SCATTERING CONFIGURATION SPACES

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ABSTRACT. For a compact manifold with boundary X we introduce the n-fold scattering stretched product X^n_{sc} which is a compact manifold with corners for each n, coinciding with the previously known cases for $n = 2, 3$. It is constructed by iterated blow up of boundary faces and boundary faces of multi-diagonals in X^n. The resulting space is shown to map smoothly, by a b-fibration, covering the usual projection, to the lower stretched products. It is anticipated that this manifold with corners, or at least its combinatorial structure, is a universal model for phenomena on asymptotically flat manifolds in which particle clusters emerge at infinity. In particular this is the case for magnetic monopoles on $\mathbb{R}^3$ in which case these spaces are closely related to compactifications of the moduli spaces with the boundary faces mapping to lower charge idealized moduli spaces.

CONTENTS

INTRODUCTION

One of the natural tools for the study of asymptotically translation-invariant phenomena on Euclidean spaces is the passage to the 'radial compactification' $X = \overline{\mathbb{R}^m} = \mathbb{R}^m \cup S^{m-1}_\infty$. This translates asymptotic behaviour to behaviour at the boundary of X and allows similar phenomena to be considered on arbitrary compact manifolds with boundary, in terms of the intrinsic scattering structure at the boundary. In this approach, emphasized in [4], typical kernels and functions, such as Euclidean distance, $d(z_1, z_2)^2 = |z_1 - z_2|^2$ which are quite singular near the corner of the compact space $X \times X$ are resolved to 'normal crossings', i.e. conormal singularities, by lifting to the the scattering stretched product X^2_{sc}. This space is obtained by iterated real blow-up of X^2. The corresponding triple product X^3_{sc} has also been discussed and here we consider the 'higher scattering products' of an arbitrary compact manifold with boundary. These inherit the permutation invariance of X^n and, apart from their construction, the most important result here is that the projections onto smaller products also lift to be smooth b-fibrations, giving

$$X^n_{\mathrm{sc}} \;\substack{\longrightarrow \\ \cdots \\ \cdots \\ \longrightarrow}\; X^{n-1}_{\mathrm{sc}} \quad \dots \quad X^4_{\mathrm{sc}} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow \\ \longrightarrow}\; X^3_{\mathrm{sc}} \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; X^2_{\mathrm{sc}} \;\substack{\longrightarrow \\ \longrightarrow}\; X. \tag{I.1}$$

It is our basic contention that the spaces X^n_{sc}, despite the apparent complexity or their definition, are universal for asymptotically translation-invariant phenomena.

There is of course a relation between the spaces considered here and the seminal work of Fulton and MacPherson, [1]. This relationship is strongest at a combinatorial level but the differences are also quite substantial. Apart from the distinction between real and complex spaces in the two settings, it should be noted that all the blow ups carried out here are in the boundary. As a result the space X^n_{sc} can be retracted to X^n and in this sense the topology has not changed at all. These spaces are designed to give 'more room' for geometric and analytic objects.

One justification for the introduction of these 'higher' stretched products is that they are anticipated to serve as at least combinatorial models for a natural compactification of the moduli space of magnetic monopoles on $\mathbb{R}^3$ and allow the detailed asymptotic description of the hyper-Kähler metric. This will be shown elsewhere and is closely related to the existence of the maps in (I.1). Given the permutation-invariance of the spaces, the existence of these maps can be reduced to one case in each dimension and then corresponds to a commutative diagramme

$$\begin{tikzcd}
X^n_{\mathrm{sc}} \arrow[r] \arrow[rr, bend left, "\pi_{n,\mathrm{sc}}"] \arrow[d] & X^{n-1}_{\mathrm{sc}} \times X \arrow[r] & X^{n-1}_{\mathrm{sc}} \arrow[d] \\
X^n \arrow[rr, "\pi_n"] & & X^{n-1}
\end{tikzcd} \tag{I.2}$$

where $\pi_{n,\mathrm{sc}}$ is the new map and π_n is projection off the last factor. In fact this lifted map is a b-fibration, meaning in particular that push-forward under it of a function with complete asymptotic expansion (so in essence 'smooth' up to the boundary faces) again has such an expansion.

In the rest of this Introduction, we give an outline of this construction, deferring the proofs to the main body of the article. Given their fundamentally combinatorial nature the constructions here also extend to the more general 'fibred-cusp' configuration spaces X^n_Φ, corresponding to the double and triple spaces introduced by Mazzeo and the first author in [2]. These arise naturally when ∂X comes with a fibration over a space Y. The scattering case appears when this fibration is the identity map $\partial X \longrightarrow \partial X$.

The main issue in what follows is the fundamental fact about iterated blow-ups, which is that *different orders generally lead to non-diffeomorphic spaces.* Thus while it is clear which submanifolds of the boundary must be blown up in the construction of X^n_{sc}, the order in which this is done has to be specified. Moreover, having determined this order, the existence of the map $\pi_{n,\mathrm{sc}}$ in (I.2) corresponds to the possibility of obtaining the same space by a performing these blow ups in a different order. In a manifold with corners, M, such as X^n and the manifolds obtained by subsequent blow up from it, admissible 'centres' of blow up, $H \subset M$, which is to say manifolds which have a collar neighbourhood, are called p-submanifolds (for product-) and are always required here to be closed. The blow up of a p-submanifold is then always possible and is denoted, with its blown-down map,

$$\beta : [M; H] \longrightarrow M. \tag{I.3}$$

Thus we are interested in the circumstances in which two p-submanifolds H_1 and H_2 of a manifold with corners *commute* in the sense that the natural identification of the complement of the preimage of $H_1 \cup H_2$ in the blown up spaces extends to a smooth diffeomorphism allowing us to identify

$$[M; H_1; H_2] = [M; H_2; H_1]. \tag{I.4}$$

Here, and throughout the paper, we have identified submanifolds with their lifts to the blow-up. The lift of H_2 to $[X; H_1]$ is the closure in $[X; H_1]$ of $\beta_1^{-1}(H_2 \setminus H_1)$ if this is non-empty and is $\beta_1^{-1}(H_2)$ in the opposite case (i.e. if $H_2 \subset H_1$). In particular to conclude (I.4) we need to know that each lifts to be a p-submanifold under the blow-up of the other. In fact, as is well-known, H_1 and H_2 commute in this sense if and only if they are either transversal (including the case that they are disjoint) or *comparable*, meaning either $H_1 \subset H_2$ or $H_2 \subset H_1$. To prove our results, we need to show that whole families of blow-ups commute and this is where the combinatorial complexity lies.

I.1. **Boundary products.** The 'scattering structure' on a manifold with corners can be identified with the intrinsic Lie algebra of 'scattering vector fields' consisting precisely of the products fV where $V \in \mathcal{V}_{\mathrm{b}}(X)$, meaning that V is a smooth vector field which is tangent to the boundary, and $f \in \mathcal{C}^\infty(X)$ vanishes at the boundary. The larger Lie algebra $\mathcal{V}_{\mathrm{b}}(X)$ is the 'boundary (b-) structure'; it can also be thought of as representing the asymptotic multiplicative structure near the boundary or geometrically as the 'cylindrical end' structure on the manifold.

Not surprisingly then, our construction begins from the corresponding n-fold stretched b-product X^n_{b}. This resolves, near the diagonal in the boundary, the pairwise distance functions (and of course much more besides) for a cylindrical-end metric, also called a b-metric, on X. Assuming, for simplicity and from now on, that ∂X is connected, X^n_{b} is obtained from X^n by the blow up of all boundary faces.

In order to describe the construction, consider the collection of all boundary faces, $\mathcal{M}(X^n)$, of X^n. Blowing up boundary hypersurfaces of a manifold with corners does noting, so let $\mathcal{B}_{\mathrm{b}} \subset \mathcal{M}(X^n)$ be the subset of boundary faces of X^n of codimension at least 2; if ∂X is not connected $\mathcal{B}_{\mathrm{b}}$ is a smaller collection.

As standard notation we shall not distinguish between boundary faces of X^n and their lifts to blown up versions of the manifold, except where absolutely necessary.

Definition I.1. The b-stretched products of X, X^n_{b} are defined to be

$$X^n_{\mathrm{b}} = [X^n; \mathcal{B}_n; \mathcal{B}_{n-1}; \ldots; \mathcal{B}_2]. \tag{I.5}$$

where $\mathcal{B}_r \subset \mathcal{M}(X^n)$ is the collection of boundary faces of codimension r.

Note the contracted notation in (I.5) for iterated blow ups. Since we have not specified an ordering of elements within each of the families $\mathcal{B}_r$, it is implicit that the result does not depend on these choices. In fact at the stage at which the elements of $\mathcal{B}_r$ are blown up they are disjoint so the order is immaterial and X^n_{b} is well-defined. It also follows from this that the permutation group lifts to X^n_{b} as diffeomorphisms.

Consider the analogue of (I.2) in this simpler setting:

$$\begin{tikzcd} X^n_{\mathrm{b}} \arrow[r] \arrow[d] \arrow[rr, bend left, "\pi_{n,\mathrm{b}}"] & X^{n-1}_{\mathrm{b}} \times X \arrow[r] & X^{n-1}_{\mathrm{b}} \arrow[d] \\ X^n \arrow[rr, "\pi_n"] & & X^{n-1} \end{tikzcd} \tag{I.6}$$

To show the existence of $\pi_{n,\mathrm{b}}$ we divide $\mathcal{B}_r$ into two pieces, the *vertical* and *non-vertical* boundary faces (with respect to the projection π_n). Namely $\mathcal{B}^{\mathrm{v}}_r$ consists of those boundary faces B of X^n of codimension r such that the n-th factor of B is X. Similarly, $\mathcal{B}^{\mathrm{nv}}_r$ consists of those boundary faces B of X^n of codimension r such that the n-th factor of B is ∂X. Thus $B \in \mathcal{B}^{\mathrm{v}}_{\mathrm{b}}$ if $B = B' \times X$ with $B' \in \mathcal{M}(X^{n-1})$, i.e. $B = \pi_n(B) \times X$. Otherwise the nth factor is necessarily ∂X and then $B \in \mathcal{B}^{\mathrm{nv}}_{\mathrm{b}}$. Equivalently, the vertical boundary faces are those that are unions of fibres of π_n.

Now observe that

$$X^{n-1}_{\mathrm{b}} \times X = [X^n; \mathcal{B}^{\mathrm{v}}_{n-1}; \dots; \mathcal{B}^{\mathrm{v}}_2] \tag{I.7}$$

since the last factor of X is unchanged throughout. Thus the existence of $\pi_{n,\mathrm{b}}$ in (I.6) follows if we show that the non-vertical boundary faces can all be commuted to come last and hence that

$$X^n_{\mathrm{b}} = [X^{n-1}_{\mathrm{b}} \times X; \mathcal{B}^{\mathrm{nv}}_n; \mathcal{B}^{\mathrm{nv}}_{n-1}; \dots; \mathcal{B}^{\mathrm{nv}}_2],$$

so exhibiting X^n_{b} is an iterated blow-up of $X^{n-1}_{\mathrm{b}} \times X$.

I.2. **Multi-diagonals.** The space X^n_{sc} is constructed from X^n_{b} by the blow up of the intersections of the (lifts of the) 'multi-diagonals' in X^n with the various boundary components of X^n_{b}, again with strong restrictions on the order in which this is done. The total diagonal $\mathrm{Diag} \subset X^m$ is diffeomorphic to X and the *simple diagonals* in X^n are the images of $\mathrm{Diag}(X^m) \times X^{n-m}$ under the factor permutation maps. The multi-diagonals (later called simply diagonals) are the intersections of these simple diagonals. Since we are assuming the boundary of X to be connected, we can identify a simple diagonal, involving equality for some collection of factors, with the boundary face $B \in \mathcal{B}_{(2)}$ which has a factor of ∂X exactly in each of the factors involving equality. Then the multi-diagonals $D_{\mathfrak{b}}$ can be identified with transversally-intersecting subsets $\mathfrak{b} \subset \mathcal{B}_{(2)}$, meaning that different elements do not have a boundary factor in common.

It is important to understand that the diagonals are not p-submanifolds in X^n. Nor, in general, are their boundary faces (which is what we are most interested in). Indeed there are always points analogous to the boundary of the diagonal in $[0,1]^2$ and hence they do not have a local product structure consistent with that of the manifold. However, the effect of the construction above is to resolve these 'singularities'.

$$\text{The lift from } X^n \text{ to } X^n_{\mathrm{b}} \text{ of each } D_{\mathfrak{b}} \text{ is a p-submanifold} \tag{I.8}$$

Now, the lift of $D_{\mathfrak{b}}$ will generally meet many boundary faces of X^n_{b}. In particular it meets the lift of every boundary face $B \in \mathcal{B}_{(2)}$ with $B \subset \cap\mathfrak{b}$ and these intersections are the 'boundary diagonals' which are to be blown up. Thus, (recalling that the lift of $B \in \mathcal{B}_{(2)}$ to X^n_{b} is also denoted simply as B) set

$$\mathcal{H} = \{H_{B,\mathfrak{b}} = B \cap D_{\mathfrak{b}} \subset X^n_{\mathrm{b}}; B \in \mathcal{B} \text{ with } B \subset \cap\mathfrak{b}\}. \tag{I.9}$$

To blow up all these submanifolds we need to choose an order and this is required to respect the 'lexicographic' partial ordering of $\mathcal{H}$

$$H_{A,\mathfrak{a}} \leqslant H_{B,\mathfrak{b}} \Longleftrightarrow D_{\mathfrak{a}} \subset D_{\mathfrak{b}} \text{ or if } D_{\mathfrak{a}} = D_{\mathfrak{b}} \text{ then } A \subset B. \tag{I.10}$$

Then the scattering configuration space X^n_{sc} is defined to be

$$X^n_{\mathrm{sc}} = [X^n_{\mathrm{b}}; \mathcal{H}]. \tag{I.11}$$

with blow up in such an order. Of course it needs to be checked that the result is independent of the choice of order consistent with (I.10). As in the case of X^n_{b} follows from the fact that the changes in order correspond to transversal intersections (including disjointness).

As noted above, these results are already known in the cases $n = 2$ and $n = 3$. Two new phenomena make the general case more complicated. The first is the necessity to blow-up multi-diagonals with the first non-trivial multi-diagonal occuring for $n = 4$. The second is the issue of the ordering of $\mathcal{H}$.

I.3. **Stretched projections.** As noted above, the existence of the 'stretched projections' in (I.1) and (I.2) is the crucial property of the spaces X^n_{sc}. The proof of their existence is in essence the same as that outlined above, see (I.7), for the simpler $\pi_{n,\mathrm{b}}$ maps but the required commutation results are necessarily more intricate. In particular we need to consider various spaces intermediate between X^n and X^n_{sc} which arise in this argument. For instance, the p-submanifold $H_{A,\mathfrak{a}} \subset X^n_{\mathrm{b}}$ actually makes sense already in $[X^n; \mathfrak{a}]$ from which it can be lifted under further blow up. These issues are discussed extensively in the article below, here we ignore such niceties to explain the procedure used later.

