SPECTRAL AND SCATTERING THEORY FOR SYMBOLIC POTENTIALS OF ORDER ZERO

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1. Introduction

In [6], Herbst initiated the spectral analysis and scattering theory for the Schrödinger operator $\Delta + V$ on $\mathbb{R}^n$ where V is a smooth, real-valued potential which is homogeneous of degree zero near infinity. Further results, reviewed below, are due to Agmon, Cruz and Herbst [1] and Herbst and Skibsted [5]. In this paper, in which the main results are restricted to the two-dimensional case, we give a rather detailed description of the generalized eigenfunctions and the structure of the scattering matrix for these operators. To obtain the most complete results we make generic assumptions on the potential, including the requirement that the restriction of the potential to the sphere at infinity is Morse. On the other hand we work in a more general context than Euclidean scattering so that our methods apply to general two-dimensional compact manifolds with boundary equipped with a scattering metric in the sense of [10]. In particular we show that the scattering matrix for this problem is a Fourier integral operator, of an appropriate type, associated with a classical scattering relation describing the asymptotic behaviour of the classical trajectories.

As can be seen already from this very brief description, even the statement of the results requires a detailed description of the associated classical problem, and the definition of classes of Fourier integral operators which match the resulting geometry in phase space. Thus, we proceed by first discussing the basic notions in scattering theory, with focus on the parameterization of tempered distributional eigenfunctions, then describe the classical problem in detail. We remark that this problem is microlocally analogous to that considered in a paper [4] of Guillemin and Schaeffer in the traditional microlocal setting. Indeed, their analysis is used in the proof of some of our results. This detailed picture will allow us to state our results, with a short interlude discussing generalizations of the isotropic calculus on vector spaces to allow different homogeneities in different directions of phase space. Finally we sketch the proof of at least some of our results. The full proofs will appear in a forthcoming manuscript.

Thus, let X be a compact manifold with boundary where, for the moment, we do not restrict the dimension. The boundary $Y = \partial X$ consists of a finite union of compact manifolds without boundary, $Y = Y_1 \cup \cdots \cup Y_N$. It is always possible to find a boundary defining function on X, $x \in \mathcal{C}^\infty(X)$ such that $x \geq 0$, $Y = \{x = 0\}$ and $dx \neq 0$ on Y. A Riemannian metric on the interior of X is a scattering metric

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if, for some choice of defining function, it takes the form

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2},\ h \in \mathcal{C}^\infty(X; S^2X),\ h_0 = h\big|_Y >> 0. \tag{1.1}$$

That is, $h = x^2(g - x^{-4}dx^2)$ is a smooth 2-cotensor on X which restricts to a metric on Y. In this more general setting we consider a real potential $V \in \mathcal{C}^\infty(X)$ and examine the spectral and scattering theory of $\Delta + V$ where Δ is the Laplace operator of a scattering metric. The Euclidean case is included since the Euclidean metric is a scattering metric on the radial compactification of $\mathbb{R}^n$ to a ball (or half-sphere) and a smooth potential which is homogeneous of degree zero near infinity is a smooth function up to the boundary of the radial compactification.

Under these assumptions the Schrödinger operator

$$P = \Delta + V \tag{1.2}$$

has continuous spectrum of infinite multiplicity (assuming $\dim X \geq 2$) occupying the interval $[\kappa, \infty)$ where

$$\kappa = \inf_Y V. \tag{1.3}$$

In addition there may be discrete spectrum in the interval $[-m, K]$,

$$m = \inf_X V,\ K = \sup_Y V \tag{1.4}$$

which is discrete in the open set $(-\infty, K) \setminus \mathrm{Crit}(V)$,

$$\mathrm{Crit}(V) = \{p \in Y; d_Y V(p) = 0\}. \tag{1.5}$$

The main interest lies in the continuous spectrum which we analyse in detail here under the assumption that $\dim X = 2$ and $V\big|_Y$ is Morse. For simplicity in these notes we outline the results under the additional assumption that Y has only one component circle and $V\big|_Y$ is perfect Morse, so only has a global maxiumum and minimum forming $\mathrm{Crit}(V)$. These restrictions are straightforward to remove but this does change the global geometry.

Our central result is the parameterization of all tempered distributions associated to the continuous spectrum; this constitutes a distributional form of 'asymptotic completeness'. Thus we examine

$$E(\lambda) = \left\{u \in \mathcal{C}^{-\infty}(X); (\Delta + V - \lambda)u = 0\right\}. \tag{1.6}$$

Note that the space of extendible distributions $\mathcal{C}^{-\infty}(X)$ reduces precisely to the space of tempered distributions in the sense of Schwartz in case X is the radial compactification of $\mathbb{R}^n$. For $\lambda < \kappa$, $E(\lambda)$ is finite-dimensional and consists of square-integrable eigenfunctions. More generally

$$E_{\mathrm{pp}}(\lambda) = E(\lambda) \cap L^2(X) \subset \dot{\mathcal{C}}^\infty(X), \tag{1.7}$$

where $L^2(X)$ is computed with respect to the Riemannian volume form, is always finite dimensional and is empty for $\lambda > K$. Since $\mathcal{C}^{-\infty}(X)$ is the dual of $\dot{\mathcal{C}}^\infty(X)$ this allows us to consider

$$\begin{gathered} E_{\mathrm{ess}}(\lambda) = \left\{u \in \mathcal{C}^{-\infty}(X); (\Delta + V - \lambda)u = 0,\ \langle u, v\rangle = 0\ \forall\ v \in E_{\mathrm{pp}}(\lambda)\right\}, \\ E(\lambda) = E_{\mathrm{ess}}(\lambda) \oplus E_{\mathrm{pp}}(\lambda). \end{gathered} \tag{1.8}$$

The structure of the space $E_{\mathrm{ess}}(\lambda)$ depends on λ; there are three distinct cases, corresponding to the values of κ, K and the additional transition point

$$\lambda_{\mathrm{Hess}} = \kappa + V''(y_{\mathrm{min}}),\ \kappa = V(y_{\mathrm{min}}) \tag{1.9}$$

where the derivatives are with respect to boundary arclength. Clearly $\lambda_{\mathrm{Hess}} > \kappa$ but the three possibilities $\lambda_{\mathrm{Hess}} < K$, $\lambda_{\mathrm{Hess}} = K$, $\lambda_{\mathrm{Hess}} > K$ may all occur for different V (or for the same V but different metrics). For any particular problem only two or three of the following four intervals can occur

$$\begin{cases} \kappa < \lambda < \min(\lambda_{\mathrm{Hess}}, K) & \text{“Near minimum”} \\ \lambda_{\mathrm{Hess}} < \lambda < K & \text{“Hessian range”} \\ K < \lambda < \lambda_{\mathrm{Hess}} & \text{“Mixed range”} \\ \max(\lambda_{\mathrm{Hess}}, K) < \lambda & \text{“Above thresholds.”} \end{cases} \tag{1.10}$$

The maximum and the minimum of the potential are thresholds corresponding to changes in the geometry of the fixed energy (i.e. characteristic) surface of the classical problem. Away from these thresholds both the maximum and the minimum correspond to critical points of the classical flow (at infinity); the Hessian transition corresponds to an energy at which there is a change in the local geometry at critical points corresponding to the minimum.

The structure of $E_{\mathrm{ess}}(\lambda)$ is closely related to the precise structure of the boundary values of the resolvent which exist for $\lambda \in (\kappa, +\infty) \setminus (\mathrm{Crit}(V) \cup \sigma_{\mathrm{pp}}(\Delta + V))$, as a consequence of an appropriate version of the Mourre estimate, or of microlocal estimates closely related to it, see [6] for the proof in the Euclidean setting. That is, for any $\delta > 0$,

$$(\Delta + V - (\lambda + i0))^{-1} = \lim_{\epsilon \to 0+} (\Delta + V - (\lambda + i\epsilon))^{-1} : x^{1/2+\delta} L^2(X) \to x^{-1/2-\delta} L^2(X). \tag{1.11}$$

For $\lambda \in \sigma_{\mathrm{pp}}(\Delta + V) \setminus \mathrm{Crit}(V)$, the limits still exist on the orthocomplement of $E_{\mathrm{pp}}(\lambda)$. Note that

$$f \in x^{1/2+\delta} L^2(X) \Longrightarrow [(\Delta + V - (\lambda + i0))^{-1} - (\Delta + V - (\lambda - i0))^{-1}] f \in E(\lambda). \tag{1.12}$$

In each of these ranges for the eigenparameter we can thus define spaces of ‘smooth’ eigenfunctions by setting

$$E^{-\infty}_{\mathrm{ess}}(\lambda) = [(\Delta + V - (\lambda + i0))^{-1} - (\Delta + V - (\lambda - i0))^{-1}](\dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda)). \tag{1.13}$$

We give a microlocal characterization of these spaces below. In all of the non-transition regions

$$E^{-\infty}_{\mathrm{ess}}(\lambda) \subset E_{\mathrm{ess}}(\lambda) \text{ is dense} \tag{1.14}$$

in the topology of $\mathcal{C}^{-\infty}(X)$. In fact, approximating sequences can be constructed rather explicitly by extending $(\Delta + V - (\lambda \pm i0))^{-1}$ to distributions satisfying a scattering wave front set condition, but we do not describe this here.

