THE INVERSE SPECTRAL PROBLEM FOR PLANAR DOMAINS

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1. Introduction

In these five lectures[1] on the spectral, and more particularly the *inverse* spectral theory of the Dirichlet problem in planar domains I wish to show how the investigation of a concrete mathematical question can draw on quite extensive areas of mathematics. Thus I hope that these talks will help to bind together various parts of the material you will have seen, or be seeing, elsewhere in this Workshop.

Each of the lectures will centre on one or two explicit 'results' which I will try to put into context. Then, as time permits, I will outline some of the proofs and discuss extensions and refinements of the various questions. For the most part I will refer elsewhere for details

In summary the topics of the five lectures will (probably) be:

I. Spectrum and resolvent of the Dirichlet problem
II. Heat kernel and heat invariants
III. Wave kernel and length spectrum
IV. Short geodesics
V. Polygonal domains and Other problems

2. Dirichlet Problem

Consider a *smooth planar domain* $\Omega \subset \mathbb{R}^2$. By this I mean a compact subset which is the closure of its interior $\mathring{\Omega}$ with boundary $\partial\Omega = \Omega \setminus \mathring{\Omega}$ consisting of a finite number of smooth simple, non-intersecting closed curves

$$\partial\Omega = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_N. \tag{2.1}$$

The first curve, γ_1, will always be taken as the 'outer curve', i.e. the unique curve which bounds a compact set containing Ω.

[1] In the end there were only four since I decided to spend the last Friday in Sydney at the beach in preparation for a return to the frozen north.

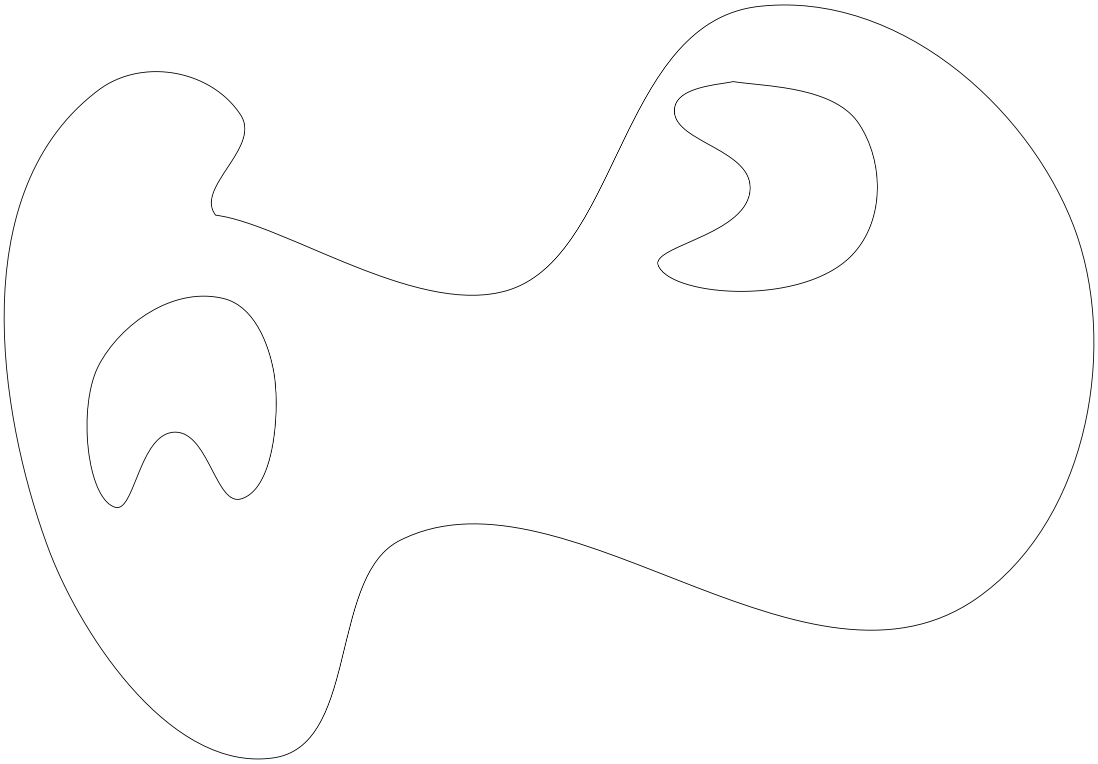

Figure 1. A planar region

The basic analytic problem I will consider is the search for solutions to

$$(\varDelta - s)u = f \text{ in } \mathring{\Omega}$$
$$u \restriction \partial\Omega = 0. \tag{2.2}$$

Here $\varDelta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}$ is the (geometer's) Laplacian, $f \in \mathcal{C}^\infty(\Omega)$ is a given function, $u \in \mathcal{C}^\infty(\Omega)$ is the unknown function and $s \in \mathbb{C}$ is a complex number. The space $\mathcal{C}^\infty(\Omega)$ deserves some discussion, it is the space of continuous functions $u : \Omega \longrightarrow \mathbb{C}$ which have derivatives of all orders

$$\frac{\partial^{j+k}}{\partial x^j \partial y^k} \text{ continuous on } \Omega. \tag{2.3}$$

3. Spectrum

The problem (2.1) is 'Fredholm' for each fixed $s \in \mathbb{C}$. This means that there are finitely many linear conditions on f which are necessary and sufficient for there to exists a solution and then there is a finite dimensional vector space of solutions, this space I will denote as $\mathrm{Eig}(s) \subset \mathcal{C}^\infty(\Omega)$. Let $D(s) = \dim \mathrm{Eig}(s)$ be its dimension. By definition s is an eigenvalue of the Dirichlet problem if $D(s) \neq 0$. The spectrum is the set of eigenvalues but I shall consider the 'augmented spectrum'

$$\mathrm{Spec}(\Omega) = \{(s, N) \in \mathbb{C} \times \mathbb{N}; \sim \in \mathbb{C} \text{ and } \mathbb{N} = \mathbb{D}(\sim) \neq \nvdash\} \tag{3.1}$$

which also contains multiplicity information.

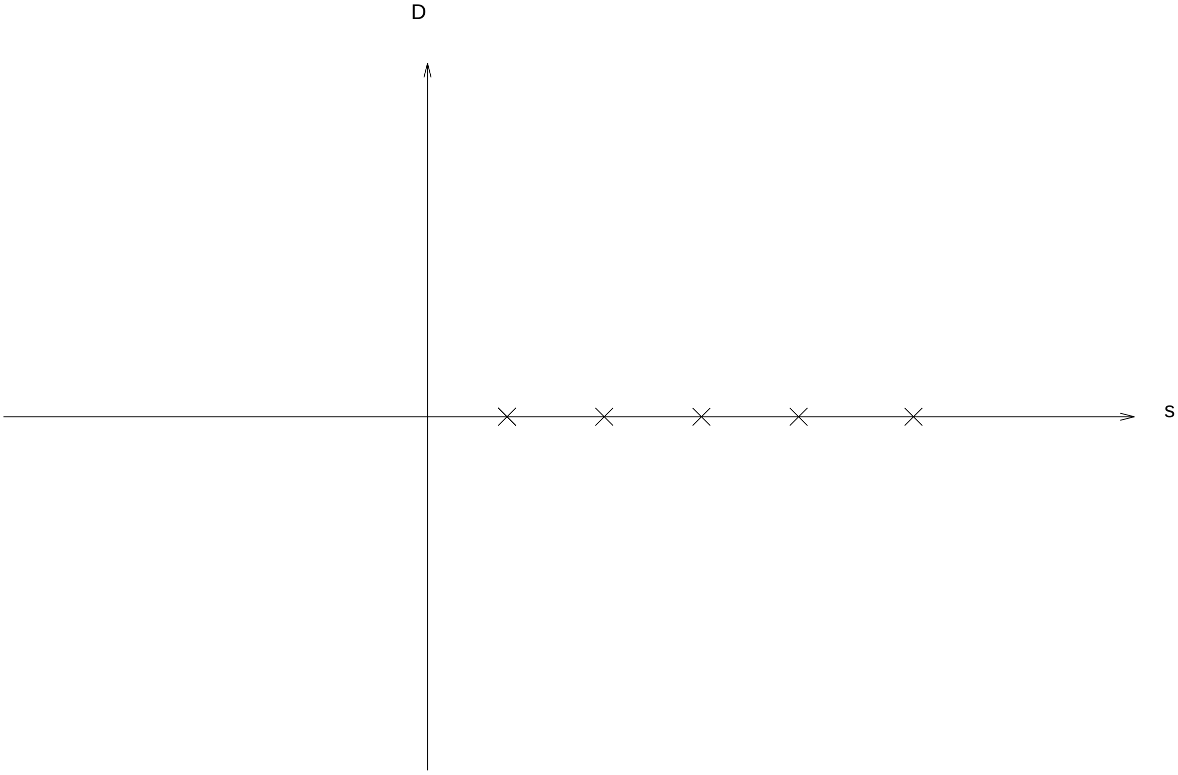

Figure 2. The augmented spectrum

Although I shall not pay too much attention to it, the Diriclet problem is self-adjoint on the appropriate domain in $L^2(\Omega)$. My intention here is to bring out the geometric aspects of these problems so I shall not make much use of such Hilbert space methods. Not only are the eigenvalues real but they form a discrete set, so another way to arrange the information in (3.1) is to write the eigenvalues as a non-decreasing sequence, with each s repeated exactly $D(s)$ times

$$0 < s_1 < s_2 \leq s_3 \leq \dots s_n \to \infty. \tag{3.2}$$

Here I have also included two additional facts, that s_1 is simple (i.e. $D(s_1) = 1$) and that there are infinitely many eigenvalues.

