THE INDEX OF PROJECTIVE FAMILIES OF ELLIPTIC OPERATORS: THE DECOMPOSABLE CASE

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Abstract. An index theory for projective families of elliptic pseudodifferential operators is developed under two conditions. First, that the twisting, i.e. Dixmier-Douady, class is in $\mathrm{H}^2(X;\mathbb{Z})\cup\mathrm{H}^1(X;\mathbb{Z})\subset\mathrm{H}^3(X;\mathbb{Z})$ and secondly that the 2-class part is trivialized on the total space of the fibration. One of the features of this special case is that the corresponding Azumaya bundle can be refined to a bundle of smoothing operators. The topological and the analytic index of a projective family of elliptic operators associated with the smooth Azumaya bundle both take values in twisted K-theory of the parameterizing space and the main result is the equality of these two notions of index. The twisted Chern character of the index class is then computed by a variant of Chern-Weil theory.

Introduction

The basic object leading to twisted K-theory for a space, X, can be taken to be a principal PU-bundle $\mathcal{P}\longrightarrow X$, where $\mathrm{PU}=\mathrm{U}(\mathcal{H})/\mathrm{U}(1)$ is the group of projective unitary operators on some separable infinite-dimensional Hilbert space $\mathcal{H}$. Circle bundles over X are classified up to isomorphism by their Chern classes in $\mathrm{H}^2(X;\mathbb{Z})$ and analogously principal PU bundles are classified by $\mathrm{H}^3(X;\mathbb{Z})$ with the element $\delta(\mathcal{P})$ being the Dixmier-Douady invariant of $\mathcal{P}$. Just as $\mathrm{K}^0(X)$, the ordinary K-theory group of X, may be identified with the group of homotopy classes of maps $X\longrightarrow\mathcal{F}(\mathcal{H})$ into the Fredholm operators on $\mathcal{H}$, the twisted K-theory group $\mathrm{K}^0(X;\mathcal{P})$ may be identified with the homotopy classes of sections of the bundle $\mathcal{P}\times_{\mathrm{PU}}\mathcal{F}$ arising from the conjugation action of PU on $\mathcal{F}$. The action of PU on the compact operators, $\mathcal{K}$, induces the *Azumaya* bundle, $\mathcal{A}$. The K-theory, in the sense of C^* algebras, of the space of continuous sections of this bundle, written $\mathrm{K}^0(X;\mathcal{A})$, is naturally identified with $\mathrm{K}^0(X;\mathcal{P})$. From an analytic viewpoint $\mathcal{A}$ is more convenient to deal with than $\mathcal{P}$ itself.

In the case of circle bundles isomorphisms are classified up to homotopy by an element of $\mathrm{H}^1(X;\mathbb{Z})$, corresponding to the homotopy class of a smooth map $X\longrightarrow\mathrm{U}(1)$. Similarly, $\delta\in\mathrm{H}^3(X;\mathbb{Z})$ determines $\mathcal{P}$ up to isomorphism with the isomorphism class determined up to homotopy by an element of $\mathrm{H}^2(X;\mathbb{Z})$, corresponding to the fact that PU is a $K(\mathbb{Z},2)$. The result is that $\mathrm{K}^0(X;\mathcal{A})$ depends as a group on the choice of Azumaya bundle with DD invariant δ up to an action of $\mathrm{H}^2(X;\mathbb{Z})$.

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In [16] we extended the index theorem for a family of elliptic operators, giving the equality of the analytic and the topological index maps in K-theory, to the case of twisted K-theory where the twisting class is a torsion element of $\mathrm{H}^3(X;\mathbb{Z})$. In this paper we prove a similar index equality in the case of twisted K-theory when the index class is decomposable

$$\delta = \alpha \cup \beta,\ \alpha \in \mathrm{H}^1(X;\mathbb{Z}),\ \beta \in \mathrm{H}^2(X;\mathbb{Z}), \tag{1}$$

and the fibration $\phi : Y \longrightarrow X$ is such that $\phi^*\beta = 0$ in $\mathrm{H}^2(Y;\mathbb{Z})$.

Under the assumption (1), that the class δ is *decomposed,* we show below that there is a choice of principal PU bundle with class δ such that the classifying map above, $c_P : X \longrightarrow K(\mathbb{Z};3)$ factors through $\mathrm{U}(1)\times\mathrm{PU}$. Twisting by a homotopically non-trivial map $\kappa : X \longrightarrow \mathrm{PU}$ does not preserve this property, so in this decomposed case there is indeed a natural choice of smooth Azumaya bundle, $\mathcal{S}$, up to homotopically trivial isomorphism and this induces a choice of twisted K-group determined by the decomposition of δ; we denote this well-defined twisted K-group by

$$\mathrm{K}^0(X;\alpha,\beta) = \mathrm{K}^0(X;\mathcal{A}), \quad \mathcal{A} = \overline{\mathcal{S}}. \tag{2}$$

The effect on smoothness of the assumption of decomposability on the Dixmier-Douady class can be appreciated by comparison with the simpler case of degree 2. Thus, if $\alpha_1 \cup \alpha_2 \in \mathrm{H}^2(X;\mathbb{Z})$ is a decomposed class, $\alpha_i \in \mathrm{H}^1(X;\mathbb{Z})$ for $i = 1, 2$, then the associated line bundle is the pull-back of the Poincaré line bundle associated to a polarization on the 2-torus under the map $u_1 \times u_2$, where the $u_i \in \mathcal{C}^\infty(X;\mathrm{U}(1))$ represent the α_i. This is to be contrasted with the general case in with the line bundle is the pull-back from a classifying space such as PU, and is only unique up to twisting by a smooth map $\kappa' : X \longrightarrow \mathrm{U}(1)$.

The data we use to define a smooth Azumaya bundle is:-

- A smooth function
$$u \in \mathcal{C}^\infty(X;\mathrm{U}(1)) \tag{3}$$
the homotopy class of which represents $\alpha \in \mathrm{H}^1(X,\mathbb{Z})$.
- A Hermitian line bundle (later with unitary connection)
$$\begin{array}{c} L \\ \big\downarrow \\ X \end{array} \tag{4}$$
with Chern class $\beta \in \mathrm{H}^2(X;\mathbb{Z})$.
- A smooth fiber bundle of compact manifolds
$$\begin{array}{ccc} Z & \text{———} & Y \\ & & \big\downarrow{\scriptstyle\phi} \\ & & X \end{array} \tag{5}$$
such that $\phi^*\beta = 0$ in $\mathrm{H}^2(Y;\mathbb{Z})$.
- An explicit global unitary trivialization
$$\gamma : \phi^*(L) \xrightarrow{\simeq} Y \times \mathbb{C}. \tag{6}$$

These hypotheses are satisfied by taking $Y = \tilde{L}$, the circle bundle of L, and then there is a natural choice of γ in (6). This corresponds to the 'natural' smooth Azumaya bundle associated to the given decomposition of $\delta = \alpha \cup \beta$ and we take $\mathrm{K}^0(X; \alpha, \beta)$ in (2) to be defined by this Azumaya bundle, discussed as a warm-up exercise in Section 1. In Appendix C it is observed that any fibration for which β is a multiple of a degree 2 characteristic class of $\phi : Y \longrightarrow X$ satisfies the hypothesis in (5).

In general, the data (3) – (6) are shown below to determine an infinite rank 'smooth Azumaya bundle', which we denote $\mathcal{S}(\gamma)$. This has fibres isomorphic to the algebra of smoothing operators on the fibre, Z, of Y with Schwartz kernels consisting of the smooth sections of a line bundle $J(\gamma)$ over Z^2. The completion of this algebra of 'smoothing operators' to a bundle with fibres modelled on the compact operators has Dixmier-Douady invariant $\alpha \cup \beta$.

In outline the construction of $\mathcal{S}(\gamma)$ proceeds as follows; details may be found in Section 3. The trivialization (6) induces a groupoid character $Y^{[2]} \longrightarrow \mathrm{U}(1)$, where $Y^{[2]}$ is the fiber product of two copies of fibration. Combined with the choice (3) this gives a map from $Y^{[2]}$ into the torus and hence by pull-back the line bundle $J = J(\gamma)$. This line bundle is *primitive* in the sense that under lifting by the three projection maps

(7) $$\tilde{L}^{[3]} \overset{\pi_S}{\underset{\pi_F}{\overset{\pi_C}{\rightrightarrows}}} \tilde{L}^{[2]}$$

(corresponding respectively to the left two, the outer two and the right two factors) there is a natural isomorphism

(8) $$\pi_S^* J \otimes \pi_F^* J = \pi_C^* J.$$

This is enough to give the space of global sections, $\mathcal{C}^\infty(Y^{[2]}; J \otimes \Omega_R)$, where Ω_R is the fiber-density bundle on the right factor, a fibrewise product isomorphic to the smoothing operators on Z. Indeed, if z represents a fiber variable then

(9) $$A \circ B(x, z, z') = \int_Z A(x, z, z'') \cdot B(x, z'', z')$$

where $\cdot$ denotes the isomorphism (8) which gives the identification

(10) $$J_{(z,z'')} \otimes J_{(z'',z')} \simeq J_{(z,z')}$$

needed to interpret the integral in (9). The naturality of the isomorphism corresponds to the associativity of this product.

Then the smooth Azumaya bundle is defined in terms of its space of global sections

(11) $$\mathcal{C}^\infty(X; \mathcal{S}(\gamma)) = \mathcal{C}^\infty(Y^{[2]}; J(\gamma)).$$

As remarked above, $J(\gamma)$, and hence also the Azumaya bundle, depends on the particular global trivialization (6). Two trivializations, γ_i, $i = 1, 2$ as in (6) determine

(12) $$\gamma_{12} : Y \longrightarrow \mathrm{U}(1),\ \gamma_{12}(y)\gamma_2(y) = \gamma_1(y)$$

which fixes an element $[\gamma_{12}] \in \mathrm{H}^1(Y; \mathbb{Z})$ and hence a line bundle K_{12} over Y with Chern class $[\gamma_{12}] \cup [\phi^* \alpha]$. Then

(13) $$J(\gamma_2) \simeq (K_{12}^{-1} \boxtimes K_{12}) \otimes J(\gamma_1)$$

with the isomorphism consistent with primitivity.

Pulling back to Y, $\phi^*\mathcal{A}(\gamma)$ is trivialized as an Azumaya bundle and this trivialization induces an isomorphism of twisted and untwisted K-theory

$$\mathrm{K}^0(Y;\phi^*\mathcal{A}(\gamma)) \xrightarrow{\simeq} \mathrm{K}^0(Y). \tag{14}$$

In fact there are stable isomorphisms between the different smooth Azumaya bundles and these induce natural and consistent isomorphisms

$$\mathrm{K}^0(X;\mathcal{A}(\gamma)) \xrightarrow{\simeq} \mathrm{K}^0(X;\alpha,\beta). \tag{15}$$

The proof may be found in Section 4.