The roughest picture of the parameterization of generalized eigenfunctions, which has little to do with our particular problem, is via the two terms of (1.13). Namely, let

$$R^{-\infty}_{\mathrm{ess},\pm}(\lambda) = [(\Delta + V - (\lambda \pm i0))^{-1}(\dot{\mathcal{C}}^\infty(X) \ominus E_{\mathrm{pp}}(\lambda))]/\dot{\mathcal{C}}^\infty(X) \tag{1.15}$$

be the range of the incoming/outgoing boundary value of the resolvent acting on Schwartz functions, modulo Schwartz functions. It is not hard to show that $(\Delta +$

$V - \lambda)\dot{\mathcal{C}}^\infty(X) \subset \dot{\mathcal{C}}^\infty(X)$ is closed, and

$$(\Delta + V - \lambda) : R^{-\infty}_{\mathrm{ess},\pm}(\lambda) \longrightarrow \dot{\mathcal{C}}^\infty(X)/(\Delta + V - \lambda)\dot{\mathcal{C}}^\infty(X) \tag{1.16}$$

is an isomorphism (for each sign). Thus the range space in (1.16) is a Fréchet space. Complex conjugation gives a conjugate linear isomorphism between the two spaces $R^{-\infty}_{\mathrm{ess},\pm}(\lambda)$. The scattering matrix may now be defined abstractly on, say, the incoming space as

$$S(\lambda) : R^{-\infty}_{\mathrm{ess},-}(\lambda) \to R^{-\infty}_{\mathrm{ess},+}(\lambda),\ S(\lambda)u = [-(\Delta + V - \lambda + i0)^{-1}(\Delta + V - \lambda)u]. \tag{1.17}$$

There is a natural pairing on $R^{-\infty}_{\mathrm{ess},+}(\lambda)$ given by

$$B(u, v) = \int_X \Big([(\Delta + V - \lambda)u]\overline{v} - u\overline{[(\Delta + V - \lambda)v]}\Big)\,; \tag{1.18}$$

the same formula also defines a pairing on $R^{-\infty}_{\mathrm{ess},-}(\lambda)$. These are easily seen to be well-defined and non-degenerate. This induces pre-Hilbert norms on $R^{-\infty}_{\mathrm{ess},\pm}(\lambda)$ with respect to which $S(\lambda)$ is unitary.

In fact, B can also be thought of as a pairing between $u \in E^{-\infty}_{\mathrm{ess}}(\lambda)$ and $v \in R^{-\infty}_{\mathrm{ess},+}(\lambda)$ (or $v \in R^{-\infty}_{\mathrm{ess},-}(\lambda)$), and then the first term of (1.18) may be dropped. This way B extends to a pairing between $u \in E_{\mathrm{ess}}(\lambda)$ and $v \in R^{-\infty}_{\mathrm{ess},+}(\lambda)$, and defines the incoming distributional boundary value of u in the dual space of $R^{-\infty}_{\mathrm{ess},+}(\lambda)$.

One of the purposes of scattering theory is to make these parameterizations of the continuous spectrum and the scattering matrix, which is the relation between them, geometric. In particular this gives a linear, rather than an anti-linear isomorphism between $R^{-\infty}_{\mathrm{ess},-}(\lambda)$ and $R^{-\infty}_{\mathrm{ess},+}(\lambda)$. To do so, we need to describe the structure of the smooth eigenfunctions in the four non-transition regions. Since this structure is intimately connected to the corresponding classical problem, we proceed to describe that first.

2. Classical problem

The classical system formally associated to the Schrödinger operator $\Delta + V$ on $\mathbb{R}^n$ is generated by the Hamiltonian function

$$p = |\zeta|^2 + V(z) \in \mathcal{C}^\infty(\mathbb{R}^n_z \times \mathbb{R}^n_\zeta). \tag{2.1}$$

In fact, only the 'large momentum' (the usual semiclassical limit) $\zeta \to \infty$ or the 'large distance' limit $z \to \infty$ are relevant to the behaviour of solutions. Since we are interested in eigenfunctions

$$(\Delta + V - \lambda)u = 0, \tag{2.2}$$

only the vicinity of the finite (fixed) energy, or characteristic, surface which can be written formally

$$\Sigma(\lambda) = \big\{|\zeta|^2 + V(z) = \lambda,\ |z| = \infty\big\} \tag{2.3}$$

is involved. Setting $|z| = \infty$ is restriction to the sphere at infinity for the radial compactification $\bar{\mathbb{R}}^n$ of $\mathbb{R}^n$ so $\Sigma(\lambda) \subset \mathbb{S}^{n-1}_\omega \times \mathbb{R}^n$, $z = |z|\omega$. This phase space at infinity is a contact manifold where the contact form is

$$\sigma = d\rho + \mu \cdot d\omega,\ \zeta = \mu \oplus \rho\omega,\ \mu \in \omega^\perp. \tag{2.4}$$

In the general case of a scattering metric on a compact manifold with boundary the corresponding phase space consists of

$$C_\partial = {}^{\mathrm{sc}}T^*_Y X, \tag{2.5}$$

the restriction of the scattering cotangent bundle to the boundary and the characteristic variety at energy λ is

$$\Sigma(\lambda) = \left\{ |(\tau,\mu)|^2_y + V(y) = \lambda \right\} \tag{2.6}$$

where $|\cdot|^2_y$ is the metric function, a fibre metric on ${}^{\mathrm{sc}}T^*X$, at the boundary, and $\tau = -\rho$ in the notation of (2.4).

Again $C_\partial = {}^{\mathrm{sc}}T^*_Y X$ is naturally a contact manifold. The contact structure arises as the 'boundary value' of the singular symplectic structure on ${}^{\mathrm{sc}}T^*X$, corresponding to the fact that the associated semiclassical model is the leading part, at the boundary, of the Hamiltonian sytem defined by the energy. Thus, in terms of local coordinates x, y near a boundary point, with x a boundary defining function, the symplectic form, arising from the identification ${}^{\mathrm{sc}}T^*X^\circ \simeq T^*X^\circ$, is

$$\widetilde{\omega} = d(\tau\frac{dx}{x^2} + \mu\cdot\frac{dy}{x}) = (d\tau + \mu\cdot dy)\wedge\frac{dx}{x^2} + \frac{d\mu\wedge dy}{x}, \tag{2.7}$$

and is hence singular at the boundary. Since the vector field $x^2\partial_x$ is well-defined at the boundary, modulo multiples and an additive term which is $O(x^2\partial_y)$, the contact form

$$\sigma = \widetilde{\omega}(x^2\partial_x,\cdot) = d\tau + \mu\cdot dy \tag{2.8}$$

defines a contact line bundle on T^*C_∂.

The leading part of the Hamilon vector field H_p, with p in (2.1), is given by the Legendre vector field W on C_∂. This is fixed in terms of the contact form by

$$d\sigma(\cdot,W) + \gamma\sigma = dp,\ \ \sigma(W) = 0, \tag{2.9}$$

for some function γ. Clearly $W_{p-\lambda} = W$ for all λ is tangent to $\Sigma(\lambda)$ whenever the latter is smooth as follows by pairing the identity with W.

Thus the semiclassical model at energy λ is the flow defined by W on $\Sigma(\lambda)$.

Lemma 2.1. *The characteristic surface $\Sigma(\lambda)$ is smooth whenever λ is not a critical value of V_Y. For regular λ the critical points of the Legendre vector field W on $\Sigma(\lambda)$ are the radial sets*

$$R_\pm(\lambda) = \{(y,\tau,\mu)\in\Sigma(\lambda); \tau = \pm\sqrt{\lambda - V(p)},\ \ \mu = 0,\ \ y = p \textit{ where } d_Y V(p) = 0\}. \tag{2.10}$$

Proof. The Legendre vector field is fixed by (2.9) so in any local coordinates y in the boundary it follows from (2.8) that

$$d\mu\wedge dy(\cdot,W)) = -\gamma(d\tau + \mu dy) + dp. \tag{2.11}$$

Given a boundary point q we may choose coordinates Riemannian normal coordinates based at q in the boundary. Thus the boundary metric is Euclidean to second order, $p = \tau^2 + |\mu|^2 + O(|y|^2) + V(y) - \lambda$ and we may easily invert (2.11) to find that at the fibre above q

$$W = 2\mu\cdot\partial_y + (-V'(y) + 2\tau\mu)\cdot\partial_\mu - 2|\mu|^2\partial_\tau,\ \ \gamma = 2\tau \tag{2.12}$$

This can only vanish when $\mu = 0$ and $V'(y) = 0$ giving (2.10). $\square$

We remark that in the Euclidean setting, (2.10) amounts to

$$R_\pm(\lambda) = \{(y,\zeta) \in \Sigma(\lambda);\ y = \mp\zeta/|\zeta|,\ y = p \text{ where } d_Y V(p) = 0\}. \tag{2.13}$$

For the remainder of this section and, unless otherwise stated, for the remainder of this note we make the hypotheses

$$\dim X = 2,\ V\big|_Y \text{ is Morse.} \tag{2.14}$$

The assumption that V is Morse on the boundary ensures that the radial sets $R_\pm(\lambda)$, defined for non-critical λ, are finite sets.