4. Kac's question

The basic subject of these lectures is

Does the Dirichlet spectrum determine the domain?

In this form the answer is obviously NO, since translations, rotations and reflections of Ω lead to (generally different) domains with the same Dirichlet spectrum, so we amend the question by adding

... up to rigid motion?

This is the famous question of Mark Kac ([15]) which was rephrased by Lipman Bers as 'Can one hear the shape of a drum?' In fact Kac mostly considered the question for polygonal domains,[2] rather than smooth ones. In that case, as I shall discuss at the end, the answer in this form has recently been shown to be no. However the main issues still remain open, for example for convex domains.[3] So in general it is only fair to warn you that I do not know the answer!

Before briefly reviewing the proof of the results on the spectrum that I have discussed above (they all date from the nineteenth or early twentieth century) let me mention some extensions and related questions. Essentially everything that I have said, or will say, remains true with only minor modifications for the Neumann problem in place of (2.2)

$$\begin{aligned} (\varDelta - s)u &= f \text{ in } \mathring{\Omega} \\ \partial_\nu u \restriction \partial\Omega &= 0 \end{aligned} \tag{4.1}$$

where ∂_ν is the inward pointing normal to $\partial\Omega$. The only formal difference is that 0 is a simple eigenvalue so the spectrum becomes

$$0 = s_1^N < s_2^N \le s_3^N \le \dots s_n^N \to \infty. \tag{4.2}$$

5. Dirichlet vs. Neumann

One standard characterization of the eigenvalues in the Dirichlet case is by the *mini-max principle*

$$s_j^D = \min_{u_1,\dots,u_j} \max_{0 \ne a \in \mathbb{R}^j} \left\{ \frac{\int_\Omega |\nabla u|^2}{\int_\Omega |u|^2}; u = \sum_{i=1}^{j} a_j u_j \right\} \tag{5.1}$$

[2]Mostly, I think, in view of the methods available at the time. It seems to me that the smooth case is more interesting than the polygonal, which is inherently finite dimensional.

[3]Whether polygonal or not. The recent progress for polygonal domains has all been in the form of examples.

where the minimum is taken over all j-tuples $u_1, \dots, u_j \in \mathcal{C}^\infty(\Omega)$ which satify $u \restriction \partial\Omega = 0$ and which are linearly independent.[4] The same characterization holds for the Neumann eigenvalues with the minimum being taken over the larger set of all independent j-tuples in $\mathcal{C}^\infty(\Omega)$. From this it follows that

$$s_j^N \leq s_j^D \ \forall \ j \in \mathbb{N} \tag{5.2}$$

for any smooth domain. More remarkable (and remarkably recent) is a result of L. Friedlander ([7]) which states that

$$s_{j+1}^N \leq s_j^D. \tag{5.3}$$

Note that this can be described as saying that for a drum with 'sliding' boundary the first positive frequency of vibration is smaller than for a drum with fixed boundary.

There is an old conjecture related to (5.3) which shows just how little we know. . Polyá conjectured (at least for convex domains) that

$$(s_j^N)^2 \leq 4\pi \frac{1}{A(\Omega)} j < (s_j^D)^2 \ \forall \ j \text{ and } \forall \ \Omega. \tag{5.4}$$

here $A(\Omega)$ is the area. It is known that (5.4) holds for 'generic' strictly convex domains for $j > J(\Omega)$.

6. Spectral problem

Although I am putting empahsis in these lectures on the inverse problem I should make clear that the main questions really concern the forward spectral problem. One can put this very bluntly

> Exactly which sequences (3.2) can occur as the eigenvalues of the Dirichlet problem for a (smooth) planar domain? (6.1)

This seems to be well beyond current understanding. Of course every time something is proved about the spectrum another necessary condition is added, the problem is to find non-trivial sufficient conditions for a sequence to arise from a domain. This is by way of a very non-linear Fourier transform.

So, to look for a moment on the positive side, what sort of things do we know? The results I shall describe are of the following types.

(1) Spectral invariants. Certain properties of the domain are known to be determined by the spectrum. Examples include the area and the boundary length.

[4] Try to use this characterization of the eigenvalues to see that the first eigenvalue is simple because it must correspond to a non-negative eigenfunction. Some small effort is required.

(2) Determined domains. Some domains are known to be distinguishable by their spectra amongst all domains. One example is the disk. Indeed I claim (but there is no proof written down so you shouldn't believe me) that all ellipses are so determined.
(3) Restricted problems. There are restricted classes of domains the elements of which can be distinguished, one from another, by their spectra.
(4) Counterexamples. There is a fair number of these! If one restricts the information in the spectrum in various ways (related to approaches to the problem) then one can give counterexamples to show that this restricted information is not enough to differentiatiate between certain domains (generally rather special ones).

7. The resolvent family

Now to the 'meat' of the Dirichlet problem. I have written the spectral information in the form (3.1) to make it into a *divisor* in the algebraic-geometric sense. This is natural because of

Proposition 1. *There is a unique (weakly) meromorphic family of operators* $R(s) : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega)$ *which gives the solution to* (2.2)*,* $u = R(s)f$*, for* $s \neq s_j$*, and which has a simple pole of rank* $D(s_j)$ *at each eigenvalue,*

$$R(s) = \frac{P_j}{s - s_j} + R'_j(s) \tag{7.1}$$

where $R'_j(s)$ *is holomorphic near* $s = s_j$ *and* P_j *is the self-adjoint projection onto* $\operatorname{Eig}(s_j)$*.*

I cannot give the complete proof of this proposition, which forms the basis for a large part of the subject of Functional Analysis, but I will describe the 'second half' of the proof, to do with analytic Fredhold theory, today and (maybe) discuss the first half tomorrow. I give this proof (very informally) because it is of a type that occurs in many places.

My starting point is the assumption (justified at least roughly tomorrow) of the existence of a *parametrix* for the problem. To describe what this is, let me first define the notion of a *smoothing operator*. A smoothing operator (on Ω) is a linear operator $A : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega)$ which is given by integration against a smooth kernel

$$Af(s) = \int_\Omega A(z, z')f(z')|dz'| \text{ where } A(\cdot, \cdot) \in \mathcal{C}^\infty(\Omega \times \Omega). \tag{7.2}$$

Here $|dz| = dxdy$ is just the usual Lebesgue measure. I will write $\Psi^{-\infty}(\Omega)$ for the space of smooth operators (considered as the space of pseudodifferential operators of order $-\infty$). Thus $\Psi^{-\infty}(\Omega)$ is just a pseudonym for the space $\mathcal{C}^\infty(\Omega\times\Omega)$ viewed as a space of operators. It is important that it is an algebra[5]

$$A, B \in \Psi^{-\infty}(\Omega) \Longrightarrow A\circ B \in \Psi^{-\infty}(\Omega). \tag{7.3}$$

Of course $\mathcal{C}^\infty(\Omega\times\Omega)$ is also an algebra under pointwise multiplication, the algebra structure in (7.3) is different, it is for instance non-commutative.

By a parametrix for (2.2) I shall mean an *entire* family of operators $E(s) : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega)$ satisfying

$$\begin{gathered}(\varDelta - s)E(s) = \operatorname{Id} + A(s) \text{ in } \overset{\circ}{\Omega} \text{ with } A(s) \in \Psi^{-\infty}(\Omega) \text{ and} \\ E(s)f \restriction \partial\Omega = 0 \ \forall\ f \in \mathcal{C}^\infty(\Omega).\end{gathered} \tag{7.4}$$

Thus, $E(s)$ solves the problem up to a smoothing error.

8. Analytic Fredholm theory

So, let us assume that such a parametrix exists and see how to construct $R(s)$. This depends really on the properties of $\Psi^{-\infty}(\Omega)$, rather than those of $E(s)$.