The transition maps for the local presentation of the smooth Azumaya bundle, $\mathcal{S}(\gamma)$, determined by the data (3) – (6), are given by multiplication by smooth functions. Thus they also preserve the corresponding spaces of differential, or pseudodifferential, operators on the fibres; the corresponding algebras of twisted fibrewise pseudodifferential operators are therefore well defined. Moreover, since the principal symbol of a pseudodifferential operator is invariant under conjugation by (non-vanishing) functions there is a well-defined symbol map from the pseudodifferential extension of the Azumaya bundle, with values in the usual symbol space on $T^*(Y/X)$ (so with no additional twisting). The trivialization of the Azumaya bundle over Y, and hence over $T^*(Y/X)$, means that the class of an elliptic element can also be interpreted as an element of $\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma))$ where $\rho : T(Y/X) \longrightarrow Y$ is the bundle projection. This leads to the analytic index map,

$$\mathrm{ind}_{\mathrm{a}} : \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma)) \longrightarrow \mathrm{K}^0(X;\mathcal{A}(\gamma)). \tag{16}$$

The topological index can be defined using the standard argument by embedding of the fibration Y into the product fibration $\pi : \mathbb{R}^N \times X \longrightarrow X$ for large N. Namely, the Azumaya bundle is trivialized over Y and this trivialization extends naturally to a fibred collar neighborhood Ω of Y embedded in $\mathbb{R}^N \times X$. Thus, the usual Thom map $\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/X)$ is trivially lifted to a map for the twisted K-theory, which then extends by excision to a map giving the topological index as the composite with Bott periodicity:-

$$\begin{aligned}\mathrm{ind}_{\mathrm{t}} : \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma)) &\longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\Omega/X);\rho^*\tilde{\pi}^*\mathcal{A}(\gamma)) \\ &\longrightarrow \mathrm{K}^0_{\mathrm{c}}(T^*(\mathbb{R}^N/X);\rho^*\pi^*\mathcal{A}(\gamma)) \longrightarrow \mathrm{K}^0(X;\mathcal{A}(\gamma)).\end{aligned} \tag{17}$$

In the proof of the equality of these two index maps we pass through an intermediate step using an index map given by semiclassical quantization of smoothing operators, rather than standard pseudodifferential quantiztion. This has the virtue of circumventing the usual problems with multiplicativity of the analytic index even though it is somewhat less familiar. A fuller treatment of this semiclassical approach can be found in [17] so only the novelties, such as they are, in the twisted case are discussed here. The more conventional route, as used in [16], is still available but is technically more demanding. In particular it is worth noting that the semiclassical index map, as defined below, is well-defined even for a general fibration – without assuming that $\phi^*\beta = 0$. Indeed, this is essential in the proof, since the product fibration $\mathbb{R}^N \times X$ does not have this property.

For a fixed fibration the index maps induced by two different trivializations γ may be compared and induce a commutative diagram

$$
(18)\qquad
\begin{array}{ccccc}
\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) & \xrightarrow{\simeq} & \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma_1)) & \xrightarrow{\mathrm{ind}(\gamma_1)} & \mathrm{K}^0(X;\mathcal{A}(\gamma_1)) \\
\Big\downarrow [K_{12}]\times & & & & \Big\downarrow \simeq \\
 & & & & \mathrm{K}^0(X;\alpha,\beta) \\
 & & & & \Big\uparrow \simeq \\
\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) & \xrightarrow{\simeq} & \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X);\rho^*\phi^*\mathcal{A}(\gamma_2)) & \xrightarrow{\mathrm{ind}(\gamma_2)} & \mathrm{K}^0(X;\mathcal{A}(\gamma_2)).
\end{array}
$$

This follows from the proof of the index theorem.

The smoothness of the Azumaya bundle here allows us to give an explicit Chern-Weil formulation for the index in twisted cohomology. We recall that the twisted deRham cohomology $\mathrm{H}^*(X;\delta)$ is obtained by deforming the deRham differential to $d+\delta\wedge$, where

$$\delta = \bar\alpha\wedge\frac{\bar\beta}{2\pi i}.$$

Here $\bar\alpha = u^*(\theta)$ is the closed 1-form on X with integral periods, where θ is the Cartan-Maurer 1-form on $\mathrm{U}(1)$ and $\bar\beta$ is the closed 2-form with integral periods which is the curvature of the hermitian connection γ on L.

We remark that our results easily extend to the case when the Dixmier-Douady class is the sum of decomposable classes, ie when it is in the $\mathbb{Z}$-span of $\mathrm{H}^2(X;\mathbb{Z})\cup\mathrm{H}^1(X;\mathbb{Z})$. The Azumya bundle in this case is the tensor product of the decomposable Azumaya bundles as defined in this paper. The case of an arbitrary, not necessarily decomposable, Dixmier-Douady invariant is postponed to a subsequent paper where the twisted index theorem is treated in full. The general case uses pseudodifferential operators valued in the Azumaya bundle, rather than the pseudodifferential operator extension of the smooth Azumaya bundle as discussed here. Again, the semiclassical index map extends without difficulty to this general case.

In outline the paper proceeds as follows. The special case of the circle bundle is discussed in §1 and §2 contains the geometry of the general decomposable case. The smooth Azumaya bundle corresponding to a decomposable Dixmier-Douady class is constructed in §3 and some examples are also given. In §4, (15) is proved. The analytic index maps is defined in §5 using spaces of projective elliptic operators but including the case of twisted families of Dirac operators. The topological index is defined in §6. The Chern-Weil representative of the twisted Chern Character is studied in §7. Semiclassical versions of the index maps are introduced in §8 and §9 contains the proof of the equality of these two indices. In §10, the Chern character of the index is computed. In Appendix A the formulation of the Dixmier-Douady invariant in terms of differential characters is explored and in Appendix B it is computed using Čech cohomology (following a similar computation by Brylinski). Appendix C contains a discussion of the conditions on a fibration under which a line bundle from the base is trivial when lifted to the total space.

1. Trivialization by the circle bundle

An element $\beta\in\mathrm{H}^2(X;\mathbb{Z})$ for a compact manifold X, represents an isomorphism class of line bundles over X. Let L be such a line bundle with Hermitian inner product h and unitary connection ∇^L. We proceed to outline the construction of the smooth Azumaya bundle in the special case, alluded to above, where $Y = \tilde{L}$ is the circle bundle of L. This is carried out separately since this case gives a *natural* choice of the smooth Azumaya bundle, and hence the twisted K-group. The corresponding twisted cohomology is also identified with the cohomology of a subcomplex of the deRham complex over $\tilde{L}$.

From $u \in \mathcal{C}^\infty(X; \mathrm{U}(1))$ construct the principal $\mathbb{Z}$-bundle

$$\begin{array}{ccc} \mathbb{Z} & \text{---} & \hat{X} \\ & & \Big\downarrow \pi \\ & & X \end{array} \tag{1.1}$$

with total space the possible values of $\frac{1}{2\pi i}\log u$ over points of X and with $\mathcal{C}^\infty$ structure determined by the smoothness of a local branch of this function. Thus $f = \frac{1}{2\pi i}\log u \in \mathcal{C}^\infty(\hat{X}; \mathbb{R})$ is a well defined smooth function and under deck transformations

$$f(\hat{x} + n) = f(\hat{x}) + n \ \forall\ \hat{x} \in \hat{X},\ n \in \mathbb{Z}. \tag{1.2}$$

Let $p : \tilde{L} \longrightarrow X$ be the circle bundle of L; pulled back to $\tilde{L}$, L is canonically trivial. If ∇^L is an Hermitian connection on L then pulled back to $\tilde{L}$ it is of the form $d + \gamma$ on the trivialization of L, with $\gamma \in \mathcal{C}^\infty(\tilde{L}, \Lambda^1)$ a principal U(1)-bundle connection form in the usual sense. That is, under the action

$$m : \mathrm{U}(1) \times \tilde{L} \longrightarrow \tilde{L}, \tag{1.3}$$

$m^*\gamma = id\theta + \gamma$. This corresponds to the 'fiber shift map' on the fiber product

$$s : \tilde{L}^{[2]} = \tilde{L} \times_X \tilde{L} \longrightarrow \mathrm{U}(1),\ \tilde{l}_1 = s(\tilde{l}_1, \tilde{l}_2)\tilde{l}_2 \text{ in } \tilde{L}_x \tag{1.4}$$

in that $d\log s = p_1^*\gamma - p_2^*\gamma$ is the difference of the pull-back of the connection form from the two factors. From the character s a bundle, J, can be constructed from the trivial bundle over the fiber product $Q = \tilde{L} \times_X \tilde{L} \times_X \hat{X}$ corresponding to the identification

$$(\tilde{l}_1, \tilde{l}_2, \hat{x} + n, z) \simeq (\tilde{l}_1, \tilde{l}_2, \hat{x}, s(\tilde{l}_1, \tilde{l}_2)^n z). \tag{1.5}$$

Thus, J is associated to Q as a principal $\mathbb{Z}$-bundle over $\tilde{L}^{[2]}$. The primitivity property (8) follows from the multiplicativity property of s, that $s(l_1, l_2)s(l_2, l_3) = s(l_1, l_2)$ for any three points in a fixed fiber, which in turn follows from (1.4). The connection $d + fd\log s$ on the trivial bundle over Q descends to a connection on J which has curvature equal to a difference

$$\omega_J = \frac{1}{2\pi i}\overline{\alpha} \wedge d\log s = \overline{\alpha} \wedge \frac{1}{2\pi i}(p_1^*\gamma - p_2^*\gamma),\ \overline{\alpha} = df. \tag{1.6}$$

By definition, the space of global sections of the smooth Azumaya bundle is

$$\mathcal{C}^\infty(X; \mathcal{A}) = \mathcal{C}^\infty(\tilde{L}^{[2]}; J), \tag{1.7}$$

where the product on the right hand side is given by composition of Schwartz kernels.

The 'Dixmier-Douady twisting', given the decomposed form, corresponds to two different trivializations of J. Over any open set $U \subset X$ where u has a smooth logarithm, J is trivial using the section of $\hat{X}$ this gives. On the other hand, over any open set $U \subset X$ over which $\tilde{L}$ has a smooth section τ, the character in (1.5) is decomposed as the product $s(\tilde{l}_1, \tilde{l}_2) = s_\tau(\tilde{l}_1) s_\tau(\tilde{l}_2)^{-1}$ where $s_\tau(\tilde{l}) = s(\tilde{l}, \tau(p(\tilde{l}))$. This allows a line bundle K to be defined over the preimage of U in $\tilde{L}$ by the identification of the trivial bundle

$$(\tilde{l}, \hat{x} + n, z) \simeq (\tilde{l}, \hat{x}, s_\tau(\tilde{l})^n z). \tag{1.8}$$

Clearly then J may be identified with $K \boxtimes K'$, where K' is the dual, over U. In terms of a local trivialization in both senses over a small open set $U \subset X$, in which $L_U = U \times \mathbb{C}$, $\tilde{L}_U = U \times \mathbb{S}$, $\hat{X}_U = U \times \mathbb{Z}$, $a(x, \theta, \theta') \in \mathcal{C}^\infty(U \times \mathbb{Z} \times \mathbb{S} \times \mathbb{S})$ satisfies

$$a(x, n, \theta, \theta') = e^{in\theta} a(x, 0, \theta, \theta') e^{-in\theta'}. \tag{1.9}$$

This twisted conjugation means that $\mathcal{A}$ is a bundle of algebras, modelled on the smoothing operators on the circle with (1.9) giving local algebra trivializations. In this case the Azumaya bundle is associated with the principal $\mathrm{U}(1) \times \mathbb{Z}$ bundle $\tilde{L} \times_X \hat{X}$ and to the projective representation of this structure group through its central extension to the Heisenberg group.

The corresponding construction in the general case is quite similar and is described in §3.

The 3-twisted cohomology on X, with twisting form $\overline{\delta} = \overline{\alpha} \wedge \overline{\beta}$, is the target for the twisted Chern character discussed below. Here $\overline{\alpha}$ is a closed 1-form and $\overline{\beta}$ is the curvature 2-form on X for the Hermitian connection on L. Thus, on $\tilde{L}$, $d\gamma = (2\pi i) p^* \overline{\beta}$. In fact the $\overline{\delta}$-twisted deRham cohomology on X can be expressed as the cohomology of a subcomplex of the ordinary (total) deRham complex on $\tilde{L}$.

Proposition 1. *The even and odd degree subspaces of $\mathcal{C}^\infty(\tilde{L}, \Lambda^*)$ fixed by the conditions with respect to the infinitesmal generator of the* $\mathrm{U}(1)$ *action on* $\tilde{L}$

$$\mathcal{L}_{\partial/\partial\theta} \tilde{v} = 0, \ \iota_{\partial/\partial\theta} \tilde{v} = \frac{p^* \overline{\alpha}}{2\pi} \wedge \tilde{v}, \ \tilde{v} \in \mathcal{C}^\infty(\tilde{L}, \Lambda^*) \tag{1.10}$$

are mapped into each other by the standard deRham differential which has cohomology groups canonically isomorphic to the $\overline{\delta}$-twisted deRham cohomology on X.