Since each of the boundary components of X is a circle we may assume that the induced metric is some multiple $(\frac{l_j}{2\pi})^2 d\theta^2$ of the standard metric on the circle on each component Y_j of Y. Only the l_j's distinguish the case of 'Euclidean ends with long range (and compact geometric) perturbation' from the general case of a two-dimensional scattering metric. Furthermore the formula (2.12) for W is valid without error in such coordinates.

If $V_i = V\big|_{Y_i}$ is perfect Morse then the component of $\Sigma(\lambda)$ above Y_i,

$$\begin{cases} \Sigma_i(\lambda) \text{ is a sphere for} & \kappa_i = \min(V\big|_{Y_i}) < \lambda < K_i = \max(V\big|_{Y_i}), \\ \Sigma_i(\lambda) \text{ is a torus for} & \lambda > K_i. \end{cases} \tag{2.15}$$

In all cases if λ is not a critical value of $V\big|_{Y_i}$ then

$$I_i(\lambda) = \Sigma_i(\lambda) \cap \{\tau = 0\} \tag{2.16}$$

is a smooth curve (generally with several components) to which W is transversal. The flow is symmetric under $\tau \to -\tau$, $\mu \to -\mu$.

Proposition 2.2. *Provided* $\dim X = 2$ *and* $V\big|_Y$ *is Morse the critical point* $P_\pm(\lambda) = (q, \pm\sqrt{\lambda - V(q)}, 0)$ *in* (2.10) *is*

$$\begin{cases} \text{a centre if } q \text{ is a minimum and } \lambda < \lambda_{\text{Hess}} \\ \text{a sink/source if } q \text{ is a minimum and } \lambda \geq \lambda_{\text{Hess}} \\ \text{a saddle if } q \text{ is a maximum.} \end{cases} \tag{2.17}$$

Proof. As already noted above, (2.12) is valid locally in geodesic boundary coordinates. If q is a critical point for V then $\mu = 0$ at the corresponding critical points of W and we may use local coordinates y, μ in the characteristic variety nearby, since $\tau = \pm\sqrt{\lambda - V(y) - \mu^2}$ is smooth. In these coordinates the linearization of W at the critical point is

$$L = 2[\mu\partial_y + (-ay + \bar\tau\mu)\partial_\mu],\ \text{where } V''(q) = 2a,\ \bar\tau = \pm\sqrt{\lambda - V(q)}. \tag{2.18}$$

The eigenvalues of L are therefore the roots of

$$(\frac{\sigma}{2})^2 - \bar\tau(\frac{\sigma}{2}) + a = 0. \tag{2.19}$$

It is convenient for future reference to write the eigenvalues σ_j as

$$\sigma_j = 2\bar\tau r_j; \tag{2.20}$$

the r_j thus satisfy

$$r^2 - r + \frac{a}{\bar\tau^2} = 0. \tag{2.21}$$

If $a < 0$, so q is a local maximum then the eigenvalues of L are real, and are given by

$$r_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{a}{\bar{\tau}^2}} < 0 < 1 < r_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{a}{\bar{\tau}^2}}, \tag{2.22}$$

and the critical point is a saddle. If $a > 0$, so q is a local minimum, then the discriminant

$$1 - \frac{4a}{\bar{\tau}^2} \begin{cases} < 0 & \text{if } \lambda < \lambda_{\text{Hess}} \\ = 0 & \text{if } \lambda = \lambda_{\text{Hess}} \\ > 0 & \text{if } \lambda > \lambda_{\text{Hess}} \end{cases}, \ \lambda_{\text{Hess}} = V(q) + 2V''(q). \tag{2.23}$$

Correspondingly the eigenvalues are of the form $2\bar{\tau}r_j$, $j = 1, 2$,

$$\begin{aligned} & r_1 = \frac{1}{2} + i\sqrt{\frac{a}{\bar{\tau}^2} - \frac{1}{4}}, \ r_2 = \frac{1}{2} - i\sqrt{\frac{a}{\bar{\tau}^2} - \frac{1}{4}}, \\ & r_1 = r_2 = 1/2, \\ & 0 < r_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{a}{\bar{\tau}^2}} < \frac{1}{2} < r_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{a}{\bar{\tau}^2}} < 1 \end{aligned} \tag{2.24}$$

and the critical point is a centre, a degenerate center, or a source/sink depending on the sign of $\bar{\tau}$. □

The main reason for considering the Legendre vector field W is its relation to the Hamiltion vector field of the energy function p with respect to the singular sympletic form in (2.7). Let ${}^{\text{sc}}H_p$ be x^{-1} times the Hamilton vector field H_p of p near the boundary of ${}^{\text{sc}}T^*X$. Then we find

$$\widetilde{\omega}(\cdot, x\,{}^{\text{sc}}H_p) = dp \Longrightarrow {}^{\text{sc}}H_p = 2\tau x\partial_x + W + xU, \ U \in \mathcal{V}_{\text{b}}({}^{\text{sc}}T^*X), \tag{2.25}$$

where $\mathcal{V}_{\text{b}}({}^{\text{sc}}T^*X)$ is the set of C^∞ vector fields on ${}^{\text{sc}}T^*X$ that are tangent to the boundary, ${}^{\text{sc}}T^*_{\partial X}X$. Thus this rescaled vector field has critical points at the same points as W on the energy surface $\Sigma(\lambda)$. The appearance of $2\tau x\partial_x$ in (2.25) is the main reason for introducing r_j in (2.21); r_j is the ratio of the eigenvalues of W (i.e. of certain eigenvalues of ${}^{\text{sc}}H_p$) to the eigenvalue

$$\sigma_0 = 2\bar{\tau} \tag{2.26}$$

corresponding to $x\partial_x$.

Suppose that $f(\tau, y, \mu)$ vanishes at $P \in \Sigma(\lambda)$, and df is an eigenvector for W on $\Sigma(\lambda)$ with eigenvalue σ. Then

$${}^{\text{sc}}H_p(x^r f) = x^r[(2r\tau + \sigma)f + q + O(x)] \tag{2.27}$$

where q is smooth and vanishes quadratically at the critical point P. We use this as a basis for the construction of positive commutators, which we sketch below.

There are some global points of interest in the dynamics of W. In particular, note that the ∂_τ component of ${}^{\text{sc}}H_p$ is $-h \le 0$, so τ is monotone decreasing along integral curves of W.

It is also easy to see that every bicharacteristic $\gamma : \mathbb{R}_t \to \Sigma(\lambda)$ tends to a point in $R_+(\lambda) \cup R_-(\lambda)$ as $t \to \pm\infty$. Indeed, $\lim_{t\to\pm\infty} \tau(\gamma(t)) = \tau_\pm$ exists by the monotonicity of τ, and any sequence $\gamma_k : [0, 1] \to \Sigma(\lambda)$, $\gamma_k(t) = \gamma(t_k + t)$, $t_k \to +\infty$, has a uniformly convergent subsequence, which is then a bicharacteristic $\tilde{\gamma}$, by standard

Hamilton flow arguments. But then τ is constant along this bicharacteristic, hence h is identically 0. In view of the ∂_μ components of ${}^{\mathrm{sc}}H_p$, $\partial_y V$ is identically 0 along $\tilde{\gamma}$, hence y is a critical point of V along $\tilde{\gamma}$. Since the set of critical points of V is discrete, it now immediately follows that $\tilde{\gamma}$ is a constant curve, $\tilde{\gamma}(t) = q$ for all t, $q \in R_+(\lambda) \cup R_-(\lambda)$, and indeed $\lim_{t\to+\infty} \gamma(t) = q$.

We need a much more precise picture of the bicharacteristic flow at the radial points, $R_+(\lambda) \cup R_-(\lambda)$; we analyze this in the next section.

3. Resolution of singularities of flows

Suppose that W is a C^∞ vector field on a manifold M, and $W(o) = 0$. Let $\Phi : M \times \mathbb{R}_t \to M$ denote the flow generated by W, and suppose that o is a sink, so there exists a neighborhood O of o such that for $p \in O$, $\lim_{t\to+\infty} \Phi(p, t) = o$. One can define the blow-up of o in M along W, which can be alternatively regarded as a compactification of $M \setminus \{o\}$ by W, as follows.

Let $S \subset O$ be a closed embedded submanifold of M transversal to W. Then $\Phi|_{S\times(0,+\infty)}$ is a diffeomorphism of $S\times(0, +\infty)$ to a punctured neighborhood O' of o in M. We compactify $S\times[0, +\infty)_t$ by making $\rho = e^{-t}$ a boundary defining function of $[0, +\infty)$ at $+\infty$, i.e. by identifying $S \times [e, +\infty)_t$ with $S \times (0, 1]_\rho$, $t = -\log \rho$, and using the smooth structure of $S\times(0, 1)_\rho$ to put a smooth structure on $\overline{S \times [0, +\infty)_t}$. Then the interior of $\overline{S \times [0, +\infty)_t}$ is diffeomorphic to $S \times (0, +\infty)$, hence to O'. The blow-up $[M; \{o\}]_W$ is then defined as the union of $M \setminus O'$ and $\overline{S \times [0, +\infty)_t}$, with $\partial O' = S$ and $S \times \{0_t\}$ identified. The interior of $[M; \{o\}]_W$ is naturally diffeomorphic to $M \setminus \{o\}$. The C^∞ structure of $[M; \{o\}]_W$ is independent of the choice of S. The boundary hypersurface $\rho = 0$ is called the front face of the blow-up, and below it is usually denoted by ff, or more precisely by $\mathrm{ff}([M; \{o\}]_W)$.

