Z;E,F);\ (1.5)\text{ holds}\right\}/x^\infty\Psi^m_{\mathrm{cu}}(Z;E,F)\xrightarrow{I}\Psi^m_{\mathrm{sus}}(\partial Z;E,F) \tag{1.6}$$

is an isomorphism. Applying Proposition 1.2 repeatedly and using this observation, *any* element of $\Psi^m_{\mathrm{cu}}(Z;E,F)$ then has a power series expansion

$$A \sim \sum_{j=0}^{\infty} x^j A_j,\ A_j \in \Psi^m_{\mathrm{cu}}(Z;E,F),\ [x^2 D_x, A_j] \in x^\infty \Psi^m_{\mathrm{cu}}(Z;E,F) \tag{1.7}$$

which determines it modulo $x^\infty\Psi^m_{\mathrm{cu}}(Z;E,F)$. Setting $I_j(A) = I(A_j)$ this gives a short exact sequence

$$\begin{gathered} x^\infty \Psi^m_{\mathrm{cu}}(Z;E,F) \longrightarrow \Psi^m_{\mathrm{cu}}(Z;E,F) \xrightarrow{I_*} \Psi^m_{\mathrm{sus}}(\partial Z;E,F)[[x]], \\ I_*(A) = \sum_{j=0}^{\infty} x^j I_j(A) \end{gathered} \tag{1.8}$$

which is multiplicative provided the image modules are given the induced product

$$I_*(A) * I_* B = \sum_{j=0}^{\infty} \frac{(ix^2)^j}{j!} (D^j_\tau I_*(A))(D^j_x I_*(B)). \tag{1.9}$$

This is equivalent to a star product although not immediately in the appropriate form because of the asymmetry inherent in (1.8); forcing the latter to be symmetric by iteratively commuting $x^{j/2}$ to the right induces an explicit star product in x^2. In contrast to Proposition 1.2, the sequence (1.8) does depend on the choice of product structure, on manifold and bundles, and the choice of the defining function.

A cusp pseudodifferential operator $A \in \Psi^m_{\mathrm{cu}}(Z;E,F)$ is said to be *fully elliptic* if it is elliptic and if its indicial family $I(A)$ is invertible in $\Psi^*_{\mathrm{sus}}(\partial Z;E,F)$; this is equivalent to the invertibility of $I(A,\tau)$ for each τ and to $I_*(A)$ with respect to the star product.

Proposition 1.3. *A cusp pseudodifferential operator is Fredholm acting on the natural cusp Sobolev spaces if and only if it is fully elliptic.*

For bundles on a compact manifold without boundary, let $G^m_{\mathrm{sus}}(Y;E,F) \subset \Psi^m_{\mathrm{sus}}(Y;E,F)$ denote the subset of elliptic and invertible elements. The η invariant of Atiyah, Patodi and Singer, after reinterpretation, is extended in [7] to a map

$$\begin{gathered} \eta : G^m_{\mathrm{sus}}(Y;E,F) \longrightarrow \mathbb{C}, \\ \eta(AB) = \eta(A) + \eta(B),\ A \in G^m_{\mathrm{sus}}(Y;F,G),\ B \in G^{m'}_{\mathrm{sus}}(Y;E,F). \end{gathered} \tag{1.10}$$

In [8] an index theorem for fully elliptic fibred cusp operators is obtained, as a generalization of the Atiyah-Patodi-Singer index theorem.

Theorem 1.4 ([8])**.** *Let $P \in \Psi^m_{\mathrm{cu}}(X;E,F)$ be a fully elliptic operator, then the index of P is given by the formula*

$$\mathrm{ind}(P) = \overline{\mathrm{AS}}(P) - \frac{1}{2}\eta(I(P)) \tag{1.11}$$

where $\overline{\mathrm{AS}}$ is a regularized integral involving only a finite number of terms in the full symbol expansion of P, $I(P) \in \Psi^m_{\mathrm{sus}}(\partial X;E)$ is the indicial family of P and η is the functional (1.10) *introduced in* [7].

Note that the ellipticity condition on the symbol of P implies that E and F are isomorphic as bundles over the boundary, since $\sigma_m(P)$ restricted to the inward-pointing normal gives such an isomorphism. Thus one can freely assume that E and F are identified near the boundary.

In the case of a Dirac operator arising from a product structure near the boundary with invertible boundary Dirac operator and spectral boundary condition, the theorem applies by adding a cylindrical end on which the Dirac operator extends to be translation-invariant, with the indicial family becoming the spectral family for the boundary Dirac operator (for pure imaginary values of the spectral parameter). The formula (1.11) then reduces to the Atiyah-Patodi-Singer index theorem.

The result (1.11) is not really in final form, since the integral $\overline{\mathrm{AS}}(P)$ is not given explicitly nor interpreted in any topological sense. However, since it is symbolic, $\overline{\mathrm{AS}}(P)$ makes sense if P is only elliptic, without assuming the invertibility of the indicial family. It therefore defines a smooth function

$$\overline{\mathrm{AS}} : \mathrm{Ell}^m_{\mathrm{cu}}(X; E, F) \longrightarrow \mathbb{C} \tag{1.12}$$

for each m. We show in Theorem 2.3 below that this function is a log-determinant for the indicial family.

Cusp operators of order $-\infty$ are in general not compact, so in particular not of trace class. Nevertheless, it is possible to define a regularized trace which will be substantially used in this paper.

Proposition 1.5. *For $A \in \Psi^{-n-1}_{\mathrm{cu}}(Z)$, $n = \dim(Z)$ and $z \in \mathbb{C}$, the function $z \mapsto \mathrm{Tr}(x^z A)$ is holomorphic for $\mathrm{Re}\, z > 1$ and has a meromorphic extension to the whole complex plane with at most simple poles at $1 - \mathbb{N}_0$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.*

For $A \in \Psi^{-n-1}_{\mathrm{cu}}(Z)$, the *boundary residue trace* of A, denoted $\mathrm{Tr}_{\mathrm{R},\partial}(A)$, is the residue at $z = 0$ of the meromorphic function $z \mapsto \mathrm{Tr}(x^z A)$. In terms of the expansion (1.7)

$$\mathrm{Tr}_{\mathrm{R},\partial}(A) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{Tr}(I_1(A, \tau)) d\tau. \tag{1.13}$$

The *regularized trace* is defined to be

$$\overline{\mathrm{Tr}}(A) = \lim_{z \to 0} \left(\mathrm{Tr}(x^z A) - \frac{\mathrm{Tr}_{\mathrm{R},\partial}(A)}{z} \right), \text{ for } A \in \Psi^{-n-1}_{\mathrm{cu}}(Z).$$

For $A \in x^2 \Psi^{-n-1}_{\mathrm{cu}}(Z)$ this reduces to the usual trace but in general it is not a trace, since it does not vanish on all commutators. Rather, there is a *trace-defect formula*

$$\overline{\mathrm{Tr}}([A, B]) = \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{Tr} \left(I(A, \tau) \frac{\partial}{\partial \tau} I(B, \tau) \right) d\tau,$$
$$A \in \Psi^m_{\mathrm{cu}}(Z),\ B \in \Psi^{m'}_{\mathrm{cu}}(Z),\ m + m' \le -n - 1. \tag{1.14}$$

The sign of this formula is correct provided we use (1.4) to define the indicial family. Notice that there is a (harmless) sign mistake in the trace-defect formula of [9], where a different convention for the indicial family is used.

2. Logarithm of the determinant

As a prelude to the discussion of the determinant bundle, we will consider the conceptually simpler situation of the principal $\mathbb{Z}$-bundle corresponding to the 1-dimensional part of the odd index. We first recall the generalization of the notion of principal bundle introduced in [10].

Definition 2.1. Let G be a smooth group (possibly infinite dimensional), then a smooth fibration $\mathcal{G} \longrightarrow B$ over a compact manifold B with typical fibre G is called a **bundle of groups** with model G if its structure group is contained in $\mathrm{Aut}(G)$, the group of smooth automorphisms of G.

Definition 2.2. Let $\phi : \mathcal{G} \longrightarrow B$ be a bundle of groups with model G, then a (right) **principal $\mathcal{G}$-bundle** is a smooth fibration $\pi : \mathcal{P} \longrightarrow B$ with typical fibre G together with a smooth fibrewise (right) group action

$$h : \mathcal{P} \times_B \mathcal{G} \ni (p, g) \longmapsto p \cdot g \in \mathcal{P}$$

which is free and transitive, where

$$\mathcal{P} \times_B \mathcal{G} = \{(p, g) \in \mathcal{P} \times \mathcal{G}; \quad \pi(p) = \phi(g)\}.$$

In particular, a principal G-bundle $\pi : \mathcal{P} \longrightarrow B$ is automatically a principal $\mathcal{G}$-bundle where $\mathcal{G}$ is the trivial bundle of groups

$$\mathcal{G} = G \times B \longrightarrow B$$

given by the projection on the right factor. In that sense, definition 2.2 is a generalization of the notion of a principal bundle.

Notice also that given a bundle of groups $\mathcal{G} \longrightarrow B$, then $\mathcal{G}$ itself is a principal $\mathcal{G}$-bundle. It is the **trivial principal $\mathcal{G}$-bundle**. More generally, we say that a principal $\mathcal{G}$-bundle $\mathcal{P} \longrightarrow B$ is **trivial** if there exists a diffeomorphism $\Psi : \mathcal{P} \longrightarrow \mathcal{G}$ which preserves the fibrewise group action:

$$\Psi(h(p, g)) = \Psi(p)g, \quad \forall (p, g) \in \mathcal{P} \times_B \mathcal{G}.$$

In this section, the type of principal $\mathcal{G}$-bundle of interest arises by considering an elliptic family $Q \in \Psi^m_{\mathrm{sus}}(M/B; E, F)$ of suspended operators over a fibration

$$\begin{tikzcd} Y \arrow[r, dash] & M \arrow[d, "\phi"] \\ & B \end{tikzcd} \tag{2.1}$$

of compact manifolds without boundary (not necessarily bounding a fibration with boundary). Namely, it is given by the smooth fibration $\mathcal{Q} \longrightarrow B$, with fibre at $b \in B$

$$\mathcal{Q}_b = \left\{Q_b + R_b; R_b \in \Psi^{-\infty}_{\mathrm{sus}}(Y_b, E_b, F_b);\ \exists\ (Q_b + R_b)^{-1} \in \Psi^{-m}_{\mathrm{sus}}(Y_b; F_b, E_b)\right\}, \tag{2.2}$$

the set of all invertible perturbations of Q_b. The fibre is non-empty and is a principal space for the action of the once-suspended smoothing group

$$G^{-\infty}_{\mathrm{sus}}(Y_b; E_b) = \left\{\mathrm{Id} + A; A \in \Psi^{-\infty}_{\mathrm{sus}}(Y; E_b),\ (\mathrm{Id} + A)^{-1} \in \mathrm{Id} + \Psi^{-\infty}_{\mathrm{sus}}(Y; E_b)\right\} \tag{2.3}$$

acting on the right. Thus, $\mathcal{Q}$ is a principal $G^{-\infty}_{\mathrm{sus}}(M/B; E)$-bundle with respect to the bundle of groups $G^{-\infty}_{\mathrm{sus}}(M/B; E) \longrightarrow B$ with fibre at $b \in B$ given by (2.3).

The structure group at each point is a classifying space for even K-theory and carries an index homomorphism

$$\mathrm{ind} : G^{-\infty}_{\mathrm{sus}}(Y_b; E_b) \longrightarrow \mathbb{Z},\ \mathrm{ind}(\mathrm{Id} + A) = \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{Tr}\left(\frac{dA(t)}{dt}(\mathrm{Id} + A(t))^{-1}\right) dt \tag{2.4}$$

labelling the components, i.e. giving the 0-dimensional cohomology. For a suspended elliptic family this induces an integral 1-class on B; namely the first Chern class of the odd index bundle of the family. This can be seen in terms of the induced principal $\mathbb{Z}$-bundle $\mathcal{Q}_{\mathbb{Z}}$ associated to $\mathcal{Q}$

$$\mathcal{Q}_{\mathbb{Z}} = \mathcal{Q} \times \mathbb{Z}/\sim, \quad (Ag, m - \mathrm{ind}(g)) \sim (A, m), \quad \forall g \in G^{-\infty}_{\mathrm{sus}}(Y; E_b). \tag{2.5}$$

Since $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a classifying space for $\mathbb{Z}$, such bundles are classified up to equivalence by the integral 1-cohomology of the base.