Lemma 1. *If $A(s)$ is an entire family of smoothing operators and ϵ, $R > 0$ are given then there is a finite rank entire smoothing operator*

$$\begin{gathered}(A'(s)f)(z) = \int A'(s; z, z')f(z')|dz'| \text{ where} \\ A'(s; z, z') = \sum_{j=1}^{M} f_j(s; z)f'_k(s; z')\end{gathered} \tag{8.1}$$

with the f_j, $f'_j \in \mathcal{C}^\infty(\Omega)$ for $j = 1, \dots, M$ depending holomorphically on s and

$$\sup_{\Omega\times\Omega, |s|\le R} |A(s; z, z') - A'(s; z, z')| \le \epsilon. \tag{8.2}$$

Proof. This can be shown using Fourier series (or many other approximation methods). □

Lemma 2. *If $B \in \Psi^{-\infty}(\Omega)$ has kernel satisfying*

$$\sup_{(z,z')\in\Omega\times\Omega} |B(z, z')| < 1/A(\Omega) \tag{8.3}$$

[5] This is really a form of Fubini's theorem; check that you see why it is true.

then $\mathrm{Id}+B : \mathcal{C}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega)$ *is invertible with inverse*

$$(\mathrm{Id}+B)^{-1} = \mathrm{Id}+G,\ \ G \in \Psi^{-\infty}(\Omega). \tag{8.4}$$

Proof. The Neumann series for G

$$G = \sum_{j=1}^{\infty} (-1)^j B^j \tag{8.5}$$

converges in $\mathcal{C}'(\Omega \times \Omega)$ because of (8.3). The sum can be seen to be in $\mathcal{C}^\infty(\Omega \times \Omega)$, i.e. $\Psi^{-\infty}(\Omega)$, since $G = -B + B^2 - B \circ G \circ B$.[6] □

Using these lemmas the resolvent family $R(s)$ can be related to the parametrix $E(s)$. First we use Lemma 1 to improve the parametrix, for $|s| < R$. Thus we can write $A(s) = A'(s) + A''(s)$ where $A'(s)$ is finite rank as in (8.1). Choosing ϵ small enough we can arrange that (8.3) holds for $A''(s) = A(s) - A'(s)$ for all $|s| < R$, so $(\mathrm{Id}+A''(s)) \circ (\mathrm{Id}+G(s)) = \mathrm{Id}$ where $G(s) \in \Psi^{-\infty}(\Omega)$. Applying the operator $\mathrm{Id}+G(s)$ on the right to both sides of the first equation in (7.4) reduces it to

$$(\mathit{\Delta} - s)E'(s) = \mathrm{Id}+F(s),\ \ F(s) = A'(s) + A'(s) \circ G(s) \in \Psi^{-\infty}(\Omega). \tag{8.6}$$

In fact $F(s)$ is finite rank.

It follows that for each $|s| < R$ (where in fact R is arbitrary) the problem is Fredholm in the sense that if f satisfies the finite number of conditions $F(s)f = 0$ then $u = E'(s)f$ solves (2.2). To see that there is only a finite dimensional space of solutions we can use duality. Suppose $v \in \mathrm{Eig}(s)$ for some s and $u \in \mathcal{C}^\infty(\Omega)$ satifies $u \restriction \partial\Omega = 0$. Integration by parts (Green's theorem) then gives

$$\int_\Omega v(z)(\mathit{\Delta} - s)u(z)|dz| = \int_\Omega \big((\mathit{\Delta} - s)v(z)\big)u(z)|dz| = 0. \tag{8.7}$$

Thus, if there is a solution, u to (2.2) for a given f then

$$\int_\Omega v(z)f(z)|dz| = 0 \ \forall\ v \in \mathrm{Eig}(s). \tag{8.8}$$

This gives $D(s) = \dim \mathrm{Eig}(s)$ independent linear constraints to be satisfied by f for (2.2) to have a solution. It follows that $\dim \mathrm{Eig}(s)$ must be finite. In fact this argument can be reversed to show that (8.8) gives all the constraints for solvability of (2.2).

[6]This is the bi-ideal property of smoothing operators. If A and B are smoothing operators and D is, for instance, a bounded operator on $L^2(\Omega)$ then $A \circ D \circ B$ is a smoothing operator.

It remains only to show that all eigenvalues are real and that they form a discrete set in the real line. The reality of the eigenvalues follows from Green's formula

$$\int_\Omega u(z)(\Delta - s)u(z)|dz| = \int_\Omega \big(|\nabla u|^2 - s|u|^2\big)|dz| = 0. \tag{8.9}$$

if $u = 0$ on $\partial\Omega$ and u is an eigenfunction. The discreteness follows from (8.6) using the finiteness of the rank of $F(s)$.

The main difficulty of the inverse spectral problem is, obviously enough, the extraction of information from the spectrum. The individual eigenvalues are functions of the domain, but it is difficult to come to grips with such functions, especially collectively. It is therefore natural to try reorganize and filter the information in some helpful way. There are various approaches and 'transforms' which have been used. The first one I will talk about is the heat transform and associated invariants.

9. Heat equation

The analytic problem on which the effectiveness of the heat transform is based is the initial value problem for the heat operator

$$\begin{gathered}(\partial_t + \Delta)v = 0 \text{ in } [0,\infty)\times\Omega \\ v \restriction \{t=0\} = v_0 \in \dot{\mathcal{C}}^\infty(\Omega),\ v\restriction [0,\infty)\times\partial\Omega = 0,\end{gathered} \tag{9.1}$$

where $v \in \mathcal{C}^\infty([0,\infty)\times\Omega)$ and $\dot{\mathcal{C}}^\infty(\Omega) \subset \mathcal{C}^\infty(\Omega)$ is the subspace of functions vanishing to infinite order at the boundary.

Proposition 2. *The initial value problem for the heat equation,* (9.1)*, has a unique solution* $v \in \mathcal{C}^\infty([0,\infty)\times\Omega)$ *for each* $v_0 \in \dot{\mathcal{C}}^\infty(\Omega)$ *and the operators so defined*

$$e^{-t\Delta} : \dot{\mathcal{C}}^\infty(\Omega) \longrightarrow \mathcal{C}^\infty(\Omega),\ e^{-t\Delta}v_0 = v(t,\cdot) \tag{9.2}$$

are, for $t > 0$*, a semigroup of smoothing operators.*

The semigroup property here is the condition $e^{-t\Delta}\circ e^{-s\Delta} = e^{-(t+s)\Delta}$, $t, s > 0$, As usual I shall only give an indication of the proof, and that towards the end of the lecture. The proof I have in mind is rather constructive and it gives quite a bit of information about the heat kernel, which is the function $H \in \mathcal{C}^\infty((0,\infty)\times\Omega\times\Omega)$ such that

$$e^{-t\Delta}v_0(z) = \int_\Omega H(t,z,z')v_0(z')|dz'|,\ t > 0, \tag{9.3}$$

the existence of which is guaranteed by the proposition.

10. Trace

Let me consider some further properties of smoothing operators. The trace of $A \in \Psi^{-\infty}(\Omega)$ is defined to be

$$\mathrm{Tr}(A) = \int_\Omega A(z,z)|dz|. \tag{10.1}$$

That is, $\mathrm{Tr}(A)$ is the integral of the restriction of the kernel of A to the diagonal. One reason for interest in the trace functional

$$\mathrm{Tr} : \Psi^{-\infty}(\Omega) \longrightarrow \mathbb{C} \tag{10.2}$$

is Lidskii's theorem which states that

$$\mathrm{Tr}(A) = \sum_j \lambda_j(A) \tag{10.3}$$

where the $\lambda_j(A)$ are the eigenvalues of A repeated according to their (algebraic) multipicity, which is finite for $\lambda_j(A) \neq 0$. If $A(z',z) = \overline{A(z,z')}$ is Hermitian symmetric then the $\lambda_j(A)$ are necessarily real and the multiplicity is just the dimension of the associated eigenspace. In particular this is the case for $e^{-t\Delta}$. Moreover it is easy to see that the eigenvalues of $e^{-t\Delta}$ are the numbers $\lambda_j = e^{-ts_j}$ where s_j are the eigenvalues of the Dirichlet problem and the multiplicity is exactly $D(s_j)$. Thus we see that the function

$$h(t) = \mathrm{Tr}\left(e^{-t\Delta}\right) = \sum_j e^{-ts_j},\ t > 0, \tag{10.4}$$

is a spectral invariant.

11. Heat invariants

As a spectral invariant $h(t)$, for any positive value of t, is not much different to the individual s_j; in brief it is difficult to say much about it. The important question is what happens as $t \downarrow 0$. Notice that $e^{-t\Delta}v_0 \to v_0$ as $t \downarrow 0$ for $v_0 \in \dot{\mathcal{C}}^\infty(\Omega)$. Thus, in some weak sense, $e^{-t\Delta} \to \mathrm{Id}$ as $t \downarrow 0$. Clearly then something has to happen to $h(t)$ because the right side of (10.4) is formally infinite at $t = 0$. The most fundamental point I want to make is that *singularities are interesting.* Here is a basic example.