In particular, inhomogeneous blow-ups can be constructed this way. Namely, let X be a 2-dimensional manifold with boundary, and let $o \in \partial X$. Choose local coordinates (x, y) such that $x \geq 0$ is a boundary defining function and o is given by $x = 0$, $y = 0$. Let $r \in (0, 1)$ be a given homogeneity. We wish to blow up o in X so that y is homogeneous of degree 1 and x is homogeneous of degree $1/r$. For this purpose, we consider the vector field $-W = r^{-1}x\partial_x + y\partial_y$, and carry out the above construction. Note that y/x^r and $x/|y|^{1/r}$ are homogeneous of degree 0 where they are bounded, so they can be regarded as variables on the transversal S. Thus, local coordinates in the lift of the region $|y/x^r| < C$ are given by y/x^r and x^r, while local coordinates in the lift of the region $x/|y|^{1/r} < C'$ are given by $x/|y|^{1/r}$ and y. Although the manifold $[X; \{o\}]$ a priori depends on the choice of W, hence on the choice of coordinates (x, y), this is in fact not so since x has the higher homogeneity. More explicitly, any change of coordinates takes the form $y' = a(x, y)y + b(x, y)x$, $x' = c(x, y)x$, $c(0, 0) > 0$, $a(0, 0) \neq 0$, so for example $y'/(x')^r = (a/c^r)(y/x^r) + (b/c^r)x^{1-r}$ which is bounded (in $x \leq x_0$, $x_0 > 0$, $|y| \leq y_0$, $y_0 > 0$) if and only if y/x^r is bounded (since $r \in (0, 1)$!). Such calculations show that the blow-ups using (x, y) and using (x', y') agree as topological manifolds. Their C^∞ structure is slightly different as can be seen from the appearance of x^{1-r} above. However, it is easy to see that both the class of conormal functions on $[X; \{o\}]$ and that of polyhomogeneous conormal functions are well-defined, regardless of the choice of local coordinates, provided that the orders (homogeneities) are appropriately adjusted. Essentially, $[X; \{o\}]$ should be thought of as a 'conormal manifold' rather than as a C^∞ manifold, i.e. the algebra of smooth functions should be replaced by the algebra of polyhomogeneous conormal functions as the basic object of interest.

As an application, we define the classical scattering relation as follows.

Definition 3.1. We say that $\xi \in \mathrm{ff}_+([\Sigma(\lambda);\zeta_{\min,-},\zeta_{\min,+}])$ is related to $\xi' \in \mathrm{ff}_-([\Sigma(\lambda);\zeta_{\min,-},\zeta_{\min,+}])$, denoted $\xi \sim \xi'$, by the classical scattering relation if there exists a bicharacteristic $\gamma : \mathbb{R}_t \to \Sigma(\lambda)$ whose lift to $[\Sigma(\lambda);\zeta_{\min,-},\zeta_{\min,+}]$ tends to ξ (resp. ξ') as $t \to +\infty$ (resp. $t \to -\infty$).

This will be used to describe the canonical relation of the scattering matrix in Section 6.

4. Structure of smooth eigenfunctions

We can now give an oscillatory function description of

$$u_\pm = R(\lambda \pm i0)f = (\Delta + V - (\lambda \pm i0))^{-1}f,\ f \in \dot{\mathcal{C}}^\infty(X). \tag{4.1}$$

This is assembled from microlocal results at the radial points, and hence most of the statements below, even as stated, do not require the assumption that $V|_Y$ is perfect Morse. Nonetheless we make this assumption in the intersets of simplicity. We denote the incoming (resp. outgoing) critical points in $\Sigma(\lambda)$ over $y_{\max}$ and $y_{\min}$ by $\zeta_{\max,+}$ and $\zeta_{\min,+}$ (resp. $\zeta_{\max,-}$ and $\zeta_{\min,-}$); the critical points over $y_{\max}$ are only present if $\lambda > K$.

We recall that $\Psi^{m,l}_{\mathrm{sc}}(X)$ is the set of scattering ps.d.o.'s of multiorder (m,l) defined by Melrose in [9]; for $X = \overline{\mathbb{R}^n}$ this is a well-known space of operators introduced originally by Shubin [13] and in Hörmander's Weyl calculus corresponds to the metric

$$\frac{dz^2}{\langle z\rangle^2} + \frac{d\zeta^2}{\langle\zeta\rangle^2} \tag{4.2}$$

and weight $\langle z\rangle^{-l}\langle\zeta\rangle^m$. The corresponding wave front set is denoted by $\mathrm{WF}_{\mathrm{sc}}$; for $u \in C^{-\infty}(X)$ with $(P-\lambda)u \in \dot{C}^\infty(X)$, $\mathrm{WF}_{\mathrm{sc}}(u)$ is contained in $\Sigma(\lambda)$ by elliptic regularity.

The microlocal estimates used in the proof of the limiting absorption principle also show that $\tau < 0$ on $\mathrm{WF}_{\mathrm{sc}}(u_+)$ and $\tau > 0$ on $\mathrm{WF}_{\mathrm{sc}}(u_-)$. By propagation of singularities, see [9], $\mathrm{WF}_{\mathrm{sc}}(u_+)$ is a subset of the 'flow-out' of $R(-\lambda)$, i.e. is a subset of

$$R(-\lambda) \cup \{\gamma(t);\ t \in \mathbb{R},\ \gamma \text{ is a bicharacteristic in } \Sigma(\lambda),\ \lim_{t' \to -\infty} \gamma(t') \in R(-\lambda)\}. \tag{4.3}$$

Thus, under the perfect Morse assumptions, $\mathrm{WF}_{\mathrm{sc}}(u_+)$ at most contains $\zeta_{\min,-}$, $\zeta_{\max,-}$, and the unique smooth Legendre submanifold $L_{\max,-}$ through $\zeta_{\max,-}$ on which $\tau \le \tau(\zeta_{\max,-}) < 0$.

We start with the simplest case, when $\mathrm{WF}_{\mathrm{sc}}(u_+) \subset \{\zeta_{\min,-}\}$. Since $\zeta_{\max,-}$ is a saddle point, one might expect on semiclassical grounds that these u_+ are fairly typical elements of $R^{-\infty}_{\mathrm{ess},+}(\lambda)$; in particular, they should suffice to parameterize the essential spectrum in an L^2 sense. Note that for $\lambda < K$, $\mathrm{WF}_{\mathrm{sc}}(u_+) \subset \{\zeta_{\min,-}\}$ is automatically satisfied.

To describe such u_+, we need to define inhomogeneous blow-ups of $y_{\min}$ in X with homogeneity given by the linearization L of W at $\zeta_{\min,-}$. More specifically, the relevant information is the real part of the smaller eigenvalue, namely $r = \operatorname{Re} r_1$ in Lemma 2.1. Note that $r \in (0, 1/2]$ at $\zeta_{\min,-}$. The desired blow-up is the r-blow-up

$$[X; y_{\min}]_r, \tag{4.4}$$

discussed above, of $y_{\min}$ where x has homogeneity $1/r$ and y has homogeneity 1. If $r < 1/2$, let $L_{\min,-}$ denote the smooth curve in ${}^{\mathrm{sc}}T^*_Y X$ through $\zeta_{\min,-}$ corresponding to the eigenvector with eigenvalue $2\bar{\tau}r_2$ that is given by the union of bicharacteristics. (This is a Legendre submanifold of $C_\partial = {}^{\mathrm{sc}}T^*_Y X$ since the contact form vanishes on it, and it has the correct dimension, namely one.) Since this is smooth, and near $y_{\min}$ it has a non-degenerate projection to X, it is given by the graph of a one-form $d(\Phi_{\min,-}/x)$, where $\Phi_{\min,-} \in C^\infty(X)$.

Theorem 4.1. *Suppose that $\lambda > \kappa$, u_+ is given by (4.1), and $\mathrm{WF}_{sc}(u_+) \subset \{\zeta_{\min,-}\}$.*

(i) *If $\lambda > \lambda_{\mathrm{Hess}}$, so $r = r_1 \in (0, 1/2)$, then*

$$u_+ = e^{i\Phi_{\min,-}/x} x^{r_2/2} x^{i\beta} a + u'_+, \ a \in C^\infty([X; y_{\min}]_r), \tag{4.5}$$

where a vanishes to infinite order at the lift of ∂X (i.e. everywhere but at the front face ff of the blow-up), β is a real constant given by the subprincipal symbol of $P - \lambda$ at $\zeta_{\min,-}$, and $u' \in x^{-1/2+\delta} L^2(X)$ for some $\delta > 0$. In fact, u'_+ has a full asymptotic expansion of the same form as the first term, but with higher powers of x.

Conversely, for any Schwartz function a_0 on the front face, there exists $f \in \dot{C}^\infty(X)$ such that $u_+ = R(\lambda + i0)f$ is of the form as above with $a|_{\mathrm{ff}} = a_0$.