Proof. The conditions in (1.10) are preserved by d since it commutes with the Lie derivative and given the first condition

$$\iota_{\partial/\partial\theta} d\tilde{v} = \mathcal{L}_{\partial/\partial\theta} \tilde{v} - d(\frac{p^* \overline{\alpha}}{2\pi} \wedge \tilde{v}) = \frac{p^* \overline{\alpha}}{2\pi} \wedge d\tilde{v}. \tag{1.11}$$

If $\tilde{v}$ satisfies (1.10) then $v' = \tilde{v} - \frac{\gamma}{2\pi i} \wedge p^* \overline{\alpha} \wedge \tilde{v}$ satisfies

$$\mathcal{L}_{\partial/\partial\theta} v' = 0, \ \iota_{\partial/\partial\theta} v' = 0 \Longrightarrow v' = p^* v, \ v \in \mathcal{C}^\infty(X; \Lambda^*). \tag{1.12}$$

Conversely if $v \in \mathcal{C}^\infty(X; \Lambda^*)$ then $\tilde{v} = p^* v + \frac{\gamma}{2\pi i} \wedge \overline{\alpha} \wedge p^* v$ satisfies (1.10). Thus every form satisfying (1.10) can be written uniquely

$$\tilde{v} = \exp\left(\frac{\gamma \wedge p^* \overline{\alpha}}{2\pi i}\right) p^* v = p^* v + \frac{\gamma \wedge p^* \overline{\alpha}}{2\pi i} \wedge p^* v. \tag{1.13}$$

Under this isomorphism d is clearly conjugated to $d + \overline{\delta}\wedge$ proving the Proposition. □

2. Geometry of the decomposed class

For a given line bundle L over X consider a fiber bundle (5) such that L is trivial when lifted to the total space. As discussed above, the circle bundle $\tilde{L}$ is an example. A more general discussion of this condition can be found in Appendix C. An explicit trivialization of the lift, γ, as in (6) is equivalent to a global section which is the preimage of 1 :

$$s' : Y \longrightarrow \phi^*(\tilde{L}). \tag{2.1}$$

Over each fiber of Y, the image is fixed so this determines a map

$$s(z_1, z_2) = s'(z_1)(s'(z_2))^{-1}$$

which is well-defined on the fiber product and is a groupoid character:

$$\begin{gathered} s : Y^{[2]} \longrightarrow \mathrm{U}(1), \\ s(z_1, z_2)s(z_2, z_3) = s(z_1, z_3) \ \forall \ z_i \in Y \text{ with } \phi(z_i) = x, \ i = 1, 2, 3, \ \forall \ x \in X. \end{gathered} \tag{2.2}$$

Conversely one can start with a unitary character s of $Y^{[2]}$ and recover L as the associated Hermitian line bundle

$$\begin{gathered} L = Y \times \mathbb{C}/ \simeq_s, \\ (z_1, t) \simeq_s (z_2, s(z_2, z_1)t) \ \forall \ t \in \mathbb{C}, \ \phi(z_1) = \phi(z_2). \end{gathered} \tag{2.3}$$

The connection on L lifts to a connection

$$\phi^* \nabla^L = d + \gamma, \quad \gamma \in \mathcal{C}^\infty(Y; \Lambda^1), \quad \pi_1^* \gamma - \pi_2^* \gamma = d \log s \text{ on } Y^{[2]} \tag{2.4}$$

on the trivial bundle $\phi^*(L)$. Conversely any 1-form on Y with this property defines a connection on L.

Now, let $Q = Y^{[2]} \times_X \hat{X}$ be the fiber product of $Y^{[2]}$ and $\hat{X}$, so as a bundle over X it has typical fiber $Z^2 \times \mathbb{Z}$; it is also a principal $\mathbb{Z}$-bundle over $Y^{[2]}$. The data above determines an action of $\mathbb{Z}$ on the trivial bundle $Q \times \mathbb{C}$ over Q, namely

$$T_n : (z_1, z_2, \hat{x}; w) \longrightarrow (z_1, z_2, \hat{x} + n, s(z_1, z_2)^{-n} w) \ \forall \ n \in \mathbb{Z}. \tag{2.5}$$

Let J be the associated line bundle over $Y^{[2]}$

$$J = (Q \times \mathbb{C})/ \simeq, \quad (z_1, z_2, \hat{x}; w) \simeq T_n(z_1, z_2, \hat{x}; w) \ \forall \ n \in \mathbb{Z}. \tag{2.6}$$

The fiber of J at $(z_1, z_2) \in Y^{[2]}$ such that $\phi(z_1) = \phi(z_2) = x$ is

$$J_{z_1, z_2} = \hat{X}_x \times \mathbb{C}/ \simeq, \quad (\hat{x} + n, w) \simeq (\hat{x}, s(z_1, z_2)^n w). \tag{2.7}$$

Lemma 1. *The connection $d + f d \log s$ on $Q \times \mathbb{C}$ descends to a connection ∇^J on J which has curvature*

$$F_{\nabla^J} = \pi_1^* \mu - \pi_2^* \mu, \ \mu = df \wedge \frac{\gamma}{2\pi i} \in \mathcal{C}^\infty(Y; \Lambda^2), \ \ Y^{[2]} \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} Y. \tag{2.8}$$

Moreover $d\mu = \phi^*(\overline{\delta})$, for the uniquely determined 3-form on X, $\overline{\delta} = \overline{\alpha} \wedge \overline{\beta} \in \mathcal{C}^\infty(X; \Lambda^3)$, where $df = \phi^*(\overline{\alpha})$ and $d\gamma = 2\pi i \phi^*(\overline{\beta})$ represent the characteristic class of $\hat{X}$ and the first Chern class of L respectively.

Proof. Clearly the 1-form $f d \log s$ has the correct transformation law under the $\mathbb{Z}$ action on $Y^{[2]} \times_X \hat{X}$ to give a connection on J. Its curvature is $\overline{\alpha} \wedge \frac{d \log s}{2\pi i}$ where $\overline{\alpha} = \frac{1}{2\pi i} d \log u$. If γ is the connection form for the trivialization of L on Y then

$$d \log s = \pi_1^* \gamma - \pi_2^* \gamma \text{ in } Y^{[2]} \tag{2.9}$$

from which (2.8), together with the remainder of the Lemma, follows. □

3. Smooth Azumaya bundle

We proceed to show how to associate to the data (3) – (6) discussed above a smooth Azumaya bundle over X. That is, we construct a locally trivial bundle with fibres modelled on the smoothing operators on the sections of a line bundle over the fibres of Y and having completion with Dixmier-Douady invariant $\alpha \cup \beta$. Note that this Azumaya bundle *does depend* on the trivialization data in (6); we will therefore denote it $J(\gamma)$. The effect of changing this trivialization is discussed in Lemma 4 below.

First consider local trivializations of the data.

Proposition 2. *A section of ϕ, over on open set $U \subset X$, $\tau : U \longrightarrow \phi^{-1}(U)$, induces a trivialization of L over U and an isomorphism of $J(\gamma)$ over the open subset $V = \phi^{-1}(U) \times_U \phi^{-1}(U)$ of $Y^{[2]}$, with*

$$J\big|_V \cong_\tau \operatorname{Hom}(K_\tau) = K_\tau \boxtimes K'_\tau \tag{3.1}$$

for a line bundle K_τ over $\phi^{-1}(U) \subset Y$, where K'_τ denotes the line bundle dual to K_τ. Another choice of section $\tau' : U \longrightarrow \phi^{-1}(U)$, determines another line bundle $K_{\tau'}$ over $\phi^{-1}(U) \subset Y$, satisfying

$$K_\tau = K_{\tau'} \otimes \phi^*(L_{\tau,\tau'}), \tag{3.2}$$

where $L_{\tau,\tau'} = (\tau,\tau')^* J$ is the fixed local line bundle over U.

Proof. A local section of ϕ induces a local trivialization of the character s,

$$s(z_1,z_2) = \chi_\tau(z_1)\chi_\tau^{-1}(z_2), \quad \chi_\tau(z) = s(z,\tau(\phi(z))) \text{ on } \phi^{-1}(U) \subset Y. \tag{3.3}$$

This trivializes L over U, identifying it with $\tau^*\mathbb{C}$ with connection $d + \tau^*\gamma$.

The line bundle K_τ over $\phi^{-1}(U)$ associated to the $\mathbb{Z}$ bundle $\phi^{-1}(U) \times_U \hat{X}_U$ by the identification $(z, \hat{x}+n, w) \simeq (z, \hat{x}, \chi_\tau(z)^n w)$ then satisfies (3.1). The line bundle $K_{\tau'}$ is similarly defined over $\phi^{-1}(U)$, satisfying (3.1) with τ' substituted for τ. The relation (3.2) follows from (3.1) and its modification with τ' substituted for τ. □

Such a section of Y will induce a local trivialization of the smooth Azumaya bundle in which it becomes the smoothing operators on the fibres of Y acting on sections of K_τ :

$$\mathcal{S}_\tau = \Psi^{-\infty}(\phi^{-1}(U)/U; K_\tau). \tag{3.4}$$

Using Proposition 2, we get the local patching,

$$\mathcal{S}_\tau = \mathcal{S}_{\tau'} \otimes \Psi^{-\infty}(\phi^{-1}(U)/U; \phi^*(L_{\tau,\tau'})). \tag{3.5}$$

Rather than use this as a definition we adopt an *a priori* global definition by trivializing over Y.

Definition 1. For any $x \in X$, the fiber of the smooth Azumaya bundle associated to the geometric data in §2 is

$$\mathcal{S}_x = \mathcal{C}^\infty(Y_x^2; J\big|_{Y_x^2} \otimes \Omega_R) \tag{3.6}$$

where Ω_R is the fiber density bundle on the right factor of Y_x^2. Globally, we have a natural identification,

$$\mathcal{C}^\infty(X, \mathcal{S}) \cong \mathcal{C}^\infty(Y^{[2]}, J \otimes \Omega_R). \tag{3.7}$$

Thus a smooth section of $\mathcal{S}$ over any open set $U \subset X$ is just a smooth section of $J \otimes \Omega_R$, where $\Omega_R = \pi_R^*\Omega$, over the preimage of U in $Y^{[2]}$.

Of course, we need to show that $\mathcal{S}$ is a bundle of algebras over X with local trivializations as indicated in (3.4). To see this globally, observe that J has the same 'primitivity' property as for $\tilde{L}$ in §1 with respect to the groupoid structure.

Lemma 2. *If*

$$Y^{[3]} \underset{\pi_F}{\overset{\pi_S}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{\pi_C}}}}} Y^{[2]} \tag{3.8}$$

are the three projections (respectively onto the two left-most, the outer two and the two right-most factors – the notation stands for 'second', 'composite' and 'first' for operator composition) then there is a natural isomorphism

$$\pi_S^* J \otimes \pi_F^* J \xrightarrow{\simeq} \pi_C^* J \tag{3.9}$$

and moreover J carries a connection ∇^J which respects this primitivity property.

Proof. The identity (3.9) is evident from the definition of J and Proposition 2. The naturality property for (3.9) corresponds to an identity on $Y^{[4]}$. Namely if J' is the dual of J then the tensor product of the pull-backs under the four projections $Y^{[4]} \longrightarrow Y^{[3]}$ of the combination $\pi_S^* J \otimes \pi_F^* J \otimes \pi_C^* J'$ over $Y^{[3]}$ is naturally trivial. That this trivialization is equal to the tensor product of the four trivializations from (3.9) follows again from the definition of J.