Proposition 3. *As $t \downarrow 0$ the heat transform has a complete asymptotic expansion*

$$h(t) \sim c_0 t^{-1} + b_0 t^{-\frac{1}{2}} + c_1 t^0 + \sum_{j>0} b_j t^{j-\frac{1}{2}} + \sum_{j>1} c_j t^{j-1}. \tag{11.1}$$

First, let me make sure that you understand the meaning of the relation $\sim$ in (11.1). The right hand side is a *formal* sum of the type

$$\sum_p a_p t^{-1+p/2} \tag{11.2}$$

for some constants a_p. The precise meaning of (11.1) is that, given any $J \in \mathbb{N}$ there is a constant C_J such that

$$|h(t) - \sum_{j<J} a_j t^{-1+j/2}| \leq C_J t^{-1+J/2} \text{ in } 0 < t < 1. \tag{11.3}$$

Here the term on the right is just the size of the first missing term on the left. In fact a stronger form of this statement is true with similar estimates for the derivates (essentially what one would get by differentiating both sides of (11.3).) The most important point is that the constants c_j and b_j are determined by (11.1). Another way to look at the meaning of (11.1), together with all the relations for the derivatives, is that the function $r^2h(r^2)$ is $\mathcal{C}^\infty$ down to $r = 0$ and (11.1) is its Taylor series expansion.

The constants are the *heat invariants* of the domain Ω. There is no known universal formula for them but their structure can be described fairly easily. The first three are

$$c_0 = c_U A(\Omega),\ b_0 = c_U L(\Omega),\ c_1 = c_U \chi(\Omega). \tag{11.4}$$

Here I am using c_U to denote various constants which are universal (independent of Ω) and non-zero. The functions A, L and χ are respectively the area, the length of the boundary and the Euler characteristic of the domain, the latter being $N-1$ where N, in (2.1), is the number of boundary components.

12. Disks are determined

Before describing the higher heat invariants let me give the first, simple, inverse spectral result (due I believe to Kac).

Theorem 1. *Amongst planar domains the disks are determined by their Dirichlet spectra.*

Proof. The proof is the isoperimetric inequality which states that for any planar domain

$$A \leq \frac{L^2}{4\pi} \tag{12.1}$$

with equality only for disks. $\square$

13. The heat invariants are not enough

This seems a good start towards the inverse spectral problem and spurs one on to look at the higher invariants. To describe these I have to briefly consider the parameterization of domains. Let me first restrict myself to simply connected domains, i.e. those having only one boundary component. The domain is determined once we know γ, its bounding curve so the simplest thing is to consider the arclength parameterization of curves. The curve is then described in terms of its curvature $\kappa \in \mathcal{C}^\infty(\mathbb{R}/\mathbb{L})$ as a function of arclength. If $\Theta(s)$ is the angle between tangent and x-axis then

$$\kappa(s) = \frac{d}{ds}\Theta(s). \tag{13.1}$$

The curve, up to rigid motion, can be recovered from κ.

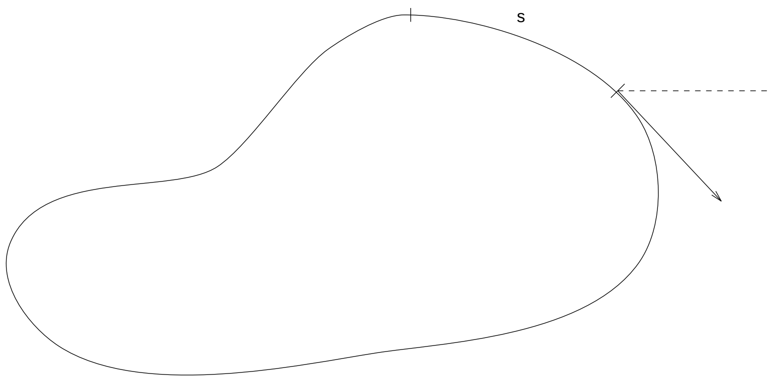

Figure 3. Tangent angle to a curve

All the heat invariants, other than c_0, are integrals over arclength of polynomials in the curvature and its derivatives. The interesting ones are the b_j which can be written

$$b_j = \int_0^L B_j(\kappa, \kappa', \dots, \kappa^{(j-1)})ds \tag{13.2}$$

for a polynomial B_j. Here $\kappa^{(p)} = d^p\kappa/ds^p$ is the pth derivative of κ with respect to arclength. Even more can be said, namely that

$$B_j(T_0, T_1, \dots, T_{j-1}) = c_U T_{j-1}^2 + B_j'(T_0, T_1, \dots, T_{j-2}). \tag{13.3}$$

There is also an overall homogeneity property in that

$$B_j(sT_0, s^{1+\frac{1}{2}}T_1, \dots, s^{1+(j-1)/2}T_{j-1}) = s^{2j}B_j(T_0, T_1, \dots, T_{j-1}) \ \forall\ s > 0. \tag{13.4}$$

The c_j, for $j \geq 1$ are similar. Before considering some positive conclusions that we can draw from these properties let me note a nasty counterexample.

Example 1. *There is a family, depending smoothly on any given finite number of parameters, of strictly convex domains no two of which are equivalent under rigid motions but on which all the heat invariants are constant.*

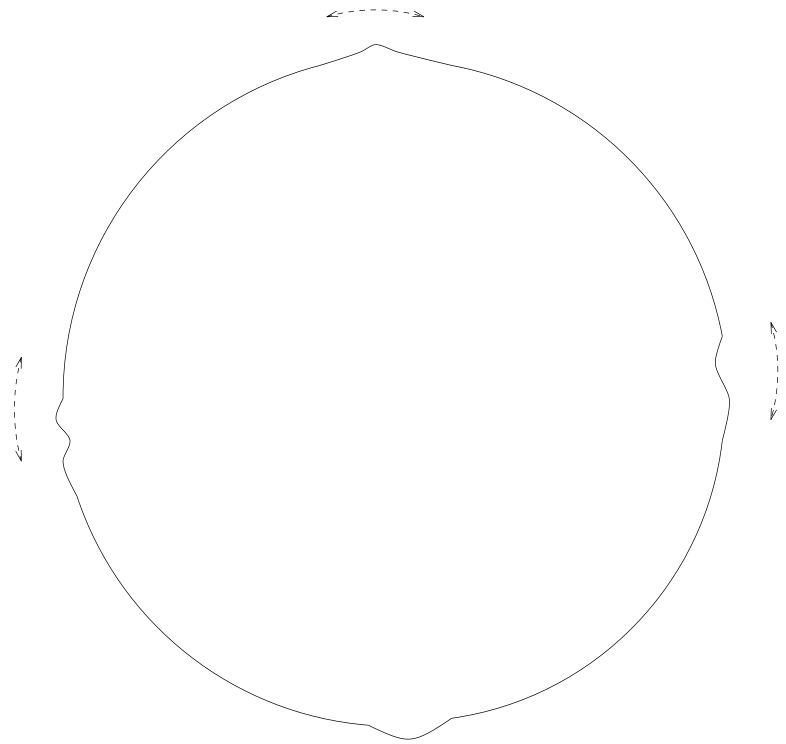

FIGURE 4. Deformation with constant heat invariants

Description. Start with a circle. Then make a finite number (one more than the number of parameters desired) of small smooth disjoint deformations of the boundary. This can be done so that the domain remains convex. Now consider the domains obtained by 'sliding' these perturbations with respect to each other around the original circle. The heat invariants are constant since the integrals (13.2) are merely being rearranged. □

These rather simple counterexamples really suggest many more questions. For example, do the heat invariants determine real analytic domains? Do they locally determine domains under perturbation from a small, generic domain (in particular one which is not the circle!) I don't know any results in this direction. The obstruction seems to be finding the coefficients of the polynomials B_j in sufficient detail.

14. Boundedness of isospectral sets in the $\mathcal{C}^\infty$ topolgy

Althought this may seem disheartening, notice that in passing from $h(t)$ (which has exactly the same information as $\operatorname{Spec}(\Omega)$) to the heat invariants we have dropped information. Next time I will describe some methods for keeping more of this lost information. Let me note one consequence of the properties of the heat invariants which can be found in [23].

Proposition 4. *For any isospectral set of domains (i.e. set of domains with the same set* $\operatorname{Spec}(\Omega)$*) there are uniform bounds on all the derivatives of the curvature (of all the components of the boundary)*

$$\sup_s |\frac{d^p\kappa(s)}{ds^p}| \le C_p \tag{14.1}$$

with C_p *independnet of* Ω *in the isospectral set.*

Proof. This follows from (13.3) and use of the Sobolev embedding theorem. □

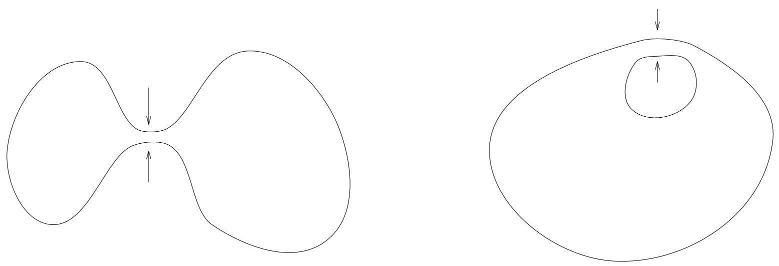

Figure 5. Non-compactness with curvature bounds

The estimates (14.1) show the 'precompactness' of the isospectral sets. They don't quite show compactness because they do not show that the isospectral sets are closed. Indeed for sets of domains satisfying uniform estimates (14.1) there is still the possibility that two distant parts of the boundary will approach each other along a sequence. If there are several boundary components it is equally possible that one boundary component might approach another. Later I will describe methods which show that the isospectral sets are indeed compact and that such degeneration cannot occur.

