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ELLIPTIC OPERATORS OF TOTALLY CHARACTERISTIC TYPE

BY

RICHARD MELROSE AND GERARDO MENDOZA

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by

Richard Melrose

Massachusetts Institute of Technology

and

Mathematical Sciences Research Institute

Gerardo Mendoza

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and

Mathematical Sciences Research Institute

1. Introduction

This paper contains proofs of most of the results announced in [9], together with some further developments. The basic analytic question addressed here concerns the mapping properties, on the natural Sobolov spaces, of elliptic totally characteristic pseudodifferential operators, on a compact C^∞ manifold with boundary. The ring, $\Psi_b^\infty(X)$, of such operators was described in detail in [8] and includes as a subring $\mathrm{Diff}_b^m(X) \subset \Psi_b^m(X)$, the ring of all totally characteristic differential operators, those given locally as the sums of products of vector fields tangent to the boundary. These differential operators are sometimes called Fuchsian differential operators by extension from the special case of ordinary differential operators with regular singular points. A representative example on the manifold with boundary $[0,\infty) \times Y$, Y compact Riemannian, is the element of $\mathrm{Diff}_b^2([0,\infty) \times Y)$

$$A = (xD_x)^2 + \Delta_Y$$

and the operators considered here are higher order (and pseudodifferential) analogues of this, near $x=0$, in that they are elliptic transversally and in the compressed vector field xD_x.

The basic results obtained are Theorems 6.4 and 6.17 which show that an elliptic element $A \in \Psi_b^m(X)$, acting on the weighted Sobolev spaces described in Section 2,

$$A\colon x^a H_b^m(X) \longrightarrow x^a L_b^2(X)$$

is Fredholm for all but a discrete set of values of $a \in \mathbb{R}$. The kernel consists of a finite dimensional space of classical conormal distributions on X, that is C^∞ functions in the interior with asymptotic expansions in terms of the distance to the boundary, and the index can be computed once it is known for one value of a , in terms of the spectral properties of an associated family of elliptic operators on the boundary. We leave open the important problem of determining an index formula for operators of this type. In Theorem 7.12 this analysis is applied to investigate the spectral properties of a self-adjoint elliptic operator in $\Psi_b^m(X)$.

These results can be considered as generalizations of well-known properties of ordinary differential operators. One motivation of this analysis is the fact that it can be applied to the Laplace-Beltrami operator on a Riemannian manifold with conic singularities. Such a connection was made in [9] and further developments, including the relation to the work of Cheeger [1], [2] will be given elsewhere. The work of Kondratév [4] on analysis on spaces with conic points is therefore also closely related to the present investigation. However, the main intention here is to give a systematic format for the treatment of such singular problems. In this regard the elliptic totally characteristic case, treated below, is the simplest of many classes of degenerate elliptic 'boundary problems' which appear on the desingularization of spaces with corners, or creases, of various codimensions. No boundary conditions as such need to be imposed on the solutions to an elliptic totally characteristic operator in order to get Fredholm properties because these would correspond geometrically to conditions on a set of dimension zero, and as shown below the kernel is finite dimensional on any space of distributions of finite regularity at the boundary. In other cases, examined in [10], this is no longer true and the theory is more complicated.

The work of R. McOwen [5], [6], [7] is even more closely related to the results below. Indeed, the introduction of polar coordinates and inversion in the radial variable reduces discussion of the Laplacian on $\mathbb{R}^n$ to that of an elliptic element of $\mathrm{Diff}_b^2(X)$, X the unit ball. McOwen admits perturbations of the Laplacian which are more general than would be permitted here, and obtains results directly comparable with Theorem 6.17 though not the structure theorem 6.4; however the methods and results below are of more general applicability. In [9] it was noted that boundary problems can be treated by the method below and McOwen has also treated certain boundary problems for infinite domains in Euclidean space. The full treatment of boundary problems outlined in [9] will appear elsewhere.

2. Totally characteristic Operators

We shall recall from [8] the main properties of the totally characteristic operators on a compact manifold with boundary. Let $C_c^{\infty}(\mathring{X})$ denote the space of C^{∞} test functions on the interior of X and $C^{-\infty}(\mathring{X})$ the corresponding space of distributions, the dual of the space $C_c^{\infty}(\mathring{X},\Omega X)$ of test densities. By Schwartz's kernel theorem any continuous linear operator

$$(2.1) \qquad A: C_c^{\infty}(\mathring{X}) \longrightarrow C^{-\infty}(\mathring{X})$$

can be represented uniquely in terms of its kernel

$$\kappa_A \in C^{-\infty}(\mathring{X}\times\mathring{X};\ \pi_2^{*}\Omega X) = C^{-\infty}(\mathring{X}\times\mathring{X}) \otimes C^{\infty}(\mathring{X}\times\mathring{X};\ \pi_2^{*}\Omega X),$$

an element of the space of distributions on the product transforming as a density in the second factor, through

$$(2.2) \qquad \langle A\phi,\Psi\rangle = \kappa_A(\Psi \otimes \phi) \qquad \forall\ \phi,\Psi \in C_c^{\infty}(\mathring{X}).$$

The product $X \times X$ is a manifold with corner and in [8] an associated manifold with corner, the stretched product of X with itself, $X \hat{\times} X$, was introduced with a natural surjective C^{∞} mapping

$$(2.3) \qquad \hat{\pi}: X \hat{\times} X \longrightarrow X \times X .$$

This map identifies the interior of the two manifolds and identifies part, $\partial_1'(X \hat{\times} X)$, of the codimension one boundary of $X \hat{\times} X$ with the codimension one boundary

$$\partial_1(X \times X) = (\partial X \times \mathring{X}) \cup (\mathring{X} \times \partial X)$$

of $X \times X$. The remaining part of $X \hat{\times} X$ is

$$\partial_1''(X \hat{\times} X) = \hat{\pi}^{-1}(\partial_2(X \times X)) = \partial X \times \partial X \times [-1,1] .$$

In $X \hat{\times} X$ the diagonal $\hat{\Delta}$ is the closure of $\hat{\pi}^{-1}(\Delta \cap (\mathring{X} \times \mathring{X}))$ and is a proper subset of $\hat{\pi}^{-1}(\Delta)$. Moreover, $\hat{\Delta}$ meets the total boundary $\partial(X \hat{\times} X)$ only in the part $\partial_1''(X \hat{\times} X)$ and meets that hypersurface transversally. If X is doubled across its boundary to give the C^∞ manifold without boundary

$$X_2 = X \cup (-X)$$

then the naturality of the construction of $X\hat{\times}X$ means that $\partial_1''((-X) \hat{\times} (-X))$ can be identified with $\partial_1''(X \hat{\times} X)$ so that

$$Y = (X \hat{\times} X) \cup ((-X) \hat{\times} (-X))$$

becomes a manifold with boundary, but without corners. The doubled diagonal $\hat{\Delta}_Y = \hat{\Delta} \cup (-\hat{\Delta}) \subset Y$ is then a compact embedded submanifold not meeting the boundary ∂Y. Thus, the space of Lagrangian or conormal distributions associated to $\hat{\Delta}_Y$,

$$I^m(Y,\hat{\Delta}_Y) \subset C^{-\infty}(Y),$$

as a subspace of the space of extendible distributions on Y, can be defined by requiring that $u \in C^{-\infty}(Y)$ satisfy (2.4) and (2.5):

In local coordinates $z_1,\ldots,z_n,t_1,\ldots,t_n$ based at $p \in \hat{\Delta}_Y$

(2.4) in which $\hat{\Delta}_Y = \{z_1 = \cdots = z_n = 0\}$ there is a symbol $a \in S^m(\mathbb{R}^n_t , \mathbb{R}^n_\zeta)$ such that $u(z,t) = (2\pi)^{-n} \int e^{iz\cdot\zeta} a(t,\zeta)\, d\zeta$ in $|t|, |z| < \epsilon, \epsilon > 0$.

(2.5) If $\rho \in C^\infty(Y)$ has $\rho = 0$ near $\hat{\Delta}_Y$, $\rho = 1$ near ∂Y then $\rho u \in \dot{C}^\infty(Y)$.

Here, (2.4) is just the coordinate version of the ususal definition of the distributions which form the Schwartz kernels of pseudodifferential operators, whereas (2.5) requires that u be smooth up to the boundary ∂Y and vanish there wih all its derivatives. If G is any C^∞ vector bundle on Y define

$$I^m(Y,\hat{\Delta}_Y; G) = I^m(Y,\hat{\Delta}_Y) \otimes C^\infty(Y;G) \text{ relative to } C^\infty(Y).$$

Consider the trivial bundle over Y , B , defined with respect to the submanifold $S = (X \hat{x} X) \cap ((-X) \hat{x} (-X))$ by taking the fibre at $y \in Y$ as

$$(2.6) \quad \begin{cases} B_y = \{f \in C^\infty(Y/S) \; ; \; tf \in C^\infty(Y) \text{ if } t \in C^\infty(Y), \; t=0 \text{ on } S\}/\underset{y}{\sim} \\ f \underset{y}{\sim} 0 \iff (tf)(y) = 0 \, . \end{cases}$$

The basic space of kernels is

$$(2.7) \quad K^m_b(X) = I^m(Y, \hat{\Delta}_Y ; \hat{\pi}^*\pi_2^*(\Omega X) \otimes B)\big|_{\mathring{X}\times\mathring{X}} \subset C^{-\infty}(\mathring{X}\times\mathring{X} ; \pi_2^*\Omega X)$$

where in the identification over $\mathring{X}\times\mathring{X}$ the isomorphism $\hat{\imath}$ and the canonical trivialization of B have been used. Then, by definition

$$(2.8) \qquad A \in \Psi_b^m(X) \iff \kappa_A \in K_b^m(X) ,$$

that is, an operator (2.1) is a totally characteristic pseudodifferential operator of order m if it is a pseudodifferential operator in the interior, of order m, and in addition its Schwartz kernel has certain specific extension properties at the boundary of $X \times X$, as contained in the definition of $K_b^m(X)$.