(ii) *If $\lambda < \lambda_{\mathrm{Hess}}$, then*

$$u_+ = e^{-i\bar{\tau}/x} e^{i\bar{\tau}y^2/(4x)} x^{1/4} \sum_{j \ge 0} x^{ic_j} a_j + u'_+, \ a_j \in C^\infty([X; y_{\min}]_{1/2}), \tag{4.6}$$

where a_j vanishes to infinite order at the lift of ∂X (i.e. everywhere but at the front face of the blow-up), and $u' \in x^{-1/2+\delta} L^2(X)$ for some $\delta > 0$. In fact, u'_+ has a full asymptotic expansion of the same form as the first term, but with higher powers of x. The a_j are multiples of normalized eigenfunctions of the harmonic oscillator with coefficients that are Schwartz in j (and this is the only restriction on them). The c_j are real, and correspond to the eigenvalues of the harmonic oscillator.

Remark 4.2. This theorem is microlocal, hence it remains valid with very minor changes in the notation, if there are several local minima of $V|_Y$.

Remark 4.3. The r-blow-up is related to the vector field $2\bar{\tau}(x\partial_x) + 2\bar{\tau}r_1(x\partial_y)$ modeling the flow 'along $L_{\min,-}$'.

If $\lambda > K$, there is a unique C^∞ Legendre submanifold $L_{\max,-}$ of ${}^{\mathrm{sc}}T^*_Y X$ which consists of $\zeta_{\max,-}$ and the two non-constant bicharacteristics that tend to $\zeta_{\max,-}$ as $t \to -\infty$. These bicharacteristics correspond to the eigenvectors of the linearization of the Legendre vector field at $\zeta_{\max,-}$ with eigenvalue $2\bar{\tau}r_2$ (recall that $r_2 > 1$). The closure of $L_{\max,-}$ is $L_{\max,-} \cup \{\zeta_{\min,-}\}$ and $L_{\max,-}$ is singular at $\zeta_{\min,-}$. First we state a result away from this singularity.

Theorem 4.4. *Suppose that $\lambda > \kappa$.*

(i) *Microlocally away from* $\zeta_{\min,-}$, u_+ *is a polyhomogeneous Legendre distribution associated to* $L_{\max,-}$. *That is, if* $Q \in \Psi_{\mathrm{sc}}^{0,0}(X)$ *and* $\zeta_{\min,-} \notin \mathrm{WF}'_{sc}(Q)$, *then* $Qu_+ \in I^{\frac{r_2-r_1}{4}}(X, L_{\max,-})$. *Microlocally near* $\zeta_{\max,-}$, u_+ *has the form*

$$u_+ \sim e^{i\Phi/x} x^{r_2/2} x^{i\beta} \sum_j \sum_{s\le j} x^{k_j} (\log x)^s a_{js}, \ \text{near } \zeta_{\max,-}, \tag{4.7}$$

a_{js} *a* C^∞ *function on* Y, $0 = k_0 < k_1 < k_2 < \ldots$, $\lim_{j\to\infty} k_j = +\infty$. *Moreover, the coefficients* a_{js} *are determined by* $a_{j's'}(y_{\max})$, $j' \le j$. *Again* β *is a real constant given by the subprincipal symbol of* $P - \lambda$ *at* $\zeta_{\min,-}$.

(ii) *There exists* $f \in \dot{\mathcal{C}}^\infty(X)$ *such that* $a_{00} \neq 0$.

As an immediate corollary we deduce that $Qu_+ \in x^s L^2(X)$ for all $s < (r_2-2)/2$. Since $r_2 > 1$, this is an improvement over $Qu_+ \in x^{-1/2}L^2(X)$. The statement $Qu_+ \in x^{-1/2}L^2(X)$ can be interpreted as the absence of channels at the maxima of $V|_Y$ and is closely related to the work of Herbst and Skibsted [5]. Thus, on the one hand, the present theorem strengthens their results in this special case by showing that u_+ has additional decay away from the minima of $V|_Y$, and gives the precise asymptotic form of u_+. On the other hand, part (ii) also shows that the decay is not rapid. One interpretation of this phenomenon is that the L^2-theory does not see the maxima of the potential (because $R(\lambda + i0)f$ is too small there), while the smooth theory (working modulo $\dot{\mathcal{C}}^\infty(X)$) does.

We can now turn to the 'ends' of $L_{\max,-}$. For simplicity we only discuss these if $\lambda > \lambda_{\mathrm{Hess}}(y_{\min})$. We blow up $\zeta_{\min,-}$ in $\Sigma(\lambda)$ using the Legendre vector field W, and denote the resulting space by $[\Sigma(\lambda); \zeta_{\min,-}]_W$ as before. Since it is given by an integral curve of W, $L_{\max,-}$ lifts to a C^∞ embedded submanifold $\hat{L}_{\max,-}$ transversal to the front face of the blow-up; it intersects the front face in two points. Near each of these two points $\hat{L}_{\max,-}$ can be parameterized near $\zeta_{-,\min}$ using a singular phase function, namely one which is polyhomogeneous conormal on $[X; y_{\min}]_r$, since $\hat{L}_{\max,-}$ has full rank projection to Y nearby. That is the phase function has the form $\Phi/x = \Phi_{\min,-}/x + \tilde{\Phi}/x$, where $\tilde{\Phi}$ is a polyhomogeneous conormal function of y, conormal to $y = 0$, with leading homogeneity $2 + \sigma$, $\sigma > 0$. Correspondingly, we define Legendre distributions u associated to $\hat{L}_{\max,-}$ of order $\frac{1}{4}[r_2(\zeta_{\max,-}) - r_1(\zeta_{\max,-})]$ at $L_{\max,-}$ and of the same asymptotics as (4.5) at $\zeta_{\min,-}$, by

$$\begin{aligned} u = u' + u'' + u''',\ u' \in I_c^{\frac{1}{4}[r_2(\zeta_{\max,-}) - r_1(\zeta_{\max,-})]}(X, L_{\max,-}),\\ u'' = e^{i\Phi/x} x^{r_2(\zeta_{\max,-})/2} a,\ u''' \text{ as in (4.5)}, \end{aligned} \tag{4.8}$$

a polyhomogeneous conormal on $[X; y_{\min}]_r$ of symbolic order 0 on the lift of ∂X, and of symbolic order $\frac{1}{2}[r_2(\zeta_{\min,-}) - r_2(\zeta_{\max,-})]$ on the front face, and $I_c^*(X, L_{\max,-})$ refers to Legendre distributions with wave front set in a compact subset of $L_{\max,-}$.

Theorem 4.5. *Suppose* $\lambda > \lambda_{\mathrm{Hess}}(y_{\min})$, $\lambda > K$. *Then for* $f \in \dot{\mathcal{C}}^\infty(X)$, $u_+ = R(\lambda + i0)f$ *has the form* (4.8).

This result implies, in particular, that near the interior of the front face of $[X; y_{\min}]_r$, u_+ takes the form $e^{i\Phi_{\min,-}/x} x^{r_2/2} x^{i\beta} a$, with a continuous up to the front face, and its restriction $a|_{\mathrm{ff}}$ to the front face is smooth. This restriction in turn determines u_+ modulo $\dot{\mathcal{C}}^\infty(X)$.

In the remainder of this section we consider only $\lambda > \lambda_{\text{Hess}}$. By Theorem 4.1, given any Schwartz function a_0 on ff, there exists $f \in \dot{\mathcal{C}}^\infty(X)$ such that the asymptotic expansion, as in (4.5), of $u_- = (\Delta + V - (\lambda - i0))^{-1} f$ has leading coefficient a_0. The Poisson operator is then the map

$$P(\lambda) : \mathcal{S}(\text{ff}) \to E_{\text{ess}}^{-\infty}(\lambda), \quad P(\lambda)a_0 = u_- - u_+, \quad u_+ = (\Delta + V - (\lambda + i0))^{-1} f. \tag{4.9}$$

The scattering matrix $S(\lambda)$, defined by (1.17), can be thus identified with the map

$$S(\lambda) : \mathcal{S}(\text{ff}) \to \mathcal{S}'(\text{ff}), \quad S(\lambda)a_0 = -a|_{\text{ff}}, \tag{4.10}$$

where $a|_{\text{ff}}$ is the leading coefficient of the asymptotics of $u_+ = (\Delta + V - (\lambda + i0))^{-1} f$.

There is a natural measure ν on the front face induced by the Riemannian density $|dh|$ on the boundary Y. In local coordinates this is given by the pull-back of $|dh|/x^{r_1}$, which gives a smooth density on ff, of the form $a|d(y/x^{r_1})|$, a a 0th order classical symbol on the front face. The boundary pairing B of (1.18) shows that that $P(\lambda)$ extends to a continuous map $P(\lambda) : L^2(\text{ff}, \nu) \to E_{\text{ess}}(\lambda)$, and $S(\lambda)$ extends to a unitary map

$$S(\lambda) : L^2(\text{ff}, \nu) \to L^2(\text{ff}, \nu). \tag{4.11}$$

To state its precise structure, it is necessary to discuss a class of pseudodifferential operators on $\mathbb{R}^k$ which we call anisotropic pseudodifferential operators of homogeneity $r \in (0, 1)$, or r-ps.d.o's for short. However, first we mention that the paper [1] of Agmon, Cruz and Herbst uses essentially semiclassical arguments to construct $P(\lambda)$ (which can also be thought of as the operator giving rise to a generalized Fourier transform) in the high energy limit. The precise structure of $P(\lambda)$, which is reflected in that of $S(\lambda)$ is not apparent in these L^2-based arguments.