15. Zeta function

The next analytic object I want to talk about is the zeta function (treated first in this sort of centext by Seeley in [26]). This is really rather closely associated to the heat transform. Indeed with $h(t)$ defined by the first equation in (10.4) the zeta function is given in terms of the Mellin transform of $h(t)$

$$\zeta(\tau) = \frac{1}{\Gamma(s)} \int_0^\infty h(t) t^{-\tau} \frac{dt}{t}. \tag{15.1}$$

Formally inserting the second part of (10.4) gives

$$\zeta(\tau) = \sum_j s_j^\tau. \tag{15.2}$$

In fact the series in (15.2) converges for $\Re\tau < -\frac{1}{2}$ and then the equality holds.

Lemma 3. *The zeta function extends to a meromorphic function with simple poles only at the points $\tau = -\frac{1}{2} + j/2$ for $j = 0, 1, 2, \ldots$ and in fact there is no pole at $\tau = 0$. The residues at the poles (together with the value at $\tau = 0$) can be expressed in terms of the heat invariants and conversely.*

16. Determinant

Thus the singularities of $\zeta(\tau)$ carry the same information as the heat invariants. However there is at least one extra invariant of some significance which can be extracted from $\zeta(\tau)$. Namely if (15.2) is formally differentiated and then evaluated at $\tau = 0$

$$\zeta'(0) \text{``}=\text{''} \sum_j \log s_j. \tag{16.1}$$

Thus it is reasonable to *define* the *determinant* of the Dirichlet problem to be

$$\det(\varDelta) = \exp(\zeta'(0)). \tag{16.2}$$

Unlike the heat invariants themselves this is not a local invariant, i.e. cannot be expressed as a local integral of the curvature and its derivatives. However its properties were analyzed by Osgood, Phillips and Sarnak ([23]) enough to see that

Proposition 5. *Any set of simply connected domains on which all the heat invariants and the determinant are fixed is compact in the sense that they are uniformly smoothly embedded.*

In particular this shows that isospectral sets are compact in the $\mathcal{C}^\infty$ topology and cannot degenerate along any sequence.

There are difficulties treating non-simply connected domains this way so I shall instead deduce this result from a discussion of the wave trace, to which I now turn.

17. Wave equation

This third and last transform of the spectral data is based on the initial value problem for the wave equation

$$\begin{aligned} (D_t^2 - \Delta)w = 0 \text{ in } \mathbb{R} \times \Omega,\ \precapprox \restriction \mathbb{R} \times \partial\Omega = \nvdash \\ w \restriction \{t = 0\} = w_0,\ D_t w \restriction \{t = 0\} = w_1. \end{aligned} \tag{17.1}$$

Here $D_t = \frac{1}{i}\partial/\partial t$. For each pair $(w_0, w_1) \in \dot{\mathcal{C}}^\infty(\Omega)$ there is a unique solution $w \in \mathcal{C}^\infty(\mathbb{R} \times \Omega)$ to (17.1). This existence and unqueness theorem can be encapsulated in the existence and properties of the *wave group*

$$U(t)\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = \begin{pmatrix} w(t, \cdot) \\ D_t w(t, \cdot) \end{pmatrix}. \tag{17.2}$$

This extends to a group of operators on the 'domain of Δ^∞' which is to say on the space of functions $\mathcal{C}^\infty_D(\Omega)^2$ where

$$\mathcal{C}^\infty_D(\Omega) = \left\{ w \in \mathcal{C}^\infty(\Omega); \Delta^j w \restriction \partial\Omega = 0\ \forall\ j = 0, 1, \dots \right\}. \tag{17.3}$$

The group property $U(r) \circ U(s) = U(r+s)$ follows from the uniqueness of solutions to (17.1). Amongst the many interesting properties of this wave group are the *finite propagation speed*

$$w_0(z), w_1(z) = 0 \text{ in } \{|z - z'| < r\} \Longrightarrow w(t, z) = 0 \text{ in } \{|z - z'| < r - |t|\}. \tag{17.4}$$

18. Trace of the wave group

As distinct from the heat semigroup the operators $U(t)$ can never be smoothing, since they are invertible! So to take the trace we need to smooth them. This can be done by averaging in the time variable. Let $\mathcal{S}(\mathbb{R})$ be Schwartz space of $\mathcal{C}^\infty$ functions of rapid decrease at infinity.

Lemma 4. *For any $\phi \in \mathcal{S}(\mathbb{R})$ the averaged wave group*

$$U(\phi) = \int_{\mathbb{R}} \phi(t) U(t) dt \tag{18.1}$$

is a 2×2 matrix of smoothing operators on Ω with trace

$$\operatorname{Tr} U(\phi) = \sum_{\lambda_j^2 = s_j} \hat{\phi}(\lambda_j) \tag{18.2}$$

where

$$\hat{\phi}(\lambda) = \int e^{-i\lambda t}\phi(t)dt \tag{18.3}$$

is the Fourier transform of ϕ.

The sum in (18.2) is over those real numbers $\lambda_j = \pm s_j^{\frac{1}{2}}$ (since the s_j are positive.) It is quite easy to see where (18.2) comes from and not too hard to prove it. Formally the wave group can be written as a function of the Laplacian with Dirichlet boundary condition

$$U(t) = \begin{pmatrix} \cos(t\sqrt{\Delta}) & \sin(t\sqrt{\Delta})/\sqrt{\Delta} \\ \sqrt{\Delta}\sin(t\sqrt{\Delta}) & \cos(t\sqrt{\Delta}) \end{pmatrix} \tag{18.4}$$

This leads one to expect that $\operatorname{Tr} U(t) = 2\operatorname{Tr}\cos(t\sqrt{\Delta})$. Thus an eigenvalue s_j of Δ should contribute a term $2\cos(t\lambda_j) = e^{i\lambda_j t} + e^{-i\lambda_j t}$ where $\lambda_j^2 = s_j$. This shows why (18.2) and (18.3) should hold.

Another way of writing (18.2) is to observe that $\operatorname{Tr} U(\phi)$ defines a (tempered) distribution which I will denote $\theta(t) \in \mathcal{S}'(\mathbb{R})$. Then (18.2) can be rewritten

$$\theta(t) = \mathcal{F}\sigma \text{ where } \mathcal{F} \text{ is the Fourier transform and}$$
$$\sigma(\lambda) = \sum_{\lambda_j^2 = s_j} \delta(\lambda - \lambda_j). \tag{18.5}$$

This is sometimes called the *Poisson formula* for (the Dirichlet problem on) Ω.

19. Poisson relation

As always the usefulness of the transform $\theta(t)$ of the spectral data lies in our ability to say something about $\theta(t)$ directly, other than using (18.5). This amounts to studying the hyperbolic problem (17.1). For the moment let me discuss, without precisely defining, the notion of a geodesic for the domain Ω. As we shall see, these curves arise in the study of $\theta(t)$. A geodesic is a curve in Ω, i.e. a continuous map

$$\chi : I \longrightarrow \Omega \tag{19.1}$$

for some non-empty interval $I \subset \mathbb{R}$ (it could be open, closed or infinite at either end but I do not want it to be either empty or just to consist of one point; more precisely it is an infinite connected subset of $\mathbb{R}$) which near each interior point (of Ω) is a straight line segment parameterized by arclength. At a boundary point of Ω the curve is supposed to satisfy 'Snell's Law' of equal angle reflection. For the moment I do not want to make this precise. A geodesic is closed if $I = \mathbb{R}$ and χ is a periodic function with some period L. Let $\mathcal{L} \subset (0, \infty)$ denote the set of lengths of closed reflected geodesics in Ω. Then the *Poisson relation* states that

$$\operatorname{singsupp}(\theta) \subset -\mathcal{L} \cup \{0\} \cup \mathcal{L}. \tag{19.2}$$

Note that, by definition, a point $\bar{t}$ is in the singular support of θ if θ is not equal to a smooth function in any open neighbourhood of $\bar{t}$.

The relation (19.2), proved in [11] (see also [10]), is part of the general principle that singularities are interesting. It is unfortunate that it is only an inclusion, but what it is *trying* to tell us is that the length spectrum is a spectral invariant of Ω. Unfortunately this is not known to be the case since there is the possibility of cancelation; in a certain sense every closed geodesic contributes a singularity to θ, all that can happen is that two (or more) such contributions from geodesics of the same length may cancel out. Under certain circumstance the inclusion in (19.2) can be reversed. Before saying something about this let me first discuss the singularity at $t = 0$.