The notion of vertical and non-vertical for boundary faces with respect to the last factor discussed above extends to the boundary diagonals $H_{A,\mathfrak{a}}$. First extend it to transversal subsets $\mathfrak{a} \subset \mathcal{B}_{(2)}$ where $\mathfrak{a}$ is vertical if and only if $\cap\mathfrak{a}$ is vertical. Thus

$$\begin{aligned} \mathcal{H}_{\mathrm{v,v}} &= \{H_{A,\mathfrak{a}} \in \mathcal{H}; \mathfrak{a} \text{ and } B \text{ are vertical}\}, \\ \mathcal{H}_{\mathrm{nv,v}} &= \{H_{A,\mathfrak{a}} \in \mathcal{H}; \mathfrak{a} \text{ is vertical but } B \text{ is non-vertical}\}, \\ \mathcal{H}_{\mathrm{nv,nv}} &= \{H_{A,\mathfrak{a}} \in \mathcal{H}; \mathfrak{a} \text{ and } B \text{ are non-vertical}\}. \end{aligned} \tag{I.12}$$

Notice that $A \subset \cap\mathfrak{a}$ in the definition of $H_{A,\mathfrak{a}}$ so if $\mathfrak{a}$ is non-vertical, then so is A. From the definitions, it is clear that we have

$$X^{n-1}_{\mathrm{sc}} \times X = [X^n; \mathcal{B}^{\mathrm{v}}; \mathcal{H}_{\mathrm{v,v}}] \tag{I.13}$$

with the appropriate order on the blow ups. So the task is to recognize X^n_{sc} as an iterated blow-up of this space. To do this, we first show that all the 'purely non-vertical' boundary diagonals can be blown down so that, as always with appropriate orders on the collections of centres,

$$\begin{aligned} X^n_{\mathrm{sc}} &= [X^n_{\mathrm{b}}; \mathcal{H}_{\mathrm{v,v}} \cup \mathcal{H}_{\mathrm{nv,v}}; \mathcal{H}_{\mathrm{nv,nv}}] \\ &= [X^{n-1}_{\mathrm{b}} \times X; \mathcal{B}^{\mathrm{nv}}; \mathcal{H}_{\mathrm{v,v}} \cup \mathcal{H}_{\mathrm{nv,v}}; \mathcal{H}_{\mathrm{nv,nv}}]. \end{aligned}$$

Thus all the boundary faces of non-vertical diagonals are first removed.

To proceed further, we remove the 'last' (which means originally largest) boundary face from $B \in \mathcal{B}^{\mathrm{nv}}$ by showing that it can be commuted past all the subsequent boundary diagonals. Then all the $H_{B,\mathfrak{a}}$ corresponding to this boundary face are commuted out and blown down. Then this procedure is iterated, at each step removing the last remaining non-vertical boundary face and *then* the boundary diagonals contained in it.

The rest of this article is devoted to providing a rigorous discussion of this outline. In §1 material on manifolds with corners and blow up is briefly recalled and in §2 the effect of the blow up of boundary faces is considered. This is extended in §3 to get a basic result on the reordering of boundary blow up, which is used extensively in the remainder of the article. In particular in §4 the results described above for the boundary configuration spaces X^n_{b} are derived. The diagonals and their resolution is examined in §5 and the properties of these submanifolds are slightly abstracted in §6 to aid the discussion of iterative blow up. Collections of boundary diagonals are described in §7 and used to construct the spaces X^n_{sc} in §8. Three results on the reordering of blow ups of boundary diagonals are given in §9 and these are used to carry out the construction of the stretched projections in §10. A simple application of these spaces in §11 is inserted to indicate why these resolutions should prove useful.

1. Manifolds with corners

Since we make heavy use of conventions for manifolds with corners , we give a brief description of the basic results which are used below. These can also be found in the [3].

1.1. **Definition and boundary faces.** By a manifold with corners we shall mean a space modelled locally on products $[0, \infty)^k \times \mathbb{R}^{n-k}$ with smooth transition maps (meaning they have smooth extensions across boundaries.) For such a space, M, $\mathcal{C}^\infty(M)$ is well-defined by localization and at each point the boundary has a definite codimension, corresponding to the number, k of functions in $\mathcal{C}^\infty(M)$ vanishing at the point which are non-negative nearby and have independent differentials. We will insist that the boundary hypersurfaces, the closures of the sets of codimension 1, be embedded. This corresponds to the existence of functions $\rho_i \in \mathcal{C}^\infty(M)$ which are everywhere non-negative and have $d\rho_i \neq 0$ on $\{\rho_i = 0\}$ and such that $\partial M = \{\prod_i \rho_i = 0\}$. The connected components of the sets $\{\rho_i = 0\}$ are the boundary hypersurfaces, the collection of which is denoted $\mathcal{M}_1(M)$. The components of the intersections of these hypersurfaces form boundary faces, all are closed and are the closures of their interiors the points of which have fixed codimension; thus $\mathcal{M}_k(M)$ consists of all the (connected) boundary faces of (interior) codimension k and $\mathcal{M}_{(k)}(M)$ denotes the collection of codimension at least k. By convention, we shall include $M \in \mathcal{M}(M)$ as a 'boundary face of codimension zero'.

Near a point of M, where the boundary has codimension k, it is generally natural to use coordinates adapted to the boundary. That is, local coordinates $x_i \geq 0$,

$i = 1, \dots, k$ and y_j, $j = 1, \dots, n-k$ where the boundary hypersurfaces through the given point are the $\{x_i = 0\}$.

If B_1, $B_2 \in \mathcal{M}(M)$ then their intersection is a boundary face (possibly empty) but their union is not. However the union is contained in a smallest boundary face which we will denote

$$B_1 \dotplus B_2 = \bigcap \{B \in \mathcal{M}(M); B \supset B_1 \cup B_2\}. \tag{1.1}$$

1.2. **p-submanifolds.** Embedded submanifolds of a manifold with corners can be rather more complicated locally than in the boundaryless case. The simplest type is a p-submanifold. This is a closed subset $Y \subset M$ which has a local product decomposition near each point, consistent with a local product decomposition of M. An interior p-submanifold (not necessarily contained in the interior) is distinguished by the fact that locally in a neighbourhood U of each of its points there are l independent functions $Z_i \in \mathcal{C}^\infty(U)$ which define it and which are independent of the local boundary defining functions, i.e. it is defined by the vanishing of interior coordinates. A general p-submanifold is an interior p-submanifold of a boundary face. Any p-submanifold Y of M can be locally put in *standard form* near a point p in the sense that there are adapted coordinates x_i, y_j, based at p such that in the coordinate neighbourhood U

$$U \cap Y = \{(x, y) \in U; x_j = 0,\ 1 \le j \le l,\ y_i = c_i \text{ for } i \in I\} \tag{1.2}$$

where I is some subset of the index set for interior coordinates and the c_i are constants.

1.3. **Blow-up and lifting of manifolds and maps.** The (radial) blow up of a p-submanifold is always well-defined and yields a new compact manifold with corners $[M; Y]$ with a blow-down map $\beta : [M; Y] \longrightarrow M$ which is a diffeomorphism from the complement of the preimage of Y to the complement of Y.

Lemma 1.1. *Under blow up of a boundary face of a manifold with corners, any p-submanifold H lifts to a p-submanifold which is contained in the lift of its boundary-hull, the smallest boundary face containing H.*

Note that the lift, also called the 'proper transform', of a subset $S \subset M$ under the blow up of a centre, $B \subset M$, which is required to be p-submanifold for this to make sense, is the subset of $[M; B]$ which is *either* the inverse image $\beta^{-1}(S)$, if $S \subset B$, or else the closure in $[M; B]$ of $\beta^{-1}(S \setminus B)$ if it is not. Of course this is a useful notion only for sets which are 'well-placed' with respect to B.

1.4. **Comparable and transversal submanifolds.** In the sequel the intersection properties of boundary faces, and later manifolds related to multi-diagonals, play an important role. The manifolds we consider here will always intersect cleanly, in the sense of Bott. That is, at each point of intersection they are modelled by linear spaces and their intersection is therefore also a manifold. This is immediate from the definition for boundary faces and almost equally obvious for the diagonal-like manifolds we consider later. We will say that two such manifolds, and this applies in particular to boundary faces, H_1 and H_2, of M are

- Comparable if $H_1 \subset H_2$ or $H_2 \subset H_1$.
- Transversal, written $H_1 \pitchfork H_2$ if N^*B_1 and N^*B_2 are linearly independent at each point of intersection.
- Neither comparable nor transversal, abbreviated to 'n.c.n.t.' otherwise.

More generally a collection of submanifolds H_i, $i = 1, \dots, J$ is transversal if at every point p of intersection of at least two of them, the conormals $N_p^* H_i$ for those i for which $p \in H_i$ are independent. This in particular implies that the intersection is a manifold.

If $B_1 \pitchfork B_2$ are two p-submanifolds of M then the lift of B_2 to $[M; B_1]$ which is defined above to be the closure of $B_2 \setminus B_1$ in the blown up manifold, is also equal to the inverse image, $\beta^{-1}(B_2)$.

1.5. **b-maps and b-fibrations.** A general class of maps between manifolds with corners which leads to a category are the b-maps. These are maps $f : M \longrightarrow M'$ which are smooth in local coordinates and have the following additional property. Let ρ_i' be a complete collection of boundary defining functions on M' and ρ_j a similar collection on M. Then there should exist non-negative integers α_{ij} and positive functions $a_i \in \mathcal{C}^\infty(M)$ such that

$$f^* \rho_i' = a_i \prod_j \rho_j^{\alpha_{ij}}. \tag{1.3}$$

Such a map is b-normal if for each j, $\alpha_{ij} \neq 0$ for at most one i. This means that no boundary hypersurfaces of M is mapped completely into a boundary face of codimension greater than 1 in M'.

The real vector fields on M which exponentiate locally to diffeomorphisms of M are the elements of $\mathcal{V}_{\mathrm{b}}(M)$, meaning smooth vector fields on M which are tangent to all boundary faces. These form all the smooth sections of a natural vector bundle ${}^{\mathrm{b}}TM$ over M and each b-map has a b-differential at each point $f_* : {}^{\mathrm{b}}T_pM \longrightarrow {}^{\mathrm{b}}T_{f(p)}M'$. A b-map is said to be a b-submersion if this map is always surjective. Blow-down maps are always b-maps and for boundary faces they are b-submersions but not for other p-submanifolds. A b-map which is both a b-submersion and b-normal is a b-fibration; blow down maps are never b-fibrations. With the notion of smoothness extended to include classical conormal functions on a manifold with corners, b-fibrations are the analogues of fibrations in the sense that such regularity is preserved under push-forward.

2. Boundary blow up

Each boundary face $B \in \mathcal{M}(M)$ of a manifold with corners is, as a consequence of the assumption that the boundary hypersurfaces are embedded, a p-submanifold. Thus, it is always permissible to blow it up. If B has codimension one (or zero), this does nothing. If $k = \operatorname{codim}(B) \geq 2$, i.e. $B \in \mathcal{M}_{(2)}(M)$, one gets a new manifold with corners, $[M; B]$, with one new boundary hypersurface $\operatorname{ff}([M; B])$ – which is the positive 2^kth part of a trivial $(k-1)$-sphere bundle over B, k being the codimension of B. This fractional-sphere bundle is the inward-pointing part of the normal sphere bundle to B and is trivialized by the choice of a defining function for each of the boundary hypersurfaces of M containing B. More generally, the other boundary hypersurfaces of $[M; B]$ are in 1-1 correspondence with the boundary hypersurfaces of M where again the assumption that the boundary hypersurfaces are embedded means that the connectedness cannot change on blow up of B. More generally, the boundary faces of $[M; B]$ not contained in $\operatorname{ff}([M; B])$ are the lifts (closures of inverse images of complements w.r.t. B) of the boundary faces of M not contained in B. The boundary faces of $[M; B]$ contained in $\operatorname{ff}([M; B])$ are either preimages (also called lifts) of boundary faces of B – hence are the restrictions of the fractional-sphere bundles to boundary faces of B – or else are proper boundary faces of the fractional balls over one of these faces (including of course B itself). The latter ones are 'new' boundary faces, not the lifts of old ones. We identify, at least notationally, each boundary face B' of M with its lift to a boundary face of $[M;B]$, even though the latter may be either a blow-up of B', if B' is not contained in B initially, or a bundle over B' if it is – in which case the dimension has increased.

For later reference we examine the effect of the blow up of a boundary face on the intersection of two others.

Lemma 2.1. *Consider two distinct boundary faces B_1 and B_2 in a manifold with corners M and their lifts to $[M;B]$ where $B \in \mathcal{M}_{(2)}(M)$:*

(i) *If $B_1 \pitchfork B_2$ then their lifts are transversal in $[M;B]$; they are disjoint if and only if $B_1 \cap B_2 \subset B$ but $B_1 \setminus B \neq \emptyset$ and $B_2 \setminus B \neq \emptyset$.*

(ii) *If $B_1 \subset B_2$ in M, then their lifts to $[M;B]$ are never disjoint, they are comparable in $[M;B]$ if and only if*

$$\begin{gathered} B_1 \setminus B \neq \emptyset, \\ B_1 \subset B \text{ and } B \pitchfork B_2 \\ \text{or } B_2 \subset B, \end{gathered} \tag{2.1}$$

the lifts are transversal if

$$B_1 \subset B \subsetneq B_2 \tag{2.2}$$

and are otherwise n.c.n.t. (that is if $B = B_2$ or $B_1 \subset B$ but B and B_2 are neither transversal nor is $B_2 \subset B$).

(iii) *If B_1 and B_2 are n.c.n.t. in M then their lifts are never comparable, are disjoint in $[M;B]$ if and only if*

$$B_1 \cap B_2 \subset B \text{ and both } B_1 \setminus B \neq \emptyset \text{ and } B_2 \setminus B \neq \emptyset, \tag{2.3}$$

they lift to meet transversally if and only if (using (1.1))

$$\text{either } B_1 \subset B \subsetneq B_1 \dotplus B_2 \text{ or } B_2 \subset B \subsetneq B_1 \dotplus B_2 \tag{2.4}$$

and otherwise their lifts are n.c.n.t..

Proof. When discussing the local effect of the blow up of a boundary face we may always choose adapted coordinates $x_i \geq 0$, y_j; in fact the 'interior coordinates' y_j play no part in the discussion here.

There exist subsets I, I_1, I_2 of $\{1, \ldots, k\}$ such that

$$B_i = \{x_j = 0 : j \in I_i\},\ i = 1, 2,\ B = \{x_j = 0 : j \in I\}. \tag{2.5}$$

The three parts of the Lemma correspond to the mutually exclusive cases where (i) $I_1 \cap I_2 = \emptyset$; (ii) $I_1 \subset I_2$ or $I_2 \subset I_1$; (iii) neither of these conditions hold. Of course we only need consider the first of the cases in (ii).

Near any point of the front face of $[X;B]$ there are adapted coordinates with the interior coordinates, y_j, lifted from X, and the boundary coordinates x_j replaced by

$$t_B = \sum_{i \in I} x_i, \text{ and } t_j = \begin{cases} x_j / t_B \text{ if } j \in I \\ x_j \text{ if } j \notin I. \end{cases} \tag{2.6}$$

Note that

$$\sum_{j\in I} t_j = 1. \tag{2.7}$$

Here t_B defines the new front face, i.e. the lift of B. The lifts $\tilde{B}_i$ of the B_i to $[X;B]$ are given by

$$\tilde{B}_i = \begin{cases} \{t_B = 0,\ t_j = 0,\ j \in I_i \setminus I\} & \text{if } B_i \subset B \Longleftrightarrow I \subset I_i \\ \{t_j = 0,\ j \in I_i\} & \text{if } B_i \setminus B \neq \emptyset, \Longleftrightarrow I \not\subset I_i. \end{cases} \tag{2.8}$$

We can now examine the intersection properties of $\tilde{B}_1$ and $\tilde{B}_2$.

(a) First assume B_1 and B_2 are both contained in B. Then from (2.8) it is clear that the lifts $\tilde{B}_1$ and $\tilde{B}_2$ are never transversal, since t_B, the defining function of the front face, vanishes on the both of them. They are clearly comparable if and only B_1 and B_2 are comparable and are otherwise n.c.n.t..

(b) Second, suppose $B_1 \subset B$ but $B_2 \setminus B \neq \emptyset$. From (2.8) we see that $\tilde{B}_1$ and $\tilde{B}_2$ meet transversally if and only if $I_1 \setminus I$ and I_2 are disjoint. The lifts can only be comparable if $\tilde{B}_1 \subset \tilde{B}_2$ and this occurs if and only if $I_1 \setminus I$ contains I_2. Otherwise, $\tilde{B}_1$ and $\tilde{B}_2$ are n.c.n.t..