5. The r-anisotropic calculus

For each $r \in (0, 1)$ we define a global calculus of pseudodifferential operators on $\mathbb{R}$ and a corresponding calculus of Fourier integral operators. For $r = \frac{1}{2}$ this calculus reduces to the 'isotropic calculus' epitomized by the harmonic oscillator.

On $\mathbb{R}^2 = T^*\mathbb{R}$, with variables y, η, consider the vector field

$$R_r = ry\partial_y + (1 - r)\eta\partial_\eta. \tag{5.1}$$

It generates an $\mathbb{R}^+$-action on $\mathbb{R}^2 \setminus \{0\}$,

$$R_r(s)(y, \eta) = (s^r y, s^{1-r}\eta). \tag{5.2}$$

Thus, the function y is homogeneous of degree r and η is homogeneous of degree $1 - r$. The 'isotropic' case occurs when the variables have the same homogeneity so $r = \frac{1}{2}$. Moreover, note that the symplectic form $\omega = d\eta \wedge dy$ is homogeneous of degree $(1 - r) + r = 1$.

The choice of radial action gives a compactification of $\mathbb{R}^2$ to a ball. Namely

$$^{r\text{-ai}}\overline{T^*\mathbb{R}} = X_r = \mathbb{R}^2 \cup \mathbb{S}_r, \ \mathbb{S}_r = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}^+. \tag{5.3}$$

The $\mathcal{C}^\infty$ structure on X_r, as a compact manifold with boundary, is given by the standard $\mathcal{C}^\infty$ structure on $\mathbb{R}^2$ together with the product structure near the boundary where the normal variable is $1/s$ for a choice of embedding of $\mathbb{S}$ as a transversal, $s = 1$, to R_r. Thus, smooth functions on $\mathbb{R}^2$ which are homogeneous of non-positive integral degree near infinity under the $\mathbb{R}^+$ action are smooth on X_r and generate the $\mathcal{C}^\infty$ structure. Let ρ be some choice of defining function for infinity, for instance a positive smooth function which is $\mathbb{R}_r$-homogeneous of degree -1 near infinity.

Note that the compactification procedure is analogous to blow-ups, especially if the latter are regarded as a compactification of a manifold with a submanifold removed!

The constant vector fields ∂_y and ∂_η extend up to the boundary of X_r to be of the form

$$\partial_y = \rho^{-r} V,\ \partial_\eta = \rho^{-(1-r)} W \tag{5.4}$$

where V and W are smooth vector fields on X_r tangent to the boundary. This follows from their homogeneity.

Proposition 5.1. *The spaces of 'classical symbols'*

$$\Psi^m_{r\text{-}ai}(\mathbb{R}) = \rho^m \mathcal{C}^\infty(X_r) \tag{5.5}$$

form order-filtered algebras of pseudodifferential operators under Weyl quantization.

Proof. These are certainly symbols in an appropriately weak sense, i.e. in the space $S^*_{\rho,\delta}(\mathbb{R})$, in the sense of Hörmander, see for instance [8], for some $\rho < 1$ and $\delta > 0$. It follows that their composition can be carried out as ordinary pseudodifferential operators. For any r it is the case that

$$\dot{\mathcal{C}}^\infty(X_r) = \mathcal{S}(\mathbb{R}^2) \tag{5.6}$$

is the 'smoothing' ideal. For finite order symbols the asymptotic composition formula, for Weyl quantization, is

$$a\#b(y,\eta) = \sum_{k=0}^{\infty} \frac{i^k}{2^k k!} (\partial_y \partial_{\eta'} - \partial_{y'} \partial_\eta)^k a(y,\eta) b(y',\eta')\big|_{y'=y,\eta'=\eta}. \tag{5.7}$$

Together with (5.4), this shows that the spaces (5.5) are preserved under composition. □

This generalizes directly to higher dimensions, but here for the sake of brevity we restrict the discussion to $\mathbb{R}$. Conjugation by the Fourier transform maps $\Psi_{r\text{-ai}}(\mathbb{R})$ onto $\Psi_{r'\text{-ai}}(\mathbb{R})$ where $r' = 1 - r$.

The contact 'phase' space for the r-calculus is the sphere at infinity in the sense of the boundary of X_r, the quotient in (5.5). Let p be a boundary point. In microlocal arguments, one of the variables, either y or η, is elliptic. Since we can pass to the first case by conjugating by the Fourier transform, we may always assume that in the some neighbourhood, where $y > 0$,

$$y^{-1/r},\ \eta/y^{(1-r)/r} \tag{5.8}$$

are bounded and are valid coordinates on the compactification X_r.

Let $\overline{\mathbb{R}}_r$ be the compactification of $\mathbb{R}$ defined by the $\mathbb{R}^+$-action corresponding $ry\partial_y$, which in turn is 'the first half' of the vector field in (5.1). Then $y^{-1/r}$ gives a coordinate in a neighborhood of the projection of p into $\overline{\mathbb{R}}_r$. The algebra ${}^{r\text{-ai}}\Psi^*(\mathbb{R})$ microlocalizes to open neighborhoods of the boundary, in the sense that linear space of functions vanishing to infinite order at the boundary in the complement of a compact subset of such an open set, modulo the smoothing operators, form an algebra (in fact a sheaf of algebras).

The local coordinates in (5.8) identify these microlocal algebras with the corresponding microlocal algebras of scattering pseudodifferential operators on $\overline{\mathbb{R}}_r$. Indeed in this microlocal region, where (5.8) are valid coordinates, $X_r = {}^{r\text{-ai}}\overline{T^*\mathbb{R}}$ can be identified with ${}^{\mathrm{sc}}T^*\overline{\mathbb{R}}_r$, or rather a subset of ${}^{\mathrm{sc}}T^*\overline{\mathbb{R}}_r$ with compact closure in ${}^{\mathrm{sc}}T^*\overline{\mathbb{R}}_r$. Namely, the latter is spanned by the differentials of functions homogeneous of degree -1 (in terms of the smooth structure, i.e. locally in terms of $x = y^{-1/r}$), so its elements are of the form $-\tau\, d(1/x) = -\tau y^{\frac{1-r}{r}}\, dy$, so $-\tau = \eta/y^{\frac{1-r}{r}}$. These microlocal identifications of the r-calculus permit us to define Fourier integral operators and the full structure of microlocal analysis by reduction to the scattering case discussed extensively in [11]. We also use these identifications to define the class of r-Legendre functions; an example of these is a function of the form $e^{icy^{1/r}}a$, $a \in C^\infty(\overline{\mathbb{R}}_r)$. Of course, going one step further and using the Fourier transform, these objects are identified microlocally with the corresponding elements in the standard theory of Fourier integral operators and Lagrangian distributions on a manifold without boundary in [7, 3].

FIGURE 1. The anisotropic and scattering cotangent spaces. All points but the two marked ones (at which $|\eta|/|y|^{(1-r)/r} \to \infty$) in $\mathbb{S}_r = \partial^{r\text{-ai}}\overline{T^*\mathbb{R}}$ correspond to points in the left or right side faces of ${}^{\mathrm{sc}}T^*\overline{\mathbb{R}}_r$, depending on whether $y > 0$ or $y < 0$.

One can also define, either directly, or via the identification, the r-anisotropic wave front set ${}^{r\text{-ai}}\,\mathrm{WF}(u) \subset \mathbb{S}_r = \partial^{r\text{-ai}}\overline{T^*\mathbb{R}}$ for $u \in \mathcal{S}'(\mathbb{R})$. As usual, this is defined by $p \in \mathbb{S}_r \setminus {}^{r\text{-ai}}\,\mathrm{WF}(u)$ if there exists $A \in \Psi^0_{r\text{-ai}}(\mathbb{R})$ which is elliptic at p (i.e. its principal symbol is non-zero there) such that $Au \in \mathcal{S}(\mathbb{R})$.

6. Structure of the scattering matrices

Here we only consider $\lambda > \lambda_{\mathrm{Hess}}$. Then for each critical point ($\zeta_{\mathrm{min},+}$ and $\zeta_{\mathrm{min},-}$) of the flow above a minimum, $\mathbb{S}_r = \partial^{r\text{-ai}}\overline{T^*\,\mathrm{ff}}$ is naturally identified with the corresponding front face of $[\Sigma(\lambda); \zeta_{\mathrm{min},+}, \zeta_{\mathrm{min},-}]_W$; we denote the two identification maps by

$$J_\pm : \mathbb{S}_r = \partial^{r\text{-ai}}\overline{T^*\,\mathrm{ff}} \to \mathrm{ff}_\pm([\Sigma(\lambda); \zeta_{\mathrm{min},+}, \zeta_{\mathrm{min},-}]_W). \tag{6.1}$$

We do not describe this identification in detail here, but note that a typical r-Legendre function on ff is $e^{icY^{1/r}} = e^{icy^{1/r}/x}$, c a constant, and multiplying this by $e^{i\Phi_{\mathrm{min},\pm}/x}$ and taking the differential of the phase, we get a map into a neighborhood of $\zeta_{\mathrm{min},\pm}$ in ${}^{\mathrm{sc}}T^*_{\partial X}X$, which is singular at $y = 0$. When performed more carefully (to reflect the structure of the bicharacteristics), this singularity exactly corresponds to that of the blow-up, giving rise to the identifications $J_\pm$. The two points in each of $J_\pm^{-1}(\hat{L}_{\mathrm{max},\pm})$ play a special role below. This identification induces a relation on $\mathbb{S}_r \times \mathbb{S}_r$ via the classical scattering relation defined in Section 3. That is, for $(\xi, \xi') \in \mathbb{S}_r \times \mathbb{S}_r$, $\xi \sim \xi'$ if $J_+(\xi) \sim J_-(\xi')$.