More generally if G, H are C^∞ vector bundles over X then the corresponding space of totally characteristic pseudodifferential operators from sections of G to sections of H is defined by tensoring the kernels with the homomorphism bundle:

$$(2.9) \qquad A \in \Psi_b^m(X;\, G,H) \iff K_A \in K_b^m(X) \otimes C^\infty(X;\, G,H) \ \text{ rel. } \ C^\infty(X).$$

As a space of Lagrangian distributions the space of kernels has a symbol mapping (see [3]). Using canonical trivializations over the diagonal this gives:

$$(2.10) \qquad \sigma_m\colon \Psi_b^m(X) \longrightarrow S^m(\tilde{T}^*X)/S^{m-1}(\tilde{T}^*X)$$

where $\tilde{T}^*X$ is the compressed cotangent bundle of X. This vector bundle is naturally isomorphic to T^*X over $\mathring{X}$ and is defined as the dual of the compressed tangent bundle, $\tilde{T}X$,

$$\tilde{T}^*X = (\tilde{T}X)^* .$$

The compressed tangent bundle is the natural vector bundle over X such that if $V(\partial X) \subset C^\infty(X,TX)$ is the $C^\infty(X)$-module of vector fields on X tangent to

∂X then

$$C^\infty(X,\tilde{T}X) = \mathcal{V}(\partial X) \ . \tag{2.11}$$

The symbol mapping gives an isomorphism

$$\sigma_m\colon \Psi_b^m(X)/\Psi_b^{m-1}(X) \longrightarrow S^m(\tilde{T}^*X)/S^{m-1}(\tilde{T}^*X) \ . \tag{2.12}$$

The basic mapping properties of totally characteristic operators are:

$$\begin{aligned} &A\colon \dot{C}^\infty(X) \longrightarrow \dot{C}^\infty(X), \quad A\colon C^\infty(X) \longrightarrow C^\infty(X) \\ &A\colon C^{-\infty}(X) \longrightarrow C^{-\infty}(X), \quad A\colon \dot{C}^{-\infty}(X) \longrightarrow \dot{C}^{-\infty}(X) \end{aligned} \qquad \forall\, A \in \Psi_b^m(X),\ \forall\, m, \tag{2.13}$$

where the second pair follows from the first by duality using the fact that if adjoints are taken with respect to a smooth non-vanishing density $\nu \in C^\infty(X,\Omega_b X)$, $\nu > 0$, $\Omega_b X = \Omega X \otimes B$, then

$$A^* \in \Psi_b^m(X) \quad \text{and} \quad \sigma_m(A^*) = \overline{\sigma_m}(A) \ . \tag{2.14}$$

In view of (2.13) composition is well-defined and

$$\Psi_b^m(X)\cdot\Psi_b^{m'}(X) \subset \Psi_b^{m+m'}(X) \quad \text{and} \quad \sigma_{m+m'}(A\cdot B) = \sigma_m(A)\cdot\sigma_{m'}(B). \tag{2.15}$$

These symbolic properties and the usual techniques of asymptotic summation allow the approximate inversion of the elliptic elements of $\Psi_b^m(X)$. An operator $A \in \Psi_b^m(X)$ is elliptic if its symbol $\sigma_m(A)$ has a representative $a \in S^m(\tilde{T}^*X)$ which is elliptic in the sense that there exists $b \in S^{-m}(\tilde{T}^*X)$ with $ab - 1 \in S^{-\infty}$. Define

$$\Psi_b^{-\infty}(X) = \bigcap_m \Psi_b^m(X).$$

(2.16) If $A \in \Psi_b^m(X)$ is elliptic then there exists $B \in \Psi_b^{-m}(X)$ such that $A \cdot B = \mathrm{Id} + R, \quad B \cdot A = \mathrm{Id} + R'$ with $R, R' \in \Psi_b^{-\infty}(X)$.

From the definition (2.8) the residual algebra $\Psi_b^{-\infty}(X)$ is easily characterized as consisting of those operators (2.1) with Schwartz kernels which lift from $\mathring{X} \times \mathring{X}$ into $X \hat{\times} X \subset Y$ to be smooth sections of the bundle $\hat{\pi}^* \pi_2^* \Omega X \otimes B$ vanishing with all derivatives at ∂Y but not necessarily at S. From this it follows that

$$\text{(2.17)} \qquad R: \dot{C}^{-\infty}(X) \longrightarrow \dot{A}(X), \quad R: C^{-\infty}(X) \longrightarrow A(X) \qquad \forall\, R \in \Psi_b^{-\infty}(X),$$

where the range spaces are the conormal distributions associated to the boundary of X, for example,

$$\text{(2.18)} \qquad A(X) = \{u \in C^{-\infty}(X)\ ;\ \exists\, s \text{ for which } V_1 V_2 \ldots V_N u \in H^s(X)$$
$$\forall\, V_1, \ldots, V_N \in \mathcal{V}(\partial X)\},$$

is the space of distributions with stable regularity, in say the Sobolev sense, under the arbitrary action of vector fields tangent to ∂X. For boundedness properties it is important to consider weighted Sobolev spaces in place of the standard ones $H^s(X)$.

Let $\nu_b \in C^\infty(X, \Omega_b X)$ be a strictly positive section, i.e. a density, such that if $r \in C^\infty(X)$ is a defining function for ∂X then $r\nu_b \in C^\infty(X, \Omega X)$ is positive everywhere on X. Thus, if B is defined as in (2.6) on X relative to ∂X then ν_b is a non-vanishing smooth section of $\Omega X \otimes B$ over X. Set

$$(2.19) \qquad L^2_b(X) = \{u \in C^{-\infty}(X) \; ; \; u \in L^1_{loc}(\mathring{X}), \int |u|^2 \nu_b < \infty \},$$

which is actually independent of the choice of ν_b with these properties. If m is a non-negative integer, define

$$(2.20) \quad H^m_b(X) = \{u \in C^{-\infty}(X); V_1 \ldots V_p u \in L^2_b(X) \text{ if } V_1, \ldots, V_p \in V(\partial X),\ p \leq m\}.$$

For a negative integer we define

$$(2.21) \quad H^m_b(X) = \{u \in C^{-\infty}(X) \; ; \; \dot{C}^\infty(X) \ni \phi \mapsto u(\phi \nu_b) \text{ extends by continuity to } H^{-m}_b(X)\}$$

where we note the fact that

$$(2.22) \qquad \dot{C}^\infty(X) \hookrightarrow H^m_b(X) \quad \text{is dense,}$$

for m a non-negative integer, and in fact for any integer. Now,

$$(2.23) \qquad A: H^{m'}_b(X) \longrightarrow H^{m'-m''}_b(X) \quad \text{if} \quad A \in \Psi^m_b(X), \text{ with } m \leq m'' .$$

It is then consistent to define

$$(2.24) \qquad H^m_b(X) = \{u \in C^{-\infty}(X) \; ; \; Au \in L^2_b(X) \quad \forall\, A \in \Psi^m_b(X)\} \quad m \in \mathbb{R}.$$

Using the properties of $\Psi^m_b(X)$ it is easily shown that, for $m > 0$, this is a Hilbert space with respect to the norm $(\|Au\|^2 + \|u\|^2)^{1/2}$, $\|\cdot\|$ a Hilbert norm on $L^2_b(X)$ and the duality, density and continuity results (2.21), (2.22) and (2.23) then hold without integrality conditions on the orders.

Another important property of the totally characteristic operators involves boundary values. Let

$$(2.25)\qquad (\ )_\partial : C^\infty(X) \longrightarrow C^\infty(\partial X)$$

be the restriction to the boundary. Then,

$$(2.26)\qquad (Au)_\partial = A_b(u_\partial), \quad u \in C^\infty(X),\ A \in \Psi_b^m(X)$$

where this defines the boundary operator of A and

$$(2.27)\qquad (\ )_b : \Psi_b^m(X) \longrightarrow \Psi^m(\partial X)$$

is a surjection from totally characteristic pseudodifferential operators on X to pseudodifferential operators on ∂X. From definition (2.11) there is a projection, and corresponding dual injection

$$(2.28)\qquad \tilde{T}_{\partial X}X \longrightarrow T\partial X\ , \quad T^*\partial X \longrightarrow \tilde{T}^*_{\partial X}X.$$

With respect to this injection the restriction map has the symbolic property

$$(2.29)\qquad \sigma_m(A_b) = \sigma_m(A)\big|_{T^*\partial X}\ .$$

It also follows from the assumption in (2.5) that the kernels vanish to all orders at $\partial_1'(X \hat{\times} X)$ that if $r \in C^\infty(X)$ is a defining function for ∂X and one considers the multiplication operator

$$r^{is} : C^{-\infty}(X) \longrightarrow C^{-\infty}(X),$$

which is an isomorphism for any $s \in \mathbb{C}$, then under conjugation,

$$(2.30)\qquad r^{-is}Ar^{is} \in \Psi_b^m(X), \quad \text{if}\ \ A \in \Psi_b^m(X)\ .$$

Using this we define the indicial family of $A \in \Psi_b^m(X)$, relative to the defining function r by

$$(2.31) \qquad I_A(s) = (r^{-is} A r^{is})_b \in \Psi^m(\partial X).$$

It is easy to see that $A(s)$ is weakly holomorphic in s .