(c) Finally, suppose that $B_1 \setminus B$ and $B_2 \setminus B$ are both non-empty. Then each $\tilde{B}_i$ is given by the vanishing of the t_j for $j \in I_i$. Now $I_1 \cup I_2 \supset I$ if and only if $B_1 \cap B_2 \subset B$. Hence $\tilde{B}_1$ and $\tilde{B}_2$ are disjoint in this case in view of (2.7). Otherwise, the transversal, comparable or n.c.n.t.according as this is the case for the original submanifolds B_1 and B_2.

It is now a simple matter to use these observations to prove the Lemma. Consider part (i) in which B_1 and B_2 are transversal, or equivalently, I_1 and I_2 are disjoint. Running through cases above, (a) cannot arise and in (b) and (c) $\tilde{B}_1$ and $\tilde{B}_2$ are transversal and are disjoint exactly as claimed.

Next consider part (ii) of the Lemma: without loss of generality, suppose $B_1 \subset B_2$, so $I_1 \supset I_2$. Then in case (a) the lifts are comparable if both are contained in B and according to (c) they are comparable if both are *not* contained in B. Comparable lifts arise in case (b) if $I_2 \subset I_1 \setminus I$, or equivalently if I and I_2 are disjoint subsets of I. Thus the lifts are comparable in this case if $B_1 \subset B_2$, $B_1 \subset B$, B and B_2 are transversal.

We also see from (b) that the lifts of $\tilde{B}_1$ and $\tilde{B}_2$ are transversal if and only if $I_1 \cap I_2 \setminus I \cap I_2 = \emptyset$ or equivalently if $I_2 \subset I$. This proves (2.2).

Finally consider part (iii) of the Lemma. Under case (a) the lifts are always n.c.n.t.. Under case (b), the lifts are transversal if and only if $(I_1 \setminus I) \cap I_2 = \emptyset$, that is, if and only if $I_1 \cap I_2 \subset I \cap I_2$. Since $I \subset I_1$, this just means that $I_1 \cap I_2 \subset I$ which gives (2.4). Otherwise, they are n.c.n.t.. Under (c), the lifts are n.c.n.t.unless the $B_1 \cap B_2 \supset B$, in which case they are disjoint, giving (2.3). □

We are most interested in the transversality of the intersections of the lifts, meaning either they are disjoint or meet transversally.

Corollary 2.2. *If B_1 and B_2 are distinct boundary faces of a manifold with corners and $B \in \mathcal{M}_{(2)}(M)$ then B_1 and B_2 are (i.e. lift to be) transversal in $[M;B]$ if and only if they are initially transversal or if not then*

$$B_1 \subset B \subsetneq B_1 \dotplus B_2 \text{ or } B_2 \subset B \subsetneq B_1 \dotplus B_2. \tag{2.9}$$

Two boundary faces lift to be disjoint if

$$B_1 \cap B_2 \subset B \text{ and both } B_1 \setminus B \neq \emptyset \text{ and } B_2 \setminus B \neq \emptyset. \tag{2.10}$$

Note that if $B_1 \pitchfork B_2$ then $B_1 \dotplus B_2 = M$.

3. Intersection-orders

Since boundary faces lift to boundary faces under blow up of any boundary face, any collection, $\mathcal{C} \subset \mathcal{M}(M)$, in any manifold with corners, can be blown up in any preassigned order, leading to a well-defined manifold with corners. Of course this actually means that after the first blow-up the lift of the second boundary face is blown up, and so on. Let the order of blow up be given by an injective function

$$o : \mathcal{C} \longrightarrow \mathbb{N} \tag{3.1}$$

where for simplicity we also assume that the range is an interval $[1, N]$ in the integers. Denote the total blow-up as $[M; \mathcal{C}, o]$; in general the choice of order does make a difference to the final result but the interior is always canonically identified with the interior of M.

From now on we will assume that the initial collection of boundary faces, $\mathcal{C}$, of M which are to be blown up is closed under non-transversal intersection in M :

$$B,\ B' \in \mathcal{C} \subset \mathcal{M}_{(2)}(M) \Longrightarrow B \pitchfork B' \text{ or } B \cap B' \in \mathcal{C}. \tag{3.2}$$

It should be noted that this is a condition in M and can fail for the lifts under blow up of boundary faces in that the intersection of the lifts may not be equal to the lift of the intersection.

Lemma 3.1. *If $\mathcal{C} \subset \mathcal{M}(M)$ is closed under non-transversal intersection then it is a disjoint union of collections $\mathcal{C}_i \subset \mathcal{C}$ which are also closed under non-transversal intersection, each contain a unique minimal element, and are such that all intersections between elements of different $\mathcal{C}_i$ are transversal.*

These $\mathcal{C}_i$ may be called the transversal components of $\mathcal{C}$.

Proof. Consider the minimal elements $A_i \in \mathcal{C}$, those which contain no other element. These must intersect transversally, since otherwise the intersection would be in $\mathcal{C}$ and they would not be minimal. Then set $\mathcal{C}_i = \{F \in \mathcal{C}; F \supset A_i\}$; these are certainly closed under non-transversal intersection. On the other hand the defining functions for elements of $\mathcal{C}_i$ are amongst the defining functions for A_i. It follows that the different $\mathcal{C}_i$ are disjoint, since their elements cannot have a defining function in common, and also that intersections between their elements are transversal. □

Definition 3.2. An order o on a collection $\mathcal{C}$ of boundary faces is an *intersection-order* if for any pair B_1 and $B_2 \in \mathcal{C}$ which are not transversal or comparable, $B_1 \cap B_2$ comes earlier than at least one of them, i.e.

$$B_1,\ B_2 \in \mathcal{C} \Longrightarrow B_1 \pitchfork B_2 \text{ or } o(B_1 \cap B_2) \leq \max(o(B_1), o(B)_2)). \tag{3.3}$$

Of course if B_1 and B_2 are comparable then the intersection is equal to one of them so (3.3) is automatic. On the other hand if B_1 and B_2 intersect non-transversally then (3.3) implies that the intersection comes strictly before the second of them with respect to the order. We will not repeatedly say that $\mathcal{C}$ is closed under non-transversal intersection, just that it has an intersection-order which is taken to imply the closure condition.

Definition 3.3. An order o on a collection $\mathcal{C}$ of boundary faces is a *size-order* if the codimension is weakly decreasing with the order, i.e.

$$o(B_1) < o(B_2) \Longrightarrow \operatorname{codim}(B_1) \geqslant \operatorname{codim}(B_2). \tag{3.4}$$

Clearly a size-order is an intersection-order since the intersection of n.c.n.t. boundary faces necessarily has larger codimension than either of them and so must occur first in the order of the three.

Lemma 3.4. *The iterated blow up in a manifold with corners M of a collection of boundary faces $\mathcal{C}$, which is closed under non-transversal intersection, with respect to any two size orders gives canonically diffeomorphic manifolds, with the diffeomorphism being the extension by continuity from the identifications of the interiors.*

Proof. The first element, B, in the order necessarily has maximal codimension so cannot contain any other. Thus all lifts of elements of $\mathcal{C}' = \mathcal{C} \setminus \{B\}$ are closures of complements with respect to B; their lifts therefore have the same dimension as before and hence in the induced order on $\mathcal{C}'$ in $[M; B]$ the codimension is weakly decreasing.

Now, we proceed to show that the lift of the elements of $\mathcal{C}'$ to $[M; B]$ is closed under non-transversal intersection. So, consider two distinct elements B_1, $B_2 \in \mathcal{C}'$. If they are comparable then B cannot contain the smaller so by, Lemma 3.1 they lift to be comparable. If they are transversal then again by Lemma 2.1 they lift to be transversal. Finally, suppose B_1 and B_2 are n.c.n.t.. Since (2.4) cannot arise here, either (2.3) holds, and hence $B_1 \cap B_2 = B$ and they lift to be disjoint, or else $B_1 \cap B_2 \setminus B \neq \emptyset$ and they lift to be n.c.n.t. with intersection the lift of $B_1 \cap B_2 \in \mathcal{C}'$.

Thus after the blow up of the first element of $\mathcal{C}$ the remaining elements lift to a collection of boundary faces closed under non-transversal intersection and in size-order. Now we can proceed by induction on the number of elements of $\mathcal{C}$ and hence assume that we already know that the result of the blow up of $\mathcal{C}'$ in $[M; B]$ is independent of the size-order. If B is the only element of maximal codimension in $\mathcal{C}$ the result follows. If there are other elements of the same codimension then by Lemma 3.1 they meet B transversally. Thus, the order of B and the second element can be exchanged. Applying discussion above twice it follows that the same manifold results from blow up in any size-order on $\mathcal{C}$. $\square$

We proceed to show that the the same manifold results from the blow up in any intersection-order.

Proposition 3.5. *The iterated blow up of M, $[M; \mathcal{C}, o]$, of an intersection-ordered collection of boundary faces is a manifold with corners independent of the choice of intersection-order in the sense that different orders give canonically diffeomorphic manifolds, with the diffeomorphism being the extension by continuity from the identifications of the interiors.*

Proof. Let o be the order in the form (3.1). For such an order we define the defect to be

$$d(o) = \sum_{J \in \mathcal{C}} o(J) \max\{(\operatorname{codim}(J) - \operatorname{codim}(I))_+ ; o(I) < o(J)\}. \tag{3.5}$$

Here the codimensions are as boundary faces of M, not after blow-up. Thus the defect is the sum over all sets of the maximum difference (if positive) between the codimensions of the 'earlier' sets and of that set, weighted by the position of the set. Thus, for a size-order the defect vanishes, because all these differences are non-positive, otherwise it is strictly positive.

For a general intersection-order take the first set, with respect to the order, I, such that its successor, J, had larger codimension in M, and consider the order o' obtained by reversing the order of I and J. We claim that this is an intersection-order and of strictly smaller defect, and that $[M, \mathcal{C}, o] = [M, \mathcal{C}, o']$.

The last point will be checked first. Certainly I cannot be the last element with respect to the order. Note also that the boundary faces up to, and including, I are in size-order, by the choice of I. Let $\mathcal{P} \subset \mathcal{C}$ be the subcollection of strict predecessors of I. Let M' be the manifold obtained from M by blowing up $\mathcal{P}$. In order to be able to commute the lifts, $\tilde{I}$ and $\tilde{J}$, of I and J to M', we need to rule out the possibility that they are n.c.n.t.; Lemma 2.1 will be used repeatedly for this.

Suppose first that I and J are comparable in M. Then $J \subset I$. According to Lemma 2.1, such a comparable pair of submanifolds can remain comparable, can become transversal, or can become n.c.n.t.in M'. If they ever become transversal, then they remain so under all subsequent blow-ups. Now comparable submanifolds $B_1 \subset B_2$ can only become n.c.n.t.under a blow-up with centre B satisfying $B_1 \subset B = B_2$ or $B_1 \subset B$, $B \setminus B_2 \neq \emptyset$. Since the centres of the blow-ups leading to M' are all of smaller dimension than I, which here plays the role of B_2, we see that these conditions can never be met by elements $B \in \mathcal{P}$. So if I and J are comparable in M, their lifts to M' cannot be n.c.n.t..

The only remaining possibility is that I and J are n.c.n.t.in M. In this case, $I \cap J$ has strictly dimension than I and so, because o is an intersection-order, this must be an element of $\mathcal{P}$. Moreover, because $\mathcal{P}$ is in a size-order, the lifts of I and J meet in the lift of $I \cap J$ until this manifold is blown up, at which point they become disjoint and then remain disjoint under all subsequent blow-ups. This shows that $\tilde{I}$ and $\tilde{J}$ commute in M'.

To show that o' is an intersection-order, consider an n.c.n.t.pair A, B. The only possibility of a failure of the intersection-order condition for o' is if I was the intersection and J the second element, in the order, of such a pair. However this means that, initially in M, $\operatorname{codim}(I) > \operatorname{codim}(J)$ and from the discussion above, this cannot occur.

Now, to compute the defect of o' observe that each of the sets which came after J initially still have the same overall collection of sets preceding them, and the same order, hence make the same contribution to the defect. The same is true for the sets which preceded I. Thus we only need to recompute the contributions from I and J after reversal. In its new position, J has one less precedent, viz. I now comes later, so the set of differences $\operatorname{codim}(J) - \operatorname{codim}(I')$ where $o(I') < o'(J)$ is smaller and the order of J has gone down, so it makes a strictly smaller contribution. The contribution of I was zero before and is again zero, since the only extra set preceding it, namely J, has larger codimension than it.

Thus this 'move' strictly decreases the defect. Repeating the procedure a finite number of times (note that after the first rearrangement, J might well be the 'new I') must reduce the defect to 0. Hence the blow up for any intersection-order is (canonically) diffeomorphic to one for which $\operatorname{codim}(B)$ is weakly decreasing, i.e. to a size-order and hence by Lemma 3.4 all intersection-orders lead to the same blown-up manifold. □

Definition 3.6. We denote by $[M,\mathcal{C}]$ the iterated blow-up of any collection of boundary faces which is closed under non-transversal intersection, with respect to any intersection-order.

One simple rearrangement result which follows from this is:

Lemma 3.7. *Suppose $\mathcal{C}_1 \subset \mathcal{C} \subset \mathcal{M}(M)$ are both closed under non-transversal intersection, then there is an intersection order on $\mathcal{C}$ in which the elements of $\mathcal{C}_1$ come before all elements of $\mathcal{C} \setminus \mathcal{C}_1$.*

Proof. Let o be a size-order on $\mathcal{C}$ and consider the new order o' on $\mathcal{C}$ defined by $o'(B) = o(B)$ if $b \in \mathcal{C}_1$, $o'(B) = o(B) + N$ otherwise, where $N = \max(o)$. Then o' has the desired property that every element of $\mathcal{C}_1$ comes before every element of $\mathcal{C} \setminus \mathcal{C}_1$. Moreover, o' must be an intersection-order. Thus we wish to show that if B_1 and B_2 are n.c.n.t.then

$$o'(B_1) < o'(B_2) < o'(B_1 \cap B_2) \tag{3.6}$$

is not possible. This certainly cannot happen unless $B_1, B_2 \in \mathcal{C}_1$, $B_1 \cap B_2 \in \mathcal{C} \setminus \mathcal{C}_1$, because o' restricts to give a size-order on each of $\mathcal{C}_1$ and $\mathcal{C} \setminus \mathcal{C}_1$. However, if B_1 and B_2 lie in $\mathcal{C}_1$ then so does $B_1 \cap B_2$ because $\mathcal{C}_1$ is closed under non-transversal intersection. Thus (3.6) is indeed impossible. □

Corollary 3.8. *If $\mathcal{C}_1 \subset \mathcal{C}_2$ are two collections of boundary faces of M, both closed under non-transversal intersection, then there is an iterated blow-down map*

$$[M;\mathcal{C}_2] \longrightarrow [M;\mathcal{C}_1]. \tag{3.7}$$

Proof. By the preceding lemma, there is an intersection-order on $\mathcal{C}_2$ with respect to which all elements of $\mathcal{C}_1$ come first. The existence of the blow-down map follows immediately from this. □

We will use the freedom to reorder blow ups frequently below. For instance if $\mathcal{C}$ is closed under non-transversal intersection then any given element is first or last in some intersection-order. In fact if the elements are first given a size-order then any one element can be moved to any other point in the order and the result is an intersection-order. Another use of the freedom to change order established above is to examine the intersection properties of boundary faces, as in Lemma 2.1, but after a sequence of boundary blow ups.