More explicitly, any principal $\mathbb{Z}$-bundle $\phi : P \longrightarrow B$ admits a 'connection' in the sense of a map $h : P \longrightarrow \mathbb{C}$ such that $h(mp) = h(p) + m$ for the action of $m \in \mathbb{Z}$. Then the integral 1-class of the principal $\mathbb{Z}$-bundle P is given by the map

$$e^{2\pi i h} : B \longrightarrow \mathbb{C}^* \tag{2.6}$$

or the cohomology class of dh seen as a 1-form on B. The triviality of the principal $\mathbb{Z}$-bundle is equivalent to the vanishing of the integral 1-class, that is, to the existence of a function $f : B \longrightarrow \mathbb{C}$ such that $h - \phi^* f$ is locally constant.

Moreover, restricted to the 'residual' subgroup $G^{-\infty}_{\mathrm{sus}}(Y; E)$, the eta functional of (1.10) reduces to twice the index

$$\eta\big|_{G^{-\infty}_{\mathrm{sus}}(Y;E)} = 2\,\mathrm{ind}\,. \tag{2.7}$$

In case the fibration is the boundary of a fibration of compact manifolds with boundary, as in [9], and the suspended family is the indicial family of an elliptic family of cusp pseudodifferential operators then we know that the whole odd index of the indicial family vanishes in odd K-theory. In particular the first Chern class vanishes and the associated principal $\mathbb{Z}$-bundle is trivial.

Theorem 2.3. *The eta invariant defines a connection $\frac{1}{2}\eta(A)$ on the principal $\mathbb{Z}$-bundle in* (2.5) *(so the first odd Chern class is $\frac{1}{2}d\eta$) and in the case of the indicial operators of a family of elliptic cusp operators, the Atiyah-Singer term in the index formula* (1.11) *is a log-determinant for the indicial family, so trivializing the $\mathbb{Z}$-bundle.*

Proof. By (2.7), the function on $\mathcal{Q} \times \mathbb{Z}$

$$h(A, m) = \frac{1}{2}\eta(A) + m \tag{2.8}$$

descends to $\mathcal{Q}_{\mathbb{Z}}$ and defines a connection on it. Thus the map

$$\tau = \exp(i\pi\eta) : B \longrightarrow \mathbb{C}^* \tag{2.9}$$

gives the classifying 1-class, the first odd Chern class in $H^1(B, \mathbb{Z})$ of the index bundle. In general this class is not trivial, but when $Q = I(Q_{\mathrm{cu}})$ is the indicial family of a family of fully elliptic cusp operators Q_{cu}, the Atiyah-Singer term $\overline{\mathrm{AS}}(Q_{\mathrm{cu}})$ is a well-defined smooth function which does not depend on the choice of the indicial family modulo $G^{-\infty}_{\mathrm{sus}}(Y; E)$. From formula (1.11)

$$h - \overline{\mathrm{AS}}(Q_{\mathrm{cu}}) = -\overline{\mathrm{AS}}(Q_{\mathrm{cu}}) + \frac{1}{2}\eta(A) + m = -\operatorname{ind}(A_{\mathrm{cu}}, b) + m \tag{2.10}$$

is locally constant. This shows that the Atiyah-Singer term explicitly trivializes the principal $\mathbb{Z}$-bundle $\mathcal{Q}_{\mathbb{Z}}$. □

3. The determinant line bundle

Consider a fibration of closed manifolds as in (2.1) and let E and F be complex vector bundles on M. Let $P \in \Psi^m(M/B; E, F)$ be a smooth family of elliptic pseudodifferential operators acting on the fibres. If the numerical index of the family vanishes, then one can, for each $b \in B$, find $Q_b \in \Psi^{-\infty}(Y_b; E_b, F_b)$ such that $P_b + Q_b$ is invertible. The families index, which is an element of the even K-theory of the base $K^0(B)$ (see [3] for a definition), is the obstruction to the existence of a smooth family of such perturbations. This obstruction can be realized as the non-triviality of the bundle with fibre

$$\mathcal{P}_b = \left\{ P_b + Q_b;\ Q_b \in \Psi^{-\infty}(Y_b, E_b, F_b),\ \exists\ (P_b + Q_b)^{-1} \in \Psi^{-k}(Y_b; F_b, E_b) \right\}. \tag{3.1}$$

As in the odd case discussed above, the fibre is non-trivial (here because the numerical index is assumed to vanish) and is a bundle of principal G-spaces for the groups

$$G^{-\infty}(Y_b; E) = \left\{ \mathrm{Id} + Q; Q \in \Psi^{-\infty}(Y_b; E),\ \exists\ (\mathrm{Id} + Q)^{-1} \in \Psi^0(Y_b; E) \right\} \tag{3.2}$$

acting on the right. Thus, $\mathcal{P} \longrightarrow B$ is a principal $G^{-\infty}(M/B; E)$-bundle for the bundle of groups $G^{-\infty}(M/B; E) \longrightarrow B$ with fibre at $b \in B$ given by (3.2).

The Fredholm determinant

$$\det : \mathrm{Id} + \Psi^{-\infty}(X; W) \longrightarrow \mathbb{C}$$

is well-defined for any compact manifold X and vector bundle W. It is multiplicative

$$\det(AB) = \det(A)\det(B)$$

and is non-vanishing precisely on the group $G^{-\infty}(X;W)$. Explicitly, it may be defined by

$$\det(B) = \exp\left(\int_{[0,1]} \gamma^* \operatorname{Tr}(A^{-1}dA)\right) \tag{3.3}$$

where $\gamma : [0,1] \longrightarrow G^{-\infty}(X;W)$ is any smooth path with $\gamma(0) = \mathrm{Id}$ and $\gamma(1) = B$. Such a path exists since $G^{-\infty}(X;W)$ is connected and the result does not depend on the choice of γ in view of the integrality of the 1-form $\frac{1}{2\pi i}\operatorname{Tr}(A^{-1}dA)$ (which gives the index for the loop group).

Definition 3.1. If $P \in \Psi^m(M/B;E,F)$ is a family of elliptic pseudodifferential operators with vanishing numerical index and $\mathcal{P} \longrightarrow B$ is the bundle given by (3.1), then the determinant line bundle $\operatorname{Det}(P) \longrightarrow B$ of P is the associated line bundle given by

$$\operatorname{Det}(P) = \mathcal{P} \times_{G^{-\infty}(M/B;E)} \mathbb{C} \tag{3.4}$$

where $G^{-\infty}(Y_b;F_b)$ acts on $\mathbb{C}$ via the determinant; thus, $\operatorname{Det}(P)$ is the space $\mathcal{P}\times\mathbb{C}$ with the equivalence relation

$$(A,c) \sim (Ag^{-1}, \det(g)c)$$

for $A \in \mathcal{P}_b$, $g \in G^{-\infty}(Y_b;F_b)$, $b \in B$ and $c \in \mathbb{C}$.

As discussed in [10], this definition is equivalent to the original spectral definition due to Quillen [12].

If $P \in \Psi^m(M/B;E,F)$ is a general elliptic family, with possibly non-vanishing numerical index, it is possible to give a similar definition but depending on some additional choices. Assuming for definiteness that the numerical index is $l \geq 0$ one can choose a trivial l-dimensional subbundle $K \subset \mathcal{C}^\infty(M/B;E)$ as a bundle over B, a Hermitian inner product on E and a volume form on B. Then the fibre in (3.1) may be replaced by

$$\mathcal{P}_{b,K} = \left\{P_b + Q_b;\ Q_b \in \Psi^{-\infty}(Y_b, E_b, F_b),\, \ker(P_b+Q_b) = K_b\right\}. \tag{3.5}$$

This fibre is non-empty and for each such choice of Q_b there is a unique element $L_b \in \Psi^{-m}(Y_b;F_b,E_b)$ which is a left inverse of P_b+Q_b with range $K_b^\perp$ at each point of B. The action of the bundle of groups $G^{-\infty}(M/B;F)$ on the left makes this into a (left) principal $G^{-\infty}(M/B;F)$-bundle. Then the fibre of the determinant bundle may be taken to be

$$\operatorname{Det}(P)_{b,K} = \mathcal{P}_{b,K} \times \mathbb{C}/\sim,\ (A,c) \sim (BA, \det(B)c). \tag{3.6}$$

In case the numerical index is negative there is a similar construction intermediate between the two cases.

4. The $2n$-suspended determinant bundle

As described in [10], it is possible to extend the notion of determinant, and hence that of the determinant line bundle, to suspended pseudodifferential operators with an even number of parameters.

Let $L \in \Psi^m_{\mathrm{s}(2n)}(M/B; E, F)$ be an elliptic family of $(2n)$-suspended pseudodifferential operators. Ellipticity (in view of the symbolic dependence on the parameters) implies that such a family is invertible near infinity in $\mathbb{R}^{2n}$. Thus the families index is well-defined as an element of the compactly supported K-theory $K_{\mathrm{c}}(\mathbb{R}^{2n}) = \mathbb{Z}$. By Bott periodicity this index may be identified with the numerical index of a family where the parameters are quantized, see the discussion in [10]. Even assuming the vanishing of this numerical index, to get an explicitly defined determinant bundle, as above, we need to introduce a formal parameter ϵ.

Let $\Psi^m_{\mathrm{s}(2n)}(Y; F)[[\epsilon]]$ denote the space of formal power series in ϵ with coefficients in $\Psi^m_{\mathrm{s}(2n)}(Y; F)$. For $A \in \Psi^m_{\mathrm{s}(2n)}(Y; F)[[\epsilon]]$ and $B \in \Psi^{m'}_{\mathrm{s}(2n)}(Y; F)[[\epsilon]]$, consider the $*$-*product* $A * B \in \Psi^{m+m'}_{\mathrm{s}(2n)}(Y; F)[[\epsilon]]$ given by

$$\begin{aligned} A * B(u) &= (\sum_{\mu=0}^{\infty} a_\mu \epsilon^\mu) * (\sum_{\nu=0}^{\infty} b_\nu \epsilon^\nu) \\ &= \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} \epsilon^{\mu+\nu} \left(\sum_{p=0}^{\infty} \frac{i^p \epsilon^p}{2^p p!} \omega(D_v, D_w)^p A(v) B(w)\big|_{v=w=u} \right) \end{aligned} \tag{4.1}$$

where $u, v, w \in \mathbb{R}^{2n}$ and ω is the standard symplectic form on $\mathbb{R}^{2n}$, $\omega(v, w) = v^T J w$ with

$$J = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}. \tag{4.2}$$

That (4.1) is an associative product follows from its identification with the usual 'Moyal product' arising as the symbolic product for pseudodifferential operators on $\mathbb{R}^n$.

Definition 4.1. The module $\Psi^m_{\mathrm{s}(2n)}(Y; E, F)[[\epsilon]]$ with $*$-product as in (4.1) will be denoted $\Psi^m_{\mathrm{s}\star(2n)}(Y; E, F)[[\epsilon]]$ and the quotient by the ideal $\epsilon^{n+1}\Psi^m_{\mathrm{s}(2n)}(Y; E, F)[[\epsilon]]$, $n = \dim(Y)$, by $\Psi^m_{\mathrm{s}\star(2n)}(Y; E, F)$.

The quotient here corresponds formally to setting

$$\epsilon^{n+1} = 0. \tag{4.3}$$

Proposition 4.2. *(Essentially from* [10]*) The group*

$$\begin{aligned} G^{-\infty}_{\mathrm{s}\star(2n)}(Y; F) = \big\{ \mathrm{Id} + S; S \in \Psi^{-\infty}_{\mathrm{s}\star(2n)}(Y; F), \\ \exists\ (\mathrm{Id} + S)^{-1} \in \Psi^0_{\mathrm{s}\star(2n)}(Y; F) \big\}, \end{aligned} \tag{4.4}$$

with composition given by the $*$-product, admits a determinant homomorphism

$$\det : G^{-\infty}_{\mathrm{s}\star(2n)}(Y; F) \longrightarrow \mathbb{C}, \ \det(A * B) = \det(A)\det(B), \tag{4.5}$$

given by

$$\det(B) = \exp\left(\int_{[0,1]} \gamma^* \alpha_n\right) \tag{4.6}$$

where α_n *is the coefficient of* ϵ^n *in the 1-form* $\operatorname{Tr}(A^{-1} * dA)$ *and* $\gamma : [0,1] \longrightarrow G^{-\infty}_{\mathrm{s}\star(2n)}(Y;F)$ *is any smooth path with* $\gamma(0) = \mathrm{Id}$ *and* $\gamma(1) = B$.