3. Mellin transform.

Consider the Mellin transform in one dimension:

$$(3.1) \qquad Mu(s) = u_M(s) = \int_0^\infty x^{-is} u(x) \, \frac{dx}{x} \, .$$

For $u \in C_c^\infty(\mathbb{R}^+)$, $\mathbb{R}^+ = (0,\infty)$, this integral coverges absolutely and $u_M(s)$ is entire in s. This can be seen easily by introducting $t = \log(x)$ as variable of integration, reducing the Mellin transform to the Fourier transform:

$$(3.2) \qquad u_M(s) = \hat{v}(s) = \int_{-\infty}^{\infty} e^{-ist} \, v(t) \, dt \, , \quad v(t) = u(x) \, , \quad x = e^t .$$

Thus, the Paley-Wiener theorem can be applied, showing that

$$(3.3) \qquad \begin{array}{l} \phi = Mu, \quad u \in C_c^\infty(\mathbb{R}^+), \quad \sup(u) \subset [a_-,a_+] \quad \Longleftrightarrow \\ \phi \text{ is entire and } \forall N \sup\limits_{\pm \mathrm{Im}(s) \geq 0} |\phi(s)|(1+|s|)^N \exp(a_\pm \mathrm{Im}(s)) < \infty. \end{array}$$

The inversion formula is therefore

$$(3.4) \qquad u(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} x^{is} u_M(s) \, ds \, .$$

Set $\mathbb{C}^+ = \{s \in \mathbb{C} \, ; \, \mathrm{Im}(s) > 0\}$. The L^2 version of the Paley-Wiener theorem shows that (3.1) extends by continuity to an isomorphism

$$M : \{u \in L_b^2(\bar{\mathbb{R}}^+)\ ;\ u(x) = 0,\ \ x \gg 0\} \ = \ L_{b,c}^2(\bar{\mathbb{R}}^+)$$

(3.5) $$\longrightarrow \{u_M : \mathbb{C}^+ \longrightarrow \mathbb{C}\ ;\ u_M \text{ is holomorphic and } \exists\, b \ \text{ s.t.}$$

$$\sup_{0<t<\infty} \int_{-\infty}^{\infty} |u_M(s+it)|^2\, e^{-bt}\, ds < \infty\}.$$

Next consider the Mellin transform of conormal distributions. Set

(3.6) $$IL_b^2(\bar{\mathbb{R}}^+) \ = \{u \in L_b^2(\bar{\mathbb{R}}^+)\ ;\ (xD_x)^k u \in L_b^2(\bar{\mathbb{R}}^+) \ \ \forall\, k \in \mathbb{N}\}.$$

(3.7) **LEMMA.** The Mellin transform gives an isomorphism

$$M: \{u \in IL_b^2(\bar{\mathbb{R}}^+)\ ;\ u(x) = 0,\ x \gg 0\} = IL_{b,c}^2(\bar{\mathbb{R}}^+)$$

$$\longrightarrow \{u_M: \mathbb{C}^+ \longrightarrow \mathbb{C}\ ;\ u_M \text{ is holomorphic and } \exists\, b \text{ s.t.}$$

$$\sup_{0<t<\infty} \int_{-\infty}^{\infty} (1+|s|)^N\, |u(s+it)|^2\, e^{-bt}\, ds < \infty \ \ \forall\, N\}\ .$$

<u>Proof</u>. The Mellin transform has the property

(3.8) $$(xD_x u)_M(s) = s\ u_M(s)\ .$$

Thus the stated estimates follow from (3.3) and continuity.

The Mellin transform also satisfies the identity, for $u \in C_c^\infty(\mathbb{R}^+)$,

(3.9) $$(x^{-it}u)_M(s) = u_M(s+t) \qquad t \in \mathbb{C}.$$

Using this it follows easity from Lemma 3.7 that for any $a \in \mathbb{R}$,

$$M: x^a IL^2_{b,c}(\bar{\mathbb{R}}^+) \longrightarrow$$

(3.10) $$\{u_M(s) \text{ is holomorphic in } \mathrm{Im}(s) > -a \text{ and } \exists\, b \text{ s.t. } \sup_{-a<t<\infty} \int_{-\infty}^{\infty} |u_M(s+it)|^2 (1+|s|)^N e^{-bt}\, ds < \infty,\ \forall N\} = \mathrm{Hol}(a)\ .$$

The element in the quotient space $\mathrm{Hol}(a)/\mathrm{Hol}(a+1)$ can be considered as the Mellin symbol of the conormal distribution $u \in x^a IL^2_{b,c}(\bar{\mathbb{R}}^+)$.

(3.11) **LEMMA.** Choose $\rho \in C^\infty_c(\mathbb{R})$ with $\rho(x) = 1$, $x < \frac{1}{2}$, $\rho(x) = 0$, $x > 1$, then the map

(3.12) $$M_a:\ x^a IL^2_b(\bar{\mathbb{R}}^+)/x^{a+1} IL^2_b(\bar{\mathbb{R}}^+) \longrightarrow \mathrm{Hol}(a)/\mathrm{Hol}(a+1)$$

defined by $M_a(u) = (\rho u)_M$ is an isomorphism independent of the choice of ρ.

<u>Proof</u>. Notice that M_a is well-defined since if ρ' is another cutoff function with the required properties then

$$(\rho u)_M - (\rho' u)_M = ((\rho - \rho')u)_M$$

is the Mellin transform of an element of $C^\infty_c(\bar{\mathbb{R}}^+)$ with compact support so is in $\mathrm{Hol}(a')$ for all $a' \in \mathbb{R}$ by (3.3). The remainder of the lemma is clear.

In fact M_a can be converted into a coordinate independent Mellin symbol map. Before doing this note that (3.10) actually includes all conormal distributions. If Y is any C^∞ manifold without boundary, set

$$IL^2_{b,c}(\bar{\mathbb{R}}^+ \times Y) = \{u \in L^2_b(\bar{\mathbb{R}}^+ \times Y) \; ; \; u(x,y) = 0 \quad \text{in} \quad x \gg 0,$$

(3.13)

$$\text{and } \forall \; V_1,\ldots,V_N \in C^\infty(Y,TY), \quad k, \quad (xD_x)^k \, V_1,\ldots,V_N \, u \in L^2_b(\bar{\mathbb{R}}^+ \times Y)\},$$

and let $A_c(\bar{\mathbb{R}}^+ \times Y) \subset A(\bar{\mathbb{R}}^+ \times Y)$ be the subspace of distributions with support compact in $\bar{\mathbb{R}}^+ \times Y$.

(3.14) LEMMA. $$A_c(\bar{\mathbb{R}}^+ \times Y) = \bigcup_{a \in \mathbb{R}} x^a IL^2_{b,c}(\bar{\mathbb{R}}^+ \times Y) \; .$$

<u>Proof</u>. Using simple commutation arguments it is easily seen that the requirement $u \in A_c(\bar{\mathbb{R}}^+ \times Y)$ is equivalent to the existence of r such that

$$D^k_x \, x^k \, (xD_x)^p \, V_1 \cdots V_N \in H^r_c(\bar{\mathbb{R}}^+ \times Y) \qquad \forall \; V_1,\ldots,V_N \in C^\infty(Y,TY) \; .$$

Sobolev's embedding theorem shows that if $k > -r+1$ this implies that $u \in x^{-k} IL^2_{b,c}(\bar{\mathbb{R}}^+ \times Y)$. The converse is similar.

Let $\mathrm{Hol}(a,Y)$ be the space of C^∞ functions on Y with values in $\mathrm{Hol}(a)$. Applying Lemma 3.11 in the x variable shows that

$$\text{(3.15)} \qquad M_a : x^a IL^2_b(\bar{\mathbb{R}}^+ \times Y)/x^{a+1} IL^2_b(\bar{\mathbb{R}}^+ \times Y) \longrightarrow \mathrm{Hol}(a,Y)/\mathrm{Hol}(a+1,Y)$$

is again an isomorphism.

Suppose that N is an oriented, trivial line bundle over Y. If N^+ is the positive half of N then the choice of a non-vanishing section $\nu \in C^\infty(Y,N^+)$ gives a linear isomorphism

$$\text{(3.16)} \qquad N^+ \cong \bar{\mathbb{R}}^+ \times Y \quad \text{with} \quad \nu(y) \longleftrightarrow (1,y) \; .$$

Such a choice of section therefore gives an isomorphism

$$x^a IL_b^2(N^+) \longleftrightarrow x^a IL_b^2(\bar{\mathbb{R}}^+ \times Y) ,$$

and so defines the Mellin symbol map, M_a on $x^a IL_b^2(N^+)/x^{a+1} IL_b^2(N^+)$. Under a change of section, to $\nu' = \phi\nu$, $\phi \in C^\infty(Y)$, $\phi > 0$, the linear coordinate x is replaced by $x' = \phi^{-1}x$. Thus the new Mellin symbol, $M_a' u$ is simply

(3.17) $$M_a' u(s) = \phi^{is}(M_a u)(s).$$

There is a natural trivial bundle over Y with sections which transform according to the law (3.17). For any $s \in \mathbb{C}$ the bundle N^{*s} is defined by taking the fibre to be

(3.18) $$N_y^{*s} = \{f: \overset{\circ}{N}{}_y^+ \rightarrow \mathbb{C} ; f(ty) = t^{is}f(y) \quad \forall\, t \in \mathbb{R}\} .$$

Here, t^{is} is the standard branch. If ν is a non-vanishing section of N^+ then there are associated sections of all the density, or power, bundles N^{*s}, namely

(3.19) $$\nu^{*s}(\nu(y)) = 1 \quad \forall\, y \in Y$$

fixes the section ν^{*s} , with the homogeneity property (3.18). If ν' is another section, as above, then

$$(\nu')^{*s}(\nu(y)) = \phi^{-is}(\nu')^{*s}(\nu'(y)) = \phi^{-is}.$$

Thus from (3.17), (3.19) we see that the section

(3.20) $$(M_a u)(s)\cdot(\nu^{*s})$$

is independent of the choice of ν. Defining $Hol(a,N^{*s}) = Hol(a,Y)$ in any trivialization of N, this proves:

(3.21) PROPOSITION. If N is an oriented, trivial line bundle over Y then

$$M_a: x^a IL^2_b(N^+)/x^{a+1} IL^2_b(N^+) \longleftrightarrow Hol(a,N^{*s})/Hol(a+1,N^{*s})$$

defined by (3.20), is independent of the choice of trivialization.