By Proposition 2, a section $\tau : U \longrightarrow Y$ of ϕ over the open subset U of X defines an isomorphism $J\big|_V \cong_\tau \operatorname{Hom}(K_\tau) = K_\tau \boxtimes K'_\tau$ where $V = \phi^{-1}(U) \times_U \phi^{-1}(U)$ is the open subset of $Y^{[2]}$. A choice of connection ∇^τ on K_τ induces a connection ∇^V on $J\big|_V$ which clearly respects the primitivity property. A global connection preserving the primitivity property can then be constructed using a partition of unity on X. □

As a consequence of Lemma 2 there is a lifting map

$$\mathcal{C}^\infty(Y^{[2]}; J \otimes \Omega_R) \xrightarrow{\pi_F^*} \mathcal{C}^\infty(Y^{[3]}; \pi_S J' \otimes \pi_C J \otimes \pi_F^* \Omega_R) \xrightarrow{\simeq} \Psi^{-\infty}(Y^{[2]}/Y; J') \tag{3.10}$$

which embeds into an algebra, namely the smoothing operators on sections of J' on the fibres of $Y^{[2]}$ as a fibration over Y (projecting onto the first factor).

Proposition 3. *Lifting $\mathcal{C}^\infty(Y^{[2]}; J \otimes \Omega_R)$ to $Y^{[3]}$ under the projection off the left-most factor (the 'first' projection in terms of composition) embeds it as a subalgebra of the smoothing operators on sections of J' as a bundle over $Y^{[2]}$ on the fibres of the projection onto the right factor such that the lift of the bundle of algebras over X is equal to the bundle of algebras over Y.*

This justifies (3.4). As discussed below it also shows that, as an Azumaya bundle, the completion of $\mathcal{S}$ is $\mathcal{A} = \mathcal{A}(\gamma)$.

Proof. It only remains to show that composition of two local sections of $\mathcal{S}$ in the algebra of fiber smoothing operators gives another section of the Azumaya bundle. However, this follows from (3.4), which in turn is a consequence of Lemma 2 applied to the local decomposition of J in (3.1). □

An infinite rank Azumaya bundle $\mathcal{A}$, over a topological space X, is a bundle of star algebras with local isomorphisms with the trivial bundle of compact operators, $\mathcal{K}(\mathcal{H})$, on a fixed separable but infinite-dimensional Hilbert space $\mathcal{H}$. The Dixmier-Douady invariant of $\mathcal{A}$ is an element of $\mathrm{H}^3(X;\mathbb{Z})$. It classifies the bundle up to stable isomorphism (i.e. after tensoring with $\mathcal{K}$) and can be realized in terms of Čech cohomology or alternatively in terms of classifying spaces as follows. The group of $*$-automorphism of $\mathcal{K}$ is $\mathrm{PU}(\mathcal{H}) = \mathrm{U}(\mathcal{H})/\,\mathrm{U}(1)$, the projective unitary group of the Hilbert space acting by conjugation. Thus the fiber trivializations of $\mathcal{A}$ form a principal $\mathrm{PU}(\mathcal{H})$-bundle over X. Since $\mathrm{PU}(\mathcal{H}) = K(\mathbb{Z},2)$ is an Eilenberg-Maclane space, this bundle, and hence $\mathcal{A}$, is classified up to isomorphism by an homotopy class of maps $X \longrightarrow B\,\mathrm{PU}(\mathcal{H}) = K(\mathbb{Z},3)$ which represents, and is equivalent to, the Dixmier-Douady invariant.

The Chern class of a line bundle L over a space X has a similar representation. Taking an Hermitian structure and passing to the associated circle bundle $\tilde{L}$ over X one can consider the Hilbert bundle $L^2(\tilde{L}/X)$ of Lebesgue square integrable functions on the fibres of the circle bundle. Each point $l \in \tilde{L}$ defines a unitary operator on the fiber through that point, namely multiplication by $U(\hat{l}) = \exp(i\theta_{\hat{l}})\times$ where the normalization is such that $\exp(i\theta_{\hat{l}})(\hat{l}) = 1$. Changing $\hat{l}$ within the fiber changes $U(\hat{l})$ to $\exp(i\theta')U(\hat{l}')$ so this defines a map

$$X \longrightarrow \mathrm{PU}(L^2(\tilde{L}/X)) \tag{3.11}$$

into the bundle of projective unitary operators on the fibres of the Hilbert bundle. By Kuiper's theorem any Hilbert bundle is trivial (in the uniform topology) and the trivialization is natural up to homotopy. Thus the map (3.11) lifts to a unique homotopy class of maps

$$X \longrightarrow \mathrm{PU}(\mathcal{H}) = K(\mathbb{Z},2) \tag{3.12}$$

and this represents, and is equivalent to, the first Chern class. This follows from the evident fact that $\tilde{L}$ is isomorphic to the pull-back of the canonical circle bunde, $\mathrm{U}(\mathcal{H})/\,\mathrm{PU}(\mathcal{H})$ over $\mathrm{PU}(\mathcal{H})$.

Now consider the decomposed case under consideration here. Over the given space X we have both a map $u \in \mathcal{C}^\infty(X;\mathrm{U}(1))$ and a line bundle L. Passing to the classifying map (3.11) this gives a unique homotopy class of maps

$$X \longrightarrow \mathrm{U}(1) \times \mathrm{PU}(\mathcal{H}). \tag{3.13}$$

Proposition 4. *The completion of the smooth Azumaya bundle $\mathcal{S}$ associated above to* (3) – (6) *to an Azumaya bundle $\mathcal{A} = \mathcal{A}(\gamma)$, has Dixmier-Douady invariant $\alpha \cup \beta \in \mathrm{H}^3(X;\mathbb{Z})$ which is represented by the composite of* (3.13) *with the classifying map $\mathrm{U}(1)\times\mathrm{PU}(\mathcal{H}) \longrightarrow K(\mathbb{Z},3)$ induced by the projectivisation of the basic representation of the Heisenberg group $\mathbb{Z}\times\mathrm{U}(1) \longrightarrow \mathrm{PU}(\mathcal{H})$.*

Proof. The classifying space BG of a topological group G is defined up to homotopy as the quotient $*/G$ of a contractible space on which G acts freely. In particular it follows that (always up to homotopy)

$$B(G_1 \times G_2) \simeq BG_1 \times BG_2 \tag{3.14}$$

and if $H \subset G$ is a closed subgroup then there is a well defined homotopy class of maps

$$BH \longrightarrow BG. \tag{3.15}$$

Recall that the basic representation of the Heisenberg group H arises from the actions of U(1) and $\mathbb{Z}$ on $L^2(\mathbb{S})$ (or $\mathcal{C}^\infty(\mathbb{S})$) respectively by translation and multiplication by $e^{in\theta}$. These commute up to scalars, which is the action of the center of H as a central extension

$$\mathrm{U}(1) \longrightarrow H \longrightarrow \mathbb{Z} \times \mathrm{U}(1) \tag{3.16}$$

and so embeds

$$\mathbb{Z} \times \mathrm{U}(1) \hookrightarrow \mathrm{PU}(\mathcal{H}) \tag{3.17}$$

as a subgroup of the projective unitary group on $L^2(\mathbb{S})$. By (3.14) and (3.15) this induces an homotopy class of continuous maps

$$\Delta : \mathrm{U}(1) \times \mathrm{PU} \simeq B(\mathbb{Z} \times \mathrm{U}(1)) \longrightarrow K(\mathbb{Z}, 3). \tag{3.18}$$

So the claim in the Proposition is that under this map the pull-back of the degree 3 generator of the cohomology of $K(\mathbb{Z}, 3)$ is the Dixmier-Douady invariant of $\mathcal{A}$ and is equal to $\alpha \cup \beta$ in $\mathrm{H}^3(X, \mathbb{Z})$.

The first statement follows from the fact that the PU bundle to which $\mathcal{A}$ is associated is obtained from the $\mathbb{Z} \times \mathrm{U}(1)$ bundle $\hat{X} \times_X \tilde{L}$ by extending the structure group using (3.17). The second statement follows from the fact that under the map (3.18) the generating 3-class $\delta_{\mathrm{DD}} \in \mathrm{H}^3(K(\mathbb{Z}, 3), \mathbb{Z})$ pulls back to $\alpha' \cup \beta'$ where $\alpha' \in \mathrm{H}^1(\mathbb{S}, \mathbb{Z})$ and $\beta' \in \mathrm{H}^2(\mathrm{PU}, \mathbb{Z})$ are the generators, that is,

$$\Delta^* \delta = \alpha' \cup \beta'. \tag{3.19}$$

Indeed, the degree 3-cohomology of $\mathrm{U}(1) \times \mathrm{PU}$ has a single generator, so (3.19) must be correct up to a multiple on the right side. Thus it is enough to check one example, to determine that the multiple is equal to one. Take $X = \mathbb{S} \times \mathbb{S}^2$ with u the identity on $\mathbb{S}$ and L the standard line bundle over the sphere. We know that the induced map (3.12) for the sphere generates the second homotopy group of PU and pulls back to the fundamental class on $\mathbb{S}^2$. Thus it suffices to note that the PU bundle over $\mathbb{S} \times \mathbb{S}^2$ with which our smooth Azumaya bundle is associated in this case is just obtained by the clutching construction from the trivial bundle over $[0, 2\pi] \times \mathbb{S}^2$ using this map. □

An interesting special case of this construction, close to the lifting to the circle bundle described in §1, arises when $\beta \in \mathrm{H}^2(X; \mathbb{Z})$ is thought of as the first Chern class of a complex vector bundle rather than a line bundle. Then Y can be taken to be the associated principal bundle

$$\begin{array}{ccc} \mathrm{U}(n) & \text{——} & P \\ & & \downarrow \\ & & X. \end{array}$$

Since the abelianization of $\mathrm{U}(n)$ is canonically isomorphic to U(1), any character (i.e. 1-dimensional unitary representation) of $\mathrm{U}(n)$ factorizes through U(1), and conversely, any character of U(1) lifts to a character of $\mathrm{U}(n)$. A U(1)-central extension of the group $\mathrm{U}(n) \times \mathbb{Z}$ arises in the form of a generalized Heisenberg group. Namely the group product on $H_n = \mathrm{U}(n) \times \mathbb{Z} \times \mathrm{U}(1)$ can be taken to be

$$(g_1, n_1, z_1)(g_2, n_2, z_2) = (g_1 g_2, n_1 + n_2, \det(g_1)^{n_2} z_1 z_2).$$

Then

$$1 \longrightarrow U(1) \longrightarrow H_n \longrightarrow U(n) \times \mathbb{Z} \longrightarrow 1$$

is a central extension.

4. Stable Azumaya isomorphism

We proceed to show that the twisted K-groups, $\mathrm{K}^0(X;\mathcal{A}(\gamma))$, defined through the possible data (3) – (6) corresponding to a fixed decomposition (1) are all naturally isomorphic, as indicated in (15). This is a consequence of the Morita invariance of the C^* K-groups and the existence of stabilized isomorphisms between the various Azumaya bundles.