20. Weyl asymptotics

The singularity of θ at $t = 0$ turns out to be a simple one. First note that it is always isolated, i.e. there cannot be a sequence of closed reflected geodesics of length $L_j > 0$ such that $L_j \to 0$. Thus the set $\mathcal{L}$ has a positive infimum $L_{\min}$. Take a smooth function $\rho \in \mathcal{C}^\infty(\mathbb{R})$ which is identically equal to 1 in a neighbourhood of 0 and which vanishes in say $|t| > \frac{1}{2}L_{\min}$. Let us take the Fourier transform of θ localized near 0. This gives a smooth function which turns out the have a complete asymptotic expansion as $\lambda \to \infty$

$$\widehat{\rho\theta}(\lambda) \sim \sum_j a_j \lambda^{2-j}, \ \lambda \to \infty. \tag{20.1}$$

The coefficients here are therefore spectral invariants. They do not depend on the choice of ρ and might therefore appear to be very interesting. Alas, they tell us nothing new because they are in reality simply the old heat invariants (slightly reorganized.)

This might indicate that the singularity of θ at $t = 0$ is of no particular interest; from the point of view of inverse spectral theory this is true since the same information can be gleened from $h(t)$. However the mere *existence* of the expansion (20.1) *does not* follow from any reasonable properties of $h(t)$. This alone has an important consequence, known as 'Weyl's law' for the growth rate of the eigenvalues. Put $N(\lambda) = \max\{j; 0 < \lambda_j \le \lambda\}$. This just counts the number of eigenvalues of the Dirichlet problem, with multiplicity, which satisfy $s_j \le \lambda^2$. Then

$$N(\lambda) = c_U A\lambda^2 + O(\lambda) \text{ as } \lambda \to \infty. \tag{20.2}$$

This notation means that there is a constant C (which depends on Ω) such that

$$|N(\lambda) - c_U A\lambda^2| \le C\lambda,\ \lambda > 1. \tag{20.3}$$

I should point out that (20.1) is not so easy to prove, basically because no way has (yet) been found to construct the singularities of the wave kernel for a general planar domain (it has been done in both the strictly convex and strictly concave cases). The result (20.1) is due to Ivrii [14]. The estimate (20.2) was first proved by Seeley, [27], [28] by a slightly different method.

The estimate (20.2) can be improved, under the condition that the set of closed geodesics has measure zero, to

$$N(\lambda) = c_U A\lambda^2 + c_U L\lambda + o(\lambda) \tag{20.4}$$

where the 'small oh' notation means that

$$\begin{gathered} \text{Given } \epsilon > 0,\ \exists\, T \text{ such that} \\ |N(\lambda) - c_U A\lambda^2 - c_U L\lambda| \le \epsilon|\lambda| \text{ for } \lambda > T. \end{gathered} \tag{20.5}$$

This is due to Ivrii, [14], extending earlier ideas of Duistermaat and Guillemin [6]. As far as I am aware, for planar domains, it is not known whether (20.5) always holds; in particular it is not known whether for smooth planar domains the measure of the set of closed reflected geodesics can be positive (the half sphere is an example among manifolds with boundary).

21. Non-communicating rooms

Later I shall talk about some positive results, including further spectral invariants, which follow from the study of the non-zero singularities of $\theta(t)$. First let me cast the usual pall on proceedings! Here the 'data' one might hope to read off from θ is concerned with its singularities. Thus two domains with wave traces which differ by a smooth functions would be troubling. Such an example was found by Michael Lifshits some years ago (but not published); see also the paper of Rauch [25].

Example 2. *There are two smooth planar domains which are not equal up to rigid motions yet whose wave traces differ by a smooth function.*

Before describing this example I have to be a little more precise about the meaning of a reflected geodesic than I have been up to now.

Definition 1. *A reflected geodesic is a curve*

$$\chi : I \longrightarrow \Omega \tag{21.1}$$

such that there is an open dense subset of the parametrizing interval $I' \subset I$ *on each maximal open interval of which* χ *is smooth, parametrized by arclength and is either a straight line segment or a segment of the boundary near each point of which* Ω *is locally convex. Furthermore the following conditions hold at points of* $I'' = I' \cap \overset{\circ}{I}$

(1) $\chi(I'') \subset \partial\Omega$.
(2) *If* $t' \in I''$ *is the end point of an interval of* I' *such that* χ *meets the boundary transversally at* $\chi(t')$ *then* t' *separates two intervals of* I' *with equal angle reflection at* $\chi(t')$.
(3) *There is no point* $t' \in I''$ *such that the region is strictly concave near* $\chi(t')$.
(4) *For any other point of* I'', *the curvature of the boundary vanishes at* $\chi(t')$ *and* χ *is differentiable and tangent to the boundary.*

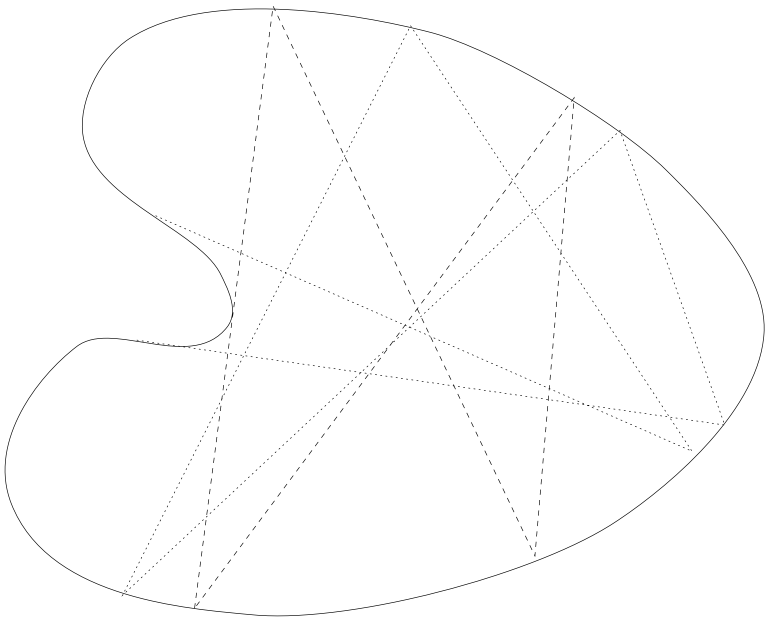

FIGURE 6. Two closed $\mathcal{C}^\infty$ geodeics

With this definition of a reflected geodesic (sometimes called a $\mathcal{C}^\infty$ geodesic) each curve χ can be extended maximally to a reflected geodesic defined on $\mathbb{R}$. On certain domains with boundary points at which the curvature vanishes to infinite order there may be non-uniqueness of such an extension (somewhat contrary to intuition), the first example was given by Taylor ([31]). This cannot happen if the domain is real analytic. The length spectrum $\mathcal{L}$ is the set of periods of maximally extended geodesics.

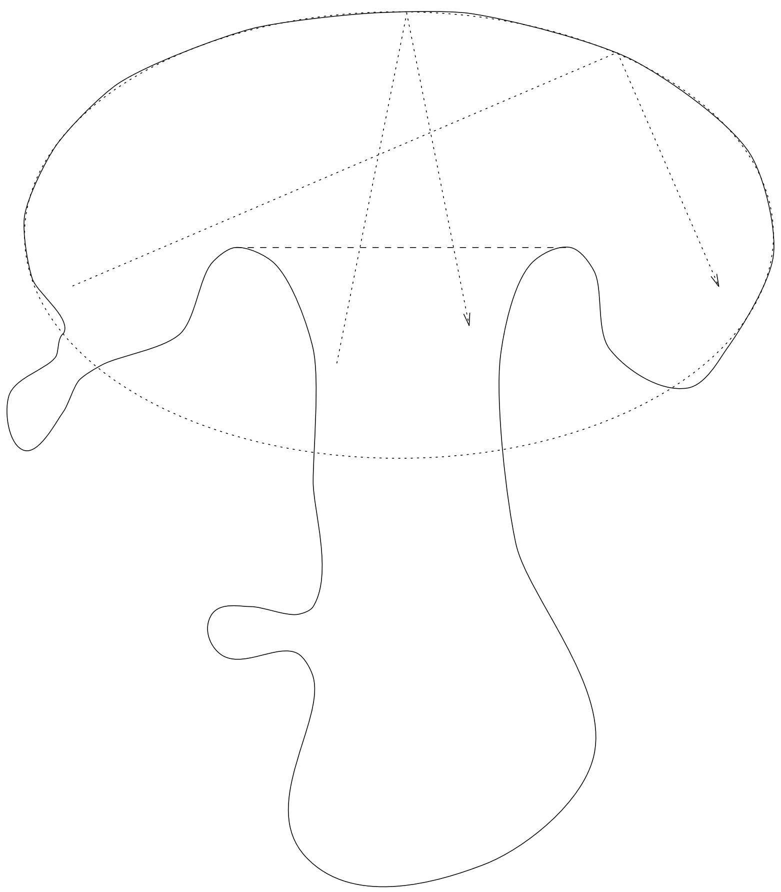

FIGURE 7. Non-communicating rooms

Description of Example 2. This construction starts with a non-circular ellipse. The ellipse is unusual in that one can describe its closed geodesics in essentially complete detail. Let me just note the fundamental property it has, which is that any geodesic passing upwards across the major semiaxis, between the foci, is reflected from the top of the ellipse back across the major semiaxis between the foci. We then make a smooth modification of the bottom half of the ellipse as pictured, adjusting the boundary curve so that it touches the major semiaxis precisely at the two foci and at which it is simply tangent.