Proposition 3.9. *If B_1, B_2 are distinct boundary faces of M and $\mathcal{C} \subset \mathcal{M}(M)$ is closed under non-transversal intersection then*

1. *The lifts of B_1 and B_2 to $[M;\mathcal{C}]$ are disjoint if they are disjoint in M or there exists $B \in \mathcal{C}$ satisfying* (2.10).
2. *The lifts of B_1 and B_2 meet transversally in $[M;\mathcal{C}]$ if they are transversal in M or there exists $B \in \mathcal{B}$ satisfying* (2.9).
3. *If $B_1 \subset B_2$ in M then this remains true for the lifts to $[M;\mathcal{C}]$ if*

$$B \in \mathcal{C},\ B_1 \subset B \Longrightarrow B_2 \subset B \text{ or } B_2 \pitchfork B. \tag{3.8}$$

Proof. We can assume that B_1 and B_2 are both proper boundary faces. If there exists an element of $\mathcal{C}$ satisfying (2.10) then, as noted above, there is an intersection-order on $\mathcal{C}$ in which a given element comes first. Lemma 2.1 shows that blowing it up first separates B_1 and B_2 which thereafter must remain disjoint. Thus shows the sufficiency of 2.1.

As above, if there is an element of $\mathcal{C}$ satisfying (2.9) then it can be blown up first in an intersection-order which makes B_1 and B_2 transversal; then Lemma 2.1 shows that persists under subsequent blow up.

In the third part of the Proposition the sufficiency of the condition follows immediately from Lemma 2.1 since the elements of $\mathcal{C}$ of codimension two or greater containing B_1, and so by hypothesis either containing B_2 or transversal to it, form a collection closed under non-transversal intersection. In fact this is separately true of those containing B_2 and those which contain B_1 but are transversal to B_2 since no intersection of the latter can contain B_2. So all these blow ups can be done first. In each case once the minimal element of a transversal component is blown up the other elements do not contain B_1 so all blow ups preserve the inclusion of B_1 in B_2. □

Lemma 3.10. *For any two boundary faces B_1, $B_2 \in \mathcal{C}$, with lifts denoted $\tilde{B}_i$, $i = 1, 2$ it is always the case that*

$$\tilde{B}_1 \cap \tilde{B}_2 \subset \widetilde{B_1 \cap B_2} \text{ in } [M;\mathcal{C}]. \tag{3.9}$$

Proof. Consider the decomposition $\mathcal{C} = \mathcal{C}' \cup \mathcal{C}''$ into the collections of elements which do not contain $B_1 \cap B_2$ and those which do contain it. These must separately be closed under non-transversal intersection. Under blow up of an element of $\mathcal{C}'$, B_1, B_2 and $B_1 \cap B_2$ all lift to the closure of their complements with respect to the centre so the lift of the intersection is the intersection of the lifts. Moreover the other elements of $\mathcal{C}'$ lift not to contain the intersection while the elements of $\mathcal{C}'$ lift to contain it. Thus after blowing up all the elements of $\mathcal{C}'$ we are reduced to the case that $\mathcal{C}'$ is empty, so we may assume that $B_1 \cap B_2$ is contained in each element of $\mathcal{C}$. Now, consider the decomposition of $\mathcal{C}$ as in Lemma 3.1. Consider the effect of the blow up of the minimal element $A_1 \in \mathcal{C}_1$. Now the lift of $B_1 \cap B_2$ to $[M, A_1]$ is its preimage, the lifts of B_1 and B_2 depend on whether they are, or are not, contained in A_1 but in any case (3.9) holds after this single blow up. If A_1 contains neither B_1 nor B_2 then by (2.10) the lifts are disjoint and we need go no further. On the other hand if $A_1 \supset B_1 \cup B_2$ then all three manifolds lift to their preimages and equality of intersection of lifts and the lift of the intersections persists. The lifts of the other elements of $\mathcal{C}_1$ contain no fibres of the front face of $[M; A_1]$ over A_1 and so cannot contain the intersection. Hence these blow ups again preserve the equality. The only case remaining is where A_1 contains one, but not both, of B_1 and B_2. We can assume that $A_1 \supset B_1$ and then all the elements of $\mathcal{C}_1$ satisfy this. Let $\mathcal{C}_1 = \mathcal{C}_1' \cup \mathcal{C}_1''$ be the decomposition into those (before blow up of A_1) which do not contain B_2 and do contain B_2, where the second collection may be empty. Blowing up in size-order for each of these subcollections, observe that after the blow up of A_1, the other elements of $\mathcal{C}_1'$ lift to the closures of their complements with respect to A_1 and hence cannot contain fibres of A_1 and hence cannot contain the lift of B_1 or the intersection. They intersection of the lifts in $[M; A_1]$ is the intersection of the lift of B_2 and the front face. No other element of $\mathcal{C}_1'$ can contain this, since then it would contain B_2 contrary to assumption. Thus the elements of $\mathcal{C}_1'$ lifted to $[M; A_1]$ do not contain the intersection of the lifts of B_1 and B_2 so after their blow up the inclusion (3.9) still holds. On the other hand the elements of $\mathcal{C}_1''$ do contain the lift of B_2 and continue to do so after all elements of $\mathcal{C}_1'$ have been blown up. They therefore contain the intersection of the lifts but cannot contain the lift of B_1. Again $\mathcal{C}_1''$ can be decomposed using Lemma 3.1 and a minimal element can be blown up. For the elements which contain this minimal one the argument now proceed as for A_1 and $\mathcal{C}_1$ above, except that the lift of B_1 can never be contained in these centres. This means that (3.9) holds at the end of the blow up of one of the transversal parts of $\mathcal{C}_1''$. However the other transversal components lift under these blow ups to their preimages, so they contain the lift of B_2 but not of B_1 and the argument can be repeated. Thus at the end of the blow up of $\mathcal{C}_1$, (3.9) holds. However the other transversal components of $\mathcal{C}$ again lift to their preimages so contain the intersection of the lift of B_1 and B_2 (and even the lift of the intersection). So the argument above for $\mathcal{C}_1$ can be repeated a finite number of times to finally conclude that (3.9) remains true in $[M;\mathcal{C}]$. □

4. Boundary configuration spaces

Let X be a compact manifold with boundary and consider $M = X^n$ for some $n \geq 2$. The boundary faces of X^n are just n-fold products with each factor either X or a component of its boundary. Define $\mathcal{B}_{\mathrm{b}} \subset \mathcal{M}_{(2)}(X^n)$ to be equal to $\mathcal{M}_{(2)}(X^n)$ if the boundary of X is connected, otherwise to be the proper subset consisting of those n-fold products where each factor is either X or the same component of the boundary in the remaining factors, and where there are at least two of these factors.

Lemma 4.1. *The collection $\mathcal{B}_{\mathrm{b}} \subset \mathcal{M}(X^n)$ is closed under non-transversal intersection.*

Proof. The intersection of two elements where all boundary factors arise from the same boundary component of X are certainly in $\mathcal{B}_{\mathrm{b}}$. So consider two elements B_1, B_2 of $\mathcal{B}_{\mathrm{b}}$ with different boundary components, $A_1 \subset X$ for the first and $A_2 \subset X$ for the second. Then $A_1 \cap A_2 = \emptyset$, since X is a manifold with boundary, so has no corners. Thus if different boundary components occur in any one factor in B_1 and B_2 then $B_1 \cap B_2 = \emptyset$. The only remaining case is when each boundary factor in B_1 corresponds to a factor of X in B_2 and then the intersection is transversal. Thus $\mathcal{B}_{\mathrm{b}}$ is closed under non-transversal intersection. □

Definition 4.2. The n-fold b-stretched product of X is defined to be

$$X^n_{\mathrm{b}} = [X^n; \mathcal{B}_{\mathrm{b}}]. \tag{4.1}$$

This definition relies on Proposition 3.5 and Definition 3.6 to make it meaningful. Boundary faces of codimension one, or indeed the whole of X^n, could be included since blow up of these 'boundary faces' is to be interpreted as the trivial operation.

Remark 4.3. We will generally concentrate on the case that X has one boundary component so (4.1) amounts to blowing up all the boundary faces; in this case $\mathcal{B}_{\mathrm{b}} = \mathcal{B}_{(2)}$. Even if the boundary of X is not connected then blowing up all elements of $\mathcal{B}_{(2)} = \mathcal{M}_{(2)}(X^n)$, in an intersection order, is perfectly possible. The result may be called the 'overblown' product

$$X^n_{\mathrm{ob}} = [X^n; \mathcal{B}_{(2)}(X^n)],\ \partial X \text{ not connected}. \tag{4.2}$$

Since we are mainly interested in considering the resolution of diagonals, the smaller manifold in (4.1) is more appropriate here.

Next we give a more significant application of Proposition 3.5.

Proposition 4.4. *If $m < n$, each of the projections off $n-m$ factors of X, $\pi : X^n \longrightarrow X^m$, fixes a unique 'b-stretched projection' π_{b} giving a commutative diagramme*

$$\begin{tikzcd} X^n_{\mathrm{b}} \arrow[r, "\pi_{\mathrm{b}}"] \arrow[d, "\beta"'] & X^m_{\mathrm{b}} \arrow[d, "\beta"] \\ X^n \arrow[r, "\pi"'] & X^m \end{tikzcd} \tag{4.3}$$

and furthermore π_{b} is a b-fibration.

Proof. The existence of π_{b} follows from Corollary 3.8. Namely, taking π to be the projection off the last $n-m$ factors for simplicity of notation, the subcollection of $\mathcal{B}_{\mathrm{b}}^{\mathrm{v}(\pi)}$ for X^n, consisting of the boundary faces of X^n in which the last $n-m$ factors consist of X, is closed under non-transversal intersection. Thus, using Corollary 3.8, there is an iterated blow-down map

$$f : [X^n; \mathcal{B}_{\mathrm{b}}] \longrightarrow [X^m; \mathcal{B}_{\mathrm{b}} \times X^{n-m}]. \tag{4.4}$$

Composing this with projection off the last $n-m$ factors gives a map π_{b} for which the diagramme (4.3) commutes. Since both the iterated blow-down map and the projection are b-submersions, so is π_{b}. To see that it is a b-fibration it suffices to show that each boundary hypersurface of X^n_{b} is mapped into either a boundary hypersurface of X^m_{b} or onto the whole manifold; this is 'b-normality'. As a b-map π_{b} maps each boundary face into a boundary face so it is enough to see what happens near the interior of each boundary hypersurface of X^n_{b}. If the boundary hypersurface in question is not the result of some blow up then π_{b} looks locally the same as π and local b-normality follows. If it is the result of blow up then π_{b} maps into the interior provided the boundary face is not the lift of a boundary face, necessarily of codimension two or greater, from X^m. If it is such a lift then π_{b} is locally the projection onto X^m_{b}, i.e. maps into the interior of the corresponding front face. □

We shall analyze more fully the structure of the boundary faces of $[X^n; \mathcal{C}]$ where $\mathcal{C} \subset \mathcal{B}_{\mathrm{b}}$ is some collection closed under non-transversal intersection. Unless otherwise stated below, although mostly for notational reasons, we will make the simplifying restriction that

(4.5) The boundary of X is connected so $\mathcal{B}_{\mathrm{b}} = \mathcal{B}_{(2)} = \mathcal{M}_{(2)}(X^n)$.

For a boundary face $B \in \mathcal{B}_{(2)}$ it is convenient to consider three distinct possibilities

(i) $B \in \mathcal{C}$
(ii) $B \notin \mathcal{C}$ but there exists $A \in \mathcal{C}$, $B \subset A$.
(iii) $B \notin \mathcal{C}$ and $A \supset B \Longrightarrow A \notin \mathcal{C}$.

In the first case

$$\mathcal{C} = \{B\} \cup \mathrm{Sm}(B) \cup \mathrm{Bi}(B) \cup \mathrm{Nc}(B) \tag{4.6}$$

is a disjoint union, where

$$\begin{gathered} \mathrm{Sm}(B) = \{B' \in \mathcal{C}; B' \subsetneq B\} \\ \mathrm{Bi}(B) = \{B' \in \mathcal{C}; B' \supsetneq B\} \\ \mathrm{Nc}(B) = \{B' \in \mathcal{C}; B \text{ and } B' \text{ are not comparable}\}. \end{gathered} \tag{4.7}$$

Proposition 4.5. *If $\mathcal{C} \subset \mathcal{B}_{\mathrm{b}}$ is closed under non-transversal intersection and $B \in \mathcal{C}$ then under any factor exchange map of X^n which corresponds to a permutation of $\{1, \ldots, n\}$ transforming B to $(\partial X)^c \times X^{n-c}$, $c = \operatorname{codim}(B)$, the lift of $B \in \mathcal{C}$ to $[X^n; \mathcal{C}]$ is diffeomorphic to*

$$(\partial X)^c \times [\mathbb{S}^{c-1,c-1}; \mathcal{C}_{\mathrm{Bi}}] \times [X^{n-c}; \mathcal{C}_{\mathrm{Sm}}] \tag{4.8}$$

where $\mathcal{C}_{\mathrm{Bi}}$ is the collection of boundary faces of the totally positive part $\mathbb{S}^{c-1,c-1}$ of the $(c-1)$-sphere corresponding to the elements of $\operatorname{Bi}(B)$ *in* (4.6) *and $\mathcal{C}_{\mathrm{Sm}}$ is the collection of boundary faces $B' \subset X^{n-c}$ arising from the elements of* $\operatorname{Sm}(B)$ *in* (4.6).

Proof. The ordering of $\mathcal{C}$ arising from (4.6), in which the pieces are size-ordered, is an intersection-order. Since $\mathcal{C}$ is closed under non-transversal intersection, each element $B' \in \operatorname{Nc}(B)$ corresponds to an element $B' \cap B \in \operatorname{Sm}(B)$. Once this is blown up the lifts of B and B' are disjoint, which accounts for the absence of terms from $\operatorname{Nc}(B)$ in (4.8). Moreover this argument shows that the result as far as B is concerned is the same if $\mathcal{C}$ is replaced by the union of the first three terms in (4.6). Relabelling the factors so that B has the boundary of X in the first c factors, the result of blowing up B in X^n is to replace a neighbourhood of it by

$$(\partial X)^c \times \mathbb{S}^{c-1,c-1} \times X^{n-c} \times [0,1) \tag{4.9}$$

where the last factor is a defining function for the front face. Moreover, the boundary faces in $\mathcal{C}$ (excluding those not comparable to B) lift either to the products of boundary faces of $\mathbb{S}^{c-1,c-1}$ with the other factors except the last, or else products of all the other factors with boundary faces of X^{n-c}. The effect of the subsequent blow-ups on the lift of B is therefore as indicated in (4.8). □

So, next suppose instead that $B \notin \mathcal{C}$. The decomposition (4.6) still exists, of course without B itself. Case (ii) above corresponds to $\operatorname{Bi}(B)$ being non-empty. Since $\operatorname{Bi}(B) \subset \mathcal{C}$ consists of those elements which contain B it is also closed under non-transversal intersection. For a fixed element $A' \in \operatorname{Bi}(B)$ no two elements contained in A' can be transversal, so this subcollection is closed under intersection and hence has a minimal element. Since $\operatorname{Bi}(B)$ is closed under non-transversal intersection, the collection of minimal elements must be transversal in pairs. Denote this collection

$$\begin{gathered}\mathfrak{b}(B) = \{A \in \mathcal{C}; B \subset A,\ B \subset A' \subset A,\ A' \in \mathcal{C} \Longrightarrow A' = A\} \text{ then} \\ \operatorname{Bi}(B) = \bigcup_{A \in \mathfrak{b}(B)} \operatorname{Bi}_A(B) \text{ is a disjoint union, where } \operatorname{Bi}_A(B) = \{A' \in \mathcal{C}; A \subset A'\}.\end{gathered} \tag{4.10}$$

Proposition 4.6. *If $B \notin \mathcal{C}$, where $\mathcal{C} \subset \mathcal{B}_{(2)}$ is closed under non-transversal intersection, but* $\operatorname{Bi}(B) \neq \emptyset$ *then $\mathfrak{b}(B) \subset \mathcal{C}$ defined by* (4.10) *is a non-empty collection of transversally intersecting boundary faces and B lifts to be a common boundary face (i.e. in the intersection of) the lifts of the elements of $\mathfrak{b}(B)$, in* (4.10), *and is diffeomorphic to*

$$[B; \operatorname{Sm}(B)] \times \prod_{A_i \in \mathfrak{b}(B)} [\mathbb{S}^{d(i)-1,d(i)-1}; \operatorname{Bi}(A_i)],\ d(i) = \operatorname{codim}(A_i), \tag{4.11}$$

where $\mathrm{Sm}(B)$ *is interpreted as a collection of boundary faces of* B *and where for each* $A_i \in \mathfrak{b}(B)$, $\mathrm{Bi}(A_i)$ *is the collection of boundary faces of* $\mathbb{S}^{d(i)-1,d(i)-1}$ *arising from the lifts of the elements of* $\mathrm{Bi}(B)$ *strictly containing* A_i.