For a smooth 1-parameter family of trivializations, as in (6), so depending smoothly on $t \in [0,1]$, the K-groups $\mathrm{K}^0(X;\mathcal{A}(\gamma(t)))$ are all naturally isomorphic. Since two such trivializations differ by a smooth map $Y \longrightarrow \mathrm{U}(1)$, the K-group can only depend on the homotopy class of this map, when the other data is fixed. It is also the case that K-theory of C^* algebras admits Morita equivalence. That is, the K-group of $\mathcal{A}$ is naturally isomorphic to the K-group of $\mathcal{A}\otimes\mathcal{K}$. Kuiper's theorem shows that the completion of the smoothing operators on any fiber bundle over a space, and acting on sections of any vector bundle, V, over the fiber bundle, is naturally isomorphic up to homotopy to the trivial Azumaya bundle $\mathcal{K}$. It follows that the 'twisted' K-theory of a space, computed with respect to such a bundle is naturally isomorphic to the untwisted K-theory. More generally, taking the smooth Azumaya bundle $\mathcal{S}(\gamma)$ and tensoring with the bundle of smoothing operators, $\Psi^{-\infty}(\psi;V)$, on any other fiber bundle $\psi: Y' \longrightarrow X$, over the same base, gives an Azumaya bundle with the same twisted K-theory, $\mathrm{K}^0(X;\mathcal{A}(\gamma))$. This proves:-

Lemma 3. *If $\psi: Y' \longrightarrow X$ is a fibration of compact manifolds and $\mathcal{S}(\gamma)$ is the Azumaya bundle associated to data* (3) – (6) *then there is a natural isomorphism of twisted K-theory*

$$\mathrm{K}^0(X;\mathcal{A}(\gamma)) \xrightarrow{\simeq} \mathrm{K}^0(X;\mathcal{A}(\gamma')) \tag{4.1}$$

where γ' is the trivialization obtained by pulling back γ to the product bundle $Y\times_X Y' \longrightarrow X$.

Applying this result to the initial Azumaya bundle in § 1 and the general case, shows that $\mathrm{K}^0(X;\alpha,\beta)$ and $\mathrm{K}^0(X;\mathcal{A}(\gamma))$ are each naturally isomorphic to some (possibly different) $\mathrm{K}^0(X;\mathcal{A}(\gamma'))$ where γ' is a trivialization of the lift of $\tilde{L}$ to $\tilde{L}\times_X Y$, obtained in the two cases by lifting the trivialization from $\tilde{L}$ or Y to the fibre product. Thus it remains to consider two different trivializations over the same fibration.

Proposition 5. *If γ_i are two trivializations of ϕ^*L over Y as in* (6) *then there is an embedding of algebras, unique up to homotopy,*

$$\mathcal{S}(\gamma_2) \longrightarrow \mathcal{S}(\gamma_1)\boxtimes\Psi^{-\infty}(\mathbb{T}^2;K) \tag{4.2}$$

for a line bundle over the 2-torus which induces natural isomorphisms

$$\mathrm{K}^0(X;\mathcal{A}(\gamma_2)) \xrightarrow{\simeq} \mathrm{K}^0(X;\mathcal{A}_{12})) \xrightarrow{\simeq} \mathrm{K}^0(X;\mathcal{A}(\gamma_1)) \tag{4.3}$$

where $\mathcal{A}_{12}$ is the completion of $\mathcal{S}(\gamma_1)\boxtimes\Psi^{-\infty}(\mathbb{T}^2;K)$.

Proof. This is really an adaptation of the proof of the index theorem via embedding. First, we recall the discussion above, which shows that the primitive line bundle $J(\gamma_2)$ is isomorphic to $J(\gamma_1)\otimes(K_{12}\boxtimes K'_{12})$ for a line bundle K_{12} over Y pulled back from a line bundle K over $\mathbb{T}$ by a smooth map $\kappa_{12}: Y \longrightarrow \mathbb{R}^2$. This map embeds

Y as a subfibration of $\phi \circ \pi_1 : Y \times \mathbb{T}^2 \longrightarrow X$. Let $N \longrightarrow Y$ be the normal bundle to this embedding. Given a metric this carries a field of harmonic oscillators on the fibres, the ground states of which give the desired embedding.

Let $v(z,\zeta)$ be the L^2-orthonormalized ground state on the fiber over $z \in Y$. Then (4.4)

$$\mathcal{C}^\infty(Y^{[2]}; J(\gamma)) \ni a(z_1,z_2) \longmapsto \tilde{a} = v(z_1,\zeta_1)a(z_1,z_2)v(z_2,\zeta_2) \in \mathcal{S}(V^{[2]}; J(\gamma_2))$$

is an embedding. Moreover, this is an embedding of algebras with the algebra structure on the right given by Schwartz-smoothing operators. Now consider the bundle $J(\gamma_1) \boxtimes K \boxtimes K'$ over $Y^{[2]} \times \mathbb{T}^2 \times \mathbb{T}^2$. Restricted to the image of the embedding of $Y^{[2]}$ given by κ_{12} acting in both fibres, this is isomorphic to $J(\gamma_2)$ since by construction K pulls back to K_{12} over Y. Now, consider an embedding of V, using the collar neighborhood theorem, as a neigborhood, $\Omega \subset Y \times \mathbb{T}^2$, of the image of $Y^{[2]}$ under this embedding. The bundle K, pulled back to $Y \times \mathbb{T}^2$ by the projection onto $\mathbb{T}^2$ can be deformed to a bundle $\tilde{K}$, which is equal over Ω to the pull back under the normal retraction of its restriction, K_{12}, to the image of Y. Then the embeddign (4.4) embeds $\mathcal{C}^\infty(Y^{[2]}; J(\gamma_2)$ as a subalgebra of $\mathcal{C}^\infty((Y')^{[2]}; J(\gamma_1) \otimes \tilde{K} \boxtimes \tilde{K}')$, $Y' = Y \times \mathbb{T}^2$. Moreover, using the full spectral expansion of the harmonic oscillator, the completion of the image is Morita equivalent to the whole subalgebra with support in the compact manifold with boundary which is the closure of $\Omega \subset Y'$. This in turn is Morita equivalent to the whole algebra and hence, after another deformation of $\tilde{K}$ back to K over $\mathbb{T}^2$ to $\mathcal{A}_{12}$ in (4.3). This gives the first isomorphism in (4.3). The second follows from stabilization by the compact operators on K over $\mathbb{T}^2$ as discussed above, completing the proof. □

Proof of (15). As noted above this is a corollary of Proposition 5 and the preceeding discussion. Namely this provides a stabilized isomorphism, unique up to homotopy, of the Azumaya bundle in §1 with that constructed over $\tilde{L} \times_X Y$ by lifting the trivialization over $\tilde{L}$ to the fiber product. The same is true by lifting the trivialization over Y to the fiber product. Then the Proposition constructs a stable isomorphism, again unique up to homotopy, of the two lifts to $\tilde{L} \times_X \times Y \times \mathbb{T}^2$. These stable isomorphisms project to a unique isomorphism of the twisted K-groups, as in (15), consistent under composition. □

Lemma 4. *The Azumaya bundle $\mathcal{S}(\gamma)$, lifted to Y, is completion isomorphic to the trivial bundle $\mathcal{K}$, with the isomorphism fixed up to homotopy, and this induces the natural isomorphisms* (14).

Proof. The primitivity condition on J shows that when lifted to the second two factors of $Y^{[3]}$ it is isomorphic to the bundle over $Y^{[3]}$ of which the elements of $\Psi(Y^{[2]}/Y; J')$, the smoothing operators on the fibers of $Y^{[2]}$ as a bundle over Y, are (density-valued) sections. As noted above, Kuiper's theorem shows that the completion of $\Psi(Y^{[2]}/Y; J')$ is naturally, up to homotopy, isomorphic to the trivial Azumaya bundle $\mathcal{K}$, from which (14) follows. □

5. Analytic index

We now proceed to define the analytic index map (16) using the constructions in §2, §3 and §4. The first step is to define the projective bundle of pseudodifferential operators. We do this by direct generalization of Definition 1. So, for any $\mathbb{Z}_2$-graded bundle $\mathbb{E} = (E_+, E_-)$ over Y set

$$\Psi^\ell(Y/X; \mathcal{A} \otimes \mathbb{E}) = I^\ell(Y^{[2]}, \mathrm{Diag}; J \otimes \mathrm{Hom}(\mathbb{E}) \otimes \Omega_R) \tag{5.1}$$

where $\mathrm{Hom}(\mathbb{E}) = E_- \boxtimes E'_+$ over $Y^{[2]}$ and I^ℓ is the space of (classical) conormal distributions. As is typical in projective index theory, the Schwartz kernel of the projective family of elliptic operators is globally defined, even though one only has local families of elliptic operators with a compatibility condition on triple overlaps given by a phase factor. More precisely, definition (5.1) means that on any open set in $Y^{[2]}$ over which J is trivialized as $\mathrm{Hom}(K_\tau)$ as in Proposition 2, the kernel is that of a family of pseudodifferential operators on the fibres of Y acting from sections of E_+ to sections of E_-. It follows from the standard case that (3.4) also extends immediately to show that if $\tau : U \longrightarrow Y$ is a section over an open set, then

$$\begin{aligned}\Psi^\ell(\phi^{-1}(U)/U; \mathcal{A} \otimes \mathbb{E}) \cong_\tau \Psi^\ell(\phi^{-1}(U)/U; K_\tau \otimes \mathbb{E})&\\ \xrightarrow{\sigma_\ell} \mathcal{C}^\infty(S^*(\phi^{-1}(U)/U); \hom(\mathbb{E}) \otimes N_\ell),&\end{aligned} \tag{5.2}$$

where we have used the fact that $\hom(K_\tau)$ is canonically trivial. The principal symbol map here is invariant under conjugation by functions and hence well-defined independent of the trivialization; N_ℓ is the trivial line bundle corresponding to functions of homogeneity ℓ on $T^*(\phi^{-1}(U)/U)$ and $\hom(\mathbb{E})$ is the bundle (over $S^*(\phi^{-1}(U)/U)$) of homomorphisms from E_+ to E_-. Thus the usual composition properties of pseudodifferential operators extend without any difficulty as do the symbolic properties. More precisely,

Lemma 5. *The spaces of smooth sections of $\Psi^\ell(Y/X; \mathcal{A} \otimes \mathbb{E})$ form graded modules under composition and the principal symbol defined through* (5.2) *is independent of τ and gives a multiplicative short exact sequence for any ℓ :*

$$\Psi^{\ell-1}(Y/X; \mathcal{A} \otimes \mathbb{E}) \hookrightarrow \Psi^\ell(Y/X; \mathcal{A} \otimes \mathbb{E}) \xrightarrow{\sigma_\ell} \mathcal{C}^\infty(S^*(Y/X; p^* \hom(\mathbb{E}) \otimes N_\ell)). \tag{5.3}$$

Proof. The theory of conormal distributional sections of a complex vector bundle with respect to a submanifold, implicit already in Hörmander's paper [15], shows that these have well-defined principal symbols which are homogeneous sections over the conormal bundle of the submanifold, in this case the fibre diagonal, of the pull-back of the bundle tensored with a density bundle. In this case, as for pseudodifferential operators, the density bundles cancel. Moreover the bundle J is canonically trivial over the (fiber) diagonal in $Y^{[2]}$ by the primitivity property of J. The symbol in (5.3) therefore does not involve any twisting – it takes values in the same space as in the untwisted case, and is a well-defined homogeneous section of the homomorphism bundle of E (hence section of that bundle tensored with the homogeneity bundle N_l) on the fibre cotangent bundle – which is canonically the conormal bundle of the fibre diagonal, as claimed. □

With the trivialization κ fixed, the symbol of a projective family of elliptic pseudodifferential operators determines an element in $\mathrm{K}^0_c(T^*(Y/X))$ We now show that the index of such a projective elliptic family is an element in twisted K-theory of the base, $\mathrm{K}^0(X, \mathcal{A})$. More precisely, let $P \in \Psi^m(Y/X; \mathcal{A} \otimes \mathbb{E})$ be a projective family of elliptic operators. This means that the symbol is invertible in the usual sense, so from the standard ellipticity construction (using iteration over ℓ in the sequence (5.3)) P has a parametrix $Q \in \Psi^{-m}(Y/X; \mathcal{A} \otimes \mathbb{E}_-)$, where $\mathbb{E}_- = (E_-, E_+)$, such that $S_0 = 1 - QP \in \Psi^{-\infty}(Y/X; \mathcal{A} \otimes E_+)$ and $S_1 = 1 - PQ \in \Psi^{-\infty}(Y/X; \mathcal{A} \otimes E_-)$. Then the index is realized using the idempotent

$$E_1 = \begin{pmatrix} 1 - S_0^2 & Q(S_1 + S_1^2) \\ S_1 P & S_1^2 \end{pmatrix} \in M_2(\Psi^{-\infty}(Y/X; \mathcal{A} \otimes \mathbb{E})^\dagger).$$

Here, † denotes the unital extension of the algebra. It is standard to verify that E_1 is an idempotent.