We also insist that the part of the bounding curve above some line parallel to the major semiaxis but below it is symmetric under the reflection around the minor semiaxis. Now, the geodesics can be divided into two classes; namely the first class consists of those which either meet the major semiaxis between the foci or lie completely in the 'tongue' between and below the foci. The second class consists of the rest.

Now consider the domain obtained by reflecting the tongue around the minor semiaxis, but not the rest of the domain. This is smooth because of the symmetry insisted on. For an appropriate choice of domain it is not rigidly equivalent to the original but it does have all geodesics of the same length. A little further investigation shows that in fact the wave traces of these two domains differ by a smooth function. The reason for this is a result which can briefly be described (somewhat crudely) as saying that the singularities of solutions to the wave equation (17.1) travel along $\mathcal{C}^\infty$ geodesics, see [20], [21] and [13]. □

Notice that in this example there are open sets within the connected region which cannot communicate with each other directly, i.e. no geodesic from one can pass into the other. One can elaborate on this example to provide arbitrarily large finite sets of domains with the corresponding θ's all equal modulo $\mathcal{C}^\infty(\mathbb{R})$ but with no two equivalent under rigid motions. [Hint, chain together many copies of examples as above.] I know of no continuous family of such domains.

22. Finer singularities

One subtlty here is the notion of smoothness. I have been talking about $\mathcal{C}^\infty$ singularities. The example above in all likelihood fails if the notion of singularity is refined, for instance to real analyticity. However it is very difficult to analyze these finer singularities for anything other than real analytic domains. There is no counterexample that I know of to the statement that for real-analytic domains the singularities of θ, modulo real analytic functions determine the domain (up to rigid motion). There is, for strictly convex domains, even some hope that this is true. For a discussion of Gevrey regularity for the wave equation see the work of Hargé and Lebeau [12] and references therein (especially [16]).

23. Billiard ball map

Now I want to draw some information from the Poisson trace formula (18.5). First I will consider strictly convex domains, then discuss two results arising from the transversally reflected closed geodesics.

So, let me first discuss strictly convex domains, those for which the curvature of the boundary is everywhere strictly negative. In this case the definition of geodesics simplifies. Clearly the case 4 of Definition 1 cannot occur. It can be shown that any geodesic is either some part of the boundary curve or else consists of transversally reflected segments, with the points of reflection forming a discrete set.

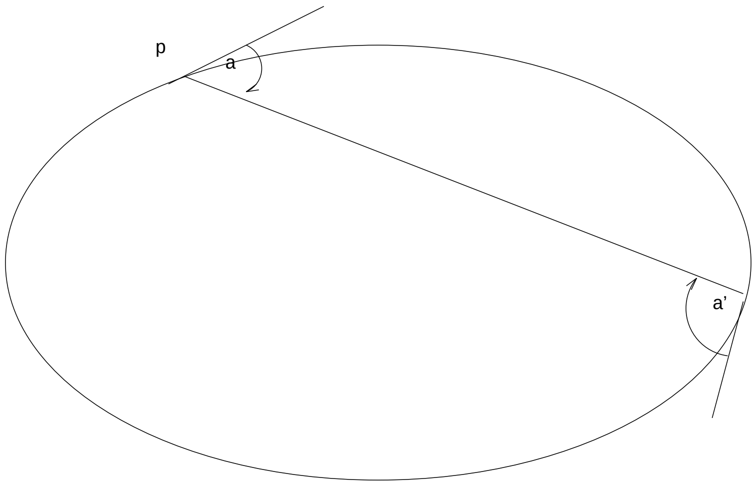

Figure 8. Billiard ball map

There is a simpler way to examine these reflected geodesics, in terms of the *billiard ball map*. Consider the annular region formed by the set

$$B = \partial\Omega \times [-\pi, \pi] \tag{23.1}$$

where for a point $(p, \alpha) \in B$, $p \in \partial\Omega$ and α should be thought of as the angle between the (anticlockwise) tangent and a line through the point p. By the strict convexity, provided $\alpha \neq \pm\pi$, the line meets the boundary at exactly one other point p' with angle $-\alpha'$. This then defines the billiard ball map

$$\beta : B \longrightarrow B. \tag{23.2}$$

Then closed geodesics (up to translation of the parameter) are in 1-1 correspondence with periodic orbits under β, i.e. finite sets q_1, $q_2, \ldots,$ q_k with $\beta q_j = q_{j+1}$, $1 \leq j < k$, $\beta q_k = q_1$.

One of the most important properties of B is that it preserves the area form

$$\omega = \cos(\alpha) ds d\alpha. \tag{23.3}$$

In terms of differential forms this can be written $\beta^*\omega = \omega$. Alternatively it means that the integral of $\cos(\alpha)dsd\alpha$ over $\beta(S)$ is the same as the integral over S for any measurable set $S \subset B$. This allows me to make precise the meaning of the condition under which (20.4) holds, that the measure of the closed geodesics is zero. In this case it just means that the measure of the set of periodic points for B is zero. Is this always true?

24. Existence of closed geodesics

Returning to the direct characterization of closed geodesics, still in the strictly convex case, let me recall a result of Birkoff ([1]).

Lemma 5. *For every integer $n \geq 2$ and every $1 \leq m < n/2$ there are at least two distinct closed geodesics making n transversal reflections and having winding number m, i.e. such that the sum over reflections of the change of tangent angle is $2\pi m$.*

Proof. Geodesics can be obtained by finding critical points for the length of a curve under variation. Look at the 'configuration space' of n ordered points in the boundary, successive points being distinct, such that the closed curve obtained by joining them by line segments, in order, has rotation number m. Denote this set $P_{n,m} \subset (\partial\Omega)^n$. Now fix the first point arbitrarily and let $P'_{n,m}(p)$ be the subset with first point p. Maximize the length of the curve over $P'_{n,m}$. The length of the curve is a continuous function on $P'_{n,m}$, this set is not compact (because successive points must be distinct) but the distance does not approach its maximum on the boundary (separating two points which have coallesced *increases* the length because of the convexity). Thus the maximim of the distance is attained in the interior. This maximum, as a function of $p \in \partial\Omega$ is continuous (in fact it is smooth). Points on $\partial\Omega$ at which this function takes critical values produce closed geodesics as desired. The maximum and minimum therefore give at least two, if they are equal then all boundary points give closed geodesics. In any case there are at least two closed geodesics as claimed. For related material see the book of Petkov and Stoyanov, [24], and papers cited there. $\square$

25. Approximation invariants

Let $l_{n,m}$ and $L_{n,m}$ be, respectively, the maximum and the minimum of the lengths of closed geodesics making n reflections and with winding number m. For fixed m one can show that for each $r > 0$ there exists C_r such that

$$L_{n,m} - l_{n,m} \leq C_r n^{-r}. \tag{25.1}$$

Furthermore one can see that

$$l_{n,m} \sim \sum_j d_j(m) n^{-2j} \text{ as } n \to \infty. \tag{25.2}$$

The numbers $d_j(m)$ are the 'approximation invariants' of the boundary. Clearly $d_0(m) = mL$, the higher invariants are more interesting.

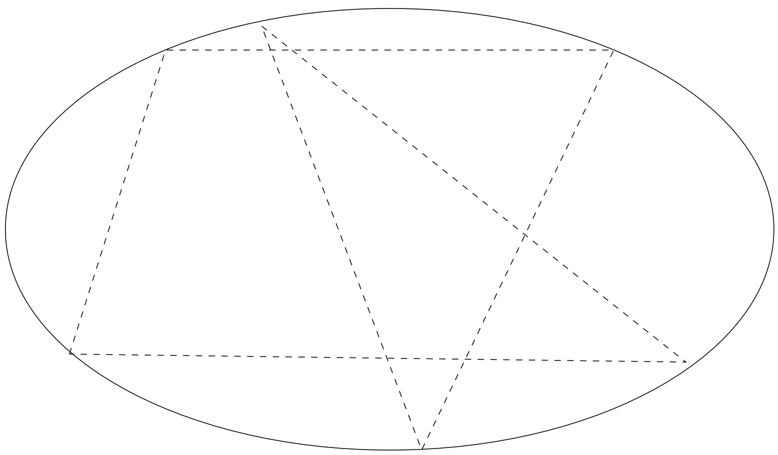

Figure 9. Closed geodesic with $n = 6$ and $m = 2$

Lemma 6. *The approximation numbers $d_j(1)$ can be written as integrals of polynomials in $\kappa^{1/3}$, $\kappa^{-1/3}$ and the derivatives of κ with respect to arclength. They are spectral invariants of nearly circular regions.*

For instance

$$d_1(1) = c_U \int_0^L \kappa^{2/3}(s) ds. \tag{25.3}$$

Using these invariants (see [17], [18]) one can for example show that there are families of strictly convex regions, with any number of parameters, which are (each) determined by their spectra amongst all regions.