Proof. Give $\mathcal{C}$ an intersection-order in which the elements of $\mathfrak{b}(B)$ come first, followed by the other elements of $\mathrm{Bi}(B)$ in a size-order, followed by the elements of $\mathrm{Sm}(B)$ in size-order, followed by the elements of $\mathrm{Nc}(B)$, also in size-order. Since the elements of $\mathfrak{b}(B)$ are transversal, they can be in any order. Since B lifts into the front face under the first blow-up and remains a boundary face of the lifts of the others it lifts to be in the intersection of the lifts of the elements of $\mathfrak{b}(B)$ and after they are blown up is of the form

$$B \times \prod_{A_i \in \mathfrak{b}(B)} \mathbb{S}^{d(i)-1,d(i)-1},\ d(i) = \mathrm{codim}(A_i). \tag{4.12}$$

The other elements of $\mathrm{Bi}(B)$ contain one of the A_i and are transversal to the others and it follows that they lift to be boundary faces of the corresponding fractional sphere, as indicated. The boundary faces in $\mathrm{Sm}(B)$ lift in the obvious way and (4.11) results from the fact that the subsequent blow ups of boundary faces not comparable to B do not affect its lift, since their intersections with B have already been blown up. ☐

The third case is then $B \notin \mathcal{C}$ and such that there is no element of $\mathcal{C}$ containing it.

Proposition 4.7. *If* $B \subset A$ *implies that* $A \notin \mathcal{C}$ *then* B *lifts to a boundary face of* $[X^n; \mathcal{C}]$ *of the same dimension which is diffeomorphic to*

$$[B; \{F \in \mathcal{M}_{(2)}(B); F = G \cap B,\ G \in \mathcal{C}\}]. \tag{4.13}$$

Proof. Give $\mathcal{C}$ the intersection order in which all the $B' \subset B$ come first (size-ordered) and then all the faces which are not comparable to B (size-ordered as well). The effect on B is then as indicated! ☐

5. Multi-diagonals

The main utility of the manifold X^n_{b} as constructed above is that it resolves the intersection with the boundary of each of the multi-diagonals in X^n. The total diagonal in X^n is the submanifold, diffeomorphic to X, which is the image of the map $X \ni p \longmapsto (p, p, p, \dots, p) \in X^n$,

$$\mathrm{Diag}(X^n) = \{m \in X^n; m = (p, p, \dots, p) \text{ for some } p \in X\}. \tag{5.1}$$

The partial diagonals in X^n are the inverse images of the total diagonal in X^k, for some $k \leq n$, pulled back to X^n by one of the projections $X^n \longrightarrow X^k$. Thus a partial diagonal involves equality in at least two factors. Since we are assuming that the boundary of X is connected, there is a 1-1 correspondence between partial diagonals, projections onto X^k for $k \geq 2$, and elements of $\mathcal{B}_{\mathrm{b}}$. Thus if $B \in \mathcal{B}_{\mathrm{b}}$ it will be convenient to denote the correspond partial diagonal as D_B and the corresponding projection as π_B, where the partial diagonal is given by equality in exactly those factors in which B has a boundary component.

Diagonals are however more complicated than boundary faces, at least when $n \geq 4$. Namely the intersection of two partial diagonals is not necessarily a partial diagonal. Indeed

$$D_B \cap D_{B'} = D_{B\cap B'} \text{ if and only if } B \cap B' \text{ is non-transversal.} \tag{5.2}$$

Non-transversality of intersection of B and B' is the condition that ∂X appears in at least one factor in common. Then equality of points in all the factors in which the boundary occurs in B and also in B' separately, implies that they are equal in all factors in $B \cap B'$ giving equality in (5.2) in the non-transversal case. Conversely, if the intersection is transversal then separate equality does not imply overall equality. Thus

Lemma 5.1. *The submanifolds of X^n arising as the intersections of collections of partial diagonals in X^n form the collection of* multi-diagonals *which are in 1-1 correspondence with the (non-empty) transversal collections of boundary faces of codimension at least two*

$$\mathfrak{b} \subset \mathcal{B}_{(2)} \text{ s.t. } B_1, B_2 \in \mathfrak{b} \Longrightarrow B_1 \pitchfork B_2. \tag{5.3}$$

The multi-diagonal corresponding to the transversal family $\mathfrak{b}$ will be denoted by $D_{\mathfrak{b}}$, so

$$D_{\mathfrak{b}} = \{m \in X^n; \pi_B(m) \in \mathrm{Diag}(X^k),\ k = \mathrm{codim}(B)\ \forall\ B \in \mathfrak{b}\}. \tag{5.4}$$

Let us now clarify the sense in which X^n_{b} resolves the multi-diagonals. Consider first the total diagonal.

Lemma 5.2. *The total diagonal in X^n is naturally diffeomorphic to X and is a b-submanifold but never a p-submanifold. Under blow-up of the boundary face of maximal codimension, $(\partial X)^n$, the total diagonal lifts to (i.e. the closure of its interior is) a p-submanifold.*

Proof. A neighbourhood of this 'maximal corner' of X^n is of the form $[0,1)^n \times (\partial X)^n$ with the coordinate in each of the first factors given by a fixed boundary defining function lifted from each factor and then denoted x_j. The total diagonal meets this in the b-submanifold $\{x_1 = \cdots = x_n\} \times \mathrm{Diag}_{\mathrm{tot}}(\partial X)^n$. After blowing up the corner a neighbourhood of the front face is of the form

$$[0,1)_\rho \times \mathbb{S}^{n-1,n-1} \times (\partial X)^n,\ \rho = x_1 + \cdots + x_n \tag{5.5}$$

to which the total diagonal lifts as

$$[0,1) \times \{\bar\omega\} \times \mathrm{Diag}_{\mathrm{tot}}(\partial X)^n,\ \bar\omega \in \mathbb{S}^{n-1,n-1} \tag{5.6}$$

being the 'centre' of the fractional sphere and hence an interior point. This shows that the total diagonal in X^n is resolved to a p-submanifold. □

The general case of a multi-diagonal $D_{\mathfrak{b}}$ is similar; the next lemma shows that it is resolved in $[X^n; \mathcal{C}]$ for any collection of boundary faces $\mathcal{C}$, closed under non-transversal intersection and containing $\mathfrak{b}$.

Proposition 5.3. *Let $\mathcal{C}$ be an intersection-ordered family of boundary faces of X^n and $\mathfrak{b} \subset \mathcal{C}$ a transversally intersecting subcollection then the lift of $D_{\mathfrak{b}}$ to $[X^n; \mathcal{C}]$ is a p-submanifold.*

Proof. Let $\mathcal{C} = \mathfrak{b} \cup \mathcal{C}'$ (disjoint union). Because $\mathcal{C}$ is closed under non-transversal intersection and $\mathfrak{b}$ is a transversal collection, there is an intersection order on $\mathcal{C}$ such that all elements of $\mathfrak{b}$ come before any element of $\mathcal{C}'$, and $\mathcal{C}'$ itself is size-ordered. We claim that the lift of $D_{\mathfrak{b}}$ to $[X^n; \mathfrak{b}]$ is a p-submanifold.

If a factor-exchange map is used to identify

$$X^n \equiv X^{n-k} \times \prod_{i=1}^{L} X^{k_i} \tag{5.7}$$

in such a way that the each element $B_i \in \mathfrak{b}$ is identified with the corner of maximal codimension in X^{k_i} then the space obtained by blow up of the elements of $\mathfrak{b}$ is identified smoothly with

$$X^{n-k} \times \prod_{i=1}^{L} [X^{k_i}; (\partial X)^{k_i}]. \tag{5.8}$$

Then Lemma 5.2 shows that $D_{\mathfrak{b}}$, which is identified with the product of X^{n-k} and the maximal diagonals in the X^{k_i}, is resolved to a p-submanifold in $[X^n; \mathfrak{b}]$.

As noted earlier, under the blow-up of boundary faces, a p-submanifold (in this case an interior p-submanifold) lifts to a p-submanifold. This proves that the lift remains a p-submanifold under subsequent blow-up of the elements of $\mathcal{C}'$. □

Let us next gather some notation and information about intersections of multi-diagonals. Since the intersection of any two multi-diagonals is another multi-diagonal, given $\mathfrak{b}_1$ and $\mathfrak{b}_2$, there exists a transversal family $\mathfrak{b}_1 \Cup \mathfrak{b}_2$ uniquely defined by the condition

$$D_{\mathfrak{b}_1} \cap D_{\mathfrak{b}_2} = D_{\mathfrak{b}_1 \Cup \mathfrak{b}_2}. \tag{5.9}$$

The family $\mathfrak{b}_1 \Cup \mathfrak{b}_2$ will be called the 'transversal union' of $\mathfrak{b}_1$ and $\mathfrak{b}_2$. It is defined as follows. Partition $\mathfrak{b}_1 \cup \mathfrak{b}_2$ into subsets where two elements lie in the same subset if and only if there is a chain of elements connecting them, each intersecting the next non-transversally. Then the elements of $\mathfrak{b}_1 \Cup \mathfrak{b}_2$ consist of the intersections over these subsets.

Clearly, if all pairs $(B_1, B_2) \in \mathfrak{b}_1 \times \mathfrak{b}_2$ meet transversally then $\mathfrak{b}_1 \Cup \mathfrak{b}_2 = \mathfrak{b}_1 \cup \mathfrak{b}_2$. Otherwise the transversal union has fewer elements than the union and they need not be elements of either collection.

It is also convenient to introduce the following notation:

- Write $\mathfrak{b}_1 \pitchfork \mathfrak{b}_2$ if $D_{\mathfrak{b}_1} \pitchfork D_{\mathfrak{b}_2}$, or equivalently if $\mathfrak{b}_1 \cup \mathfrak{b}_2 = \mathfrak{b}_1 \Cup \mathfrak{b}_2$.
- Say $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are comparable if $\mathfrak{b}_1 \Cup \mathfrak{b}_2 = \mathfrak{b}_2$ or $\mathfrak{b}_1 \Cup \mathfrak{b}_2 = \mathfrak{b}_1$ which is equivalent to $D_{\mathfrak{b}_2} \subset D_{\mathfrak{b}_1}$ or $D_{\mathfrak{b}_1} \subset D_{\mathfrak{b}_2}$.
- Otherwise say $\mathfrak{b}_1$ and $\mathfrak{b}_2$ are n.c.n.t.: this is equivalent to $D_{\mathfrak{b}_1}$ and $D_{\mathfrak{b}_2}$ being n.c.n.t., or combinatorially, to the condition that $\mathfrak{b}_1 \Cup \mathfrak{b}_2$ is neither the union nor either of the individual sets of boundary faces.

To motivate the discussion of the next section, let us give a local coordinate description of the multi-diagonal $D_{\mathfrak{b}}$ and its lift to $[X^n; \mathfrak{b}]$. Explicitly, there is a set $(I_1, \dots, I_L)$ of disjoint subsets of $\{1, \dots, n\}$, each of cardinality $\geqslant 2$, such that

$$D_{\mathfrak{b}} = \cap_{r=1}^{L} \{z_k = z_l \text{ for all } k, l \in I_r\}. \tag{5.10}$$

Using adapted local coordinates $z = (x, y)$ near the boundary of X, a full set of local boundary defining functions for $[X^n; \mathfrak{b}]$ are given by the t_r, $B_r \in \mathfrak{b}$, which are the sums

$$T_r = \sum_{j \in I_r} x_j \tag{5.11}$$

of the local defining functions for B_r and the $t_j = x_j/T_r$ if $j \in B_r$ for some r and and $t_j = x_j$ otherwise. Interior coordinates lift to interior coordinates.

In order to describe the lift of $D_{\mathfrak{b}}$ to $[M;\mathfrak{b}]$ introduce new variables

$$s_j = \log t_j,\ u_j = y_j - y_{a_r} \text{ if } j \in J_r \text{ for some } r. \tag{5.12}$$

Then the variables

$$(s_j, u_j, t_k, y_k) \text{ for } j \notin J,\ k \in J, \text{ where } t_k \geqslant 0 \tag{5.13}$$

($J = J_1 \cup \ldots \cup J_L$) form an adapted local coordinate system on $[X^n;\mathfrak{b}]$ with respect to which

$$\tilde{D}_{\mathfrak{b}} = \{s_j = 0, u_j = 0; j \in J\}. \tag{5.14}$$

Let us now consider the family $\mathcal{D}(\mathfrak{b})$ of all multi-diagonals containing $D_{\mathfrak{b}}$. It is clear that if $D \in \mathcal{D}(\mathfrak{b})$, then D must be an intersection of the form

$$D = \{s_j = 0, u_j = 0; j \in K_0\} \cap \cap_{r=1}^{M} \{s_i = s_j, u_i = u_j \text{ for all } i, j \in K_r\}. \tag{5.15}$$

where $K_0, K_1, \ldots, K_r$ are disjoint subsets. In fact, for each $r = 1, \ldots, M$, K_r must be contained in one of $J_1, \ldots, J_L$.

This discussion shows that any given multi-diagonal lifts to a p-submanifold, but that there is no single system of adapted coordinates which put all elements of $\mathcal{D}$ in standard form. The notion of a d-collection, which we introduce in the next section, is designed to capture the local structure of families like $\mathcal{D}$.

6. D-collections

Next we introduce a notation for collections of p-submanifolds which includes the resolutions of diagonals.

Any p-submanifold is locally of the form (1.2) in adapted coordinates. The *interior codimension* is $d = |I|$; Y is an interior p-submanifold if $k = 0$, i.e. no boundary variables are involved in its definition, otherwise it is contained in a unique boundary face of codimension k (its *boundary hull* B(Y)). If N is such that $d + N$ is less than or equal to the number of interior variables then Y can alternatively be brought to a *local diagonal* form relative to the adapted coordinates in the sense that

$$U \cap Y = \{(x,y) \in U; x_i = 0,\ 1 \le i \le k,\ y_i = y_j \text{ for all } i, j \in \mathfrak{l}_l,\ l = 1, \ldots, N\} \tag{6.1}$$

where the $\mathfrak{l}_l$ are some disjoint subsets (including possibly none) each having cardinality $\geqslant 2$.

Of course to get (6.1) one just needs to divide the interior coordinates into groups and subtract one of N of the remaining interior variables from each element of each set. The interior codimension of Y (which is constant, since Y is connected by assumption) is $\sum\limits_{l}(|\mathfrak{l}_l| - 1)$.

The important property of boundary diagonals that we wish to capture in the notion of a d-collection is that they can simultaneously brought to such diagonal form near any point.

Definition 6.1. A collection $\mathcal{E}$ of p-submanifolds (if not connected then each must have fixed dimension) in M is called a *d-collection* if for each point $p \in M$ there is one set of adapted coordinates based at that point in terms of which all the elements of $\mathcal{E}$ through that point take the form (6.1).