Then, as in the usual case, the analytic index of P expressed in terms of idempotents is

$$\begin{gathered} \operatorname{ind}_a(P) = [E_1 - E_0] \in \mathrm{K}_0(\Psi^{-\infty}(Y/X; \mathcal{A} \otimes \mathbb{E})) \text{ where} \\ E_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(\Psi^{-\infty}(Y/X; \mathcal{A} \otimes \mathbb{E})^\dagger). \end{gathered} \tag{5.4}$$

That $\operatorname{ind}_a(P)$ is a well-defined element in the K-theory follows from invariance of K-theory under Morita equivalence of algebras. Thus, the inclusion

$$\mathcal{C}^\infty(X, \mathcal{A}) = \Psi^{-\infty}(Y/X; \mathcal{A}) \hookrightarrow \Psi^{-\infty}(Y/X; \mathcal{A} \otimes \mathbb{E}),$$

induces a natural isomorphism of $\mathrm{K}_0(\Psi^{-\infty}(Y/X; \mathcal{A} \otimes \mathbb{E}))$ and $\mathrm{K}^0(X; \mathcal{A})$. Therefore we have defined the *analytic index* of any projective family of elliptic pseudodifferential operators.

To see that this fixes the map,

$$\operatorname{ind}_a : \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X); \rho^*\phi^*\mathcal{A}) \longrightarrow \mathrm{K}^0(X, \mathcal{A}) \tag{5.5}$$

we need, as usual, to check homotopy invariance, invariance under bundle isomorphisms and stability. However, this all follows as in the standard case.

Of particular geometric interest are examples arising from projective families of (twisted) Dirac operators. If the fibres of Y are even-dimensional and consistently oriented, let $\mathrm{Cl}(Y/X)$ denote the bundle of Clifford algebras associated to some family of fiber metrics and let $\mathbb{E}$ be a $\mathbb{Z}_2$-graded hermitian Clifford module over Y with unitary Clifford connection $\nabla^{\mathbb{E}}$.

This data determines a family of (twisted) Dirac operators $\eth_{\mathbb{E}}$ acting fibrewise on the sections of $\mathbb{E}$. We can further twist $\eth_{\mathbb{E}}$ by a connection ∇^τ of the line bundle K_τ over $\phi^{-1}(U) \subset Y$ for contractible open subsets $U \subset X$. In this way, we get a projective family of (twisted) Dirac operators $\eth_{\mathcal{A}\otimes\mathbb{E}} \in \Psi^1(Y/X; \mathbb{E} \otimes \mathcal{A})$ which can be viewed as a family of twisted Dirac operators acting on a projective Hilbert bundle $\mathbb{P}(\phi_*(\mathbb{E} \otimes K_\tau))$ over X. Here the local bundle $\phi_*(\mathbb{E} \otimes K_\tau)$ is given by $U \times L^2(\phi^{-1}(U)/U; \mathbb{E} \otimes K_\tau)$ for contractible open subsets $U \subset X$.

The above projective Dirac family can be globally defined as follows. Consider the delta distributional section $\delta_Z^{\mathbb{E},J} \in I^\bullet(Y^{[2]}, J \otimes \mathrm{Hom}(\mathbb{E}) \otimes \Omega_R)$, which is supported on the fibrewise diagonal in $Y^{[2]}$. Let ${}^L\nabla^{\mathbb{E}}$ denote the unitary Clifford connection acting on the left variables, and ∇^J a connection on J which is compatible with the primitive property of J. Then

$$(1 \otimes {}^L\nabla^{\mathbb{E}} + \nabla^J \otimes 1)\delta_Z^{\mathbb{E},J} \in I^{\bullet-1}(Y^{[2]}, J \otimes \mathrm{Hom}(\mathbb{E}) \otimes T^*(Y/X) \otimes \Omega_R),$$

and composition with the contraction given by Clifford multiplication gives

$$c \circ (1 \otimes {}^L\nabla^{\mathbb{E}} + \nabla^J \otimes 1)\delta_Z^{\mathbb{E},J} \in I^{\bullet-1}(Y^{[2]}, J \otimes \mathrm{Hom}(\mathbb{E}) \otimes \Omega_R),$$

which represents the Schwartz kernels of the projective family of (twisted) Dirac operators denoted above by $\eth_{\mathcal{A}\otimes\mathbb{E}}$.

6. The topological index

In this section we define the *topological index* map for the setup in the previous section,

$$\mathrm{ind}_t : \mathrm{K}^0_c(T(Y/X); \rho^*\phi^*\mathcal{A}) \longrightarrow \mathrm{K}^0(X; \mathcal{A}). \tag{6.1}$$

It is defined in terms of Gysin maps in twisted K-theory, which have been studied in the case of torsion twists in [16], which extends routinely to the general case as in [6, 8]. In the particular case that we consider here, there are several simplifications that we shall highlight.

We first recall some functorial properties of twisted K-theory. Let $F : Z \longrightarrow X$ be a smooth map between compact manifolds. Then the pullback map,

$$F^! : \mathrm{K}^0(X, \mathcal{A}) \longrightarrow \mathrm{K}^0(Z, F^*\mathcal{A}),$$

is well defined.

Lemma 6. *There is a canonical isomorphism,*

$$j_! : \mathrm{K}^0(X, \mathcal{A}) \cong \mathrm{K}^0_c(X \times \mathbb{R}^{2N}, \pi_1^*\mathcal{A}),$$

determined by Bott periodicity, where the inclusion $j : X \to X \times \mathbb{R}^{2N}$ is onto the zero section. Here $\pi_1 : X \times \mathbb{R}^{2N} \to X$ is the projection onto the first factor.

Proof. First notice that $\mathrm{K}^\bullet(X, \mathcal{A}) = \mathrm{K}_\bullet(C^\infty(X, \mathcal{A}))$ and $\mathrm{K}^\bullet_c(X \times \mathbb{R}^{2N}, \pi_1^*\mathcal{A}) = \mathrm{K}_\bullet(C^\infty_c(X \times \mathbb{R}^{2N}; \pi_1^*\mathcal{A}))$. But $C^\infty_c(X \times \mathbb{R}^{2N}; \pi_1^*\mathcal{A}) = C^\infty(X, \mathcal{A}) \otimes C^\infty_c(\mathbb{R}^{2N})$. So the lemma follows from Bott periodicity for the K-theory of (smooth) operators algebras. □

For the fiber bundle $\phi : Y \longrightarrow X$ of compact manifolds, we know that there is an embedding $i : Y \longrightarrow X \times \mathbb{R}^N$, cf. [4] §3. The fibrewise differential is an embedding $Di : T(Y/X) \longrightarrow X \times \mathbb{R}^{2N}$ with complex normal bundle $\mathcal{N}$.

Let $\mathcal{A}$ be the smooth Azumaya algebra over X as defined earlier in §3; there is a fixed trivialization of $\phi^*\mathcal{A}$. Let $\mathcal{A}_\mathcal{N}$ be the lift of $\phi^*\mathcal{A}$ to $\mathcal{N}$. Let $\rho : T(Y/X) \longrightarrow Y$ be the projection map. Then since $\phi^*\mathcal{A}$ is trivialized, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{K}^0_c(T(Y/X), \rho^*\phi^*\mathcal{A}) & \xrightarrow{Di_!} & \mathrm{K}^0_c(\mathcal{N}, \mathcal{A}_\mathcal{N}) \\ \cong_{\kappa^{-1}} \big\downarrow & & \big\downarrow \cong \\ \mathrm{K}^0_c(T(Y/X)) & \xrightarrow{Di_!} & \mathrm{K}^0_c(\mathcal{N}), \end{array} \tag{6.2}$$

where $Di_!$ in the lower horizontal arrow is given by $\xi = (\xi^+, \xi^-, G) \longmapsto \pi^*\xi \otimes (\pi^*\mathcal{S}^+, \pi^*\mathcal{S}^-, c(v))$. Here $\xi = (\xi^+, \xi^-)$ is pair of vector bundles over $T(Y/X)$, $G : \xi^+ \longrightarrow \xi^-$ a bundle map between them which is an isomorphism outside a compact subset and $(\pi^*\mathcal{S}^+, \pi^*\mathcal{S}^-, c(v))$ is the usual Thom class of the complex vector bundle $\mathcal{N}$, where π is the projection map of $\mathcal{N}$ and $\mathcal{S}^\pm$ denotes the bundle of half spinors on $\mathcal{N}$. On the the right hand side the the graded pair of vector bundle data is

$$(\pi^*\xi^+ \otimes \pi^*\mathcal{S}^+ \oplus \pi^*\xi^- \otimes \pi^*\mathcal{S}^-, \pi^*\xi^+ \otimes \pi^*\mathcal{S}^- \oplus \pi^*\xi^- \otimes \pi^*\mathcal{S}^+)$$

with map between them being

$$\begin{bmatrix} G & c(v) \\ c(v) & G \end{bmatrix}, \ v \in \mathcal{N}.$$

This is an isomorphism outside a compact subset of $\mathcal{N}$ and defines a class in $\mathrm{K}^0_{\mathrm{c}}(\mathcal{N})$ which is independent of choices, provided the trivialization of $\phi^*(\mathcal{A})$ is kept fixed. Then the usual Thom isomorphism theorem asserts that $Di_!$ is an isomorphism. The upper horizontal arrow is defined in the same way by tensoring with the same Thom class.

Now, $\mathcal{N}$ is diffeomorphic to a tubular neighborhood $\mathcal{U}$ of the image of Y; let $\Phi : \mathcal{U} \longrightarrow \mathcal{N}$ denote this diffeomorphism. Then the induced map in K-theory gives isomorphisms,

$$\Phi^* : \mathrm{K}^0_c(\mathcal{N}) \cong \mathrm{K}^0_c(\mathcal{U}), \qquad \Phi^* : \mathrm{K}^0_c(\mathcal{N}, \mathcal{A}_{\mathcal{N}}) \cong \mathrm{K}^0_c(\mathcal{U}, \Phi^*(\mathcal{A}_{\mathcal{N}})).$$

We will next show that the inclusion $i_{\mathcal{U}} : \mathcal{U} \to X \times R^{2N}$ of the open set $\mathcal{U}$ in $X \times R^{2N}$ induces a natural extension map

$$(i_{\mathcal{U}})_! : \mathrm{K}^0_{\mathrm{c}}(\mathcal{U}, \Phi^*(\mathcal{A}_{\mathcal{N}})) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X \times R^{2N}, \pi_1^*\mathcal{A})$$

To see this, we need to show that the restriction $i^*_{\mathcal{U}}\pi_1^*\mathcal{A}$ is trivialized. Note that $\phi \circ \tau = \pi_1 \circ i_{\mathcal{U}}$, where $\tau : \mathcal{U} \longrightarrow Y$ is equal to the composition, $\lambda \circ \Phi$, and $\lambda : \mathcal{N} \longrightarrow Y$ the projection map. Since $(\phi \circ \tau)^*\mathcal{A} = \tau^*\phi^*\mathcal{A}$ is trivializable because $\phi^*\mathcal{A}$ is trivialized, it follows that $i^*_{\mathcal{U}}\pi_1^*\mathcal{A}$ is trivialized.