If $f: N^+ \to N^+$ is a diffeomorphism of a neighbourhood of the zero section onto its range with the properties

(3.22) $$f: 0_N \to 0_N \text{ is the identity}$$

(3.33) $$f_*: T_0N \to T_0N \text{ is the identity}$$

and if $x: N^+ \to \bar{R}^+$ is a non-trivial linear function on the fibres, i.e. $N^+ \mapsto (x,y)$ is a trivialization, then

$$f^*x = x(1 + xb(x,y)),$$

where b is C^∞ near 0_N. It follows that if f is used as a change of variable of integration in (3.1), with u supported sufficiently near 0, then

$$u_M(s) = \int_0^\infty x^{-is} f^*u(x)\frac{dx}{x} + \int_0^\infty x^{-is} c(x,s) x f^*u(x)\frac{dx}{x}$$

where $c(x,s)$ is C^∞ in x and entire in s. If $u \in x^a IL^2_{b,c}(N^+)$ it follows that

$$(3.24) \qquad M_a u = M_a f^* u \quad \text{if} \quad f \quad \text{satisfies (3.22), (3.23).}$$

Suppose that X is a compact C^∞ manifold with boundary. Then there always exists a normal fibration of X near ∂X, i.e. a tubular neighbourhood of ∂X in X. Thus, there exists a diffeomorphism of a neighbourhood, G, of ∂X in X

$$(3.25) \qquad g: G \longrightarrow N^+\partial X$$

with a neighbourhood of 0 in the positive, i.e. inward pointing, half of the normal bundle to ∂X with the additional properties:

$$(3.26) \qquad g: \partial X \longrightarrow 0_{N\partial X} \quad \text{is the natural identification}$$

$$(3.27) \qquad g_*: N\partial X \longrightarrow N0_{N\partial X} \quad \text{is the natural identification.}$$

Recall that $N_x\partial X = T_xX/T_x\partial X$ for each $x \in \partial X$, so $N\partial X$ is a trivial line bundle over ∂X. Thus there is a natural identification of the zero section of $N\partial X$ and the base ∂X, this is the meaning of (3.26). Since $g: \partial X \longrightarrow 0_{N\partial X}$ the Jacobian $g_*; T_x\partial X \longrightarrow T_{x'}0_{N\partial X}$ is an isomorphism. Thus g_* projects to a map as claimed in (3.27). Since $N\partial X$ is a vector bundle there is a natural identification of T_xN_x with N_x for each $x \in \partial X$ and hence of the quotient

$$N_x0_x = T_x(N\partial X)/T_x0_N \cong N_x$$

it is this identification which is involved in (3.27).

It is easy to see that if g_1, g_2 are two maps as in (3.26), (3.27), then $f = g_1 g_2^{-1}$ satisfies (3.22) and (3.23). Thus the invariance result (3.24) gives the invariance of the Mellin symbol.

(3.28) THEOREM. If X is a compact C^∞ manifold with boundary the Mellin symbol, defined by reference to a normal fibration (3.25) - (3.27) gives an isomorphism independent of choice:

$$(3.29) \qquad M_a : x^a IL^2_b(X)/x^{a+1} IL^2_b(X) \longrightarrow \mathrm{Hol}(a, N^{*s})/\mathrm{Hol}(a+1, N^{*s}) \ .$$

As well as functions holomorphic in a half-space we need to consider, briefly, functions in a strip. If $a_2 < a_1$, let $\mathrm{Hol}(a_1, a_2, \partial X)$ be the space of holomorphic functions

$$f : \{s \in \mathbb{C};\ -a_1 < \mathrm{Im}(s) < -a_2\} \longrightarrow C^\infty(\partial X)$$

satisfying estimates analogous to (3.5):

$$(3.30) \qquad \sup_{a_2 < t < a_1} \int \|f(s-it)\|^2_{H^p(\partial X)} (1 + |s|)^N \, ds < \infty \quad \forall\ p, N.$$

The inversion formula (3.4) shows that $f = u_M$ where $u \in x^a IL^2_b(\bar{\mathbb{R}}^+ \times \partial X)$ for every $a \in [a_2, a_1]$. Choose $\rho \in C^\infty_c(\mathbb{R})$ with $\rho(x) = 1$ in $|x| < 1$ and consider the map

$$(3.31) \qquad \mathrm{Com} : \mathrm{Hol}(a_1, a_2, \partial X) \ni f \longmapsto (\rho u)_M \in \mathrm{Hol}(a, \partial X) \ .$$

Similarly, if $\mathrm{Hol}(a_1, a_2, N^{*s})$ is the space of holomorphic sections of the power bundles, reducing to $\mathrm{Hol}(a_1, a_2, \partial X)$ when the N^{*s} are simultaneously trivialized by the choice of a trivialization of N, then replacing ρ by an element of $C_c^\infty(N^+\partial X)$ equal to one near the zero section, gives

$$\text{(3.32)} \qquad \mathrm{Com}\colon \mathrm{Hol}(a_1,a_2,N^{*s}) \longrightarrow \mathrm{Hol}(a_1,N^{*s}) \qquad \text{any } a_2 < a_1 .$$

(3.3) **LEMMA.** The compactification map (3.32) has the property

$$M_a(Au) = \mathrm{Com}(I_A(s)f) \ \mathrm{mod}\ \mathrm{Hol}(a+1,\, N^{*s})$$

if $A \in \Psi_b^m(X)$, $f \in \mathrm{Hol}(a,a',N^{*s})$ with $a' < a$, and $M_a(u) = \mathrm{Com}(f)\ \mathrm{mod}\ \mathrm{Hol}(a+1,\, N^{*s})$.

<u>Proof</u>. If $u \in x^a IL_b^2(X)$ then by assumption, $M_a(Au) = I_A(s)M_a(u) = I_A(s)\mathrm{Com}(f)\ \mathrm{mod}\ \mathrm{Hol}(a+1,\, N^{*s})$. Now, $g_1 = \mathrm{Com}(f) - f$ is the Mellin transform of an element of $x^{a'} IL_b^2(N^+\partial X)$ with support disjoint from the zero section. Thus, g_1 satisfies estimates (3.30) with $a_2 = a'$ and for any a_1 , say $a_1 = a+1$. Similarly for $g_2 = \mathrm{Com}(I_A(s)f) - I_A(s)f$. Thus, $h = -I_A(s)g_1 + g_2$ satisfies the estimates (3.30) with $a_1 = a+1$, $a_2 = a'$ and is holomorphic in the corresponding strip. In fact, $h = \mathrm{Com}(I_A(s)f) - I_A(s)\mathrm{Com}(f) \in \mathrm{Hol}(a,N^{*s})$, is therefore holomorphic throughout the half-space $\mathrm{Im}(s) > a-1$, and so is an element of $\mathrm{Hol}(a+1,N^{*s})$, proving the Lemma.

A similar argument shows that

$$\text{(3.34)} \qquad \mathrm{Com}\colon \mathrm{Hol}(a,N^{*s}) \longrightarrow \mathrm{Hol}(a,N^{*s}) \ \text{ is the identity mod } \mathrm{Hol}(a+1,N^{*s}).$$

4. Boundary spectrum.

If $A \in \Psi_b^m(X)$, X a compact manifold with boundary, then the family of pseudodifferential operators on the boundary of X given by (2.31) is the indicial family of A. To make this family coordinate free we shall consider each element as acting on sections of the power bundle N^{*s}:

$$I_A(s) \in \Psi^m(\partial X, N^{*s}) \text{ is entire in } s \in \mathbb{C}. \tag{4.1}$$

The boundary spectrum of A is the set

$$\operatorname{spec}_b(A) = \{s \in \mathbb{C} \;;\; I_A(s) : H^m(\partial X) \longrightarrow L^2(\partial X) \text{ is not an isomorphism}\}. \tag{4.2}$$

(4.3) THEOREM: If $A \in \Psi_b^m(X)$ is elliptic and X is compact then $\operatorname{spec}_b(A)$ is a discrete subset of the complex plane wih

$$\operatorname{spec}_b(A) \cap \{s \in \mathbb{C} \;;\; |\operatorname{Im}(s)| < R\} \text{ finite for every } R \in \mathbb{R}, \tag{4.4}$$

$$\text{i.e. if } s_j \in \operatorname{spec}_b(A) \text{ and } |\operatorname{Re}(s_j)| \longrightarrow \infty \text{ then } |\operatorname{Im}(s)| \longrightarrow \infty.$$

Proof. Under a change of section of N, $I_A(s)$ is conjugated by an entire non-vanishing function ϕ^{is} which does not alter $\operatorname{spec}_b(A)$, so it is certainly enough to consider $I_A(s)$ in the form (2.31). In proving (4.4) it is enough to show that for each $R \in \mathbb{R}$ there exists $R' \in \mathbb{R}$ such that if $|\operatorname{Im}(s)| < R$ and $|\operatorname{Re}(s)| > R'$ then $I_A(s)$ is invertible. Indeed, once this is done it follows that $I_A(s)$, as an entire family of elliptic operators on the compact manifold ∂X , has constant index zero; hence the set of points of non-invertibility is discrete and satisfies (4.4) as claimed.

Now, consider the parametrix B of A as in (2.16). Passing to the corresponding indicial operators shows that

$$(4.5) \qquad I_B(s)\cdot I_A(s) = \mathrm{Id} + I_R(s), \quad I_A(s)\cdot I_B(s) = \mathrm{Id} + I_{R'}(s) .$$

For $R \in \Psi_b^{-\infty}(X)$ the indicial family $I_R(s) \in \Psi^{-\infty}(\partial X)$ has Schwartz' kernel obtained as

$$(4.6) \qquad \kappa(s,y,y')dy' = \int_0^\infty \alpha(0,t,y,y')\, t^{is}\frac{dt}{t}\, dy'$$

where $\alpha(0,t,y,y') \in C^\infty(\bar{\mathbb{R}}^+ \times \partial X \times \partial X)$ is rapidly decreasing with all derivatives as $t \to \infty$ and vanishes to all order at $t=0$. As $|\mathrm{Re}(s)| \longrightarrow \infty$, with $\mathrm{Im}(s)$ bounded, $\kappa(s,y,y')$ tends to 0 rapidly in $C^\infty(\partial X \times \partial X)$. Thus, given $R \in \mathbb{R}$ there exists $R' \in \mathbb{R}$ such that if $|\mathrm{Im}(s)| < R$, $|\mathrm{Re}(s)| > R'$ then

$$\|I_R(s)\| , \|I_{R'}(s)\| < \tfrac{1}{2} \quad \text{on } L^2(\partial X).$$

Thus the invertibility of $I_A(s)$ follows from (4.5), completing the proof of the theorem.