Clearly this condition is void locally at any point not contained in one of the elements of $\mathcal{E}$ and for any single p-submanifold which does not have maximal interior codimension. Any collection of boundary faces can be added to a d-collection and it will remain a d-collection since they are automatically of the form (6.1) (for any adapted coordinates) for the empty collection of disjoint sets $\mathfrak{l}_l$.

We will decompose a d-collection into the subcollections of elements which are and those which are not boundary faces.

$$\mathcal{E} = \mathcal{E}_{\mathrm{b}} \cup \mathcal{E}', \ \mathcal{E}_{\mathrm{b}} = \mathcal{E} \cap \mathcal{M}(M). \tag{6.2}$$

As usual, including a boundary hypersurface in a given collection $\mathcal{E}$ is a matter of convention; for the sake of definiteness we exclude hyerpsurfaces.

Lemma 6.2. *Any subcollection of the boundary faces of the elements of a d-collection (with the addition of any of the non-hypersurface boundary faces of the manifold) is a d-collection.*

Proof. Immediate from the definition. □

As already mentioned, the lifts of the diagonals give examples of d-collections provided the appropriate boundary faces have been blown up.

Proposition 6.3. *If $\mathcal{C} \subset \mathcal{B}_{\mathrm{b}}$ is closed under non-transversal intersection then all the diagonals $D_{\mathfrak{b}}$ with $\mathfrak{b} \subset \mathcal{C}$ lift from X^n to $[X^n;\mathcal{C}]$ to interior p-submanifolds which form a d-collection.*

Proof. This follows from the discussion at the end of the preceding section since the same coordinates work for all diagonals. □

For a d-collection $\mathcal{E}$ of p-submanifolds we consider a closure condition corresponding to the index sets in (6.1) that define them. Let $\mathfrak{l}$ and $\mathfrak{l}'$ be two subpartitions of the index set (of interior coordinates). Then as in §I.2 we write

$$\begin{gathered} \mathfrak{l} \Subset \mathfrak{l}' \text{ if each set } \mathfrak{l}_i \in \mathfrak{l} \text{ is contained in one of the } \mathfrak{l}'_j \\ \mathfrak{l} \pitchfork \mathfrak{l}' \text{ all sets } \mathfrak{l}_i \text{ are disjoint from all sets } \mathfrak{l}_j. \end{gathered} \tag{6.3}$$

Thus in the second case $\mathfrak{l} \cup \mathfrak{l}'$ is still a subpartition.

Now the condition we impose on $\mathcal{E}$ concerns the elements which are not boundary faces, and which pass through a given point

$$\begin{gathered} \forall\ E,\ E' \in \mathcal{E}' \text{ and } p \in E \cap E' \text{ if neither condition in (6.3) holds} \\ \text{for the index sets defining them then} \\ \exists\ F \in \mathcal{E}',\ p \in F \text{ with the same index set as } E \cap E', \\ \text{containing it and with boundary hull in } B(E) \dotplus B(E'). \end{gathered} \tag{6.4}$$

Proposition 6.4. *If $\mathcal{E}$ is a d-collection of p-submanifolds all contained in (or equal to) proper boundary faces of a manifold with corners M for which the closure condition* (6.4) *holds then on the blow up of an element $G \in \mathcal{E}_{\mathrm{b}}$ or an element $E \in \mathcal{E}'$ of maximal interior codimension the elements of $\mathcal{E} \setminus \{E\}$ lift to a d-collection in $[M;E]$ which again satisfies the closure condition.*

Proof. Certainly the blow up of E is well-defined, since it is a p-submanifold by assumption.

The simplest case is if $G \in \mathcal{E}_{\mathrm{b}}$ is actually a boundary face. Then all the other p-submanifolds certainly lift to p-submanifolds. The other elements of $\mathcal{E}_{\mathrm{b}}$ lift to boundary faces, the elements of $\mathcal{E}'$ which are not contained in G lift to the closures of the complements with respect to G and the elements of $\mathcal{E}'$ which are contained in G lift to their preimages. Away from G nothing has changed and near it, the boundary defining functions $x_1, \dots, x_k$ which define it are replace by their sum T_G and $t_j = x_j/T_G$. The defining conditions involving interior variables are unchanged by the blow up. The changes of intersections of elements of $\mathcal{E}'$ correspond to whether they are contained in G or not and so (6.1) persists everywhere locally, with only the boundary functions changing. The closure condition also persists at every point of intersection after blow up, since the only problem would from E, $E' \subset G$ but $F \setminus G \neq \emptyset$, since then the lift of F would not contain the lift of the intersection of E and E'. The last condition in (6.4), on the hulls, prevents this from happening, since if E $E' \subset G$ both then the boundary hull of F must also be contained in G and the boundary hull of its lift must be contained in the hull of the lifts of E and E'.

So next consider the blow up of an element $Y \in \mathcal{E}'$ with boundary hull B. The condition of maximality of its interior codimension, i.e. the number of equations defining it within B, means, by (6.4) that the only other elements of $\mathcal{E}'$ it meets must satisfy one of the conditions in (6.3), since otherwise the F whose existence is demanded by (6.4) would have larger interior codimension. In particular the only way another element of $\mathcal{E}'$ can be contained in Y is if it is a boundary face of Y, hence has the same index set but boundary hull which is a boundary face of the boundary hull of Y. On blow up of Y these p-submanifolds lift to boundary faces of the front face produced by the blow up (which is one good reason boundary faces are allowed in the definition of d-collections). So consider elements of $\mathcal{E}'$ which meet, but are not contained in Y. By the maximality of the interior codimension of Y, these corresponds to index sets in one of the two cases in (6.3) so fall into two classes, those with interior defining conditions implied by the defining conditions for Y and those involving variables which are completely independent of those defining Y. The latter clearly lift to have the same defining conditions and with hull simply the lift of the previous hull.

The blow up of Y can be made explicit locally by choosing one of the elements labelled by the $\mathfrak{l}_i$ and subtracting it from the others. This changes the defining conditions for Y into the vanishing of interior variables and boundary variables, so locally the blow up corresponds to polar coordinates in these variables. All the elements of $\mathcal{E}'$ meeting Y, but not contained in it, and corresponding to the first case in (6.3) have lifts defined by some of the new boundary variables (not including ρ_{ff}) and the vanishing of some of the interior polar variables. It follows that the same intersection property (6.4) results by simply adding the lifted 'extra variable' to each of these lifted interior variables. Thus it follows that the the d-submanifold condition holds for the lift of the elements of $\mathcal{E} \setminus \{Y\}$.

It remains only to check the closure condition, but this persists from the same arguments since the local index sets have not changed. □

Notice that the interior codimension of each element of the lift of the d-collection on the blow up of a boundary face is the same as before blow up. On blow up of an element of $Y \in \mathcal{E}'$ with maximal interior codimension, all the lifts have the same interior codimension except for any which are boundary faces of E itself, which lift to boundary faces of $[M; E]$ and hence are subsequently boundary faces, for which the notion of an interior codimension is not defined.

7. Boundary diagonals

In Proposition 5.3 it is shown that the diagonal $D_{\mathfrak{b}}$ associated to a transversal subset $\mathfrak{b} \subset \mathcal{B}_{(2)}$ lifts to a p-submanifold of $[X^n; \mathcal{C}]$ provided $\mathfrak{b} \subset \mathcal{C}$. If $A \in \mathcal{M}(\cap \mathfrak{b})$ is a boundary face of the intersection of the elements of $\mathfrak{b}$ and $\tilde{A}$ is its lift to $[X^n; \mathfrak{b}]$ we denote the intersection by

$$H_{A,\mathfrak{b}} = \tilde{A} \cap D_{\mathfrak{b}}\ A \subset \cap \mathfrak{b}. \tag{7.1}$$

These are all p-submanifolds, indeed they are each interior p-submanifolds of the corresponding boundary face $\tilde{A}$, henceforth denoted A again, which is the boundary-hull, i.e. $H_{A,\mathfrak{b}}$ is contained in no smaller boundary face than A. As such their lifts are always well-defined under blow up of boundary faces (for us only lifted from X^n) and $H_{A,\mathfrak{b}}$ remains an interior p-submanifolds of the lift of A. Thus we conclude that

$$\begin{gathered} \text{Provided } \mathfrak{b} \subset \mathcal{C},\ H_{A,\mathfrak{b}} \subset [X^n; \mathcal{C}] \text{ is an interior p-submanifold of} \\ A \text{ lifted to } [X^n; \mathcal{C}],\ \forall\ A \subset B(\mathfrak{b}) = \cap \mathfrak{b}. \end{gathered} \tag{7.2}$$

If fact if $\mathfrak{b} \subset \mathcal{C}$ then (7.1) still holds in after further blow ups:

$$H_{A,\mathfrak{b}} = A \cap D_{\mathfrak{b}} \text{ in } [X^n; \mathcal{C}],\ A \subset \cap \mathfrak{b}. \tag{7.3}$$

This allows us to compute the intersection of $H_{A,\mathfrak{b}}$ and $H_{A',\mathfrak{b}'}$ in any $[X^n; \mathcal{C}]$ in which they are both defined, i.e. if $\mathfrak{b} \cup \mathfrak{b}' \subset \mathcal{C}$, $A \subset \cap \mathfrak{b}$ and $A' \subset \cap \mathfrak{b}'$.

First, if $\mathfrak{b}$, $\mathfrak{b}' \subset \mathcal{B}_{(2)}$ are each transversal subsets their 'transversal union' is defined by, and following, (5.9).

Consider the boundary diagonals which lie in a given boundary face. As already noted, if $A \in \mathcal{B}_{(2)}$ then $H_{A,\mathfrak{b}}$ is a p-submanifold of the lift of A to $[X^n; \mathcal{C}]$ provided $\mathfrak{b} \subset \mathcal{C}$ and $A \subset \cap \mathfrak{b}$. We need to blow up all these submanifolds, as A and $\mathfrak{b}$ vary over all such possibilities. Notice that if $\mathfrak{b}$, $\mathfrak{b}' \subset \mathcal{C}$, with the latter closed under non-transversal intersection, then $\mathfrak{b} \Cup \mathfrak{b}' \subset \mathcal{C}$ since the elements are all non-transversal intersections of elements of $\mathfrak{b}$ and $\mathfrak{b}'$.

From (5.9) we conclude:

Lemma 7.1. *If $\mathfrak{b}_i \subset \mathcal{C} \subset \mathcal{B}_{(2)}$ are two transversal subsets for $i = 1, 2$ and $A \subset (\cap \mathfrak{b}_1) \cap (\cap \mathfrak{b}_2)$ is a common boundary face then $\mathfrak{b}_1 \Cup \mathfrak{b}_2 \subset \mathcal{C}$, $A \subset \cap(\mathfrak{b}_1 \Cup \mathfrak{b}_2)$ and*

$$H_{A,\mathfrak{b}_1} \cap H_{A,\mathfrak{b}_2} = H_{A,\mathfrak{b}_1 \Cup \mathfrak{b}_2} \text{ in } [X^n; \mathcal{C}]. \tag{7.4}$$

Proof. This follows from the fact that we can identify $D_{\mathfrak{b}_1}$ in $[X^n; \mathfrak{b}_1]$, then blow up the elements of $\mathfrak{b}_1 \Cup \mathfrak{b}_2$ and then $\mathfrak{b}_2$. There is an intersection order of $\mathcal{C}$ in which these are the first blow-ups and are in this order; it then follows that (7.4) holds in general. □

Definition 7.2. The codimension of a transversal collection $\mathfrak{b} \subset \mathcal{B}_{(2)}$ is the sum of the codimensions of its elements, so is the codimension of their intersection. A size-order on such transversal collections is an order in which this total codimension is weakly decreasing.

In the second stage of the construction of the scattering n-fold product we need to blow up all of the $H_{A,\mathfrak{b}}$. Since this has to be done step by step we consider a closure condition under intersection on a collection of the submanifolds which is enough to allow them all to be blown up unambiguously.

Definition 7.3. A collection $\mathcal{G} \subset \{H_{A,\mathfrak{b}}\}$ in $[X^n; \mathcal{C}]$, so by assumption $H_{A,\mathfrak{b}} \in \mathcal{G}$ implies $\mathfrak{b} \subset \mathcal{C}$, is *intersection-closed* if

$$H_{A_i,\mathfrak{b}_i} \in \mathcal{G},\ i = 1, 2 \Longrightarrow H_{A,\mathfrak{b}} \in \mathcal{G},\ A = A_1 \cap A_2,\ \mathfrak{b} = \mathfrak{b}_1 \Cup \mathfrak{b}_2. \tag{7.5}$$

A *chain-order* on $\mathcal{G}$ is an order in which each $\mathfrak{b}$ which occurs does so only in an uninterrupted interval with the codimension of $\mathfrak{b}$ weakly decreasing overall and when $\mathfrak{b}$ is unchanging, the codimension of A is weakly decreasing.

Thus this is a 'lexicographic order' in which $\mathfrak{b}$ is the first 'letter' and A the second.

Proposition 7.4. *An intersection-closed collection, $\mathcal{G}$, of boundary diagonals in $[X^n; \mathcal{C}]$ can be blown up in any chain-order (so under such blow ups all later elements lift to p-submanifolds) and the resulting manifold is independent of the chain-order chosen.*

Proof. It follows from Proposition 6.3 that the elements of $\mathcal{G}$ form a d-collection of p-submanifolds in $[X^n; \mathcal{C}]$. To apply Proposition 6.4 we need to check the closure condition (6.4). Consider two elements $H_{A_i,\mathfrak{b}_i}$ of $\mathcal{G}$; by assumption $H_{A,\mathfrak{b}}$ in (7.5) is also an element of $\mathcal{G}$. Applying Lemma 24.5.2008.150 to A_1 and A_2 shows that

$$A_1 \cap A_2 \subset A \text{ in } [X^n; \mathcal{C}]. \tag{7.6}$$

However from (7.3) we conclude that

$$H_{A_1,\mathfrak{b}_1} \cap H_{A_2,\mathfrak{b}_2} \subset H_{A,\mathfrak{b}} \text{ in } [X^n; \mathcal{C}]. \tag{7.7}$$

Since the index sets as a d-collection are just the $\mathfrak{b}$'s this shows that the closure condition (6.4) for $\mathcal{G}$ follows from (7.5).

Thus Proposition 6.4 shows that the elements of $\mathcal{G}$ can be blown up in any order so when blown up each element has maximal interior codimension or is a boundary face. Clearly a chain-order as defined above has this property.

To complete the proof of the Proposition it remains to show that different chain-orders lead to the same blown up manifold. To see this means first showing that two neighbouring elements $H_{A_i,\mathfrak{b}}$ with the same $\mathfrak{b}$ and with A_i of the same original codimension in X^n can be interchanged. By (7.5), $H_{A_1 \cap A_2,\mathfrak{b}} \in \mathcal{G}$ must already have been blown up. As follows from (7.7), before it is blown up this contains the intersection of the $H_{A_i,\mathfrak{b}}$ but cannot contain either of them. It follows as in the case of boundary faces that after this boundary diagonal has been blown up these two are disjoint and hence can be interchanged. It is also necessary see that the ordering amongst the $\mathfrak{b}$ can be changed, subject to the decrease of codimension of $\mathfrak{b}$. That this is possible follows from the next result which completes the proof of the Proposition. ☐

Definition 7.5. Let $\mathcal{H}_{*,\mathcal{C}}$ be the collection of the $H_{A,\mathfrak{b}} \subset [X^n; \mathcal{C}]$ where $\mathfrak{b} \subset \mathcal{C}$ is a transversal subcollection of boundary faces and $A \subset \cap\mathfrak{b}$. A collection $\mathcal{G} \subset \mathcal{H}_{*,\mathcal{C}}$ is *face–closed* if $H_{A,\mathfrak{b}_1 \Cup \mathfrak{b}_2} \in \mathcal{G}$ whenever $H_{A,\mathfrak{b}_i} \in \mathcal{G}$ for $i = 1, 2$. Thus each of the subcollections which have a given boundary face as boundary hull is closed under intersection.