Theorem 6.1. *Suppose $\lambda > \lambda_{\mathrm{Hess}} = \lambda_{\mathrm{Hess}}(y_{\mathrm{min}})$. Then $S(\lambda)$ extends by continuity to $a \in C^{-\infty}(\mathrm{ff})$ satisfying ${}^{r\text{-}ai}\,\mathrm{WF}(a) \cap J_+^{-1}(\hat{L}_{\mathrm{max},+}) = \emptyset$. Moreover,*

$${}^{r\text{-}ai}\,\mathrm{WF}(S(\lambda)a) \subset J_-^{-1}(\hat{L}_{\mathrm{max},-}) \cup \{\xi \in \mathbb{S}_r :\ \exists\, \xi' \in {}^{r\text{-}ai}\,\mathrm{WF}(a),\ \xi \sim \xi'\}. \tag{6.2}$$

Note that if $\lambda < K$, the statements involving $J_\pm^{-1}(\hat{L}_{\mathrm{max},\pm})$ can be dropped.

This wave front set mapping property corresponds to a canonical relation consisting of Lagrangians singular at $J_\pm^{-1}(\hat{L}_{\mathrm{max},\pm})$. Rather than defining a corresponding class of singular anisotropic FIO's, we simply state the precise structure theorem away from these points.

Theorem 6.2. *Suppose $\lambda > \lambda_{\mathrm{Hess}}$. Let $Q_\pm \in \Psi_{r\text{-}ai}(\mathrm{ff})$ with ${}^{r\text{-}ai}\mathrm{WF}'(Q_\pm) \cap J_\pm^{-1}(\hat{L}_{\max,\pm}) = \emptyset$. Then $Q_- S(\lambda) Q_+$ is an r-anisotropic FIO corresponding to the classical scattering relation.*

7. Sketch of proofs

The results on the structure of $R(\lambda + i0)f$, $f \in \dot{\mathcal{C}}^\infty(X)$ are based on positive commutator estimates, while those on $S(\lambda)$ by reduction of the operator to model form microlocally near the radial point, as in the paper of Guillemin and Schaeffer [4]. Here we briefly discuss the positive commutator arguments.

The first step towards showing that u with

$$(P - \lambda)u = (\Delta + V - \lambda)u = f \in \dot{\mathcal{C}}^\infty(X) \tag{7.1}$$

has a special structure is showing that $u \in H^{\infty,s}_{\mathrm{sc}}(X)$ remains in this class under iterated application of certain scattering pseudo-differential operators corresponding to the geometry of the bicharacteristic flow. The ps.d.o.'s we consider always form a module over $\Psi^{0,0}_{\mathrm{sc}}(X)$. Due to the non-commutativity of the ps.d.o. calculus, we need to impose a condition on the module in order to make the subsequent definitions reasonable. Below by $\Psi^{*,*}_{\mathrm{sc}}(X)$ we denote polyhomogeneous scattering pseudo-differential operators.

Definition 7.1. An admissible module M is a $\Psi^{0,0}_{\mathrm{sc}}(X)$-module which is a subset of $\Psi^{-\infty,-1}_{\mathrm{sc}}(X)$, is closed under commutators, finitely generated as a $\Psi^{0,0}_{\mathrm{sc}}(X)$-module, and includes $\Psi^{0,0}_{\mathrm{sc}}(X)$.

Remark 7.2. Since $M \subset \Psi^{-\infty,-1}_{\mathrm{sc}}(X)$, it follows that $A \in M$, and $C \in \Psi^{-\infty,0}_{\mathrm{sc}}(X) \subset M$ implies that $[A, C] \in \Psi^{-\infty,0}_{\mathrm{sc}}(X)$.

Recall that $H^{r,s}_{\mathrm{sc}}(X)$ are the scattering weighted Sobolev spaces; if X is the radial compactification of $\mathbb{R}^n$, these are the standard weighted Sobolev spaces $\langle z\rangle^{-s} H^r(\mathbb{R}^n)$.

Definition 7.3. Let M be a $\Psi^{0,0}_{\mathrm{sc}}(X)$-module as above. We define $I^{(s)}_{\mathrm{sc}}(X, M)$ to be the subset of $H^{\infty,s}_{\mathrm{sc}}(X)$ consisting of $u \in H^{\infty,s}_{\mathrm{sc}}(X)$ such that for all m and for all $B_i \in M$, $i = 1, \ldots, m$, we have $\prod_{i=1}^m B_i u \in H^{\infty,s}_{\mathrm{sc}}(X)$.

We thus prove that, under appropriate microlocal conditions expressing that there are no incoming singularities, $(P - \lambda)u = f \in \dot{\mathcal{C}}^\infty(X)$ implies that $u \in I^{(s)}_{\mathrm{sc}}(X, M)$ microlocally near the radial point for all $s < -1/2$. The specific module depends on the Legendre flow as follows.

(i) Source/sink: $r_1, r_2 \in \mathbb{R}$, $0 < r_1 \le r_2 < 1$. There are unique smooth canonical Legendre submanifolds L_j corresponding to the eigenvectors of the linearization. Then, in view of (2.27), we can gain x along $\Sigma(\lambda)$, x^{r_1} along L_1, x^{r_2} along L_2, i.e. more along L_2. The relevant chain of inclusions is thus:
$$\{\zeta\} \subset L_2 \subset \Sigma(\lambda) \subset {}^{\mathrm{sc}}T^*_{\partial X} X.$$
Hence we consider the $\Psi^{0,0}_{\mathrm{sc}}(X)$-submodule M of $\Psi^{0,-1}_{\mathrm{sc}}(X)$ consisting of operators of the form

$$B_0 + B_1 + B_2 + B_3,\ B_0 \in \Psi^{0,0}_{\mathrm{sc}}(X),\ B_1 \in \Psi^{0,-r_1}_{\mathrm{sc}}(X),$$
$$B_2 \in \Psi^{0,-r_2}_{\mathrm{sc}}(X),\ B_3 \in \Psi^{0,-1}_{\mathrm{sc}}(X),$$

satisfying that the principal symbols of B_3, B_2, resp. B_1, vanish on $\Sigma(\lambda)$, L_2, resp. at ζ. The first term, B_0, can be skipped, since it can be included in B_3.

(ii) Saddle: $r_1, r_2 \in \mathbb{R}$, $r_1 < 0$, $r_2 > 1$. Again there are unique smooth canonical Legendre submanifolds L_j corresponding to the eigenvectors of the linearization. Using (2.27), we can gain x along L_2, but nothing along L_1. Hence our module M is simply the subset of $\Psi_{\mathrm{sc}}^{0,-1}(X)$ consisting of operators with principal symbol vanishing at L_2, and the corresponding distributions are Legendre along L_2.

(iii) Center: $r_1, r_2 \notin \mathbb{R}$, $\operatorname{Re} r_1 = \operatorname{Re} r_2 = 1/2$, $r_2 = \overline{r_1}$. Now r_1 and r_2 play a symmetric role. The relevant module M is the subset of $\Psi_{\mathrm{sc}}^{0,-1}(X)$ generated by operators in $\Psi_{\mathrm{sc}}^{0,-1/2}(X)$ with principal symbol vanishing at ζ and by operators in $\Psi_{\mathrm{sc}}^{0,-1}(X)$ with principal symbol vanishing on $\Sigma(\lambda)$.

Below we sketch the positive commutator construction for sources/sinks.

Lemma 7.4. *Suppose that $\zeta \in R_+(\lambda) \cup R_-(\lambda)$ is a source/sink, $u \in C^{-\infty}(X)$, $(P-\lambda)u \in \dot{C}^\infty(X)$, and there exists a neighborhood U of ζ such that $\mathrm{WF}_{sc}(u) \cap U \subset \{\zeta\}$. Then $u \in I_{\mathrm{sc}}^{(-1/2-\epsilon)}(X, M)$ for all $\epsilon > 0$.*

Proof. (*Sketch.*) We can arrange that near ζ, L_j, $j = 1, 2$, is given by $\tau = -\Phi_j(y)$, $\mu = \Phi_j'(y)$, so inside $\Sigma(\lambda)$, $\mu - \Phi_j'(y)$ is a defining function for L_j. We take $A_0 = x^{-1}(P-\lambda)$, $\sigma_\partial(A_2) = x^{-r_2}(\mu - \Phi_2'(y))$, $\sigma_\partial(A_1) = x^{-r_1}(\mu - \Phi_1'(y))$, $A_3 = \mathrm{Id}$. Then M is generated by these over $\Psi_{\mathrm{sc}}^{0,0}(X)$. The operator A_0 is special, since it is closely related to the Hamiltonian $P - \lambda$; we ignore it here. Since ${}^{\mathrm{sc}}H_p$ is tangent to L_j, and the principal symbol of A_j vanishes on L_j, $j = 1, 2$, the same holds for the principal symbol ${}^{\mathrm{sc}}H_p a_j$ of $-i[A_j, P-\lambda]$. Moreover, for $j = 1, 2$, $d(x^{r_j} a_j)$ corresponds to an eigenvector of W of eigenvalue $2\tau r_j$, hence of ${}^{\mathrm{sc}}H_p$, so ${}^{\mathrm{sc}}H_p a_j$ can be computed up to a term $x^{-r_j} e$, where e vanishes at ζ quadratically. As we have seen, it also vanishes along L_2, so

$$ {}^{\mathrm{sc}}H_p a_j = x^{-r_j}(2(-r_j)\tau + 2\tau r_j + f) a_j, \; f(\zeta) = 0, \tag{7.2} $$

cf. (2.27). Thus, the A_j 'almost' commute with $P - \lambda$. Note that it is here that the r_j determine the order of gain of decay along the Legendrians L_j.