Proof. In [10] the determinant is defined via (4.6) for the full formal power series algebra with $*$-product. Since the 1-form α_n only depends on the term of order n in the formal power series, and this term for a product only depends on the first n terms of the factors, we can work in the quotient and (4.5) follows. □

For the group $G^{-\infty}_{\mathrm{s}\star(2)}(X;E)$, the form α_2 can be computed explicitly.

Proposition 4.3. *On* $G^{-\infty}_{\mathrm{s}\star(2)}(X;E)$

$$\alpha_2 = i\pi d\mu(a) - \frac{1}{4\pi i}\int_{\mathbb{R}^2} \operatorname{Tr}\big((a_0^{-1}\frac{\partial a_0}{\partial t})(a_0^{-1}\frac{\partial a_0}{\partial \tau})a_0^{-1}da_0 \\ - (a_0^{-1}\frac{\partial a_0}{\partial \tau})(a_0^{-1}\frac{\partial a_0}{\partial t})a_0^{-1}da_0\big)dtd\tau, \tag{4.7}$$

where

$$\mu(a) = \frac{1}{2\pi^2 i}\int_{\mathbb{R}^2} \operatorname{Tr}(a_0^{-1}a_1)dtd\tau. \tag{4.8}$$

Proof. For $a = a_0 + \epsilon a_1 \in G^{-\infty}_{\mathrm{s}\star(2)}(X;E)$, the inverse a^{-1} of a with respect to the $*$-product is

$$a^{-1} = a_0^{-1} - \epsilon(a_0^{-1}a_1a_0^{-1} - \frac{i}{2}\{a_0^{-1}, a_0\}a_0^{-1})\ , \tag{4.9}$$

where a_0^{-1} is the inverse of a_0 in $G^{-\infty}_{\mathrm{s}(2)}(X;E)$ and

$$\{a,b\} = D_t a D_\tau b - D_\tau a D_t b = \partial_\tau a \partial_t b - \partial_t a \partial_\tau b$$

is the Poisson Bracket. Hence,

$$\begin{aligned}\operatorname{Tr}(a^{-1} * da) &= \operatorname{Tr}\Bigg(a_0^{-1}da_0 + \epsilon(-\frac{i}{2}\{a_0^{-1}, da_0\} - a_0^{-1}a_1a_0^{-1}da_0 \\ &\qquad + \frac{i}{2}\{a_0^{-1}, a_0\}a_0^{-1}da_0 + a_0^{-1}da_1)\Bigg) \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\Bigg(\operatorname{Tr}(a_0^{-1}da_0) + \epsilon\operatorname{Tr}(-\frac{i}{2}\{a_0^{-1}, da_0\} \\ &\qquad - a_0^{-1}a_1a_0^{-1}da_0 + \frac{i}{2}\{a_0^{-1}, a_0\}a_0^{-1}da_0 + a_0^{-1}da_1)\Bigg)dtd\tau\ .\end{aligned} \tag{4.10}$$

So

$$\alpha_2(a) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}\Big(-\frac{i}{2}\{a_0^{-1}, da_0\} - a_0^{-1}a_1a_0^{-1}da_0 \\ + \frac{i}{2}\{a_0^{-1}, a_0\}a_0^{-1}da_0 + a_0^{-1}da_1\Big)dtd\tau\ . \tag{4.11}$$

On the right hand side of (4.11), the first term vanishes since it is the integral of the trace of a Poisson bracket. Indeed, integrating by parts one of the terms with respect to t and τ,

$$\begin{aligned}\int_{\mathbb{R}^2} \mathrm{Tr}(\{a_0^{-1}, da_0\})dtd\tau &= \int_{\mathbb{R}^2} \mathrm{Tr}(D_t a_0^{-1} D_\tau(da_0) - D_\tau a_0^{-1} D_t(da_0))dtd\tau \\ &= \int_{\mathbb{R}^2} \mathrm{Tr}(D_t a_0^{-1} D_\tau(da_0) + D_t D_\tau a_0^{-1}(da_0))dtd\tau \\ &= \int_{\mathbb{R}^2} \mathrm{Tr}(D_t a_0^{-1} D_\tau(da_0) - D_t a_0^{-1} D_\tau(da_0))dtd\tau \\ &= 0.\end{aligned} \tag{4.12}$$

Hence,

$$\alpha_2(a) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \mathrm{Tr}(\frac{i}{2}\{a_0^{-1}, a_0\}a_0^{-1}da_0 + a_0^{-1}da_1 - a_0^{-1}a_1a_0^{-1}da_0)dtd\tau. \tag{4.13}$$

The last two terms on the right combine to give $i\pi d\mu(a)$. Writing out the Poisson bracket in terms of t and τ, gives (4.7). □

Proposition 4.4. *Integration of the 1-form α_2 gives an isomorphism*

$$\phi : \pi_1\left(G^{-\infty}_{s\star(2)}(X;E)\right) \ni f \longmapsto \frac{1}{2\pi i}\int_{\mathbb{S}^1} f^*\alpha_2 \in \mathbb{Z}. \tag{4.14}$$

Proof. By the previous proposition and Stokes' theorem,

$$\begin{aligned}\int_{\mathbb{S}^1} f^*\alpha_2 &= \frac{1}{12\pi i}\int_{\mathbb{S}^3} g^*(\mathrm{Tr}((a^{-1}da)^3)) \\ &= 2\pi i \int_{\mathbb{S}^3} g^*\beta_2^{odd},\ \forall\ f : \mathbb{S}^1 \longrightarrow G^{-\infty}_{s(2)}(X;E),\end{aligned} \tag{4.15}$$

where the map g is defined by

$$g : \mathbb{S}^3 \ni (s,\tau,t) \longmapsto f(s)(t,\tau) \in G^{-\infty}(X;E), \tag{4.16}$$

and $\beta_2^{odd} = \frac{1}{6(2\pi i)^2}\mathrm{Tr}((a^{-1}da)^3)$. Our convention is that the orientation on $\mathbb{R}^2$ is given by the symplectic form $\omega = d\tau \wedge dt$. The 3-form β_2^{odd} on $G^{-\infty}(X;E)$ is such that

$$\lambda : \pi_3(G^{-\infty}(X;E)) \ni h \longmapsto \textstyle\int_{\mathbb{S}^3} h^*\beta_2^{odd} \in \mathbb{Z} \tag{4.17}$$

is an isomorphism (see [9]) where

$$G^{-\infty}(X;E) = \{\mathrm{Id} + S; S \in \Psi^{-\infty}(X;E), ((\mathrm{Id} + S)^{-1} \in \Psi^0(X;E)\}\ . \tag{4.18}$$

Up to homotopy, $G^{-\infty}_{\mathrm{s}(2)}(X;E) \cong [\mathbb{S}^2, G^{-\infty}(X;E)]$, so the map $f \mapsto g$ is an isomorphism $\pi_1(G^{-\infty}_{\mathrm{s}(2)}(X;E)) \cong \pi_3(G^{-\infty}(X;E))$. Hence the proposition follows from (4.15) and (4.17). $\square$

We may identify

$$\Psi^m_{\mathrm{s}(2n)}(M/B;E,F) \subset \Psi^m_{\mathrm{s}\star(2n)}(M/B;E,F) \tag{4.19}$$

as the subspace of elements independent of ϵ. For an elliptic family L with vanishing numerical index one can then consider in the same way as above the (non-empty) principal $G^{-\infty}_{\mathrm{s}\star(2n)}(Y_b;F_b)$ spaces

$$\begin{aligned}\mathcal{L}_b = \{L_b + S_b; S_b \in \Psi^{-\infty}_{\mathrm{s}\star(2n)}(Y_b;E_b,F_b)&\\ \exists\ (L_b+S_b)^{-1} \in \Psi^{-m}_{\mathrm{s}\star(2n)}(Y_b;F_b,E_b)\}&\end{aligned} \tag{4.20}$$

forming a smooth infinite-dimensional bundle over B.

Definition 4.5. For an elliptic family $L \in \Psi^m_{\mathrm{s}\star(2n)}(M/B;E,F)$ with vanishing numerical index, the determinant line bundle is given by

$$\operatorname{Det}(L) = \mathcal{L} \times_{G^{-\infty}_{\mathrm{s}\star(2n)}(M/B;E)} \mathbb{C} \tag{4.21}$$

where each fibre of $G^{-\infty}_{\mathrm{s}\star(2n)}(M/B;E)$ acts on $\mathbb{C}$ via the determinant of Proposition 4.2.

5. Cobordism invariance of the index

Suppose that the fibration (2.1) arises as the boundary of a fibration where the fibre is a compact manifold with boundary:

$$\begin{tikzcd}
& \partial Z \arrow[r, dash] \arrow[dl, hook, "\partial"'] & \partial M \arrow[dl, hook, "\partial"'] \arrow[dd, "\partial\phi"] \\
Z \arrow[r, dash] & M \arrow[dr, "\phi"'] & \\
& & B,
\end{tikzcd} \tag{5.1}$$

so Z and M are compact manifolds with boundary. Let E and F be complex vector bundles over the manifold M. Suspending the short exact sequence of Proposition 1.2 one arrives at the short exact sequence

$$x\Psi^m_{\mathrm{cs}(k)}(Z;E,F) \longrightarrow \Psi^m_{\mathrm{cs}(k)}(Z;E,F) \xrightarrow{I} \Psi^m_{\mathrm{s}(k+1)}(\partial Z;E,F),\ k \in \mathbb{N}. \tag{5.2}$$

Theorem 5.1. *Let $L \in \Psi^m_{\mathrm{s}(2n)}(\partial M/B;E,F)$ be an elliptic family of $2n$-suspended pseudodifferential operators and suppose that the fibration arises as the boundary of a fibration as in* (5.1) *and that L is the indicial family $L = I(P)$ of an elliptic* *family $P \in \Psi^m_{\mathrm{cs}(2n-1)}(M/B;E,F)$ of $(2n-1)$-suspended cusp pseudodifferential operators, then the index bundle of* (4.20) *is trivial.*

Proof. Given $b \in B$, we claim that P_b can be perturbed by

$$Q_b \in \Psi^{-\infty}_{\mathrm{cs}(2n-1)}(M_b;E_b,F_b)$$

to become invertible. Indeed, we may think of P_b as a family of cusp operators on $\mathbb{R}^{2n-1}$. To this family we can associate the bundle $\mathcal{I}_b$ over $\mathbb{R}^{2n-1}$ of invertible perturbations by elements in $\Psi^{-\infty}_{\mathrm{cu}}(M_b;E_b,F_b)$. This bundle is well-defined in the sense that invertible perturbations exist for all $t \in \mathbb{R}^{2n-1}$ by Theorem 5.2 of [9]. The ellipticity of P_b ensures that there exists $R > 0$ such that $P_b(t)$ is invertible for $|t| \geq R$. By the contractibility result of [9], there exists an invertible section $P_b(t) + Q_b(t)$ of $\mathcal{I}_b$ such that $Q_b(t) = 0$ for $|t| > R$. In particular Q_b is an element of $\Psi^{-\infty}_{\mathrm{cs}(2n-1)}(M_b;E_b,F_b)$, and so $P_b + Q_b$ is the desired invertible perturbation.