Such a collection is *chain–closed* if $H_{A,\mathfrak{b}} \in \mathcal{G} \subset \mathcal{H}_{*,\mathcal{C}}$ and $A' \subset A \subset \cap\mathfrak{b}$, implies $H_{A',\mathfrak{b}} \in \mathcal{G}$.

A collection $\mathcal{G} \subset \mathcal{H}_{*,\mathcal{C}}$ is *fc-closed* if it is both face-closed and chain-closed.

Now, for a collection $\mathcal{G} \subset \mathcal{H}_{*,\mathcal{C}}$ let $\beta(\mathcal{G})$ be the collection of transversal boundary faces which occurs, that is, $\mathfrak{b} \in \beta(\mathcal{G})$ if and only if $H_{A,\mathfrak{b}} \in \mathcal{G}$ for some A.

Lemma 7.6. *If $\mathcal{C} \subset \mathcal{B}_{(2)}$ is closed under non-transversal intersection then $\mathcal{G} \subset \mathcal{H}_{*,\mathcal{B}}$ is fc-closed if and only if it is intersection-closed in the sense of* (7.5) *and*

$$\begin{aligned}&\mathfrak{b} \in \beta(\mathcal{G}) \Longrightarrow \\ &\quad \{A \in \mathcal{B}_{(2)}; H_{A,\mathfrak{b}} \in \mathcal{G}\} \text{ is closed under passage to boundary faces.}\end{aligned} \tag{7.8}$$

Proof. Indeed, (7.8) is just a restatement of the chain-closure condition. The intersection-closure property (7.5) implies the face-closure condition by applying it with $A_1 = A_2$. Conversely (7.5) must always hold for an fc-closed collection in the sense defined above since $H_{A_i,\mathfrak{b}_i} \in \mathcal{G}$ for $i = 1, 2$ implies $H_{A_1 \cap A_2,\mathfrak{b}_i} \in \mathcal{G}$ for $i = 1, 2$ by chain-closure and then $H_{A_1 \cap A_2,\mathfrak{b}_1 \Cup \mathfrak{b}_2} \in \mathcal{G}$ by face-closure. □

Remark 7.7. Thus Proposition 7.4 applies to an fc-closed collection of boundary diagonals. It also follows that the part of an fc-closed collection $\mathcal{G} \subset \mathcal{H}_{*,\mathcal{C}}$ which occurs before any given point in a chain-order is also fc-closed and chain-ordered, since the elements the existence of which is required by (9.2) and (7.8) must occur earlier in the chain-order.

8. Scattering configuration spaces

In this section we complete the definition of the scattering configuration space X^n_{sc}. This is defined by blowing up the boundary d-collection of boundary multi-diagonals in X^n_{b}.

Definition 8.1. The n-fold scattering configuration space (or stretched product) of a compact manifold with boundary is defined to be

$$X^n_{\mathrm{sc}} = [X^n_{\mathrm{b}}; H_{*,\mathcal{B}_{\mathrm{b}}}] \tag{8.1}$$

where the boundary diagonals are to be blown up in a chain-order.

That this manifold exists and is independent of choice of the chain-order chosen follows from the fact that Proposition 7.4 certainly applies to the collection of all boundary diagonals when $\mathcal{C} = \mathcal{B}_{\mathrm{b}}$. Moreover the same argument applies to show the symmetry of the resulting object.

Proposition 8.2. *The permutation group Σ_n acts on X^n_{sc} as the lifts of the factor exchange diffeomorphisms of X^n.*

Proof. This just amounts to carrying out the blow up in (8.1) in a different chain-order. □

This means that to construct all the maps $X^n_{\mathrm{sc}} \longrightarrow X^m_{\mathrm{sc}}$, $m < n$, covering the projections off various factors, it suffices to consider the case $m = n-1$ with the last factor projected off and then apply permutations and compose. This is discussed in detail in §10 where the arguments depend on more complicated commutation results which we proceed to discuss.

9. Reordering blow-ups

From Proposition 7.4 it follows that the blow up in $[X^n; \mathcal{C}]$ of an fc-closed subcollection $\mathcal{G} \subset \mathcal{H}_{*,\mathcal{C}}$, where $\mathcal{C} \subset \mathcal{B}_{\mathrm{b}}$ is closed under non-transversal intersection, is iteratively defined with respect to any chain-order and the final result is a manifold with corners which is independent of the chain-order. Thus, under these conditions, $[X^n; \mathcal{C}; \mathcal{G}]$ is well-defined. In this section we give three results which relate these manifolds under blow-down.

Remark 9.1. In the blow up of an fc-closed collection $\mathcal{G}$ of boundary diagonals the order is such that all lifts are closures of inverse images of complements with respect to the centre. It follows that the intersection properties of any two elements, meaning whether they are transversal, comparable or n.c.n.t., remain unchanged unless their (original and hence persisting) intersection is blown up in which case they become disjoint.

Proposition 9.2. *Let $\mathcal{C} \subset \mathcal{B}_{(2)}$ be closed under non-transversal intersection and suppose $\mathcal{G} \subset \mathcal{H}_{*,\mathcal{C}}$ is an fc-closed subcollection of boundary diagonals such that*

$$H_{A,\mathfrak{a}} \in \mathcal{G} \Longrightarrow A \in \mathcal{C} \tag{9.1}$$

and consider a particular element $H_{C,\mathfrak{c}} \in \mathcal{G}$ with the additional properties

$$\begin{gathered}\{H_{C,\mathfrak{b}} \in \mathcal{G}; \mathfrak{b} \neq \mathfrak{c}\} \text{ is closed under intersections and} \\ \mathcal{C} \ni A \supsetneq C \Longrightarrow H_{A,\mathfrak{c}} \notin \mathcal{G}\end{gathered} \tag{9.2}$$

then there is a blow-down map

$$[X^n; \mathcal{C}; \mathcal{G}] \longrightarrow [X^n; \mathcal{C}; \mathcal{G}'], \ \mathcal{G}' = \mathcal{G} \setminus \{H_{C,\mathfrak{c}}\}. \tag{9.3}$$

Proof. First observe that the collection $\mathcal{G}'$ is fc-closed. The chain-closure condition is just (7.8) and this holds for $\mathfrak{b} \neq \mathfrak{c}$ since it holds for $\mathcal{G}$. For $\mathfrak{b} = \mathfrak{c}$ the only danger is that the element in question is $H_{C,\mathfrak{c}}$. However, the second assumption in (9.2) shows that C cannot be a boundary face of A with $H_{A,\mathfrak{c}} \in \mathcal{G}'$. Similarly if $H_{A_i,\mathfrak{b}_i} \in \mathcal{G}'$ for $i = 1$, 2, the element required in (7.5), which exists in $\mathcal{G}$ by hypothesis, is certainly in $\mathcal{G}'$ unless $A_1 \cap A_2 = C$ and $\mathfrak{c} = \mathfrak{b}_1 \Cup \mathfrak{b}_2$. Thus C must then be a boundary face of both A_1 and A_2. It cannot be that $\mathfrak{b}_i = \mathfrak{c}$ for either $i = 1$ or 2 since this would mean $A_i = C$ by the second part of (9.2) and hence the corresponding boundary diagonal would not be in $\mathcal{G}'$. Thus $\mathfrak{b}_i \neq \mathfrak{c}$ and then (7.8) implies that $H_{C,\mathfrak{b}_1}$ and $H_{C,\mathfrak{b}_2} \in \mathcal{G}$. Their intersection being $H_{C,\mathfrak{c}}$ then violates the first condition in (9.2) so (7.5) does hold for $\mathcal{G}'$.

Thus the right side of (9.3) is indeed defined. The body of the proof below is devoted to showing that if $H_{A,\mathfrak{a}}$ is the last element in $\mathcal{G}'$ with respect to a chosen chain-order and $\widetilde{\mathcal{G}}$ is $\mathcal{G}'$ with this removed then

$$[X^n; \mathcal{C}; \widetilde{\mathcal{G}}; H_{C,\mathfrak{c}}; H_{A,\mathfrak{a}}] = [X^n; \mathcal{C}; \widetilde{\mathcal{G}}; H_{A,\mathfrak{a}}; H_{C,\mathfrak{c}}] \tag{9.4}$$

including of course showing that both are defined.

To see that this identity proves the Proposition, observe, following Remark 7.7, that the $\mathcal{G}_j$ obtained from $\mathcal{G}$ by dropping the last j terms for $1 \le j \le j'$, where $H_{C,\mathfrak{c}}$ is $j' + 1$ terms from the end, are fc-closed and chain-ordered. Moreover, the conditions of the Proposition hold for all these j. Then iterating (9.4) shows that all the manifolds obtained by blowing up $H_{C,\mathfrak{c}}$ at some later point are all canonically diffeomorphic and hence there is a blow-down map (9.3).

Thus we are reduced to showing (9.4) under the assumptions of the Proposition. To do so we consider that the intersection properties of $H_{C,\mathfrak{c}}$ and $H_{A,\mathfrak{a}}$ (which is the last element of $\mathcal{G}$) in the manifold $M_1 = [X^n; \mathcal{C}]$ and subsequently in the manifold M_2 which is M_1 with all elements preceding $H_{C,\mathfrak{c}}$ blown up.

All boundary faces B forming the boundary-hull of elements $H_{B,\mathfrak{b}} \in \mathcal{G}$ are, by assumption in (9.1), in $\mathcal{C}$ and hence have already been blown up in M_1. In particular if the intersection $A \cap C$ is n.c.n.t., then $A \cap C$ has been blown up, the lifts of A and C are disjoint and hence so are $H_{C,\mathfrak{c}}$ and $H_{A,\mathfrak{a}}$. On the other hand, if $A \pitchfork C$ in X^n then $\cap\mathfrak{a}$ and $\cap\mathfrak{c}$, which contain them, must also be transversal so in fact $\mathfrak{a}$

and $\mathfrak{c}$ must meet transversally as subsets of $\mathcal{B}_{(2)}$. It follows that $H_{A,\mathfrak{a}}$ and $H_{C,\mathfrak{c}}$ are transversal in M_1. By Remark 9.1 they remain transversal in M_2.

So we need consider the intersection properties of $H_{C,\mathfrak{c}}$ and a later $H_{A,\mathfrak{a}}$ with A and C comparable. The chain-order condition on blow-ups, in which $H_{A,\mathfrak{a}}$ comes after $H_{C,\mathfrak{c}}$ implies that $\mathfrak{a} > \mathfrak{c}$ since the second part of (9.2) implies that $H_{C,\mathfrak{c}}$ is the last element corresponding to the diagonal $D_{\mathfrak{c}}$. It follows that $\mathfrak{a} \Cup \mathfrak{c} < \mathfrak{c}$ unless $\mathfrak{c} \Subset \mathfrak{a}$. In the first case $H_{A\cap C,\mathfrak{a}\Cup\mathfrak{c}}$ precedes $H_{C,\mathfrak{c}}$ and so has already been blown up, and hence by Remark 9.1, $H_{C,\mathfrak{c}}$ and $H_{A,\mathfrak{a}}$ are disjoint at some point before $H_{C,\mathfrak{c}}$ in the order of $\mathcal{G}$ and so remain so in M_2.

Summarizing we see that

$$\begin{aligned} &\text{Either } A \text{ and } C \text{ not comparable or } \mathfrak{a} \not\Subset \mathfrak{c} \Longrightarrow \\ &H_{C,\mathfrak{c}} \text{ and } H_{A,\mathfrak{a}} \text{ are transversal (or disjoint) in } M_2. \end{aligned} \tag{9.5}$$

Thus it remains to consider the cases in which A and C are comparable and the diagonals $D_{\mathfrak{a}}$ and $D_{\mathfrak{c}}$ are also comparable.

Suppose first that $A \subset C$ with strict inclusion, before blow up. Then $H_{A,\mathfrak{c}}$ has been blown up earlier and

$$A \subsetneq C, \mathfrak{c} \Subset \mathfrak{a} \Longrightarrow H_{C,\mathfrak{c}} \text{ and } H_{A,\mathfrak{a}} \text{ are disjoint in } M_2 \tag{9.6}$$

by Remark 9.1. Next

$$A = C,\ \mathfrak{c} \Subset \mathfrak{a} \Longrightarrow H_{C,\mathfrak{c}} \subset H_{C,\mathfrak{a}} \text{ in } M_2 \tag{9.7}$$

since this is true in M_1 and not affected by subsequent blow ups. Finally

$$C \subsetneq A,\ \mathfrak{c} \Subset \mathfrak{a} \Longrightarrow H_{C,\mathfrak{c}} \subset H_{A,\mathfrak{a}} \text{ in } M_2 \text{ and } H_{C,\mathfrak{a}} \in \mathcal{G}. \tag{9.8}$$

Combining (9.5), (9.6), (9.7) and (9.8) we see that in M_2, i.e. immediately before $H_{C,\mathfrak{c}}$ is to be blown up in $\mathcal{G}$, all the subsequent lifted boundary diagonals are transversal (including disjoint) or closed else $H_{C,\mathfrak{c}}$ is contained in $H_{A,\mathfrak{c}}$ with $H_{C,\mathfrak{a}}$ coming earlier if $C \subsetneq A$. From this (9.4) follows.

This completes the proof of (9.4) and hence of the Proposition. □

Next we consider a result which allows a boundary face to be blown down without blowing down the boundary diagonals which are (or rather were) contained in it provided the diagonals do not 'involve' the given boundary face.

Proposition 9.3. *Let $B \in \mathcal{C} \subset \mathcal{B}_{(2)}$ be such that both $\mathcal{C}$ and $\mathcal{C}' = \mathcal{C} \setminus \{B\}$ are closed under non-transversal intersections and suppose $\mathcal{G} \subset \mathcal{H}_{*,\mathcal{C}'}$ is an fc-closed subcollection of boundary diagonals such that in addition*

$$\begin{aligned} &B \text{ comes last in some intersection-order of } \mathcal{C} \text{ and} \\ &H_{A,\mathfrak{b}} \in \mathcal{G} \text{ implies } A \in \mathcal{C} \end{aligned} \tag{9.9}$$

then there is a blow-down map

$$[X^n;\mathcal{C};\mathcal{G}] \longrightarrow [X^n;\mathcal{C}';\mathcal{G}]. \tag{9.10}$$

Proof. By hypothesis both sides of (9.10) are well defined and at least at a formal level differ by the blow-up of B. Furthermore, the hypotheses continue to hold if the 'tail' of $\mathcal{G}$ is cut off at any point, with respect to a given chain-order. Thus we only need to show that under the given hypotheses, blowing up (the lift of) B in $[X^n;\mathcal{C}';\mathcal{G}]$ gives the same manifold as blowing it up before the last element of $\mathcal{G}$ since then we can use induction over the number of elements in $\mathcal{G}$ to prove (9.10) in the general case.

So, let the last element of $\mathcal{G}$ be $H_{A,\mathfrak{a}}$. Thus $\mathfrak{a} \subset \mathcal{C}'$ is a transversal collection of boundary faces and by the second hypothesis, either $A = B$ or else $A \in \mathcal{C}'$. Now, if B and A are n.c.n.t., then $A \cap B$ must already have been blown up and as a result B and $H_{A,\mathfrak{a}}$ are disjoint in $[X^n; \mathcal{C}']$ and this must remain true at the point of interest, so commutation is possible. If A and B are transversal, then so are $H_{A,\mathfrak{a}}$ and B for any $\mathfrak{a}$ so the same conclusion follows.

Thus we need only consider the case that A and B are comparable. Suppose first that $B \subset A$ is a strict inclusion. Then, by the chain condition on $\mathcal{G}$, $H_{B,\mathfrak{a}} \in \mathcal{G}$ must already have been blown up. However, initially, this is the intersection of $H_{A,\mathfrak{a}}$ and B and by Remark 9.1 remains so until it is blown up. Thus the manifolds $H_{A,\mathfrak{a}}$ and B must be disjoint at the point of interest and commutation is trivial.