We have the following commutative diagram.

$$\begin{array}{ccccc}
\mathrm{K}^0_c(T(Y/X)) & \xrightarrow[\cong]{Di_!} & \mathrm{K}^0_c(\mathcal{N}) & \xrightarrow[\cong]{\Phi^*} & \mathrm{K}^0_c(\mathcal{U}) \\
\Big\downarrow{\scriptstyle \rho^*\kappa_*\ \cong} & & \Big\downarrow{\scriptstyle \cong} & & \Big\downarrow{\scriptstyle \cong} \\
\mathrm{K}^0_c(T(Y/X), \rho^*\phi^*\mathcal{A}) & \xrightarrow[\cong]{Di_!} & \mathrm{K}^0_c(\mathcal{N}, \mathcal{A}_{\mathcal{N}}) & \xrightarrow[\cong]{\Phi^*} & \mathrm{K}^0_c(\mathcal{U}, \Phi^*(\mathcal{A}_{\mathcal{N}})) \\
 & \searrow & & & \Big\downarrow{\scriptstyle (i_{\mathcal{U}})_!} \\
 & & & & \mathrm{K}^0_{\mathrm{c}}(X \times R^{2N}, \pi_1^*\mathcal{A}) \\
 & & \widetilde{\mathrm{ind}}_t & \searrow & \Big\downarrow{\scriptstyle \cong\ j_!^{-1}} \\
 & & & & \mathrm{K}^0_{\mathrm{c}}(X, \mathcal{A}).
\end{array} \tag{6.3}$$

The composition of the maps in the diagram above defines the Gysin map in twisted K-theory,

$$Di_! : \mathrm{K}^0_{\mathrm{c}}(T(Y/X)) \longrightarrow \mathrm{K}^0_{\mathrm{c}}(X \times \mathbb{R}^{2N}, \pi_1^*\mathcal{A}).$$

Here we have used the fact that since $\pi = \pi_1 \circ i$ it follows that $Di^*\pi_1^*\mathcal{A} = \rho^*\phi^*\mathcal{A}$ is trivialized. Now define the ***topological index***, as the map

$$\mathrm{ind}_t = j_!^{-1} \circ Di_! : \mathrm{K}^0_{\mathrm{c}}(T(Y/X)) \longrightarrow \mathrm{K}^0(X; \mathcal{A}), \tag{6.4}$$

where we apply the Thom isomorphism in Lemma 6 to see that the inverse $j_!^{-1}$ exists. We also note that $\widetilde{\mathrm{ind}}_t \circ \rho^*\kappa_* = \mathrm{ind}_t$, consistent with the corresponding analytic indices.

The source is untwisted since $\mathcal{A}$ is trivialized by κ, as an Azumaya bundle, when pulled back to Y. The identification of twisted and untwisted K-theory in (16) depends on the choice of trivialization (6) but then so does the Azumaya bundle and these choices do not change the index map $\widetilde{\operatorname{ind}}_{\mathrm{t}}$.

7. Twisted Chern character

First we recall an explicit formula for the odd Chern character in the untwisted case. For any compact manifold (of positive dimension), Z, the group of invertible, smoothing, perturbations of the identity operator

$$G^{-\infty}(Z) = \{a \in \Psi^{-\infty}(Z); \exists\ (\operatorname{Id}+a)^{-1} = \operatorname{Id}+b,\ b \in \Psi^{-\infty}(Z)\} \tag{7.1}$$

is classifying for odd K-theory. So there is a canonical identification of the odd K-theory of a compact manifold X with the (smooth) homotopy classes of (smooth) maps

$$\mathrm{K}^1(X) = [X; G^{-\infty}(Z)]. \tag{7.2}$$

The odd Chern character is then represented in deRham cohomology by the pull-back of the universal Chern character on $G^{-\infty}(Z)$:

$$\begin{aligned} \operatorname{Ch} &= \sum_{k=0}^{\infty} c_k \operatorname{tr}\left((A^{-1}dA)^{2k+1}\right) \\ &= -\frac{1}{2\pi i}\int_0^1 \operatorname{tr}\left((A^{-1}dA)\exp(\frac{t(1-t)}{2\pi i}(A^{-1}dA)^2)\right) dt,\ A = \operatorname{Id}+a. \end{aligned} \tag{7.3}$$

Here $dA = da$, as for finite dimensional Lie groups, is the natural identification of $T_aG^{-\infty}$ with $\Psi^{-\infty}(Z)$ coming from the fact that $G^{-\infty}(Z)$ is an open (and dense) set in $\Psi^{-\infty}(Z)$. Thus for an odd K-class

$$\begin{gathered} a : X \longrightarrow G^{-\infty}(Z),\ \operatorname{Ch}([a]) = [a^* \operatorname{Ch}] \in \mathrm{H}^{\mathrm{odd}}(X), \\ a^* \operatorname{Ch} = -\frac{1}{2\pi i}\int_0^1 \operatorname{tr}\left((\operatorname{Id}+a)^{-1}da \exp(\frac{t(1-t)}{2\pi i}((\operatorname{Id}+a)^{-1}da)^2\right) dt \end{gathered} \tag{7.4}$$

where now the differential can be interpreted in the usual way for functions valued in the fixed vector space $\Psi^{-\infty}(Z)$.

For any fiber bundle $\phi : Y \longrightarrow X$, with typical fiber Z, $\mathrm{K}^1(X)$ is also naturally identified with the abelian group of homotopy classes of sections of the bundle of groups over X with fiber $G^{-\infty}(Z_x)$ at $x \in X$. That is, the twisting by the diffeomorphism group does not affect this property. The formula (7.4) can be extended to this geometric setting by choosing a connection on ϕ, i.e. a lift of vector fields from X to Y. Indeed, such a connection can be identified as a connection on the bundle $\mathcal{C}^\infty(Y/X)$, with fibres $\mathcal{C}^\infty(Z_x)$ (and space of sections $\mathcal{C}^\infty(Y)$), as a differential operator

$$\begin{gathered} \nabla : \mathcal{C}^\infty(Y) \longrightarrow \mathcal{C}^\infty(Y; \phi^*T^*X), \\ \nabla(hg) = (dh)g + h\nabla g,\ h \in \mathcal{C}^\infty(X),\ g \in \mathcal{C}^\infty(Y). \end{gathered} \tag{7.5}$$

The curvature of such a connection (extended to a superconnection), is a first order differential operator on the fibres $w = \nabla^2/2\pi i \in \operatorname{Diff}^1(Y/X; \mathbb{C}, \phi^*\Lambda^2 X)$ from the trivial bundle to the 2-form bundle lifted from the base. The connection on Y

induces a connection on $\Psi^{-\infty}(Y/X)$, as a bundle of operators on $\mathcal{C}^\infty(Y/X)$, acting by conjugation and then (7.4) is replaced by

$$\begin{gathered}\operatorname{Ch}(A) = \\ -\frac{1}{2\pi i}\int_0^1 \operatorname{tr}\left((A^{-1}\nabla A)\exp\left((1-t)w + tA^{-1}wA + \frac{t(1-t)}{2\pi i}(A^{-1}\nabla A)^2\right)\right)dt, \\ A : X \longmapsto G^{-\infty}(Y/X),\ \pi\circ A = \operatorname{Id}.\end{gathered}\tag{7.6}$$

Note that any such section is homotopic to a section which is a finite rank perturbation of the identity, in which case (7.6) becomes the more familiar formula. The same conclusions, and formula hold, if the bundle of groups of smoothing perturbations of the identity acting on a vector bundle over Y, $G^{-\infty}(Y/X;E)$, is considered, provided the connection (and curvature) are lifted to a connection on E.

Note that (7.6) can also be considered as the pull-back of a universal form on the total space of the fibration $G^{-\infty}(Y/X)$. It then has the property that restricted to a fiber, so that the curvature vanishes, one recovers the original form in (7.3).

The case of immediate interest arises from a circle bundle $p : \tilde{L} \longrightarrow X$. As explained in §3 we consider the fiber product $\tilde{L}^{[2]}$ fibred over $\tilde{L}$ with the fibres taken to be in the second factor, with the smoothing operators acting on sections of J. Of course these operators are acting on the restriction of this line bundle to each fiber, which is a circle, so they can always be identified on each fiber with ordinary smoothing operators. On the other hand J has the primitivity property of Lemma 2 which allows us to identify the smoothing operators on sections of J on the fibres of $\tilde{L}^{[2]}$ with $\mathcal{C}^\infty(\tilde{L}^{[3]};\pi_F^*J)$ as in Proposition 3. An explicit fiber density factor is not needed since this is supplied naturally by the Hermitian structure.

Proposition 6. *Suppose $a \in \mathcal{C}^\infty(\tilde{L}^{[2]};J)$ is such that $A = \operatorname{Id}+a$ is everywhere invertible over X. Then the odd Chern character of* $\operatorname{Id}+a$, *as a form on $\tilde{L}$ computed with respect to a unitary connection on L and the primitive connection of Lemma 1 on J, with combined curvature Ω,*

$$\begin{gathered}\operatorname{Ch}_{\mathcal{A}}(A) = \\ -\frac{1}{2\pi i}\int_0^1 \operatorname{tr}\left((A^{-1}\nabla A)\exp\left((1-t)\Omega + tA^{-1}\Omega A + t(1-t)(A^{-1}\nabla A)^2\right)\right)dt \\ \in \mathcal{C}^\infty(\tilde{L};\Lambda^{odd})\end{gathered}\tag{7.7}$$

is closed and satisfies the conditions in (1.10).

Proof. That the Chern form (7.7) is closed follows from the standard properties. To see the other stated properties, we choose a section of $\tilde{L}$ over an open set $U \subset X$ over which u has a smooth logarithm and set $f = \frac{1}{2\pi i}\log u$. In terms of the induced trivializations

$$\tilde{L}_U = U\times\mathbb{S},\ \tilde{L}_U^{[2]} = U\times\mathbb{S}\times\mathbb{S},\tag{7.8}$$

let the fiber variables be θ_1 and θ_2. The operators are acting in the θ_2 variable and the lifted connection on $\tilde{L}$ as a fibration over $\tilde{L}$ is therefore

$$\nabla = d_x + d_{\theta_1} + \overline{\gamma}\partial_{\theta_2}\tag{7.9}$$

where $\overline{\gamma} \in \mathcal{C}^\infty(U;\Lambda)$ is the local connection form for L. The corresponding connection on $\mathcal{C}^\infty(\tilde{L}^{[2]}/\tilde{L};J)$ in terms of this trivialization and of the connection on J from

Lemma 1

$$\nabla_J = d_x + d_{\theta_1} + \overline{\gamma}\partial_{\theta_2} + f d\theta_1 - \overline{\gamma} f. \tag{7.10}$$

The curvature is

$$\Omega = \nabla_J^2/2\pi i = \overline{\beta}(\partial_{\theta_2} - f) + \frac{1}{2\pi i}\overline{\alpha}\wedge\gamma,\ \gamma = d\theta_1 + \overline{\gamma}. \tag{7.11}$$

In terms of this local trivialization $a = a(x, \theta_2, \theta_3)$ is independent of the first (parameter) fiber variable. Inserting (7.11) into (7.7) observe that the two terms in (7.11) commute so

$$\mathrm{Ch}_{\mathcal{A}}(A) = e^{\frac{\overline{\alpha}\wedge\gamma}{2\pi i}} v,\ v \in \mathcal{C}^\infty(U; \Lambda^{\mathrm{odd}}) \tag{7.12}$$

satisfies the conditions of (1.10). □

Note that under a deck transformation of $\hat{X}$, i.e. integral shift of f by $n \in \mathbb{N}$, each term undergoes conjugation by $\exp(in\theta)$ and the Chern form itself is therefore unchanged.