It follows that there exists $S_b \in \Psi^{-\infty}_{\mathrm{s}(2n)}(\partial Z_b;E_b,F_b)$ such that $I(P_b) = L_b$ is invertible. This could also have been seen directly using K-theory and the cobordism invariance of the index. In any case, this shows that the family P gives rise to a bundle $\mathcal{P}_{\mathrm{cs}(2n-1)}$ on the manifold B with fibre at $b \in B$

$$\begin{aligned}\mathcal{P}_{\mathrm{cs}(2n-1),b} = \{P_b + Q_b;\ Q_b \in \Psi^{-\infty}_{\mathrm{cs}(2n-1)}(Z_b;E_b,F_b),& \\ (P_b+Q_b)^{-1} \in \Psi^{-k}_{cs(2n-1)}(Z_b;F_b,E_b)\}.& \quad (5.3)\end{aligned}$$

If we consider the bundle of groups $G^{-\infty}_{\mathrm{cs}(2n-1)}(M/B;E) \longrightarrow B$ with fibre at $b \in B$

$$\begin{aligned}G^{-\infty}_{\mathrm{cs}(2n-1)}(Z_b;E_b) = \{\mathrm{Id} + Q_b;\ Q_b \in \Psi^{-\infty}_{\mathrm{cs}(2n-1)}(Z_b;E_b),& \\ (\mathrm{Id} + Q_b)^{-1} \in \Psi^{0}_{\mathrm{cs}(2n-1)}(Z_b;E_b)\},& \quad (5.4)\end{aligned}$$

then $\mathcal{P}_{cs(2n-1)}$ may be thought as a principal $G^{-\infty}_{\mathrm{cs}(2n-1)}(M/B;E)$-bundle, where the group $G^{-\infty}_{\mathrm{cs}(2n-1)}(Z_b;E_b)$ acts on the right in the obvious way. From [9] it follows $G^{-\infty}_{\mathrm{cs}(2n-1)}(Z_b;F_b)$, is weakly contractible. Hence, $\mathcal{P}_{\mathrm{cs}(2n-1)}$ has a global section defined over B, so is trivial as a principal $G^{-\infty}_{\mathrm{cs}(2n-1)}(M/B;E)$-bundle. Taking the indicial family of this global section gives a global section of the bundle $\mathcal{L}$ which is therefore trivial as a principal $G^{-\infty}_{\mathrm{s}\star(2n)}(M/B;E)$-bundle. □

As an immediate consequence, the determinant bundle of a $2n$-suspended family which arises as the indicial family of elliptic cusp operators is necessarily trivial. Indeed, it is an associated bundle to the index bundle, which is trivial in that case. In the case of a twice-suspended family we will give an explicit trivialization in terms of the extended τ invariant of the elliptic cusp family. To do so we first need to define the η invariant in this context. As for the determinant of a suspended family discussed in [10] and in §4 above, the extended η invariant is only defined on the $\star$-extended operators which we discuss first.

6. Suspended cusp $\star$-algebra

On a compact manifold Z with boundary, consider, for a given boundary defining function x, the space of formal power series

$$\mathcal{A}^m(Z;E) = \sum_{j=0}^{\infty} \varepsilon^j x^j \Psi^m_{\mathrm{cs}}(Z;E) \tag{6.1}$$

in which the coefficients have increasing order of vanishing at the boundary. The exterior derivations D_t (differentiation with respect to the suspending parameter) and $D_{\log x}$ can be combined to give an exterior derivative $D = (D_t, D_{\log x})$ valued in $\mathbb{R}^2$. Here, the derivation $D_{\log x}$ is defined to be

$$D_{\log x} A = \frac{d}{dz} x^z A x^{-z}\big|_{z=0}.$$

for $A \in \Psi^m_{\mathrm{cs}}(Z;E)$. It is such that (cf. [9] where a different convention is used for the indicial family)

$$I(D_{\log x} A) = 0, \quad I(\frac{1}{x} D_{\log x} A)) = D_\tau I(A)$$

where τ is the suspension variable for the indicial family. Combining this with the symplectic form on $\mathbb{R}^2$ gives a star product

$$\begin{aligned} A * B &= \sum_{j,k,p} \varepsilon^{j+k+p} \frac{i^p}{2^p p!} \omega(D_A, D_B)^p A_j B_k, \\ A &= \sum_j \varepsilon^j A_j, \;\; B = \sum_k \varepsilon^k B_k. \end{aligned} \tag{6.2}$$

Here, the differential operator $\omega(D_A, D_B)^p$ is first to be expanded out, with D_A being D acting on A and D_B being D acting on B and then the product is taken in $\Psi^m_{\mathrm{cs}}(Z;E)$. Note that

$$D_{\log x} : x^p \Psi^m_{\mathrm{cs}}(Z;E) \longrightarrow x^{p+1} \Psi^{m-1}_{\mathrm{cs}}(Z;E)$$

so the series in (6.2) does lie in the space (6.1). The same formal argument as in the usual case shows that this is an associative product. We take the quotient by the ideal spanned by $(\varepsilon x)^2$ and denote the resulting algebra $\Psi^m_{\mathrm{cs}\star}(Z;E)$. Its elements are sums $A + \varepsilon A'$, $A' \in x\Psi^m_{\mathrm{cs}}(Z;E)$ and the product is just

$$\begin{aligned} (A + \varepsilon A') * (B + \varepsilon B') &= AB + \varepsilon(AB' + A'B) \\ &\quad - \frac{i\varepsilon}{2} (D_t A D_{\log x} B - D_{\log x} A D_t B) \quad \operatorname{mod} \varepsilon^2 x^2. \end{aligned} \tag{6.3}$$

The minus sign comes from our definition of the symplectic form (4.2). The boundary asymptotic expansion (1.8), now for suspended operators, extends to the power series to give a map into triangular, doubly-suspended, double power series

$$I_* : \mathcal{A}^m(Z;E) \longrightarrow \Psi^m_{\mathrm{s}(2)}(\partial Z;E)[[\varepsilon x, x]]. \tag{6.4}$$

To relate this more directly to the earlier discussion of star products on the suspended algebras we take the quotient by the ideal generated by x^2 giving a map

$$\tilde{I}:\Psi^m_{\mathrm{cs}\star}(Z;E)\longrightarrow\Big\{a_0+xe+\varepsilon xa_1,\ a_0,\ e,a_1\in\Psi^m_{\mathrm{s}(2)}(\partial Z;E)\Big\}. \tag{6.5}$$

The surjectivity of the indicial map shows that this map too is surjective and so induces a product on the image.

Proposition 6.1. *The surjective map $\tilde{I}$ in* (6.5) *is an algebra homomorphism for the product generated by*

$$a_0\tilde{\star}b_0=a_0b_0-\varepsilon x\frac{i}{2}\left(D_ta_0D_\tau b_0-D_\tau a_0D_tb_0\right),\ a_0,\ b_0\in\Psi^m_{\mathrm{s}(2)}(\partial Z;E) \tag{6.6}$$

extending formally over the parameters εx, x to the range in (6.5).

Proof. First observe that in terms of the expansions (1.7) for $A\in\Psi^m_{\mathrm{cs}}(Z;E)$ and $B\in\Psi^{m'}_{\mathrm{cs}}(Z;E)$ at the boundary

$$\begin{gathered}I_*(AB)=A_0B_0+x(A_0F+EB_0)+O(x^2),\\ A=A_0+xE+\mathcal{O}(x^2),\quad B=B_0+xF+\mathcal{O}(x^2).\end{gathered}$$

It follows that for

$$A=A_0+xE+\varepsilon xA_1,\ B=B_0+xF+\varepsilon xB_1$$

the image of the product is

$$\tilde{I}(A*B)=\tilde{I}(A)\tilde{I}(B)-\varepsilon x\frac{i}{2}(D_ta_0D_\tau b_0-D_\tau a_0D_tb_0), \tag{6.7}$$

where $a_0=I(A_0)$ and $b_0=I(B_0)$. This is precisely what is claimed. □

For any manifold without boundary Y we will denote by $\Psi^m_{\mathrm{s}\tilde{\star}(2)}(Y;E)$ the corresponding algebra with the product coming from (6.7) so that (6.5) becomes the homomorphism of algebras

$$\tilde{I}:\Psi^m_{\mathrm{cs}\star}(Z;E)\longrightarrow\Psi^m_{\mathrm{s}\tilde{\star}(2)}(\partial Z;E). \tag{6.8}$$

As the notation indicates, this algebra is closely related to $\Psi^m_{\mathrm{s}\star(2)}(Y;E)$ discussed in §4. Namely, by identifying the parameter as $\epsilon=\varepsilon x$ the latter may be identified with the quotient by the ideal

$$x\Psi^m_{\mathrm{s}(2)}\longrightarrow\Psi^m_{\mathrm{s}\tilde{\star}(2)}(Y;E)\xrightarrow{\epsilon=\varepsilon x}\Psi^m_{\mathrm{s}\star(2)}(Y;E). \tag{6.9}$$

Similarly, for the invertible elements of order zero,

$$\widetilde{G}^0_{\mathrm{s}(2)}(Y;E)\longrightarrow G^0_{\mathrm{s}\tilde{\star}(2)}(Y;E)\xrightarrow{\epsilon=\varepsilon x}G^0_{\mathrm{s}\star(2)}(Y;E) \tag{6.10}$$

is exact, where

$$\widetilde{G}^0_{\mathrm{s}(2)}(Y;E)=\{\mathrm{Id}+Q\in G^0_{\mathrm{s}\tilde{\star}(2)}(Y;E);Q\in x\Psi^0_{\mathrm{s}(2)}(Y;E)\}.$$

7. Lifting the determinant from the boundary

As a special case of (6.10) the groups of order $-\infty$ perturbations of the identity are related in the same way:

$$\begin{aligned}\widetilde{G}^{-\infty}_{\mathrm{s}(2)}(Y;E) \longrightarrow G^{-\infty}_{\mathrm{s}\tilde{\star}(2)}(Y;E) = \\ \Big\{\mathrm{Id}+A_0+xE+\varepsilon xA_1,\ A_0,\ E,\ A_1\in\Psi^{-\infty}_{\mathrm{s}(2)}(Y;E);\mathrm{Id}+A_0\in G^0_{\mathrm{s}(2)}(Y;E)\Big\} \\ \longrightarrow G^{-\infty}_{\mathrm{s}\star(2)}(Y;E),\end{aligned} \tag{7.1}$$

where

$$\widetilde{G}^{-\infty}_{\mathrm{s}(2)}(Y;E) = \{\mathrm{Id}+Q\in G^{-\infty}_{\mathrm{s}(2)}(Y;E); Q\in x\Psi^{-\infty}_{\mathrm{s}(2)}(Y;E)\}.$$

The determinant defined on the quotient group lifts to a homomorphism on the larger group with essentially the same properties. In fact, it can be defined directly as

$$\det(b) = \exp\left(\int_0^1 \gamma^*\tilde{\alpha}_2\right),\ b\in G^{-\infty}_{\mathrm{s}\tilde{\star}(2)}(Y;E) \tag{7.2}$$

where $\tilde{\alpha}_2$ is the coefficient of εx in the expansion of $a^{-1}\tilde{\star}da$ and γ is a curve from the identity to b. Since the normal subgroup in (7.1) is affine, the larger group is contractible to the smaller. Certainly the pull-back of $\tilde{\alpha}_2$ to the subgroup is α_2, with ϵ replaced by εx, so Proposition 4.4 holds for the larger group as well. Indeed a minor extension of the computations in the proof of Proposition 4.3 shows that at $a=(a_0+xe_2+\varepsilon xa_1)\in G^{-\infty}_{\mathrm{s}\tilde{\star}(2)}(Y;E)$

$$\begin{aligned} a^{-1} = a_0^{-1} - x\left(a_0^{-1}ea_0^{-1}\right) + \varepsilon x\left(a_0^{-1}a_1a_0^{-1} + \frac{i}{2}\{a_0^{-1},a_0\}a_0^{-1}\right) \\ \Longrightarrow \tilde{\alpha}_2 = \alpha_2\end{aligned} \tag{7.3}$$

in terms of formula (4.7).

Since the group

$$G^{-\infty}_{\mathrm{cs}\star}(Z;E) = \big\{\mathrm{Id}+A,\ A\in\Psi^{-\infty}_{\mathrm{cs}\star}(Z;E);\ \exists\ (\mathrm{Id}+A)^{-1}\in\mathrm{Id}+\Psi^{-\infty}_{\mathrm{cs}\star}(Z;E)\big\} \tag{7.4}$$

is homotopic to its principal part, and hence is contractible, the lift of $d\log\det$ under $\tilde{I}$ must be exact; we compute an explicit formula for the lift of the determinant.