(4.7) COROLLARY. Under the hypotheses of Theorem 4.3, $I_A^{-1}(s)$ is a meromorphic family of elliptic pseudodifferential operators acting on secions of $N^{*s}\partial X$, with poles only at $\mathrm{spec}_b(A)$ and with residues there smoothing operators, of finite rank. With respect to any trivialization of N , and for $\mathrm{Im}(s)$ uniformly bounded, $I_A^{-1}(s)\colon H^k(\partial X) \longrightarrow H^{k+m}(\partial X)$ has norm at most polynomially growing as $|\mathrm{Re}(s)| \longrightarrow \infty$.

5. Graded conormal distributions.

The simplest conormal distribution on a C^∞ manifold with boundary, X, apart from the elements of $C^\infty(X)$ are those having complete asymptotic expansions at the boundary. Thus, if r is a defining function for ∂X with $r \leqslant \frac{1}{2}$ and $\phi \in C^\infty(X)$ then

$$r^{is} \log^p r \cdot \phi \in A(X)$$

for any $s \in \mathbb{C}$ and any integer p. More generally if $s(j)$ form a sequence in $\mathbb{C}$ with $\operatorname{Im}(s_j) \to -\infty$ as $j \to \infty$ and $J(j)$ is a sequence in $\mathbb{N}$ then for any sequence $\phi_{j,p} \in C^\infty(X)$ the formal series

$$u = \sum_{0 \leqslant p \leqslant J(j)} \phi_{j,p} \cdot r^{is(j)} \log^p r \tag{5.1}$$

determines a unique class in $A(X)/\dot{C}^\infty(X)$. In fact, if $\psi \in C^\infty(\mathbb{R})$ has $\psi(r) = 1$ in $|r| \leqslant 1$ and $\epsilon(j)\downarrow 0$ tend to zero sufficiently rapidly as $j \to \infty$ the series

$$u = \sum_{0 \leqslant p \leqslant J(j)} \psi(r/\epsilon(j))\, \phi_{j,p} \cdot r^{is(j)} \log^p r \tag{5.2}$$

converges to $u \in A(X)$. More precisely each of the series as (5.2) with j restricted to the range $j \geqslant j'$ converges in some $r^{t(j')}C^{t(j')}(X)$ for j' large and as $j' \to \infty$, $t(j') \to \infty$. The class $[u]$ of (5.1) is fixed analogously as consisting of those $u \in A(X)$ with the property that for N sufficiently large

$$u - \sum_{\substack{0 \leqslant j \leqslant N \\ 0 \leqslant p \leqslant J(j)}} \phi_{j,p}\, r^{is(j)} \log^p r \in r^{T(N)} C^{T(N)}(X) \tag{5.3}$$

where $T(N) \longrightarrow \infty$ as $N \longrightarrow \infty$. Elements of $A(X)$ with the property (5.3) will be caled graded elements, the subspace of graded elements will be denoted

$$A_{gr}(X) \subset A(X) \ ,$$

and similarly,

$$x^a I_{gr} L^2_b(X) \ = \ x^a I L^2_b(X) \cap A_{gr}(X) \ .$$

For each $a \in \mathbb{R}$ let $Mer(a)$ be the space of meromorphic functions, u_M, on the plane with poles p_j satisfying

(5.4) $\quad \operatorname{Im}(p_j) < -a \ , \ \ \operatorname{Im}(P_j) \longrightarrow -\infty \ , \ \ u_M\big|_{\operatorname{Im}(s)>-a} \in Hol(a)$

(5.5) $\quad |u_M(s)| \ (1+|s|)^N \longrightarrow 0$ as $|\operatorname{Re}(s)| \longrightarrow \infty$ with $\operatorname{Im}(s)$ bounded, $\forall N$.

In particular, $Mer(a) \subset Hol(a)$.

(5.6) LEMMA. The Mellin transform (3.1) is an isomorphism of $x^a I_{gr} L^2_{b,c}(\bar{\mathbb{R}}^+)$ onto $Mer(a)$ for every $a \in \mathbb{R}$.

__Proof__. If $\phi \in C^\infty_c(\mathbb{R})$ has $\phi(r) = 1$ for $|r| < \frac{1}{2}$ then

(5.7) $$r^{i\bar{s}} \log^p r \cdot \phi(r) \ \in \ x^a I L^2_{b,c}(\bar{\mathbb{R}}^+) \iff \operatorname{Im}(s) < -a \ .$$

The Mellin transform of this function is easily seen to be in $Mer(a)$ with a single pole, of multiplicity p, at $s = \bar{s}$. Thus, if $v \in Mer(a)$ it can be written, for any $b > a$, as a sum

$$v = v_b + \sum_{\text{finite}} (u_j)_M$$

where the u_j are of the form (5.7) and $v_b \in \mathrm{Hol}(b)$. This proves that $v = u_M$ with u of the form (5.3). The converse is similar.

Of course this lemma applies equally in the case that $X = \bar{\mathbb{R}}^+ \times Y$, or X is $\mathbb{N}^+$ corresponding to an oriented trivial line boundle over Y, with the obvious changes, namely $\mathrm{Mer}(a)$ should be replaced by $\mathrm{Mer}(a,Y)$, the space of C^∞ functions on Y with values in $\mathrm{Mer}(a)$, so in particular the poles p_j are independent of $y \in Y$, or by the space $\mathrm{Mer}(a,N^{*s})$ of sections of N^{*s} of the appropriate type.

For $a, a' \in \mathbb{R}$ let $\mathrm{Mer}(a,a')$ be the space of meromorphic functions, v, in $\mathrm{Im}(s) > -a'$ holomorphic in $\mathrm{Im}(s) > -a$ and such that for some finite set $u_1,\dots,u_N$ of elements of the form (5.7), $v - \sum_{1 \leqslant j \leqslant N} (u_j)_M \in \mathrm{Hol}(a')$.

Similarly let $\mathrm{Mer}(a,a',Y)$ and $\mathrm{Mer}(a,a',N^{*s})$ be the corresponding spaces with C^∞ coefficients in Y and power transition laws. Thus we have immediately,

(5.8) **PROPOSITION.** If $u \in x^a IL^2(X)$ has $u_M \in \mathrm{Mer}(a,a',N^{*s}) \subset \mathrm{Hol}(a,N^{*s})/\mathrm{Hol}(a',N^{*s})$ modulo $\mathrm{Hol}(a',N^{*s})$, for $a \leqslant a' \leqslant a+1$ then

$$u \in x^a I_{gr} L^2_b(X) + x^{a'} IL^2_b(X).$$

Note that graded connormal distributions are acted upon by arbitrary totally characteristic pseudodifferential operators.

(5.9) PROPOSITION. If $A \in \Psi_b^m(X)$, X a compact manifold with boundary then

$$A: x^a I_{gr} L_b^2(X) \longrightarrow x^a I_{gr} L_b^2(X) \quad \forall\, a \in \mathbb{R} .$$

<u>Proof</u>. This can easily be seen to follow from (2.3). Thus, $r^{-is} A r^{is}$ is a totally characteristic pseudodifferential operator depending holomorphically on s. Differentiating gives:

$$A\cdot\log(r) - \log(r)\cdot A = A_1 \in \Psi_b^m(X). \tag{5.10}$$

Iterating this commutation result, and using (2.30) again shows that

$$A\cdot r^{is}\log^p(r) = \sum_{0\leq j\leq p} r^{is}\log^j r\cdot A_j\,, \quad A_j \in \Psi_b^m(X) .$$

Since $\Psi_b^m(X)$ acts on $C^\infty(X)$ it follows that if $\phi \in C^\infty(X)$,

$$A\cdot r^{is}\log^p r\cdot\phi = \sum_{0\leq j\leq p} r^{is}(\phi_j\cdot\log^j r)\,, \quad \phi_j \in C^\infty(X). \tag{5.11}$$

Since, for any $a \in \mathbb{R}$, $u \in A_{gr}(X)$ is a finite sum of such terms modulo $x^a I L_b^2(X)$, which is preserved by A, it follows from (5.11) that A maps $A_{gr}(X)$ into itself and in view of (5.7), $x^a I_{gr} L_b^2(X)$ into itself for any a, proving the proposition.

Note that if f is a rational function, with values in $C^\infty(\partial X)$, then f is not the Mellin symbol of a conormal distribution, unless it is identically zero. However, if Rat(a) is the space of rational functions, with values in $C^\infty(\partial X)$ and no poles in the half-space $\mathrm{Im}(s) > -a$ we can define a compactification map:

$$\text{(5.12)} \qquad \mathrm{Cm}: \mathrm{Rat}(a) \longrightarrow \bigcap_{a'<a} x^{a'} IL^2_b(X)$$

as follows. First choose a trivialization of $N\partial X$, and a normal fibration, F , of X near ∂X. Now, for $b \gg 0$, $q(s) = \exp(-(-is-b)^{1/2}) \in \mathrm{Hol}(a+1)$ with the square root taken as the principal branch. Since q has no zeros in $\mathrm{Im}(s) > -a-1$ we can define

$$(F^*\mathrm{Cm}(f))_M = q(s)f ,$$

and so ensure that

$$M_{a'}(\mathrm{Cm}(f)) = q(s)f \bmod \mathrm{Hol}(a'+1, \partial X), \quad a' > a$$

with respect to the same trivialization, gives an injection from $\mathrm{Rat}(a)/\mathrm{Rat}(a+1)$.

6. Fredholm properties.

The basic tool used to analyse the kernel of an elliptic element of $\Psi_b^m(X)$ is the indicial operator, which arises in the following manner.

(6.1) PROPOSITION. If $u \in x^a IL_b^2(X)$ and $A \in \Psi_b^m(X)$ then $M_a(Au) = I_A(s)M_a u$.