For $s < -1/2$, consider

$$ A = x^{-s-1/2} A_2^\alpha A_1^\beta Q $$

where $Q \in \Psi_{\mathrm{sc}}^{0,0}(X)$ and $\mathrm{WF}_{\mathrm{sc}}'(Q)$ lies in a small neighborhood of ζ but $\zeta \notin \mathrm{WF}_{\mathrm{sc}}'(\mathrm{Id} - Q)$, i.e. Q is a microlocal cut-off near ζ. Thus, the commutator near $\mathrm{WF}_{\mathrm{sc}}'(\mathrm{Id} - Q)$ is irrelevant. The key point is that we can also neglect the commutators with A_1 and A_2.

Here we only make this precise if $|\alpha| + |\beta| = 1$, say $\alpha = 1$, $\beta = 0$. Then, away from $\mathrm{WF}_{\mathrm{sc}}'(\mathrm{Id} - Q)$

$$ -i\sigma_\partial([A, P-\lambda]) = (2\tau(-s - 1/2) + f) a_2, \tag{7.3} $$

where f vanishes at ζ. Hence the principal symbol $-i\sigma_\partial([A^*A, P - \lambda])$ of the commutator has a fixed sign microlocally near ζ. We can then apply the standard positive commutator argument, see e.g. [9], to deduce $Au \in H_{\mathrm{sc}}^{\infty,s}(X)$. Iterating this argument proves the lemma. $\square$

Notice that we may take $A_1 = x^{-r_1}(xD_y - \Phi_1'(y))$ and $A_2 = x^{-r_2}(xD_y - \Phi_2'(y))$. Since $\tau + \Phi_2(y)$ also vanishes on L_2, it follows that $x^{-r_2}(x^2D_x + \Phi_2(y))u \in M$ as well.

Let $v = e^{-i\Phi_2(y)/x}u$, so $u = e^{i\Phi_2(y)/x}v$. Regularity properties of u then correspond to regularity properties of v under the module $e^{-i\Phi_2(y)/x}Me^{i\Phi_2(y)/x}$. This conjugation corresponds to the contact map $\Phi_2^\sharp : (y, \tau, \mu) \mapsto (y, \tau+\Phi_2(y), \mu-\Phi_2'(y))$, under which ζ is mapped to $(y(\zeta), 0, 0)$ (we take $y(\zeta) = 0$ for convenience from now on), and L_2 is mapped to the zero section of ${}^{\mathrm{sc}}T^*_{\partial X}X$, i.e. to $\tau = 0$, $\mu = 0$. The pull-back of $p - \lambda$ under this map will thus vanish on the zero section, and its differential is a multiple of the contact form at $(0, 0, 0)$, namely $\lambda_0\, d\tau$. Hence, the pull back p_{Φ_2} of $p - \lambda$ is $\lambda_0\tau + \tilde{p}$, where $\tilde{p}$ and its first derivatives vanish at $(0, 0, 0)$, and $\tilde{p}$ vanishes identically on the zero section. Since the conjugate of $x^{-r_2}(xD_y - \Phi_2'(y))$ under $e^{i\Phi_2(y)/x}$ is $x^{-r_2}(xD_y) = x^{r_1}D_y$, we deduce that v is stable under the module generated by $x^{-1}e^{-i\Phi_2/x}(P - \lambda)e^{i\Phi_2/x}$, $x^{-r_1}y$ and $x^{-r_2}(xD_y) = x^{r_1}D_y$. Note that $x^{-r_2}(x^2D_x)$ is in M since τ also vanishes on the zero section. In particular, $\tilde{p} = \tau\tilde{p}_1 + \mu\tilde{p}_2$, $\tilde{p}_1$ and $\tilde{p}_2$ vanish at $(0, 0, 0)$, so

$$x^{-1}\tilde{P} = x^{-1}e^{-i\Phi_2/x}(P - \lambda)e^{i\Phi_2/x} - x^{-1}(\lambda_0x^2D_x)$$

preserves the decay of v. Since $x^{-1}e^{-i\Phi_2/x}(P - \lambda)e^{i\Phi_2/x}$ does the same, we deduce that xD_x also preserves the decay of v.

The structure of $\tilde{p}$ can be described more concretely. Namely, ∂_y is an eigenvector of the linearization $L(p_{\Phi_2})$ of p_{Φ_2}, of eigenvalue σ_2, i.e. $d\mu$ is an eigenvector of the corresponding linearized Hamilton vector field of eigenvalue σ_2. Substituting in the Taylor series of $\tilde{p}$ to calculate $L(p_{\Phi_2})$, shows that $p_{\Phi_2} = \sigma_0\tau + (\sigma_0 - \sigma_2)y\mu + p_2$, where any quantization of p_2 improves the decay of v by $x^{\min\{2r_2,1+r_1\}}$, hence by $x^{1+\delta}$ for $\delta > 0$ small, apart from the subprincipal terms. Note that the second coefficient is σ_1.

To demostrate the polyhomogeneous behavior of u, as usual, the subprincipal term of $P - \lambda$ must also be taken into account. However, this dependence is quite simple. Indeed, for $P_1 \in \Psi^{\infty,1}_{\mathrm{sc}}(X)$, $P_1v - \sigma_{\partial,1}(P_1)(0, 0, 0)v \in x^{1/2+r_1-\epsilon}L^2_{\mathrm{sc}}(X)$ for all $\epsilon > 0$, so it can be ignored for our purposes. Thus, P_1 may be simply replaced by β_1x, $\beta_1 = \sigma_{\partial,1}(P_1)(0, 0, 0)$ is a constant.

It is useful to translate the microlocal picture at $(0, 0, 0)$ into conormal statements on a blown-up space (4.4). Introducing $Y = y/x^{r_1}$ and $X = x$, so

$$xD_y = X^{1-r_1}D_Y,\ xD_x = XD_X - r_1YD_Y,$$

we deduce that v is stable under XD_X, D_Y and Y. This means that v is conormal on an inhomogeneous blow up of $x = 0$, $y = y(\zeta)$, and it is rapidly decreasing off the front face.

Moreover,

$$e^{-i\Phi_2/x}(P - \lambda)e^{i\Phi_2/x}v - (\sigma_0x^2D_x + \sigma_1xyD_y + \beta x)v \in x^{1/2+\delta}L^2_{\mathrm{sc}}(X)$$

$\delta > 0$ small, β appropriately chosen. In fact, $\operatorname{Im}\beta = \sigma_0r_2/2$. Although this again follows from a direct calculation, note that the conjugate of the self-adjoint operator $P - \lambda$ is formally self-adjoint, and $x^2D_x + \frac{ix}{2}$, $xyD_y - \frac{ix}{2}$ are such, so $\operatorname{Im}\beta$ is in fact determined by formal self-adjointness. Since $(P - \lambda)u \in \dot{C}^\infty(X)$, we deduce that

$$(XD_X + \beta/\sigma_0)v \in x^{-1/2+\delta}L^2_{\mathrm{sc}}(X).$$

Writing $v = X^{-i\beta/\sigma_0}\tilde{v}$, this yields

$$D_X\tilde{v} \in x^{-3/2+\delta-r_2/2}L^2_{\mathrm{sc}}(X).$$

Taking into account the smoothness in Y, and changing the measure, this means for each Y,

$$\begin{aligned}\partial_X\tilde{v} \in X^{-3/2+\delta-r_2/2+(2-r_1)/2}L^2([0,1)_X;dX) =&X^{-1/2+\delta}L^2([0,1)_X;dX)\\ &\subset X^{\delta'}L^1([0,1)_X;dX),\ \delta'>0.\end{aligned}$$

But that implies that $\tilde{v}$ is continuous to $X = 0$, and after subtracting $\tilde{v}(0,Y)$, the result is bounded by $CX^{\delta'}$. This shows that, modulo $x^{-1/2+\epsilon'}L^2$, $\epsilon' > 0$ small and microlocally near the critical point u has the form $X^{-i\beta/\sigma_0}e^{i\Phi_2(y)/x}u_0$ with u_0 smooth on the blown-up space (4.4) and rapidly vanishing off the front face. A simple asymptotic series construction then yields the asymptotic series described before, and standard uniqueness result shows that f is actually given by such a series. This finishes the proof of the first half of Theorem 4.1.

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