Theorem 7.1. *On $G^{-\infty}_{\mathrm{cs}\star}(Z;F)$,*

$$\det(\tilde{I}(A)) = e^{i\pi\eta_{\mathrm{cu}}(A)} \tag{7.5}$$

where

$$\eta_{\mathrm{cu}}(A) = \frac{1}{2\pi i}\int_{\mathbb{R}}\overline{\mathrm{Tr}}\left(A_0^{-1}\frac{\partial A_0}{\partial t} + \frac{\partial A_0}{\partial t}A_0^{-1}\right)dt + \mu(\tilde{I}(A)), \tag{7.6}$$

with μ defined in Proposition 4.3.

Proof. We proceed to compute $d\eta_{\mathrm{cu}}$,

$$d(\eta_{\mathrm{cu}}-\mu)(A)=\frac{1}{2\pi i}\int_{\mathbb{R}}\overline{\mathrm{Tr}}\big(-(A_0^{-1}dA_0)(A_0^{-1}\frac{\partial A_0}{\partial t})+A_0^{-1}\frac{\partial dA_0}{\partial t}\\ +\frac{\partial dA_0}{\partial t}A_0^{-1}-(\frac{\partial A_0}{\partial t}A_0^{-1})(dA_0A_0^{-1})\big)dt. \quad (7.7)$$

Integrating by parts in the second and third terms gives

$$d(\eta_{\mathrm{cu}}-\mu)(A)=\frac{1}{2\pi i}\int_{\mathbb{R}}\overline{\mathrm{Tr}}\left([A_0^{-1}\frac{\partial A_0}{\partial t},A_0^{-1}dA_0]-[\frac{\partial A_0}{\partial t}A_0^{-1},dA_0A_0^{-1}]\right)dt. \quad (7.8)$$

Using the trace-defect formula, this becomes

$$d(\eta_{\mathrm{cu}}-\mu)(A)=-\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\mathrm{Tr}\,\big(a_0^{-1}\frac{\partial a_0}{\partial t}\frac{\partial}{\partial\tau}\left(a_0^{-1}da_0\right)\\ -\frac{\partial a_0}{\partial t}a_0^{-1}\frac{\partial}{\partial\tau}\left(da_0a_0^{-1}\right)\big)dtd\tau, \quad (7.9)$$

where $a_0=I(A_0)$. Expanding out the derivative with respect to τ and simplifying

$$d(\eta_{\mathrm{cu}}-\mu)(A)=\frac{1}{4\pi^2}\int_{\mathbb{R}^2}\mathrm{Tr}\,\big((a_0^{-1}\frac{\partial a_0}{\partial t})(a_0^{-1}\frac{\partial a_0}{\partial\tau})a_0^{-1}da_0\\ -(a_0^{-1}\frac{\partial a_0}{\partial\tau})(a_0^{-1}\frac{\partial a_0}{\partial t})a_0^{-1}da_0\big)dtd\tau, \quad (7.10)$$

which shows that $i\pi d\eta_{\mathrm{cu}}(A)=d\log det(\tilde{I}(A))$. □

Now, consider the subgroup

$$G^{-\infty}_{\mathrm{cs}\star,\tilde{I}=\mathrm{Id}}(Z;E)\subset G^{-\infty}_{\mathrm{cs}\star}(Z;E) \quad (7.11)$$

consisting of elements of the form $\mathrm{Id}+Q$ with $Q\in\Psi^{-\infty}_{\mathrm{cs}}(Z;F)$ and $\tilde{I}(Q)=0$. In particular

$$\mathrm{Id}+Q\in G^{-\infty}_{\mathrm{cs}\star,\tilde{I}=\mathrm{Id}}(Z;E)\Longrightarrow I(Q_0)=0. \quad (7.12)$$

Proposition 7.2. *In the commutative diagramme*

$$\begin{tikzcd} G^{-\infty}_{\mathrm{cs}\star,\tilde{I}=\mathrm{Id}}(Z;F) \arrow[r] \arrow[d, "\mathrm{ind}"] & G^{-\infty}_{\mathrm{cs}\star}(Z;F) \arrow[r, "\tilde{I}"] \arrow[d, "\frac{1}{2}\eta_{\mathrm{cu}}"] & G^{-\infty}_{\mathrm{s}\tilde{\star}(2)}(\partial Z;F) \arrow[d, "\det"] \\ \mathbb{Z} \arrow[r] & \mathbb{C} \arrow[r, "\exp(2\pi i\cdot)"] & \mathbb{C}^* \end{tikzcd} \quad (7.13)$$

the top row is an even-odd classifying sequence for K-theory.

Proof. We already know the contractibility of the central group, and the end groups are contractible to their principal parts, which are classifying for even and odd K-theory respectively. For $A=\mathrm{Id}+Q\in G^{-\infty}_{\mathrm{cs}\star,\tilde{I}=\mathrm{Id}}(Z;E)$, $Q=Q_0+\epsilon Q_1$, and the condition $\tilde{I}(Q) = 0$ reduces to Q_0, $Q_1 \in x^2\Psi^{-\infty}_{\mathrm{cs}}(Z;F)$ so are all of trace class. Then

$$\frac{1}{2}\eta_{\mathrm{cu}}(A) = \frac{1}{2\pi i}\int_{\mathbb{R}} \mathrm{Tr}(A_0^{-1}\frac{\partial A_0}{\partial t})dt \tag{7.14}$$

which is the formula for the odd index (2.4). □

8. The extended η invariant

Next we show that the cusp η-invariant defined in (7.6) can be extended to a function on the elliptic invertible elements of $\Psi^m_{\mathrm{cs}\star}(Z;E)$. To do so the boundary-regularized trace $\overline{\mathrm{Tr}}(A)$, defined on operators of order $-\dim(Z)-1$, is replaced by a fully regularized trace functional on $\Psi^{\mathbb{Z}}_{\mathrm{cs}}(Z;E)$ following the same approach as in [7].

For $m \in \mathbb{Z}$ arbitrary and $A \in \Psi^m_{\mathrm{cs}}(Z;E)$,

$$\frac{d^pA(t)}{dt^p} \in \Psi^{m-p}_{\mathrm{cs}}(Z;E), \tag{8.1}$$

so the function

$$h_p(t) = \overline{\mathrm{Tr}}\left(\frac{d^pA(t)}{dt^p}\right) \in \mathcal{C}^\infty(\mathbb{R}), \tag{8.2}$$

is well-defined for $p > m + \dim(Z) + 1$. Since the regularization in the trace functional is in the normal variable to the boundary, $h_p(t)$ has, as in the boundaryless case, a complete asymptotic expansion as $t \longrightarrow \pm\infty$,

$$h_p(t) \sim \sum_{l\geq 0} h^{\pm}_{p,l}|t|^{m-p+\dim(Z)-l}. \tag{8.3}$$

So

$$g_p(t) = \int_{-t}^{t}\int_0^{t_p}\cdots\int_0^{t_1} h_p(r)dr dt_1\ldots dt_p \tag{8.4}$$

also has an asymptotic expansion as $t \to \infty$,

$$g_p(t) \sim \sum_{j\geq 0} g_{p,j}t^{m+1+\dim(Z)-j} + g'_p(t) + g''_p(t)\log t, \tag{8.5}$$

where $g'_p(t)$ and $g''_p(t)$ are polynomials of degree at most p. Increasing p to $p+1$ involves an additional derivative in (8.2) and an additional integral in (8.4). This changes the integrand of the final integral in (8.4) by a polynomial so $g_{p+1}(t) - g_p(t)$ is a polynomial without constant term. This justifies

Definition 8.1. The *doubly regularized trace* is the continuous linear map

$$\overline{\overline{\mathrm{Tr}}} : \Psi^{\mathbb{Z}}_{\mathrm{cs}}(Z;E) \longrightarrow \mathbb{C} \tag{8.6}$$

given by the coefficient of t^0 in the expansion (8.5).

When $m < -1 - \dim(Z)$, this reduces to the integral of the boundary-regularized trace

$$\overline{\overline{\mathrm{Tr}}}(A) = \int_{\mathbb{R}} \overline{\mathrm{Tr}}(A(t))dt. \tag{8.7}$$

In general, the doubly regularized trace does not vanish on commutators. However, it does vanish on commutators where one factor vanishes to high order at the boundary so the associated trace-defect can only involve boundary terms.

The trace-defect formula involves a similar regularization of the trace functional on the boundary for doubly suspended operators. So for a vector bundle over a compact manifold without boundary consider

$$\int_{\mathbb{R}^2} \mathrm{Tr}(b)dtd\tau,\ b \in \Psi^m_{\mathrm{s}(2)}(Y;E),\ m < -\dim(Y) - 2. \tag{8.8}$$

For general $b \in \Psi^{\mathbb{Z}}_{\mathrm{s}(2)}(Y;E)$ set

$$\tilde{h}_p(t) = \int_{\mathbb{R}} \mathrm{Tr}\left(\frac{\partial^p b(t,\tau)}{\partial t^p}\right) d\tau,\ p > m + \dim(Y) + 2. \tag{8.9}$$

As $t \to \pm\infty$, there is again a complete asymptotic expansion

$$\tilde{h}_p(t) \sim \sum_{l \geq 0} h^{\pm}_{p,l} |t|^{m+1+\dim(Y)-p-l} \tag{8.10}$$

so

$$\tilde{g}_p(t) = \int_{-t}^{t} \int_0^{t_p} \cdots \int_0^{t_1} \tilde{h}_p(r) dr dt_1 \dots dt_p \tag{8.11}$$

has an asymptotic expansion as $t \to +\infty$

$$\tilde{g}_p(t) \sim \sum_{j \geq 0} \tilde{g}_{p,j} t^{m+2+\dim(Y)-j} + \tilde{g}'_p(t) + \tilde{g}''_p(t) \log t, \tag{8.12}$$

where $\tilde{g}'_p(t)$ and $\tilde{g}''_p(t)$ are polynomials of degree at most p.

Proposition 8.2. *For $a \in \Psi^{\mathbb{Z}}_{\mathrm{s}(2)}(Y)$ the* regularized trace $\overline{\mathrm{Tr}}_{\mathrm{s}(2)}(a)$, *defined as the coefficient of t^0 in the expansion* (8.12) *is a well-defined trace functional, reducing to*

$$\overline{\mathrm{Tr}}_{\mathrm{s}(2)}(a) = \int_{\mathbb{R}^2} \mathrm{Tr}(a) dtd\tau,\ \textit{when } m < -\dim(Y) - 2 \tag{8.13}$$

and it satisfies

$$\overline{\mathrm{Tr}}_{\mathrm{s}(2)}(\frac{\partial a}{\partial \tau}) = 0. \tag{8.14}$$

Proof. That $\overline{\mathrm{Tr}}_{\mathrm{s}(2)}(a)$ is well-defined follows from the discussion above. That it vanishes on commutators follows from the same arguments as in [7]. Namely, the derivatives of a commutator, $\frac{d^p}{dt^p}[A,B]$, are themselves commutators and the sums of the orders of the operators decreases as p increases. Thus, for large p and for a commutator, the function $\tilde{h}_p(t)$ vanishes. The identity (8.14) follows similarly. □

Proposition 8.3 (Trace-defect formula). *For* $A,\ B \in \Psi^{\mathbb{Z}}_{\mathrm{cs}}(Z)$,

$$\overline{\overline{\mathrm{Tr}}}([A,B]) = \frac{1}{2\pi i}\,\overline{\mathrm{Tr}}_{\mathrm{s}(2)}\left(I(A,\tau)\frac{\partial I(B,\tau)}{\partial\tau}\right) = -\frac{1}{2\pi i}\,\overline{\mathrm{Tr}}_{\mathrm{s}(2)}\left(I(B,\tau)\frac{\partial I(A,\tau)}{\partial\tau}\right). \tag{8.15}$$

Proof. For $p \in \mathbb{N}$ large enough, we can apply the trace-defect formula (1.14) to get

$$\overline{\mathrm{Tr}}\left(\frac{\partial^p}{\partial t^p}[A,B]\right) = \frac{1}{2\pi i}\int_{\mathbb{R}} \mathrm{Tr}\left(\frac{\partial^p}{\partial t^p}\left(I(A,\tau)\frac{\partial}{\partial\tau}I(B,\tau)\right)\right)d\tau,$$

from which the result follows. □

Using the regularized trace functional, μ may be extended from $G^{-\infty}_{\mathrm{s}\tilde{\star}(2)}(Y;E,F)$ to $G^{m}_{\mathrm{s}\tilde{\star}(2)}(Y;E)$ by setting

$$\mu(a) = \frac{1}{2\pi^2 i}\,\overline{\mathrm{Tr}}_{\mathrm{s}(2)}(a_0^{-1}a_1),\ \ a = (a_0 + xe + \varepsilon x a_1) \in G^{m}_{\mathrm{s}\tilde{\star}(2)}(Y;E). \tag{8.16}$$