Proof. Using a normal fibration of X near ∂X the computation can be transferred to $N^+\partial X$. On $N^+\partial X$ the $\mathbb{R}^+$-action singles out the class of $\mathbb{R}^+$-invariant elements of $\Psi_b^m(N^+\partial X)$. From the form of the kernel of $A \in \Psi_b^m(N^+\partial X)$,

$$\lim_{t\downarrow 0} m_t^* \cdot A \cdot m_{1/t}^* = A_0$$

exists in $\Psi_b^m(X)$ and is $\mathbb{R}^+$-invariant, where $m_t\colon N^+\partial X \longrightarrow N^+\partial X$ is the $\mathbb{R}^+$ action. Moreover, $(A_0)_b = A_b$ so

$$A - A_0 = xQ, \qquad Q \in \Psi_b^m(N^+\partial X). \tag{6.2}$$

Since $x\colon x^a IL_{b,c}^2(N^+\partial X) \longrightarrow x^{a+1} IL_{b,c}^2(N^+\partial X)$,

$$M_a(Au) = M_a(A_0 u'), \qquad u' = f^* u , \tag{6.3}$$

f being a normal fibration. From the inversion formula (3.4),

$$u'(x,y) = (2\pi)^{-1} \int_{\mathrm{Im}(s)=-a+\epsilon} x^{is} u'_M(s,y)\, \frac{ds}{s} \qquad \epsilon > 0$$

in terms of some section of $N^{+}\partial X$ and corresponding linear coordinate x. Thus,

$$A_0 u'(x,y) = (2\pi)^{-1} \int x^{is}(x^{-is} A_0 x^{is}) u'_M(s,y)\, \frac{ds}{s}.$$

Since $x^{-is}A_0x^{is}$ is itself $\mathbb{R}^{+}$-invariant, $x^{-is}A_0x^{is}v(y) = (x^{-is}Ax^{is})_b v(y) = I_A(s)v(y)$. Thus, $(A_0u')_M = I_A(s)u'_M$, which together with (6.3) proves the proposition.

Using the results of Section 4 this leads to structure theorem for elements of the kernel of an elliptic totally characteristic operator.

(6.4) THEOREM. If $A \in \Psi_b^m(X)$ is elliptic on a compact manifold with boundary, X, and $u \in C^{-\infty}(X)$ satisfies $Au \in A_{gr}(X)$ then $u \in A_{gr}(X)$.

<u>Proof</u>. The existence, see (2.16), of a parametrix modulo $\Psi_b^{-\infty}(X)$ shows that $u = B\cdot Au - R'u \in A(X)$. From Lemma 3.14, $u \in x^a IL_b^2(X)$ for some $a \in \mathbb{R}$. Thus, if $Au = v \in A_{gr}(X)$ then by Proposition 6.1,

$$I_A(s)M_a u(s) = M_a v(s) \in \mathrm{Mer}(a)/\mathrm{Hol}(a+1)$$

since $v \in x^a IL_b^2(X) \cap A_{gr}(X) = x^a I_{gr} L_b^2(X)$. The inversion properties of $I_A(s)$, described in Section 4 show that $M_a u(s) = I_A(s)^{-1} M_a v(s)$ is holomorphic in $\mathrm{Im}(s) > -a$ and meromorphic in $\mathrm{Im}(s) > a-1$, with only finitely many poles there. There may be a pole of either $M_a v(s)$ or

$I_A(s)^{-1}$ on $Im(s) = -a-1$ but for $\epsilon > 0$ sufficiently small,

$$M_a u \in Mer(a,a') , \qquad a' = a + 1 - \epsilon .$$

From Proposition 5.8 it follows that $u = u_1 + u_2$, $u_1 \in x^a I_{gr} L^2_b(X)$, $u_2 \in x^{a'} I L^2_b(X)$. Thus,

$$Au_2 = Au - Au_1 = v - Au_1 \in A_{gr}(X)$$

by Proposition 5.9. Proceeding by induction it follows that

$$u \in x^a I_{gr} L^2_b(X) + x^{a'} I L^2_b(X)$$

for any $a < a'$, hence $u \in A_{gr}(X)$ proving the Theorem.

(6.5) COROLLARY. If $u \in x^a H_b^{-\infty}(X)$ satisfies $Au = 0$, $A \in \Psi_b^m(X)$ elliptic and X compact then $M_a u \in Mer(a,a',N^{*s})$ for any $a' < a+1$ has poles only at

$$spec_b(A) \cap \{s \in \mathbb{C} ; -a > Im(s) > -a'\}.$$

As usual the Fredholm properties of operators are related to embedding properties of associated spaces.

(6.6) LEMMA. If X is a compact C^∞ manifold with boundary then for any $a' > a$, $m' > m$ the inclusion $x^{a'} H_b^{m'}(X) \hookrightarrow x^a H_b^m(X)$ is compact.

Proof. Since multiplication with the power of a defining function gives an isomorphism, $r^b: x^aH_b^m(X) \longrightarrow x^{a+b}H_b^m(X)$ it is enough to consider the case $a = 0$ or $a' = 0$ for each pair m, m'. In fact it is enough to consider the case $m' > m \geq 0$, since if $m' > 0$, $m < 0$ then compactness follows from the case $m = 0$ and if $0 \geq m' > m$ it follows by duality. Now, assuming $a' > 0$, $a = 0$, $m' > m \geq 0$ it is immediate that on a norm bounded set in $x^{a'}H_b^{m'}(X)$ the function Bu, $B \in \Psi_b^m(X)$, has uniformly small L^2 norm near ∂X, and is uniformly equicontinuous in the L^2-mean over any compact subset of the interior, and similarly for u itself. Thus, from the L^2 form of the Ascoli-Arzela theorem such a bounded set is precompact in $H_b^m(X)$, proving the Lemma.

(6.7) **PROPOSITION.** If $A \in \Psi_b^m(X)$ is elliptic and X is compact then for any $a \in \mathbb{R}$ and $m \in \mathbb{R}$ the kernel of A in $x^aH_b^m(X)$ is finite dimensional, and is independent of m.

Proof. From (2.16) if $u \in x^aH_b^m(X)$ satisfies $Au = 0$ then $u \in x^aIL_b^2(X) = \bigcap_m x^aH_b^m(X)$. Thus,

$$\mathrm{Ker}_a(A) = \{u \in x^aH_b^m(X)\ ;\ Au = 0\} \subset x^aIL^2(X)$$

is independent of m. From Corollary 6.5 it follows that if $u \in \mathrm{Ker}_a(A)$ and $\epsilon > 0$ is sufficiently small, independent of u, then $M_a(u) \in \mathrm{Hol}(a+\epsilon, N^{*s})$. Thus, from Theorem 3.28 it follows that $\mathrm{Ker}_a(A) \subset x^{a+\epsilon}IL_b^2(X)$. Either directly, or from the closed graph theorem the first inclusion in

$$\mathrm{Ker}_a(A) \hookrightarrow x^{a+\epsilon}H_b^1(X) \hookrightarrow x^aL_b^2(X)$$

is continuous, so the composite is compact. As a closed subspace of $x^a L_b^2(X)$, $\mathrm{Ker}_a(A)$ is therefore finite dimensional.

We proceed to construct a parametrix for $A \in \Psi_b^m(X)$, elliptic, modulo a finer remainder than $\Psi_b^{-\infty}(X)$. Fixing a, suppose $g \in x^a IL_b^2(X)$. Choosing a normal fibration F and a cut-off function $\rho \in C_c^\infty(N^+\partial X)$, restricting supports to near $0_{N\partial X}$, consider the map

$$g \mapsto (\rho F^* g)_M \in \mathrm{Hol}(a, N^{*s}).$$

Suppose that

$$(6.8) \qquad a \notin -\mathrm{Im}\ \mathrm{spec}_b(A) \ .$$

Then, $I_A(s)^{-1}$ has no pole on the line $\mathrm{Im}(s) = -a$. Thus, for some $a' < a$, $f = I_A^{-1}(s)(\rho F^* g)_M \doteq \mathrm{Hol}(a,a',N^{*s})$ as defined following (3.31). Indeed the estimates (3.30) follow from (6.8) and the estimates on $I_A^{-1}(s)$ of Section 4. Applying the compactification map (3.32), set

$$(6.9) \qquad E_a(g) = F^{-1*}(\rho v) \in x^a IL_b^2(X) \quad \text{if} \quad v_M = \mathrm{Com}(f).$$

Then,

$$M_a(AE_a(g)) = I_A \cdot M_a(E_a(g)) = I_A \cdot \mathrm{Com}(f) = \mathrm{Com}(I_A \cdot f)$$

by Lemma 3.33. Since, using Lemma 3.34

$$\mathrm{Com}(I_A \cdot f) = \mathrm{Com}(\rho F^* g)_M) = M_a(g) \bmod \mathrm{Hol}(a+1, N^{*s})$$

we conclude that

$$(6.10) \qquad A\cdot E_a = \mathrm{Id} + S_a, \quad S_a\colon x^a IL^2_b(X) \longrightarrow x^{a+\epsilon} IL^2_b(X), \quad \epsilon > 0$$

with S_a certainly continuous. Thus we have proved:

(6.11) LEMMA. If $A \in \Psi^m_b(X)$ is elliptic, X compact, then if (6.8) holds the map

$$(6.12) \qquad G_a = B - E_a\cdot R : \ x^a H^k_b(X) \longrightarrow x^a H^{k+m}_b(X) \qquad \forall\, a,$$

where $B \in \Psi^{-m}_b(X)$ is a parametrix as in (2.16), satisfies

$$(6.13) \qquad A\cdot G_a = \mathrm{Id} + T_a, \quad T_a\colon x^a H^k_b(X) \longrightarrow x^{a+\epsilon} IL^2_b(X) \quad \forall\, k, \text{ with } \epsilon > 0.$$

We also need some more information on the kernel of A.