Thus we may suppose conversely that $B \supset A$, at first strictly. Then $A \in \mathcal{C}'$ has been blown up, and in doing so it becomes transversal to B and this implies that B is transversal to $H_{A,\mathfrak{a}}$ so this remains true at the end of the blow up of $\mathcal{G}$ and commutation of the blow up of B and $H_{A,\mathfrak{a}}$ is again possible.

Thus only the case $B = A$ remains. Then $A = B \subset \cap\mathfrak{a}$ and indeed $H_{A,\mathfrak{a}} = H_{B,\mathfrak{a}} \subset B$. By Remark 9.1, this must still be true at the point of interest since no $H_{B,\mathfrak{c}}$ containing $B_{B,\mathfrak{a}}$ can have been blown up (since $\mathcal{G}$ is chain-ordered), so commutation is again possible.

This proves, inductively, that there is a blow-down map (9.10). □

The third result in this section corresponds to blowing down boundary diagonals with boundary hull which has not been blown up.

Proposition 9.4. *Suppose $\mathcal{C} \subset \mathcal{B}_{(2)}$ and $B \in \mathcal{C}$ is such that both $\mathcal{C}$ and $\mathcal{C}' = \mathcal{C} \setminus \{B\}$ are closed under passage to boundary faces (in X^n) and $\mathcal{G} \subset \mathcal{H}_{*,\mathcal{C}'}$ is fc-closed and such that*

$$H_{A,\mathfrak{b}} \in \mathcal{G} \Longrightarrow A \in \mathcal{C} \tag{9.11}$$

then there is an iterated blow-down map

$$[X^n; \mathcal{C}'; \mathcal{G}] \longrightarrow [X^n; \mathcal{C}'; \mathcal{G}'],\ \mathcal{G}' = \{H_{A,\mathfrak{b}} \in \mathcal{G}; A \neq B\}. \tag{9.12}$$

Proof. Under the hypotheses of the Proposition it suffices to show that there is a blow-down map

$$[X^n; \mathcal{C}'; \mathcal{G}] \longrightarrow [X^n; \mathcal{C}'; \mathcal{L}] \tag{9.13}$$

where $\mathcal{G} = \mathcal{L} \cup \{H_{B,\mathfrak{b}}\}$ with $H_{B,\mathfrak{b}} \in \mathcal{G}$ where $\mathfrak{b}$ is of minimal codimension for the existence of such an element. The general case then follows by iteration. Thus we need to show that $H_{B,\mathfrak{b}}$ can be blown down, and to do this it needs to be commuted through the boundary diagonal faces $H_{A,\mathfrak{a}}$ which come after it. We can take the chain-order of $\mathcal{G}$ so that $H_{B,\mathfrak{b}}$ is followed only by elements $H_{C,\mathfrak{b}}$ with $\operatorname{codim} C < \operatorname{codim} B$ or by $H_{A,\mathfrak{a}}$ with $\mathfrak{a}$ of smaller codimension than $\mathfrak{b}$.

Consider the intersection properties of $H_{B,\mathfrak{b}}$ and $H_{A,\mathfrak{a}}$ in $M = [X^n; \mathcal{C}']$ with the boundary diagonals preceding $H_{B,\mathfrak{b}}$ also blown up. Certainly neither A nor C can contain B since then $\mathcal{C}'$ would not contain the boundary faces of all its elements. It follows that $H_{C\cap B,\mathfrak{b}}$ has already been blown up so $H_{B,\mathfrak{b}}$ and $H_{C,\mathfrak{b}}$ are disjoint, by Remark 9.1. So, consider a later element $H_{A,\mathfrak{a}} \in \mathcal{G}$ where $\operatorname{codim}(\mathfrak{a}) < \operatorname{codim}(\mathfrak{b})$. As already noted, $A \supset B$ is not possible. Thus either A and B are not comparable or $A \subset B$ strictly. In either first case, $A \cap B \in \mathcal{C}'$ has been blown up so $H_{B,\mathfrak{b}}$ and $H_{A,\mathfrak{a}}$

are disjoint. In the second, $H_{A,\mathfrak{a}\uplus\mathfrak{b}}$ has been blown up and again these manifolds are disjoint in M.

Thus all the boundary diagonals following $H_{B,\mathfrak{b}}$ are disjoint from it and commutation is possible, proving (9.13) and hence the Proposition. □

10. Scattering stretched projections

For any compact manifold X with connected boundary the n-fold scattering space is defined has been defined in §8.

Theorem 10.1. *The stretched boundary projection off any $n-l$ factors lifts to a uniquely defined smooth map which is a b-fibration giving a commutative diagramme with the vertical blow-down maps*

$$\begin{tikzcd} X^n_{\mathrm{sc}} \arrow[r, "\pi_{\mathrm{sc}}"] \arrow[d] & X^l_{\mathrm{sc}} \arrow[d] \\ X^n_{\mathrm{b}} \arrow[r, "\pi_{\mathrm{b}}"] \arrow[d] & X^l_{\mathrm{b}} \arrow[d] \\ X^n \arrow[r, "\pi"] & X^l. \end{tikzcd} \tag{10.1}$$

Proof. It suffices to prove this in case $l=n-1$ with the last factor projected off and then iterate; the lower part of the diagramme has already been constructed in Proposition 4.4. We proceed to show there is an iterated blow-down map

$$\beta : X^n_{\mathrm{sc}} \longrightarrow X^{n-1}_{\mathrm{sc}} \times X \tag{10.2}$$

starting from the corresponding map for the boundary stretched products

$$\beta : X^n_{\mathrm{b}} \longrightarrow X^{n-1}_{\mathrm{b}} \times X. \tag{10.3}$$

The scattering n-fold stretched product is obtained from X^n_{b} by blowing up, in chain-order, all the $H_{A,\mathfrak{b}}$. Each $\mathfrak{b} \subset \mathcal{B}_{(2)}$ either has $\cap\mathfrak{b} \subset X^{n-1} \times \partial X$, or not, and similarly either $A \subset X^{n-1} \times \partial X$ or not. This leads to the decomposition

$$\mathcal{H}_{*,*} = \mathcal{H}_{\mathrm{v,v}} \cup \mathcal{H}_{\mathrm{nv,v}} \cup \mathcal{H}_{\mathrm{nv,nv}} \tag{10.4}$$

where the first part corresponds to those $H_{A,\mathfrak{b}}$ with $\cap\mathfrak{b} \subset X^{n-1} \times \partial X$ (and hence also $A \subset X^{n-1} \times \partial X$, the second part to those with $A \subset X^{n-1} \times \partial X$ but $\cap\mathfrak{b} \cap X^{n-1} \times (X \setminus \partial X) \neq \emptyset$ and the third part being the remainder. Note that the implied order here is very far from a chain-order.

Since $\mathcal{H}_{\mathrm{v,v}}$ is just the lift from X^{n-1}_{b} of all the boundary diagonals there

$$[X^n; \mathcal{B}^{\mathrm{v}}_{(2)}; \mathcal{H}_{\mathrm{v,v}}] = X^n_{\mathrm{sc}} \times X \tag{10.5}$$

which is the space we wish to map to. Thus we need to show that the elements of $\mathcal{H}_{\mathrm{nv,nv}}$ and $\mathcal{H}_{\mathrm{nv,v}}$ can all be blown down.

Proposition 9.2 applies directly to the elements of $\mathcal{H}_{\mathrm{nv,nv}}$. Thus we define successive subsets of the set of all boundary diagonals by dropping the elements of $\mathcal{H}_{\mathrm{nv,nv}}$, starting with the last, in an overall chain-order. Then (9.1) holds for each $\mathcal{G}$ so constructed as do the two conditions in (9.2). Thus, the elements of $\mathcal{H}_{\mathrm{nv,nv}}$ can indeed be blown down in reverse order and we have shown the existence of an iterated blow down map

$$X^n_{\mathrm{sc}} \longrightarrow [X^n; \mathcal{B}_{(2)}; \mathcal{H}_{\mathrm{v,v}} \cup \mathcal{H}_{\mathrm{nv,v}}]. \tag{10.6}$$

Now, we order the boundary faces of X^n by taking first the elements of $\mathcal{B}^{\mathrm{v}}_{(2)}$, products of the boundary faces of X^{n-1} with X, in size-order, and then the remainder which form $\mathcal{B}^{\mathrm{nv}}_{(2)}$ in a size-order. Overall this is an intersection order so $X^n_{\mathrm{b}} = [X^{n-1}_{\mathrm{b}} \times X; \mathcal{B}^{\mathrm{v}}_{(2)}]$.

Propositions 9.3 and 9.4 can now be applied to show the existence of successive blow-down maps

$$[X^{n-1}_{\mathrm{b}} \times X; \mathcal{B}; \mathcal{H}_{\mathrm{v,v}}; \mathcal{H}_{\mathrm{nv,v}}] \longrightarrow [X^{n-1}_{\mathrm{b}} \times X; \mathcal{B}^{\mathrm{nv}}(j); \mathcal{H}_{\mathrm{v,v}}; \mathcal{H}_{\mathrm{nv,v}}(j)] \tag{10.7}$$

where $\mathcal{B}^{\mathrm{nv}}(j) \subset \mathcal{B}^{\mathrm{nv}}$ is obtained by removing the last element j elements and $\mathcal{H}_{\mathrm{nv,v}}(j) \subset \mathcal{H}_{\mathrm{nv,v}}$ is obtained by removing all those $H_{B,\mathrm{v}}$ corresponding to the $B \in \mathcal{B}^{\mathrm{nv}} \setminus \mathcal{B}^{\mathrm{nv}}(j)$. This can be accomplished inductively, so it suffices to show that we can pass from the space on the right in (10.7) for j to the same space for $j+1$. Thus, the objective is to remove the last element of $\mathcal{B}^{\mathrm{nv}}_{(2)}$; Proposition 9.3 shows that this is possible. Thus there is a blow down map

$$[X^{n-1}_{\mathrm{b}} \times X; \mathcal{B}^{\mathrm{nv}}(j); \mathcal{H}_{\mathrm{v,v}}; \mathcal{H}_{\mathrm{nv,v}}(j)] \longrightarrow [X^{n-1}_{\mathrm{b}} \times X; \mathcal{B}^{\mathrm{nv}}(j+1); \mathcal{H}_{\mathrm{v,v}}; \mathcal{H}_{\mathrm{nv,v}}(j)] \tag{10.8}$$

for each j. Now, Proposition 9.4 can then be applied to 'remove,' i.e. blow down, all elements of $\mathcal{H}_{\mathrm{nv,v}}(j) \setminus \mathcal{H}_{\mathrm{nv,v}}(j)$; these are of the form $H_{B,\mathfrak{b}}$ where $\mathcal{B}^{\mathrm{nv}}_{(2)}(j+1) = \mathcal{B}^{\mathrm{nv}}_{(2)}(j) \cup \{B\}$. Thus, by alternating between Propositions 9.3 and 9.4 we have shown the existence of an iterated blow-down map

$$X^n_{\mathrm{sc}} \longrightarrow [X^{n-1}_{\mathrm{b}} \times X; \mathcal{H}_{\mathrm{v,v}}] = X^{n-1}_{\mathrm{sc}} \times X \tag{10.9}$$

as desired. Composing with the projection shows the existence of a b-map π_{sc} giving the commutative diagramme of smooth maps (10.1).

By construction the stretched projection is a b-map. To show that it is a b-submersion we proceed backwards through the construction above. Starting from $X^{n-1}_{\mathrm{sc}} \times X$ all of the blow-ups of elements of $\mathcal{H}_{\mathrm{nv,v}}$ and $\mathcal{B}^{\mathrm{nv}}_{(2)}$ are of boundary faces. Thus these blow-ups are indeed b-submersions. However, the remaining blow-ups, of the elements of $\mathcal{H}_{\mathrm{nv,nv}}$ are not of boundary faces and so these are not b-submersions. Rather it is only the map back to X^{n-1}_{sc}, i.e. after the projection off the factor of X that is a b-submersion. □

11. Vector spaces

As one indication of the claimed 'universality' of these scattering-stretched products alluded to in the Introduction we consider here the case of a vector space, V over $\mathbb{R}$. As noted in the Introduction the 'model' asymptotic translation-invariance structure for a general manifold with boundary is the radial compactification $\overline{V}$ of V to a ball. This has useful analytic properties, for instance the space $S^0(V)$ of classical (1-step) symbols on V of order 0 is identified with $\mathcal{C}^\infty(\overline{V})$. The question we consider here is the extension of the difference map

$$\delta : V \times V \ni (u,v) \longmapsto u - v \in \overline{V} \tag{11.1}$$

and its higher variants

$$\delta_{ij} : V^n \xrightarrow{\pi_{ij}} V \times V \xrightarrow{\delta} \overline{V}. \tag{11.2}$$

Proposition 11.1. *The difference map* (11.1) *extends to a b-fibration* $\tilde{\delta} : (\overline{V})^2_{\mathrm{sc}} \longrightarrow \overline{V}$ *and the higher maps* (11.2) *similarly extend to b-fibrations*

$$\tilde{\delta}_{ij} : (\overline{V})^n_{\mathrm{sc}} \longrightarrow \overline{V} \tag{11.3}$$

for all $i \neq j$.

Proof. The second result follows from the first and the properties of $(\overline{V})^n_{\mathrm{sc}}$ since $\tilde{\delta}_{ij} = \tilde{\delta} \circ \pi_{ij,\mathrm{sc}}$ where $\pi_{ij,\mathrm{sc}}$ is the scattering stretched projection corresponding to π_{ij}. As the product of two b-fibrations it is also a b-fibration. The proof that δ extends to a b-fibration is elementary. First, the difference extends smoothly to either $\overline{V} \times V$ or $V \times \overline{V}$ so it suffices to consider a small neighbourhood of $(\partial\overline{V})^2$ in $(\overline{V})^2$. This is of the form $[0,1)^2 \times \mathbb{S}^{n-1}$ with the boundary variables being inverted radial variables, so the difference, written in terms of inverted polar coordinates in $\overline{V}$ is

$$((x,\omega),(s,\theta)) \longmapsto (|s\omega - x\theta|, \frac{\omega/x - \theta/s}{|\omega/x - \theta/s|}). \tag{11.4}$$

The blow up to $(\overline{V})^2_{\mathrm{b}}$ replaces x, s by $T = x + s$ and $y = \frac{x-s}{x+s}$ so the difference map becomes

$$X = T|(1-y)\omega - (1+y)\theta|,\ \phi = \frac{(1-y)\omega - (1+y)\theta}{|(1-y)\omega - (1+y)\theta|} \in \mathbb{S}^{n-1}. \tag{11.5}$$

This is smooth away from $T = 0$, $\omega = \theta$ and the scattering blow up, precisely of this set, resolves the singularity. □

As an example consider the following result on 'multilinear convolution' examining integrals of the form

$$\int_{V^n} a_{0,1}(z_0-z_1)\cdots a_{0,n}(z_0-z_n)a_{1,2}(z_1-z_2)\cdots a_{n-1,n}(z_{n-1}-z_n)dz_1\dots dz_n. \tag{11.6}$$

Corollary 11.2. *If* $a_{i,j} \in \rho^{n+1}\mathcal{C}^\infty(\overline{V})$ *are classical symbols of order* $-n-1$ *for all* $0 \leq i < j \leq n$ *and*

$$A = \prod_{0\leq i<j\leq n} \tilde{\delta}^*_{ij} a_{i,j} \in \mathcal{C}^\infty((\overline{V})^{n+1}_{\mathrm{sc}}) \tag{11.7}$$

then A *pushes forward to a classical symbol, with logs, on* $\overline{V}$.

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