Proposition 8.4. *For* $A = A_0 + \varepsilon x A_1 \in G^m_{\mathrm{cs}\star}(Z;E,F)$, *the set of invertible elements of* $\Psi^m_{\mathrm{cs}\star}(Z;E,F)$,

$$\eta_{\mathrm{cu}}(A) := \frac{1}{2\pi i}\overline{\overline{\mathrm{Tr}}}\left(A_0^{-1}\frac{\partial A_0}{\partial t} + \frac{\partial A_0}{\partial t}A_0^{-1}\right) + \mu(\tilde{I}(A)), \tag{8.17}$$

is log-multiplicative under composition

$$\eta_{\mathrm{cu}}(A*B) = \eta_{\mathrm{cu}}(A) + \eta_{\mathrm{cu}}(B),\ \ \forall\ A \in G^m_{\mathrm{cs}\star}(Z;E,F),\ \ B \in G^{m'}_{\mathrm{cs}\star}(Z;F,G). \tag{8.18}$$

Proof. If $a = \tilde{I}(A,\tau)$ and $b = \tilde{I}(B,\tau)$ denote the associated boundary operators, a straightforward calculation shows that

$$\mu(a*b) = \mu(a) + \mu(b) - \frac{1}{4\pi^2}\,\overline{\mathrm{Tr}}_{\mathrm{s}(2)}(b_0^{-1}a_0^{-1}\{a_0,b_0\}). \tag{8.19}$$

On the other hand,

$$\begin{aligned}\overline{\overline{\mathrm{Tr}}}&\left((A_0B_0)^{-1}\frac{\partial(A_0B_0)}{\partial t} + \frac{\partial(A_0B_0)}{\partial t}(A_0B_0)^{-1}\right) = \\ &\overline{\overline{\mathrm{Tr}}}\left(A_0^{-1}\frac{\partial A_0}{\partial t} + \frac{\partial A_0}{\partial t}A_0^{-1}\right) + \overline{\overline{\mathrm{Tr}}}\left(B_0^{-1}\frac{\partial B_0}{\partial \tau} + \frac{\partial B_0}{\partial \tau}B_0^{-1}\right) + \alpha,\end{aligned} \tag{8.20}$$

where

$$\alpha = \overline{\overline{\mathrm{Tr}}}\left([B_0^{-1}A_0^{-1}\frac{\partial A_0}{\partial t}, B_0] + [A_0, \frac{\partial B_0}{\partial t}B_0^{-1}A_0^{-1}]\right). \tag{8.21}$$

Using the trace-defect formula (8.15),

$$\begin{aligned}\alpha =& \frac{1}{2\pi i}\,\overline{\mathrm{Tr}}_{\mathrm{s}(2)}\left(b_0^{-1}a_0^{-1}\frac{\partial a_0}{\partial t}\frac{\partial b_0}{\partial\tau} - \frac{\partial a_0}{\partial\tau}\left(\frac{\partial b_0}{\partial t}b_0^{-1}a_0^{-1}\right)\right) \\ =& -\frac{1}{2\pi i}\,\overline{\mathrm{Tr}}_{\mathrm{s}(2)}\left(b_0^{-1}a_0^{-1}\{a_0,b_0\}\right).\end{aligned} \tag{8.22}$$

Combining (8.19), (8.20) and (8.22) gives (8.18). □

9. Trivialization of the determinant bundle

In §4 the determinant bundle is defined for a family of elliptic, doubly-suspended, pseudodifferential operators on the fibres of a fibration of compact manifolds without boundary. When the family arises as the indicial family of a family of once-suspended elliptic cusp pseudodifferential operators on the fibres of fibration, (5.1), the determinant bundle is necessarily trivial, following the discussion in §5, as a bundle associated to a trivial bundle.

Theorem 9.1. *If $P \in \Psi^m_{\mathrm{cs}}(M/B; E, F)$ is an elliptic family of once-suspended cusp pseudodifferential operators and $\mathcal{P}$ is the bundle of invertible perturbations by elements of $\Psi^{-\infty}_{\mathrm{cs}\star}(M/B; E, F)$ then the τ invariant*

$$\tau = \exp(i\pi\eta_{\mathrm{cu}}) : \mathcal{P} \longrightarrow \mathbb{C}^* \tag{9.1}$$

descends to a non-vanishing linear function on the determinant bundle of the indicial family

$$\tau : \mathrm{Det}(I(P)) \longrightarrow \mathbb{C}. \tag{9.2}$$

Proof. As discussed in §5, the bundle $\mathcal{P}$ has non-empty fibres and is a principal $G^{-\infty}_{\mathrm{cs}\star}(M/B; E)$-bundle. The cusp eta invariant, defined by (8.17) is a well-defined function

$$\eta_{\mathrm{cu}} : \mathcal{P} \longrightarrow \mathbb{C}. \tag{9.3}$$

Moreover, under the action of the normal subgroup $G^{-\infty}_{\mathrm{cs}\star,\tilde{I}=\mathrm{Id}}(M/B; E)$ in (7.13) it follows that the exponential, τ, of η_{cu} in (9.1) is constant. Thus

$$\tau : \mathcal{P}' \longrightarrow \mathbb{C}^* \tag{9.4}$$

is well-defined where $\mathcal{P}' = \mathcal{P}/G^{-\infty}_{\mathrm{cs}\star,\tilde{I}=\mathrm{Id}}(M/B; E)$ is the quotient bundle with fibres which are principal spaces for the action of the quotient group $G^{-\infty}_{\mathrm{s}\tilde{\star}(2)}(\partial; E)$ in (2.9). In fact $\tilde{I}$ identifies the fibres of $\mathcal{P}'$ with the bundle $\mathcal{L}$ of invertible perturbations by $\Psi^{-\infty}_{\mathrm{s}\tilde{\star}(2)}(\partial Z; E, F)$ of the indicial family of the original family P, so

$$\tau : \mathcal{L} \longrightarrow \mathbb{C}^*. \tag{9.5}$$

Now, the additivity of the cusp η invariant in (8.19) and the identification in (7.13) of τ with the determinant on the structure group shows that τ transforms precisely as a linear function on $\mathrm{Det}(I(P))$:

$$\tau : \mathrm{Det}(I(P)) \longrightarrow \mathbb{C}.$$

□

10. Dirac families

In this section, we show that Theorem 9.1 can be interpreted as a generalization of a theorem of Dai and Freed in [5] for Dirac operators defined on odd dimensional Riemannian manifolds with boundary. This essentially amounts to two things. First, that the eta functional defined in(8.17) corresponds to the usual eta invariant in the Dirac case, which is established in Proposition 10.2 below. Since we are only defining this eta functional for invertible operators, $e^{i\pi\eta}$ really corresponds to the τ functional which trivializes the inverse determinant line bundle in [5]. Secondly, that in the Dirac case, the determinant bundle $\det(I(P))$ is isomorphic to the determinant line bundle of the associated family of boundary Dirac operators, which is the content of Proposition 10.3 below.

As a first step, let us recall the usual definition of the eta function on a manifold with boundary (see [11]). Let X be a Riemannian manifold with nonempty boundary $\partial X = Y$. Near the boundary, suppose that the Riemannian metric is of product type, so there is a neighborhood $Y \times [0,1) \subset X$ of the boundary in which the metric takes the form

$$g = du^2 + h_Y \tag{10.1}$$

where $u \in [0,1)$ is the coordinate normal to the boundary and h_Y is the pull-back of a metric on Y via the projection $Y \times [0,1) \longrightarrow Y$. Let S be a Hermitian vector bundle over X and let $D : \mathcal{C}^\infty(X,S) \longrightarrow \mathcal{C}^\infty(X,S)$ be a first order elliptic differential operator on X which is formally selfadjoint with respect to the inner product defined by the fibre metric of S and the metric on X. In the neighborhood $Y \times [0,1) \subset X$ of the boundary described above, assume that the operator D takes the form

$$D = \gamma \left(\frac{\partial}{\partial u} + A \right) \tag{10.2}$$

where $\gamma : S\big|_Y \longrightarrow S\big|_Y$ is a bundle isomorphism and $A : \mathcal{C}^\infty(Y, S\big|_Y) \longrightarrow \mathcal{C}^\infty(Y, S\big|_Y)$ is a first order elliptic operator on Y such that

$$\gamma^2 = -\operatorname{Id},\ \gamma^* = -\gamma,\ A\gamma = -\gamma A,\ A^* = A. \tag{10.3}$$

Here, A^* is the formal adjoint of A. Notice in particular that this includes the case of a compatible Dirac operator when S is a Clifford module and $\gamma = cl(du)$ is the Clifford multiplication by du. If $\ker A = \{0\}$, consider the spectral boundary condition

$$\varphi \in \mathcal{C}^\infty(X,S),\ \Pi_-(\varphi\big|_Y) = 0, \tag{10.4}$$

where Π_- is the projection onto the positive spectrum of A. In the case where $\ker A \neq \{0\}$, a unitary involution $\sigma : \ker A \longrightarrow \ker A$ should be chosen such that $\sigma\gamma = -\gamma\sigma$ (such an involution exists), and the boundary condition is then modified to

$$\varphi \in \mathcal{C}^\infty(X,S),\ (\Pi_- + P_-)(\varphi\big|_Y) = 0, \tag{10.5}$$

where P_- is the orthogonal projection onto $\ker(\sigma + \operatorname{Id})$. The associated operator D_σ is selfadjoint and has pure point spectrum. For this operator, the eta invariant is

$$\eta_X(D_\sigma) = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-\frac{1}{2}} \operatorname{Tr}(D_\sigma e^{-sD_\sigma^2}) ds. \tag{10.6}$$

To make a link with the cusp calculus, we need to enlarge X by attaching the half-cylinder $\mathbb{R}^+ \times Y$ to the boundary Y of X. The product metric near the boundary extends to this half-cylinder, which makes the resulting manifold a complete Riemannian manifold. Similarly, the operator D has a natural extension to M using its product structure near the boundary. Denote its L^2 extension (on M) by $\mathcal{D}$. The operator $\mathcal{D}$ is selfadjoint. The eta invariant of $\mathcal{D}$ is

$$\eta_M(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-\frac{1}{2}} \int_M tr(E(z,z,s)) dz ds, \tag{10.7}$$

where $E(z_1, z_2, s)$ is the kernel of $\mathcal{D}e^{-s\mathcal{D}^2}$. One of the main result of [11] is to establish a correspondence between the eta invariants (10.6) and (10.7).

Theorem 10.1 (Müller). *Let $D : \mathcal{C}^\infty(X,S) \longrightarrow \mathcal{C}^\infty(X,S)$ be a compatible Dirac operator which, on a neighborhood $Y \times [0,1)$ of Y in X, takes the form* (10.2)*, let $C(\lambda) : \ker A \longrightarrow \ker A$ be the associated scattering matrix (see* [11] *for a definition) in the range $|\lambda| < \mu_1$, where μ_1 is the smallest positive eigenvalue of A and put $\sigma = C(0)$, then*

$$\eta_X(D_\sigma) = \eta_M(\mathcal{D}).$$

Now, on M, it is possible to relate D to a cusp operator. Extending the variable u to the negative reals gives a neighborhood $Y \times (-\infty, 1) \subset M$ of ∂X in M. The variable

$$x = -\frac{1}{u} \tag{10.8}$$

takes value in $(0,1)$ and by extending it to $x = 0$, gives a manifold with boundary $\overline{M}$, with x as a boundary defining function so fixing a cusp structure. Denote by D_c the natural extension of D to $\overline{M}$. Near the boundary of $\overline{M}$,

$$D_c = \gamma \left(x^2 \frac{\partial}{\partial x} + A \right) \tag{10.9}$$

and so is clearly a cusp differential operator. If $S = S^+ \oplus S^-$ is the decomposition of S as a superspace, then

$$\hat{D}_{\mathrm{cs}}(t) = D_c + it \in \Psi^1_{\mathrm{cs}\star}(\bar{M}; S) \tag{10.10}$$

is a suspended cusp operator, where there are no x and εx terms. When D_c is invertible, $\hat{D}_{\mathrm{cs}}$ is invertible as well and $\eta_{\mathrm{cu}}(\hat{D}_{\mathrm{cs}})$ is well-defined.