(6.14) LEMMA. Suppose $A \in \Psi^m_b(X)$, is elliptic X compact, and $a \in -\mathrm{Im}\ \mathrm{spec}_b(A)$. If $\epsilon > 0$ is sufficiently small then there exists a finite dimensional subspace $F_a \subset x^{a-\epsilon} IL^2_b(X)$, with $F_a \cap x^{a+\epsilon} IL^2_b(X) = \{0\}$, such that for any k,

$$u \in x^{a+\epsilon} H^k_b(X) \text{ has } Au \in x^{a+\epsilon} H^{k-m}_b(X) \iff u \in x^{a+\epsilon} H^k_b(X) \oplus F.$$

__Proof__. Select a trivialization of $N^*\partial X$ and consider $I_A(s)$ as acting on $C^\infty(\partial X)$. By assumption $a \in -\mathrm{Im}\ \mathrm{spec}_b(A)$, so for each $\bar{s} \in \mathrm{spec}_b(A)$ with $\mathrm{Im}(\bar{s}) = -a$, let $J(\bar{s})$ be the order of the pole of $I_A(s)^{-1}$ at $\bar{s}$, where a simple pole is taken to have order zero, and set

$$F_{\bar{s}} = \{f\colon E \to C^\infty(\partial X)\ ;\ f \text{ is a polynomial of degree } J(\bar{s}) \text{ and} \tag{6.15}$$
$$I_A(s)f(s) \text{ vanishes at } \bar{s} \text{ with its first } J(\bar{s}) \text{ derivatives}\}.$$

The ellipticity of I_A implies that each $F_{\bar{s}}$ is finite dimensional. Recalling the compactification map of (5.12), consider

$$F_a = \bigoplus_{\bar{s}} \mathrm{Cm}\{s^{-J(\bar{s})}F_{\bar{s}}\ ;\ \bar{s} \in \mathrm{spec}_b(A),\ \mathrm{Im}(\bar{s}) = -a\} \subset x^{a-\epsilon} IL_b^2(X) \quad \forall\ \epsilon > 0. \tag{6.16}$$

If $u \in F_a$ then $M_a(u) = q \sum_{\bar{s}} s^{-J(\bar{s})} f_{\bar{s}}$, $f_{\bar{s}} \in F_{\bar{s}}$, $q \in \mathrm{Hol}(a', N^{*s})$ $a' \gg a$. Thus, $M_a(Au) = 0$ modulo $\mathrm{Hol}(a+\epsilon, N^{*s})$, from the definition of $F_{\bar{s}}$, so $Au \in x^{a+\epsilon} IL_b^2(X)$, and therefore $A(x^{a+\epsilon}H_b^k(X) \oplus F) \subset x^{a+\epsilon}H_b^{k-m}(X)$. The converse is similar.

(6.17) THEOREM. Suppose $A \in \Psi_b^m(X)$ is elliptic and X is a compact manifold with boundary. As a map $A\colon x^a H_b^k(X) \to x^a H_b^{k-m}(X)$, A is Fredholm if and only if

$$a \notin -\mathrm{Im}\ \mathrm{spec}_b(A) \tag{6.18}$$

The index of $\mathrm{Ind}(a)$ is independent of k and if $a' > a$ both satisfy (6.18),

$$\mathrm{Ind}(a) - \mathrm{Ind}(a') = \sum_{a' > -\mathrm{Im}(\bar{s}) > a} N_{\bar{s}} , \tag{6.19}$$

where for $\bar{s} \in \mathrm{spec}_b(A)$, $N_{\bar{s}}$ is the dimension of the space $F_{\bar{s}}$ in (6.15).

<u>Proof</u>. In Proposition 6.7 it was shown that the kernel of A was finite dimensional, without any assumption on a. Given condition (6.18), Lemma 6.11 provides a right parametrix modulo the map T_a, which is compact since the inclusion $x^{a'}H_b^{k'}(X) \hookrightarrow x^a H_b^k(X)$ is compact whenever $a' > a$, $k' > k$ by Lemma 6.6. Thus $(Id + T_a)$ is Fredholm and hence so is A. Since the kernel of A lies in $x^a IL_b^2(X)$, and there is always a cokernel in this space, ind(a) is indeed independent of k.

The formula (6.19) for the jump in the index as a crosses $-\mathrm{Im}\ \mathrm{spec}_b(A)$ follows from Lemma 6.14. Thus, $\mathrm{Ind}(a) = \mathrm{Ind}(x^{-a}Ax^a)$ acting as $H_b^k(X)$. This is a continuous family of Fredholm operators, provided a does not cross $-\mathrm{Im}\ \mathrm{spec}_b(A)$, so the index is constant and it suffices to consider $\mathrm{Ind}(a-\epsilon) - \mathrm{Ind}(a+\epsilon)$ when $a \in -\mathrm{Im}\ \mathrm{spec}_b(A)$ for $\epsilon > 0$ small. Then,

$$\ker(a-\epsilon) = \ker(a+\epsilon) \oplus H_1 \oplus H_2 ,$$

where H_1 is the kernel of A on F_a, and

$$\dim(H_2) = \dim(A(F_a) \cap A(x^{a+\epsilon}H_b^m(X))) .$$

Thus, if $f \in F_a$ has $Af = Ag$, $g \in x^{a+\epsilon}H_b^m(X)$, then $(f-g) \in H_2$, $f \notin x^{a-\epsilon}H_b^m(X)$. Similarly, if $H_3 \subset A(F)$ is a complement to $A(F) \cap A(x^{a+\epsilon}H_b^m(X))$ and K is a complement in $x^{a-\epsilon}L_b^2(X)$ to the range of A on $x^{a-\epsilon}H_b^m(X)$ then $H_3 \oplus K$ is a complement in $x^{a+\epsilon}L_b^2(X)$ to the range on $x^{a+\epsilon}H_b^m(X)$. Since A, on F, has finite rank, $\dim(F_a) = \dim(H_1) + \dim(H_2) + \dim(H_3)$. This proves (6.19) since the sum on the right is, in this case, $\dim(F_a)$.

Finally, to complete the proof of the theorem note that A cannot be Fredholm from $x^a H_b^k(X)$ to $x^a H_b^{k-m}(X)$ if $a \in -\mathrm{Im}\ \mathrm{spec}_b(A)$. Indeed, if it were Fredholm the index would be constant in a across $-\mathrm{Im}\ \mathrm{spec}_b(A)$, which contradicts (6.19). Thus, when (6.18) does not hold the range of A is not closed.

(6.20) **THEOREM.** Suppose that $A \in \Psi_b^m(X)$ is elliptic and X is compact and that A is formally self-adjoint with respect to a density $\nu \in C^\infty(\mathring{X},\Omega X)$ such that if r is a defining function for ∂X then $r^{-2b-1}\nu \in C^\infty(X,\Omega X)$ is positive definite. Then for any a satisfying (6.18),

$$\text{(6.21)} \qquad \operatorname{Ind}(a) = \sum_{-\operatorname{Im}(\bar{s})\in(a,b)} N_{\bar{s}} + \frac{1}{2} \sum_{-\operatorname{Im}(\bar{s})=b} N_{\bar{s}} .$$

Proof. $r^bAr^{-b} \in \Psi_b^m(X)$ is formally sef-adjoint with respect to a density $r^{-1}\omega$, $\omega \in C^\infty(X,\Omega X)$ positive definite. This density defines a Hilbert norm on $L_b^2(X)$, with respect to which r^bAr^{-b} is an unbounded self-adjoint operator with domain $H_b^m(X)$. If $b \notin -\operatorname{Im}\operatorname{spec}_b(A)$, then r^bAr^{-b} is Fredholm so has $\operatorname{Ind}(b) = 0$. This gives (6.21) under this additional hypothesis.

Thus, we can assume that $A \in \Psi_b^m(X)$ is formally self-adjoint on $L_b^2(X)$ but then, by examining kernels and cokernels, it is clear that $\operatorname{Ind}(a) = -\operatorname{Ind}(-a)$. Thus, (6.21) follows from (6.19).

7. Spectral theory.

As an application of the ideas and constructions above we shall examine the structure of the spectrum of a formally self-adjoint elliptic totally characteristic operator, of positive order, on a compact manifold. As noted above, if $A \in \Psi_b^m(X)$ is formally self-adjoint with respect to a positive density $\nu \in C^\infty(X,\Omega_b X)$ then

$$(7.1) \qquad A\colon H_b^m(X) \longrightarrow L_b^2(X)$$

is self-adjoint with respect to ν. We shall simply say A is self-adjoint leaving ν understood. Thus we wish to investigate

$$\operatorname{spec}(A) = \mathbb{C}\setminus\{\lambda \in \mathbb{C}\ ;\ A - \lambda\colon H_b^m(X) \longrightarrow L_b^2(X) \text{ has dense range and is 1-1 with inverse } (A-\lambda)^{-1} \text{ bounded on } L_b^2(X)\}.$$

Since A is elliptic, and continuous on $L_b^2(X)$ with range in $H_b^{-m}(X)$, if $u = (A - \lambda)^{-1}f = R(A,\lambda)f \in L_b^2(X)$, $f \in L_b^2(X)$ then $(A - \lambda)u = f$ in $H_b^{-m}(X)$ shows that $(A - \lambda)u \in L_b^2(X)$ so $u \in H_b^m(X)$ and $R(A,\lambda)\colon L_b^2(X) \longrightarrow H_b^m(X)$ must be bounded by the closed graph theorem. In particular,

$$(7.2) \qquad \operatorname{spec}(A) = \{\lambda \in \mathbb{C}\ ;\ A-\lambda\colon H_b^m(X) \longrightarrow L_b^2(X) \text{ is not an isomorphism}\} \subset \mathbb{R}.$$

(7.3) LEMMA. If $A \in \Psi_b^m(X)$, $m > 0$, is elliptic and formally self-adjoint on the compact manifold X and in addition $\sigma_m(A) \in S^m(T^*X)/S^{m-1}(T^*X)$ has a positive representative then $\operatorname{spec}(A)$ is bounded below.