26. Isospectral compactness again

Next let me return to the alternative proof of isospectral compactness that I promised. This is based on the following result.

Proposition 6. *If Ω is a planar region such that the infimum, $L_{\min}$, of $\mathcal{L}$ is attained as a local minimum and only by closed geodesics making two reflections then $L_{\min}$ is a singular point of* $\operatorname{Tr} U(t)$.

Consider a family of domains on which all the heat invariants are constant. Suppose there is a sequence in the family which is not uniformly embedded. As noted earlier, the only way for embedding to fail in these circumstances is for points on the boundary which are either in different component curves, or are bounded away from each other in terms of arclength on the boundary, to approach each other. A subsequence then satisfies the hypothesis of Lemma 6 and has $L_{\min} \to 0$. The family cannot be isospectral since Proposition 6 would lead to a contradiction of the fact that, for any domain, 0 is an isolated point in the singular support of $\operatorname{Tr} U(t)$. Indeed, far enough along the sequence $L_{\min}$ is a singular point of $\operatorname{Tr} U(t)$, which is by assumption independent of the element.

27. Analytic domains

Let me mention another result which was obtained, using the Poisson relation, by Colin de Verdière ([5])

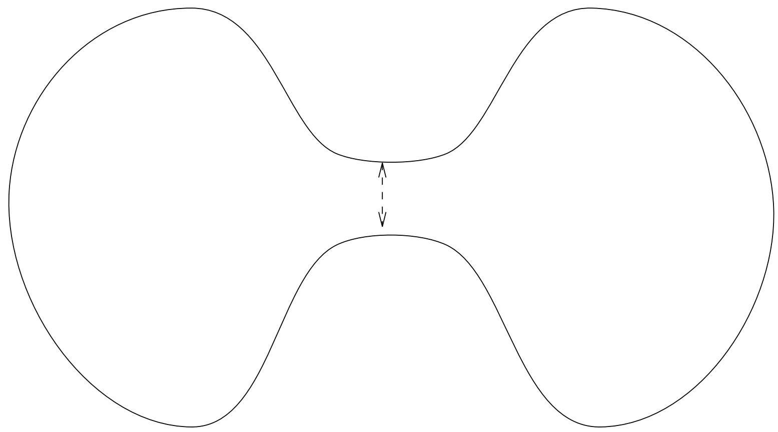

Figure 10. Short geodesic

Consider a domain satisfying a strengthened form of the hypothesis of Proposition 6 as illustrated in Figure 10.

$$\text{The infimim of the length spectrum is attained for exactly one geodesic, the minimum is non-degenerate and the domain has a reflection symmetry about the orthogonal bisector of the geodesic segment.} \tag{27.1}$$

Proposition 7. *There can be no one-parameter real analytic family of real analytic domains all satisfying* (27.1).

Proof. Using invariants constructed from the singularity arising from the shortest geodesic, the Taylor series of the variation of the curvature at each end point of the geodesic (the symmetry forces them to be the same) can be shown to vanish. The assumed real analyticity therefore forces the variation to be trivial. □

28. Triangles

In this last talk[7] I wanted to describe some result for polygonal domains, i.e. rather than consider Ω to be a smooth planar domain suppose instead that it is bounded by a (non-reentrant) polygon. There are two basic results I know of, the first one is in the positive direction.

Proposition 8. *Any two triagular domains can be distinguished by their Dirichlet spectra.*

This was shown by F.G. Friedlander and C. Durso.

Although the disussion above of the spectral theory for the Dirichlet problem and the heat and wave equations does not carry over *verbatim*, especially as regards smoothness, similar results can be obtained in the case of polygonal domains (in fact this case is in most senses simpler). For instance the heat invariants are defined. Unfortunately there are only two which are non-trivial, namely the area and the length of the bounding polygon. Clearly a triangle, up to rigid motion, has three independent parameters. These can be taken to be, for instance, the length of one side, the height from that side and the opposing angle. The triangle can be recovered from area, boundary length and height. The main problem is to show that the height can be recovered, under appropriate conditions, from the Poisson formula.

29. Isospectral polygonal domains

Let me now briefly describe what are the only known examples of isospectral planar domains which are not equivalent under rigid motions. The origin of their construction lies in group theory. The basic

[7]The notes are very brief, since the talk was not given.

idea was formulated by Sunada ([29]) (although there were some examples in higher dimensions before). The two-dimensional case was analysed by Buser ([2], [3]) who produced examples of two-dimensional flat manifolds with corners (including 'interior corners') which are isospectral but not isometric. These were finally shown to be embeddable in the plane (actually immersible in such a way as still to give isospectral domains) by Gordon, Webb and Wolpert ([9]). See [8] for more details and extensions.

See also the recent article of Buser, Conway, Doyle and Semmler [4] in which an example of this type is shown to have the stronger isospectrality property that the two domains are 'homophonic'. Not only are the eigenvalues the same, but there is a point in each of the domains such that the expansions of delta functions at these points, in terms of the eigenfunctions for the corresponding domain, have the same coefficients. This means that the 'drums' struck at the appropriate points sound exactly the same.

30. Other things

Finally let me finish these five lectures by describing some other 'related' ideas. I have talked about the resolvent family, the heat equation, the zeta function and the wave equation in relation to planar domains. These same objects appear in many different settings. For instance

Direct generalization. There are all sorts of things one can do to simply generalize the problem and ask all the same questions. For instance one can replace the Laplace operator $\varDelta$ by some other operator or pass to higher dimensions (where generally less is known). For instance if $V \in \mathcal{C}^\infty(\Omega)$ one can think of it as a potential and consider instead of $\varDelta$ the operator $\varDelta + V$. The analysis goes through with fairly minor changes. If V is real then the eigenvalues may become negative. The heat invariants involve V as well and so on. If V becomes complex then the eigenvalues can become complex too (but only a little bit). One virtue of the method I have described briefly above to construct the resolvent family is that it is not seriously affected by allowing V to become complex. Somethings change however, for example the Poisson trace formula must be modified.

Boundary measurement. There is an inverse problem which is more 'analytic' than the spectral problem that I have discussed. To say that 0 is not in the spectrum of the Dirichlet problem for $\varDelta + V$ is to say that the boundary problem

$$(\varDelta + V)u = f,\ u \restriction \partial\Omega = 0 \tag{30.1}$$

has a unique solution $u \in \mathcal{C}^\infty(\Omega)$ for each $f \in \mathcal{C}^\infty(\Omega)$. This actually implies that the Dirichlet problem itself

$$(\Delta + V)u = 0,\ u \restriction \partial\Omega = u_0 \tag{30.2}$$

also has a unique solution for each $u_0 \in \mathcal{C}^\infty(\partial\Omega)$. Indeed, to solve (30.2), take any function $u' \in \mathcal{C}^\infty(\Omega)$ with $u' \restriction \partial\Omega = u_0$. Then set $f = (\Delta + V)u'$ and let u'' be the solution of (30.1) for that f. Then $u = u' - u''$ solves (30.2). The uniqueness follows from the uniqueness of (30.1).

The map (sometimes called the *Dirichlet to Neumann*, or just the *Neumann,* map)

$$N : \mathcal{C}^\infty(\partial\Omega) \ni u_0 \longmapsto \partial_\nu u \restriction \partial\Omega \in \mathcal{C}^\infty(\partial\Omega) \tag{30.3}$$

is one of the basic examples of a *pseudodifferential operator* (it is close to being $|D_s|$). One can ask the question

Does knowledge of N (and Ω) determine V?

(30.4)

Indeed it does if V is small enough in L^∞ norm. See the work of Sylvester and Uhlmann [30] and Nachman [22].

Scattering theory. Suppose one considers the exterior problem instead of the interior problem I have been discussing. Thus suppose that Ω is the exterior of a smooth simple closed curve. Then the nature of the spectrum of Δ changes considerably; its spectrum is continuous rather than discrete as in the compact case discussed above. There are still inverse problems (solved in part but there are lots of interesting questions which are open). There is a lot of mathematical activity in this general direction involving spectral and scattering theory on non-compact spaces, often much more general than Euclidean space. You may (indeed I hope you do) find [19] a useful introduction to this area – see also the references listed there.

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