Proposition 10.2. *Let $X, Y, M, \overline{M}$ be as above and let D be a compatible Dirac operator for some Clifford module S on X, which, on a neighborhood $Y \times [0,1)$ of Y takes the form* (10.2)*, suppose that D is invertible, and let D_c be its extension to $\overline{M}$, then*

$$\eta_X(D_\sigma) = \eta_{\mathrm{cu}}(\hat{D}_{\mathrm{cs}}),$$

where $\hat{D}_{\mathrm{cs}} = D_c + it \in \Psi^1_{\mathrm{cs}\star}(\overline{M}; S)$ and σ is trivial since A is invertible.

Proof. By the theorem of Müller, it suffices to show that $\eta_{\mathrm{cu}}(\hat{D}_{\mathrm{cs}}) = \eta_M(\mathcal{D})$. In order to do this, we closely follow the proof of Proposition 5 in [7], which is the same statement but in the case of a manifold without boundary.

Let $E(z_1, z_2, s)$ denote the kernel of $\mathcal{D}e^{-s\mathcal{D}^2}$, where $\mathcal{D}$ is the L^2 extension of $\eth$ on M. In [11], it is shown that $\operatorname{tr}(E(z,z,s))$ is absolutely integrable on M, so set

$$h(s) = \int_M \operatorname{tr}(E(z,z,s))dz \, , s \in [0,\infty). \tag{10.11}$$

Then, (see [11]) for $n = \dim(X)$ even, $h(s) \in \mathcal{C}^\infty([0,\infty))$, while for $n = \dim(X)$ odd, $h(s) \in s^{\frac{1}{2}}C^\infty([0,\infty))$. Moreover, since $\ker \mathcal{D} = \{0\}$, h is exponentially decreasing as $s \to +\infty$. As in [7], consider

$$g(v,t) = \int_v^\infty e^{-st^2} h(s)ds \, , v \geq 0. \tag{10.12}$$

This is a smooth function of $v^{\frac{1}{2}}$ in $v \geq 0$ and $t \in \mathbb{R}$, and as $|t| \to \infty$, it is rapidly decreasing if $v > 0$. From the fact that $h(s) \in C^\infty([0,\infty))$ for n even, $h(s) \in s^{\frac{1}{2}}C^\infty([0,\infty))$ for n odd, and the exponential decrease, we get

$$\left| \left(t\frac{\partial}{\partial t} \right)^p g(v,t) \right| \leq \frac{C_p}{1+t^2} \quad , v \geq 0, t \in \mathbb{R}. \tag{10.13}$$

So g is uniformly a symbol of order -2 in t as v approaches 0. In fact, when n is odd, it is uniformly a symbol of order -3. Now,using the identity

$$1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} s^{\frac{1}{2}} \exp(-st^2)dt \quad , s > 0, \tag{10.14}$$

$\eta_M(\mathcal{D})$ may be written as a double integral

$$\eta_M(\mathcal{D}) = \frac{1}{\sqrt{\pi}} \int_0^\infty s^{-\frac{1}{2}} h(s)ds = \frac{1}{\pi} \int_0^\infty \left(\int_{-\infty}^{+\infty} e^{-st^2} h(s)dt \right) ds. \tag{10.15}$$

The uniform estimate (10.13) allows the limit and integral to be exchanged so

$$\eta_M(\mathcal{D}) = \frac{1}{\pi} \lim_{v \to 0} \int_{-\infty}^{+\infty} g(v,t)dt. \tag{10.16}$$

For $p \in \mathbb{N}_0$

$$g_p(v,t) := \int_{-t}^{t} \int_0^{t_p} \cdots \int_0^{t_1} \frac{\partial^p}{\partial r^p} g(v,r)dr dt_1 \ldots dt_p, \tag{10.17}$$

has a uniform asymptotic expansion as $t \to \infty$ and $\eta_M(\mathcal{D})$ is just the limit as $v \to 0$ of the coefficient of t^0 in this expansion.

The kernel $E(z_1, z_2, s)$ can also be thought as the kernel of $D_c e^{-sD_c}$, a cusp operator of order $-\infty$ on $\overline{M}$. This can be checked directly from the explicit construction of $E(z_1, z_2, s)$ given in [11]. Therefore,

$$h(s) = \overline{\operatorname{Tr}}(D_c e^{-sD_c^2}), \tag{10.18}$$

where $\overline{\mathrm{Tr}}$ is the regularized trace defined in [9]. Note however that in this case, it is just the usual trace, that is, the integral of the kernel along the diagonal, since the residue trace vanishes. Consider now the (cusp product-suspended) operator

$$\hat{A}(t) = \int_0^\infty e^{-st^2} D_c e^{-sD_c^2} ds = \frac{D_c}{t^2 + D_c^2} . \tag{10.19}$$

Then, $\overline{\overline{\mathrm{Tr}}}(\hat{A})$ is the coefficient of t^0 in the asymptotic expansion as $t \to \infty$ of

$$\int_{-t}^{t} \int_0^{t_p} \cdots \int_0^{t_1} \overline{\mathrm{Tr}}(\frac{d^p}{dr^p} \hat{A}(r)) dr dt_1 \dots dt_p \tag{10.20}$$

for $p > n = dim(\overline{M})$ and

$$\begin{aligned} \overline{\mathrm{Tr}}\left(\frac{d^p}{dt^p}\hat{A}(t)\right) &= \overline{\mathrm{Tr}}\left(\frac{d^p}{dt^p}\int_0^\infty e^{-st^2} D_c e^{-sD_c^2} ds\right) \\ &= \frac{d^p}{dt^p}\int_0^\infty e^{-st^2}\overline{\mathrm{Tr}}(D_c e^{-sD_c^2}) ds \\ &= \frac{d^p}{dt^p} g(0,t), \end{aligned} \tag{10.21}$$

so $\eta_M(\mathcal{D}) = \frac{1}{\pi}\overline{\overline{\mathrm{Tr}}}(\hat{A})$. Instead of $\hat{A}$, consider

$$\hat{B}(t) = \int_0^\infty e^{-st^2}(D_c - it) e^{-sD_c^2} ds = \frac{1}{it + D_c} = (\hat{D}_{\mathrm{cs}})^{-1} . \tag{10.22}$$

Since $\hat{B}(t) - \hat{A}(t)$ is odd in t, $\overline{\overline{\mathrm{Tr}}}(\hat{B} - \hat{A}) = 0$, so finally

$$\begin{aligned} \eta_M(\mathcal{D}) &= \frac{1}{\pi}\overline{\overline{\mathrm{Tr}}}((\hat{D}_{\mathrm{cs}})^{-1}) = \frac{1}{2\pi i}\overline{\overline{\mathrm{Tr}}}\left(\frac{\partial}{\partial t}(\hat{D}_{\mathrm{cs}})(\hat{D}_{\mathrm{cs}})^{-1} + (\hat{D}_{\mathrm{cs}})^{-1}\frac{\partial}{\partial t}(\hat{D}_{\mathrm{cs}})\right) \\ &= \eta_{\mathrm{cu}}(\hat{D}_{\mathrm{cs}}). \end{aligned} \tag{10.23}$$

□

Let $\eth$ be some compatible Dirac operator as in Proposition 10.2. Then near the boundary of $\overline{M}$, its cusp version $\eth_c$ takes the form

$$\eth_c = \gamma(x^2 \frac{\partial}{\partial x} + A). \tag{10.24}$$

Here, it is tacitly assumed that near the boundary, S is identified with the pull-back of $S\big|_{\partial \overline{M}}$ via the projection $\partial \overline{M} \times [0,1) \longrightarrow \partial \overline{M}$. Since the map

$$T^*(\partial \overline{M}) \ni \xi \longmapsto cl(du)cl(\xi) \in \mathrm{Cl}(\overline{M}),\ \gamma = cl(du), \quad x = -\frac{1}{u}, \tag{10.25}$$

extends to an isomorphism of algebras

$$\mathrm{Cl}(\partial \overline{M}) \longrightarrow \mathrm{Cl}^+(\overline{M})\big|_{\partial \overline{M}}, \tag{10.26}$$

where $\mathrm{Cl}(\partial\overline{M})$ and $\mathrm{Cl}(\overline{M})$ are the Clifford algebras of $\partial\overline{M}$ and $\overline{M}$, this gives an action of $\mathrm{Cl}(\partial\overline{M})$ on $S^0 = S^+|_{\partial\overline{M}}$. If $\nu^+ : S^+_{\partial\overline{M}} \longrightarrow S^0$ denotes this identification, $S^-_{\partial\overline{M}}$ can be identified with S^0 via the map

$$\nu^- = \nu^+ \circ cl(du) : S^-\big|_{\partial\overline{M}} \longrightarrow S^0. \tag{10.27}$$

The combined identification $\nu : S|_{\partial\overline{M}} \longrightarrow S^0 \oplus S^0$ allows us to write $\eth_c$ and γ as

$$\eth_c = \begin{pmatrix} 0 & \eth_0 + x^2\frac{\partial}{\partial x} \\ \eth_0 - x^2\frac{\partial}{\partial x} & 0 \end{pmatrix} \; \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{10.28}$$

acting on $S^0 \oplus S^0$, where $\eth_0$ is the Dirac operator associated to the $\mathrm{Cl}(\partial\overline{M})$-module S^0. If instead we decompose the bundle $S^0 \oplus S^0$ in terms of the $\pm i$ eigenspaces $S^\pm$ of γ, then $\eth_c$ and γ take the form

$$\eth_c = \begin{pmatrix} ix^2\frac{\partial}{\partial x} & \eth_0^- \\ \eth_0^+ & -ix^2\frac{\partial}{\partial x} \end{pmatrix}, \; \gamma = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{10.29}$$

with $\eth_0^\pm = \pm i\eth_0$, so that $\eth_0^+$ and $\eth_0^-$ are the adjoint of each other. In this notation, the suspended operator $\hat{\eth}_{\mathrm{cs}}(t)$ can be written as

$$\hat{\eth}_{\mathrm{cs}}(t) = \begin{pmatrix} it + ix^2\frac{\partial}{\partial x} & \eth_0^- \\ \eth_0^+ & it - ix^2\frac{\partial}{\partial x} \end{pmatrix}. \tag{10.30}$$

Thus, its indicial operator $\hat{\eth}_{s(2)}(t,\tau)$ is

$$\hat{\eth}_{s(2)}(t,\tau) = e^{\frac{i\tau}{x}} \hat{D}_c(t) e^{-\frac{i\tau}{x}}\Big|_{x=0} = \begin{pmatrix} it - \tau & \eth_0^- \\ \eth_0^+ & it + \tau \end{pmatrix}. \tag{10.31}$$

Note that

$$\hat{\eth}^*_{s(2)}\hat{\eth}_{s(2)} = \begin{pmatrix} t^2 + \tau^2 + \eth_0^-\eth_0^+ & 0 \\ 0 & t^2 + \tau^2 + \eth_0^+\eth_0^- \end{pmatrix} \tag{10.32}$$

which is invertible everywhere except possibly at $t = \tau = 0$ so $\hat{\eth}_{s(2)}$ is invertible for every $t, \tau \in \mathbb{R}$ if and only if $\eth_0^+$ (and consequently $\eth_0^-$) is invertible.

Now, we wish to relate the determinant bundle associated to the family $\hat{\eth}_{s(2)}$ with the determinant bundle of the boundary Dirac family $\eth_0^+$ using the periodicity of the determinant line bundle discussed in [10]. For Dirac operators on a closed manifold, this can be formulated as follows.