Proof. Since A is formally self-adjoint and has a positive elliptic principal symbol an approximate square-root can be constructed by standard symbolic methods

$$(7.4)\qquad A = B^2 + G\ ,\quad B \in \Psi_b^{\frac{1}{2}m}(X)\ ,\quad B^* = B\ ,\quad G \in \Psi_b^{-\infty}(X).$$

Now, G is bounded on $L_b^2(X)$, so with $\bar{\lambda} = -\ \|G\|$,

$$(7.5)\qquad A - \lambda \ \text{ has no kernel in } \ H_b^m(X) \ \text{ for } \ \lambda < \bar{\lambda}.$$

Moreover, consider the indicial operator. For $s \in \mathbb{R}$, $I_A(s) \in \Psi^m(\partial X)$ is self-adjoint with respect to $\nu' \in C^\infty(\partial X,\Omega\partial X)$, $\nu = \frac{dr}{r}\nu'$ at ∂X. Passing from (7.4) to the corresponding indicial operators gives

$$(7.6)\qquad I_A(s) = I_B(s)^2 + I_G(s)\ ,$$

where $I_G(s)$ is rapidly decreasing in $\Psi^{-\infty}(\partial X)$ as $|\mathrm{Re}(s)| \to \infty$, with $\mathrm{Im}(s)$ bounded. In particular, when s is real, $I_B(s)^* = I_B(s)$, so it follows from the fact that $I_A(s)$ is Fredholm and (7.6) that for some $\bar{\lambda} \in \mathbb{R}$,

$$(7.7)\qquad \mathrm{spec}_b(A - \lambda) \cap \mathbb{R} = \emptyset \quad \text{if } \ \mathbb{R} \ni \lambda < \bar{\lambda}\ .$$

Thus, from Theorem 6.17, $A - \lambda$ is Fredholm as an operator (7.1) for $\lambda < \bar{\lambda}$, and so an isomorphism if $\lambda \ll 0$, by (7.5) , proving the lemma.

Consider the set

$$(7.8) \qquad \Lambda_{bad}(A) = \{\lambda \in \mathbb{R} \; ; \; \exists \, \lambda_j \longrightarrow \lambda \text{ in } \mathbb{R} \text{ and } s_j \in \mathrm{spec}_b(A - \lambda_j) \text{ with } \mathrm{Im}(s_j) \neq 0, \; \mathrm{Im}(s_j) \longrightarrow 0\} ,$$

of values of λ at which the number of real points, with multiplicity, in $\mathrm{spec}_b(A - \lambda)$ changes.

(7.9) LEMMA. If $m > 0$ and $A \in \Psi_b^m(X)$ is formally self-adjoint and elliptic in the compact manifold, X, then $\Lambda_{bad}(A) \subset \mathbb{R}$ is discrete and on $\mathbb{R}\setminus\Lambda_{bad}$, $\mathrm{spec}_b(A) \cap \mathbb{R}$ is the image of a locally constant number, $N = N(\lambda)$, of real analytic functions of λ.

<u>Proof</u>. Near any point $\bar{\lambda} \in \mathbb{C}$ the points $s \in \mathrm{spec}_b(A - \lambda)$ near $\bar{s} \in \mathrm{spec}_b(A - \lambda)$ can be obtained as the zeros of a polynomial

$$(7.10) \qquad p(s,\lambda) = s^r + \sum_{j=1}^{r} c_j(\lambda) s^{r-j}$$

where the coefficients $c_j(\lambda)$ are real analytic in λ. Indeed, let $H^m(\partial X) = K_1 \oplus R_1$, $L^2(\partial X) = K_2 \oplus R_2$ be a decomposition with K_1 the kernel of $I_A(\bar{s}) - \bar{\lambda}$, R_2 the range of $I_A(\bar{s}) - \bar{\Lambda}$. In the corresponding decomposition

$$I_A(s) - \lambda = \begin{pmatrix} T_1(s) - \lambda & T_2(s) \\ T_3(s) & T_4(s) \; \lambda \end{pmatrix}$$

$T_4(s) - \lambda$ is invertible for s near $\bar{s}$ and λ near $\bar{\lambda}$. Thus, $I_A(s) - \lambda$ is invertible if and only if

$$P(s,\lambda) = T_1(s) - \lambda - T_2(s)\cdot(T_4(s) - \lambda)^{-1}\cdot T_3(s)$$

is invertible. $P(s,\lambda)$ is an operator on K_1, so taking $p_1(s,\lambda)$ to be its determinant gives an analytic function with $spec_b(A - \lambda)$ as its zero set near $(\bar{s},\bar{\lambda})$. The fact that $I_A(s) - \bar{\lambda}$ has a finite pole at $\bar{s}$ allows the Weierstrass preparation theorem to be applied to reduce p_1 to a polynomial, as in (7.10), in s, with r the order of zero of $p(s,\lambda)$ at $s = \bar{s}$.

By assumption, if $\bar{\lambda} \in \mathbb{R}\backslash\Lambda_{bad}$ then $p(s,\lambda)$ has no complex zeros near $\bar{s} \in spec_b(A - \lambda) \cap \mathbb{R}$. By the use of Pusieux series it is immediately seen that these zeros must be analytic functions of λ, proving the Lemma.

If $s_q(\lambda) \in spec_b(A - \lambda) \cap \mathbb{R}$ are the points near $\bar{s} \in spec_b(A - \bar{\lambda})$, with $\bar{\lambda} \in \mathbb{R}\backslash\Lambda_{bad}$ then use of Puiseux series shows that the orthogonal projections $\pi_q(\lambda)$, for λ real, onto the eigenspaces of $I_A(s_q(\lambda))$ are also analytic in λ. In particular the generalized eigenspaces

$$E_q(\lambda) = \text{range } \pi_q(\lambda)$$

consist just of eigenvectors, even for $\lambda \in \mathbb{C}$, $|\lambda - \bar{\lambda}| < \epsilon$. This also means that the spaces $F_{\bar{s}}$, of (6.15), after slight reorganization, have analytic bases. Notice that if $J(\bar{s})$ is replaced by $J(\bar{s}) + N$, $N \in \mathbb{N}$, in (6.15) the resulting space of polynomials $F'_{\bar{s}}$ is naturally isomorphic to $F_{\bar{s}}$:

$$F'_{\bar{s}} \ni f' = (s-\bar{s})^N f + g \ , \quad \deg(g) < N \ \mapsto f \in F_{\bar{s}},$$

and replacing $F_{\bar{s}}$ by $F'_{\bar{s}}$ in (6.16) leaves the identity unchanged.

Now, suppose $\bar{\lambda} \in \mathbb{R}\backslash\Lambda_{bad}$ and

$$spec_b(A - \bar{\lambda}) \cap \mathbb{R} = \{\bar{s}_1,\bar{s}_2,\ldots,\bar{s}_K\}. \qquad (7.11)$$

For each $j=1,\ldots,k$ let $s_q^j(\lambda)$, $\pi_q^j(\lambda)$, be the corresponding points in $\mathrm{spec}_b(A - \lambda)$, with $s_q^j(\bar\lambda) = \bar s_j$, $q=1,\ldots,n_j$, and orthogonal projections onto the eigenspaces, as above selected to be analytic in λ. If F is a normal fibration of X near ∂X , $r \in C^\infty(X)$ is a defining function for the boundary and $f \in C_c^\infty(\mathbb{R})$ has $\rho(x) = 1$ near $x = 0$ and is such that $\rho(r)$ has support in the range of F , consider the kernel obtained by transferring to $\mathring{X} \times \mathring{X}$ the Schwartz kernel of

$$(7.12) \qquad \sum_j \sum_q r^{is_q(\lambda)} \pi_q(\lambda) (r')^{-is_q(\lambda)} \operatorname{sgn}\left(\frac{ds_q(\lambda)}{d\lambda}\right) \rho(r)\, \rho(r')\, \frac{dr'}{r'} \cdot |d\lambda| .$$

Let $dE_1(\lambda)$ be the operator-valued measure obtained on $\mathbb{R}\setminus\Lambda_{bad}$ in this way.

(7.13) **THEOREM.** If $m > 0$ and $A \in \Psi_b^m(X)$ is formally self-adjoint with respect to some positive section of $C^\infty(X,\Omega_b X)$ and is elliptic on the compact manifold with boundary, X , then as an unbounded operator on $L_b^2(X)$, as in (7.1), the spectrum of A in $\mathbb{R}\setminus\Lambda_{bad}$ consists of a discrete set of eigenvalues of finite multiplicity and continuous spectrum of uniform multiplicity equal to the cardinality of $\mathrm{spec}_b(A - \lambda) \cap \mathbb{R}$, counted with multiplicity. Moreover, on the complement of the eigenspaces the spectral measure dE_{ess} , satisfies

$$(7.14) \qquad dE_{ess}(\lambda) = dE_1(\lambda) + dR(\lambda) \quad \text{on } \mathbb{R}\setminus\Lambda_{bad}$$

where $R(\lambda)$ is real-analytic on $\mathbb{R}\setminus\Lambda_{bad}$ with values in the compact operators on $L_b^2(X)$.

<u>Proof</u>. Notice that $ds_q/d\lambda \neq 0$ for all roots $s_q(\lambda)$ near $\lambda \in \mathbb{R}\setminus\Lambda_{bad}$. The spectral measure $dE(\lambda)$ can be constructed as the jump of the resolvent

$R(A,\lambda) = (A - \lambda)^{-1}$ across the real axis, so consider the behavior of the parametrix

$$G_a = B - E_a \cdot R$$

of Lemma 6.11, for $(A-\lambda)$ as $\operatorname{Im} \lambda \downarrow 0$. When $\operatorname{Im} \lambda > 0$, $\operatorname{spec}_b(A - \lambda) \cap \mathbb{R} = \emptyset$, since $I_A(s)$ is self-adjoint for $s \in \mathbb{R}$. Thus, with $a = 0$, the only points in $\operatorname{spec}_b(A - \lambda)$ which need contribute to the correction term E_0 in the parametrix are those with $\operatorname{Im}(s) > 0$, $\operatorname{Im}(s) \downarrow 0$ as $\operatorname{Im} \lambda \downarrow 0$, i.e. the $s_q(\lambda)$ with $ds_q/d\lambda > 0$. The construction of E_0 and a simple residue argument shows that

$$E' = \lim_{\operatorname{Im} \lambda \downarrow 0} E_0 - \lim_{\operatorname{Im} \lambda \uparrow 0} E_0$$

has kernel as in (7.12). Furthermore,

$$R(A,\lambda) = (B - E_0 \cdot R)(Id + S_\lambda)$$

where S_λ is meromorphic in λ with compact values. From this (7.14) follows easily.

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