Proposition 10.3. *If $\eth_0^+ \in \mathrm{Diff}^1(N/B; S)$ is a family of Dirac type operators on a fibration of closed manifold $N \longrightarrow B$ with vanishing numerical index then the determinant bundle of $\eth_0^+$ is naturally isomorphic to the determinant bundle of the associated family of twice suspended operators*

$$\eth_{s(2)}(t,\tau) = \begin{pmatrix} it - \tau & \eth_0^- \\ \eth_0^+ & it + \tau \end{pmatrix} \in \Psi^1_{\mathrm{s}(2)}(N/B; S \oplus S)$$

where $\eth_0^- = (\eth_0^+)^*$.

In [10], this periodicity is formulated in terms of product-suspended operators instead of suspended operators, since in general, given $P \in \Psi^1(Y;S)$, the family of operators

$$P_{s(2)}(t,\tau) = \begin{pmatrix} it-\tau & P^* \\ P & it+\tau \end{pmatrix}$$

is a twice product-suspended operator but not a suspended operator unless P is a differential operator. In this latter case, which includes Dirac operators, the periodicity of the determinant line bundle can be formulated using only suspended operators.

11. Generalization to product-suspended operators

As written, Theorem 9.1 applies to elliptic families of once-suspended cusp pseudodifferential operators. As we discussed in §10, this includes the result of Dai and Freed [5] for a family of self-adjoint Dirac operators D on a manifold with boundary by passing to the associated family of elliptic cusp operators and then to the elliptic family of once-suspended cusp operators $D + it$, the suspension parameter being t.

More generally, one can consider the case of an arbitrary elliptic family of first order self-adjoint cusp pseudodifferential operators P. Then $P+it$ is not in general a once-suspended family of cusp operators. Instead we pass to the larger algebra, and related modules $\Psi^{k,l}_{\mathrm{cps}}(M/B;E,F)$, of product-suspended cusp pseudodifferential operators since then

$$P + it \in \Psi^{1,1}_{\mathrm{cps}}(M/B;E). \tag{11.1}$$

The ellipticity of P again implies the corresponding full ellipticity of $P + it$.

Enough of the properties of suspended (cusp) operators extend to the product-suspended case to allow the various definitions of regularized traces and the eta invariant to carry over to the more general case. In this context, the proof of Proposition 8.4 still applies, so the eta invariant is also multiplicative under composition of invertible fully elliptic cusp product-suspended operators.

So, given an elliptic family of cusp product-suspended operators (see Definition .2 in the Appendix), consider the bundle $\mathcal{P}$ of invertible perturbations by elements in

$$\Psi^{-\infty,-\infty}_{\mathrm{cps}}(M/B;E,F) = \Psi^{-\infty}_{\mathrm{cs}}(M/B;E,F).$$

Again the fibres are non-empty since the full ellipticity of P_b implies that it is invertible for large values of the suspension parameter t. Then the same argument as in the proof of Theorem 7.1 of [9] applies, using the contractibility of $G^{-\infty}_{\mathrm{cu}}(M_b;E)$, to show the existence of an invertible perturbation $P_b + Q_b$. At the same time, this shows the existence of an invertible perturbation of the indicial family $I(P_b) \in \Psi^{k,l}_{\mathrm{ps}(2)}(M_b;E_b,F_b)$, and so the associated index bundle and determinant bundle of the indicial family are also well-defined, in the latter case using the $*$-product as before.

Consequently, we can formulate the following generalization of Theorem 9.1 with the proof essentially unchanged.

Theorem 11.1. *If $P \in \Psi^{k,l}_{\mathrm{cps}}(M/B; E, F)$ is a fully elliptic family of cusp product-suspended pseudodifferential operators and $\mathcal{P}$ is the bundle of invertible perturbations by elements of $\Psi^{-\infty}_{\mathrm{cs}\star}(M/B : E, F)$, then the τ invariant*

$$\tau = \exp(i\pi\eta_{\mathrm{cu}}) : \mathcal{P} \longrightarrow \mathbb{C}^*$$

descends to a non-vanishing linear function on the determinant line bundle of the indicial family $\tau : \mathrm{Det}(I(P)) \longrightarrow \mathbb{C}$.

As a special case, Theorem 11.1 includes elliptic families of self-adjoint first order cusp pseudodifferential operators $P \in \Psi^1_{\mathrm{cu}}(M/B; E, F)$ by considering the cusp product suspended family

$$P + it \in \Psi^{1,1}_{\mathrm{cps}}(M/B; E, F).$$

Appendix. Product-Suspended Operators

In this appendix, we will briefly review the main properties of product-suspended pseudodifferential operators and then discuss the steps needed to extend this notion to the case of the cusp algebra of pseudodifferential operators on a compact manifold with boundary as used in §11. For a more detailed discussion on product-suspended operators see [10].

For the case of a compact manifold without boundary, product-suspended operators are, formally, generalizations of the suspended operators by relaxing the conditions on the the full symbols. This is achieved by replacing the radial compactification $\overline{\mathbb{R}^p \times T^*X}$ by the following blown-up version of it

$$^X\overline{\mathbb{R}^p \times T^*X} = [\overline{\mathbb{R}^p \times T^*X}; \partial(\overline{\mathbb{R}^p} \times X)] \tag{A.2}$$

where X is understood as the zero section of T^*X. If ρ_r and ρ_s denote boundary defining functions for the 'old' boundary and the 'new' boundary (arising from the blow-up) then set

$$S^{z,z'}(^X\overline{\mathbb{R}^p \times T^*X}; \hom(E, F)) = \rho_r^{-z}\rho_s^{-z'}\mathcal{C}^\infty(^X\overline{\mathbb{R}^p \times T^*X}; \hom(E, F)). \tag{A.3}$$

This is the space of 'full symbols' of product-suspended pseudodifferential operators (with possibly complex multiorders). After choosing appropriate metrics and connections, Weyl quantization gives families of operators on X which we interpret as elements of $\Psi^{z,z'}_{\mathrm{ps}(p)}(X; E, F)$, the space of product p-suspended operators of order (z, z'). However this map is not surjective modulo rapidly decaying smoothing operators as is the case for ordinary (suspended) pseudodifferential operators. Rather we need to allow as a subspace

$$\Psi^{-\infty,z'}_{\mathrm{ps}(p)}(X; E, F) = \rho_s^{-z'}\mathcal{C}^\infty(\overline{\mathbb{R}^p} \times X^2; \mathrm{Hom}(E, F), \Omega_R)) \subset \Psi^{-\infty,z'}_{\mathrm{ps}(p)}(X; E, F) \tag{A.4}$$

considered as smoothing operators on X with parameters in $\mathbb{R}^p$. The image of the symbol space (A.3) under Weyl quantization is, modulo such terms, independent of the choices made in its definition. The product-suspended operators form a bigraded algebra.

Property 1. For all k, k', l and l' and bundles E, F and G,

$$\Psi^{k,k'}_{\mathrm{ps}(p)}(X;F,G)\circ\Psi^{l,l'}_{\mathrm{ps}(p)}(X;E,F)\subset\Psi^{k+l,k'+l'}_{\mathrm{ps}(p)}(X;E,F).$$

There are two symbol maps. The usual symbol coming from the leading part of the full symbol in (A.3) at the 'old' boundary (B_σ) and the 'base family' which involves both the leading term of this symbol at the 'new' boundary and the leading term of the smoothing part in (A.4). These symbols are related by a compatibility condition just corresponding to the leading part of the full symbol at the corner.

Property 2. The two symbols give short exact sequences

$$\Psi^{k-1,k'}_{\mathrm{ps}(p)}(X;E,F)\longrightarrow\Psi^{k,k'}_{\mathrm{ps}(p)}(X;E,F)\xrightarrow{\sigma}\rho^{-k'}\mathcal{C}^\infty(B_\sigma;\hom(E,F)),\tag{A.5}$$

and

$$\Psi^{k,k'-1}_{\mathrm{ps}(p)}(X;E,F)\longrightarrow\Psi^{k,k'}_{\mathrm{ps}(p)}(X;E,F)\xrightarrow{\beta}\Psi^k((X\times\mathbb{S}^{p-1})/\mathbb{S}^{p-1};E,F)\tag{A.6}$$

and the joint range is limited only by the condition

$$\sigma(\beta)=\sigma|_{\partial B_\sigma}.\tag{A.7}$$

Ellipticity of A is the condition of invertibility of $\sigma(A)$ and full ellipticity is in addition the invertibility of $\beta(A)$.

Property 3. If a fully elliptic product-suspended operator $Q\in\Psi^{k,k'}_{\mathrm{ps}(p)}(X;E,F)$ is invertible (i.e. is bijective from $\mathcal{C}^\infty(X;E)$ to $\mathcal{C}^\infty(X;F)$) then its inverse is an element of $\Psi^{-k,-k'}_{\mathrm{ps}(p)}(X;F,E)$.

In §11, we also make use of the following important properties.

Property 4. For all $k\in\mathbb{Z}$,

$$\Psi^k_{\mathrm{s}(p)}(X;E,F)\subset\Psi^{k,k}_{\mathrm{ps}(p)}(X;E,F),$$
$$\Psi^{-\infty}_{\mathrm{s}(p)}(X;E,F)=\Psi^{-\infty,-\infty}_{\mathrm{ps}(p)}(X;E,F).$$

Property 5. If $P\in\Psi^1(X;E,F)$ is any first order pseudodifferential operator, then

$$P+it\in\Psi^{1,1}_{\mathrm{ps}(1)}(X;E,F)$$

where t is the suspension parameter.

Property 6. Given $Q\in\Psi^{k,l}_{\mathrm{ps}(p)}(X;E,F)$,

$$\left(\frac{\partial Q}{\partial t_i}\right)\in\Psi^{k-1,l-1}_{\mathrm{ps}(p)}(X;E,F),$$

where $t=(t_1,\dots,t_p)$ is the suspension parameter and $i\in\{1,\cdots,p\}$.

Next we extend this to a construction of cusp product-suspended pseudodifferential operators on a compact manifold with boundary Z. Again, for suspended cusp operators, there is a Weyl quantization map from the appropriate space of classical symbols $\rho^{-z}\mathcal{C}^\infty(\overline{\mathbb{R}^p \times {}^{\mathrm{cu}}T^*Z}; \hom(E,F))$ which is surjective modulo cusp operators of order $-\infty$ decaying rapidly in the parameters. To capture the product-suspended case consider the spaces of symbols analogous to (A.3)

$$\rho_r^{-z}\rho_s^{-z'}\mathcal{C}^\infty([\overline{\mathbb{R}^p \times {}^{\mathrm{cu}}T^*Z}; \partial(\overline{\mathbb{R}^p \times Z})]; \hom(E,F))$$

with the corresponding 'old' and 'new' boundaries. Then an element of the space

$$A \in \Psi^{z,z'}_{\mathrm{cps}(p)}(Z; E, F)$$

is the sum of the Weyl quantization (for the cusp algebra) of an element of (11) plus an element of the residual space

$$\Psi^{-\infty,z'}_{\mathrm{cps}(p)}(Z; E, F) = \rho^{-z'}\mathcal{C}^\infty(\overline{\mathbb{R}^p}; \Psi^{-\infty}_{\mathrm{cu}}(Z; E, F)). \tag{A.8}$$

Now, with this definition the properties above carry over to the boundary setting. Property 1 is essentially unchanged. The same two homomorphisms are defined, the symbol and base family, with the latter taking values in families of cusp operators. In addition the indicial family for cusp operators leads to a third homomorphism giving a short exact sequence

$$x\Psi^{m,m'}_{\mathrm{cps}(p)}(Z; E, F) \longrightarrow \Psi^{m,m'}_{\mathrm{cps}(p)}(Z; E, F) \xrightarrow{I_{\mathrm{cu}}} \Psi^{k,l}_{\mathrm{ps}(p),\mathrm{sus}}(\partial Z; E, F)$$

where the image space has the same p product-suspended variables but taking values in the suspend operators on ∂Z. Since the suspended algebra may be realized in terms of ordinary pseudodifferential operators on $\mathbb{R} \times \partial Z$ there is no difficulty in considering these 'mixed-suspended' operators.

Definition .2. A cusp product-suspended operator $A \in \Psi^{m,m'}_{\mathrm{cps}(p)}(Z; E, F)$ is said to be *elliptic* if both its symbol $\sigma(A)$ and its base family $\beta(A)$ are invertible; it is *fully elliptic* if its symbol $\sigma(A)$, its base family $\beta(A)$ and its indicial family are all invertible.

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