It follows from Proposition 1 that $\mathrm{Ch}_{\mathcal{A}}(A)$ defines an element in the twisted cohomology of X, given explicitly by the form v in (7.12). Although the proof above is written out for sections of J over $\tilde{L}^{[2]}$ the passage to matrix-valued sections is merely notational, so it applies essentially unchanged to elements of

$$\begin{gathered} G(X; \mathcal{A} \otimes M(N, \mathbb{C})) = \\ \{a \in \mathcal{C}^\infty(\tilde{L}^{[2]}; J \otimes M(N, \mathbb{C}); \mathrm{Id}_{N\times N} + a(x) \text{ is invertible for all } x \in X\}. \end{gathered} \tag{7.13}$$

Lemma 7. *The Chern form* (7.7) *descends to represent the twisted Chern character*

$$G(X; \mathcal{A} \otimes M(N, \mathbb{C}))/\!\sim\, = \mathrm{K}^1(X; \mathcal{A}) \longrightarrow \mathrm{H}^{odd}(X; \delta) \tag{7.14}$$

where the equivalence relation on invertible matrix-valued sections of the Azumaya bundle is homotopy and stability.

Proof. The invariance of the twisted cohomology class under stabilization follows directly from the definition. Invariance under homotopy follows as usual from the fact that the construction is universal and the form is closed, so is closed for a homotopy when interpreted as a family over $X \times [0, 1]$ and this proves the invariance of the cohomology class.

It also follows directly from the definition that the twisted Chern character behaves appropriately under the Thom isomorphism for a complex (or symplectic) vector bundle $w : W \longrightarrow X$. That is, there is a commutative diagram with horizontal isomorphisms

$$\begin{array}{ccc} \mathrm{K}^1(X; \mathcal{A}) & \xrightarrow{\boxtimes b} & \mathrm{K}^1(W; w^*\mathcal{A}) \\ {\scriptstyle \mathrm{Ch}_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \mathrm{Ch}_{\mathcal{A}}} \\ \mathrm{H}^{\mathrm{odd}}(X; \delta) & \xrightarrow{\wedge\,\mathrm{Td}} & \mathrm{H}^{\mathrm{odd}}_{\mathrm{c}}(W; w^*\delta) \end{array} \tag{7.15}$$

As in the case of the bundle of groups $G^{-\infty}(Y/X)$ the form (7.7) is again the pull-back from the total space of the bundle of groups $G^{-\infty}(X; \mathcal{A})$ of invertible sections of the unital extension of the Azumaya bundle and then restricting this universal form to a fiber one again recovers the standard odd Chern character in (7.3). This is enough to show that the Chern form here does represent the twisted

Chern character as widely discussed in the literature, for instance recently by Atiyah and Segal in [3]. Namely they remark that the Chern character as they describe it (in the even case), which is determined by universality under pull-back from the twisted PU bundle over $K(\mathbb{Z},3)$, is actually determined by its pull-back to the 3-sphere. The PU bundle over $\mathbb{S}^3$ with generating DD class is trivial over points and so can be transferred to $(0,\pi)\times\mathbb{S}^2$ to be trivial outside a compact set and thence to $\mathbb{S}\times\mathbb{S}^2$ where it reduces to the twisted bundle again with generating DD class. As shown in [3], the universal twisted Chern character over the 3-sphere is determined by multiplicativity and the fact that it restricts to the standard Chern character on the fibres over points. The odd case follows by suspension so the deRham version of the Chern character above does correspond with more topological definitions. □

We need some extensions of this discussion of the odd twisted Chern character. In particular we need to discuss the even case. However, the context needs to be broadened to cover operators on the fibres of a trivializing bundle Y as in (3) – (6). Finally the relative case is needed for the discussion of the Chern character of the symbol and the index formula in twisted cohomology. Fortunately these are all straightforward generalizations of the untwisted case.

We start with the extension of the odd twisted Chern character to the more general geometric case under discussion here. Thus, instead of being over $\tilde{L}^{[2]}$, the bundle J is defined over $Y^{[2]}$. Still, when lifted to the fiber product

$$\tilde{Y}^{[2]} = \tilde{L}\times_X Y^{[2]}, \tag{7.16}$$

J is reduced to a the exterior tensor product

$$p^*J = \tilde{J}\boxtimes\tilde{J}' \text{ over } \tilde{Y}^{[2]} \tag{7.17}$$

where $\tilde{J}$ is a line bundle over $\tilde{Y}=\tilde{L}\times_X Y$. Namely, there is a character property for $s: Y^{[2]}\longrightarrow \mathrm{U}(1)$, which is determined by the trivialization of L over Y, when lifted to $\tilde{Y}^{[2]}$:

$$s(z_1,z_2)=\tilde{s}(z_1,\tilde{l})\tilde{s}(z_2,\tilde{l})^{-1},\ \ \tilde{s}:\tilde{Y}\longrightarrow \mathrm{U}(1). \tag{7.18}$$

Here $\tilde{s}$ is fixed by the demand that it intertwine the trivializations over $\tilde{L}$ and Y. Thus, using $\tilde{s}$ to define $\tilde{J}$ by the same procedure as previously used to define J, (7.17) follows.

From this point the discussion proceeds as before. That is, the Azumaya bundle $\mathcal{A}_Y$ acting on the fibres of Y over X lifts to $\tilde{Y}$, acting on the same fibres but now over $\tilde{L}$, into a subalgebra of $\Psi^{-\infty}(\tilde{Y}/\tilde{L};\tilde{J})$. Then, as above, the odd Chern character for invertible sections of the unital extension of the Azumaya bundle is a differential form on $\tilde{L}$ satisfying (1.10) and so defines the twisted odd Chern character in this more general geometric setting.

Next, consider the even twisted Chern character. To do so, recall that for a complex vector bundle E embedded as a subbundle of some trivial $\mathbb{C}^N$ over a manifold X the curvature, and Chern character, can be written in terms of an idempotent e projecting onto the range as the 2-form valued homomorphism $\omega_E = e(de)(1-e)(de)e/2\pi i$. There is a similar formula if E is embedded in a possibly non-trivial bundle F with connection ∇_F which is projected onto E using e. The same formula applies in the case of a subbundle of $\mathcal{C}^\infty(Y/X;E)$ given by a family of idempotents e. In the untwisted case, the K-theory of X can be represented by formal differences of finite rank idempotents in the fibres of $\mathcal{C}^\infty(Y/X;E)$, giving finite dimensional bundles. In general, in the twisted case, the K-theory is interpreted as the C^* K-theory of a non-unital algebra (the completion of $\mathcal{A}$ in the compacts), it is necessary to take pairs of infinite rank idempotents in $\mathbb{C}^N \otimes \mathcal{A}^\dagger$ with differences valued in $\mathbb{C}^N \otimes \mathcal{A}$. In fact it is enough to take single idempotents in $\mathbb{C}^N \otimes \mathcal{A}^\dagger$ with constant unital part $e_0 \in M(N, \mathbb{C})$ and consider the formal difference $e \ominus e_0$ to generate the K-theory. For the untwisted case, the usual Chern character is given by

$$\operatorname{tr}(\exp(\frac{\omega_E^2}{2\pi i}) - e_0) \tag{7.19}$$

as can be shown by suspension from the odd case if desired. Here all terms in $\Lambda^{>0}$ involve a derivative of e and hence are smoothing, as is the normalized term of form degree zero, so the trace functional can be applied.

To carry this discussion of the even Chern character to the twisted case, we can proceed precisely as above. Namely, given an idempotent section, e, of $\mathbb{C}^N \otimes \mathcal{A}_Y^\dagger$ as a bundle over X with constant unital term e_0 one can compute the Chern form (7.19) after lifting the idempotent to $\mathbb{C}^N \otimes \Psi^{-\infty}(\tilde{Y}/\tilde{L}; \tilde{J} \otimes \mathbb{C}^N)$ as discussed above. Then, for the same reason, the form satisfies (1.10) and defines the even Chern character as a twisted deRham form on X.

The final extension is to the relative case to handle the Chern character of the symbol of a pseudodifferential operator. As discussed in [1] for any real vector bundle $W \longrightarrow Y$ (here applied to $T^*(Y/X)$) the compactly supported cohomology of W can be obtained directly as from the relative deRham complex of SW, the sphere bundle of W, and Y. This involves the same odd Chern class on SW (which is no longer closed) and the even Chern class on Y which 'corrects' the failure of the odd form to be closed. The extension to the twisted case just combines the two cases discussed above; this is briefly considered in §10.

8. Semiclassical quantization

To avoid the usual complications which arise in the proof of the index theorem, especially concerning the multiplicativity of the analytic index – although they are no worse in the present twisted setting than the standard one – we introduce another definition of the index map using semiclassical pseudodifferential operators. This approach is discussed in more detail in [17] but the underlying notion of a semiclassical family of pseudodifferential operators is well established in the literature [12]. The method of 'asymptotic morphism' of Connes and Higson is closely related to the notion of semiclassical limit.

Proposition 7. *Let $\psi : M \longrightarrow B$ be a fibre bundle of possibly non-compact manifolds then the modules*

$$\Psi^{\ell}_{\mathrm{c,scl}}(M/B; \mathbb{E}) \subset \mathcal{C}^\infty((0,1)_\epsilon; \Psi^{\ell}_{\mathrm{c}}(M/B; \mathbb{E}))$$

of semiclassical families of classical, uniformly compactly-supported, pseudodifferential operators on the fibres of ψ are well defined for any $\mathbb{Z}_2$-graded bundle $\mathbb{E}$, have a global multiplicative exact symbol sequence

$$0 \longrightarrow \epsilon\Psi^{\ell}_{\mathrm{c,scl}}(M/B; \mathbb{E}) \hookrightarrow \Psi^{\ell}_{\mathrm{c,scl}}(M/B; \mathbb{E}) \xrightarrow{\sigma_{\mathrm{scl}}} S^{\ell}_{\mathrm{c}}(T^*(M/B); \hom(\mathbb{E})) \longrightarrow 0 \tag{8.1}$$

and completeness property

$$\bigcap_j \epsilon^j \Psi^\ell_{\mathrm{c,scl}}(M/B;\mathbb{E}) = \dot{\mathcal{C}}^\infty([0,1);\Psi^\ell_{\mathrm{c}}(M/B;\mathbb{E})). \tag{8.2}$$

Note that the space of functions on the right in (8.1) consists of the *global* classical symbols on $T^*(M/B)$, with compact support in the base M, not the quotient by the symbols of order $\ell-1$. The space on the right in (8.2) consists of the smooth families of pseudodifferential operators with uniformly compact support in the usual sense, depending smoothly on the additional parameter $\epsilon \in [0,1)$ down to $\epsilon = 0$ where they vanish with all derivatives. Thus, by iteration, the semiclassical symbol in (8.1) captures the complete behaviour of these operators as $\epsilon \downarrow 0$.

To define the semiclassical index maps, one for each parity, we only need the smoothing operators of this type, for $\ell = -\infty$; indeed this is the key to their utility. In this special case the Schwartz kernels of the operators are easily described explicitly. Namely they correspond to the subspace of $\mathcal{C}^\infty((0,1)\times M^{[2]}_\psi;\mathrm{Hom}(\mathbb{E})\otimes\Omega_R))$ consisting of those functions which have support in some set $(0,1)\times K$ with $K \subset M^{[2]}_\psi$ compact, which tend to 0 rapidly with all derivatives as $\epsilon \downarrow 0$ in any closed set in $M^{[2]}_\psi$ disjoint from the diagonal and which near each point of the diagonal take the from

$$\epsilon^{-d} K(\epsilon, b, z, z', \frac{z-z'}{\epsilon})|dz'| \tag{8.3}$$

where K is a smooth bundle homomorphism which is uniformly Schwartz in the last variable and d is the fiber dimension.

As with usual pseudodifferential operators, there is no obstruction to defining $\Psi^\ell_{\mathrm{scl}}(Y/X;\mathcal{A}\otimes\mathbb{E})$ either by transferring the kernels directly to sections of $J\otimes\mathrm{Hom}(\mathbb{E})$ over $Y^{[2]}$ or by using the local form (3.4).

Proposition 8. *The space of invertible elements in the unital extension of the semiclassical twisted smoothing operators defines an odd index map via the diagram*
(8.4)

$$\begin{array}{ccc} & \bigcup_N \left\{(A,B)\in\Psi^{-\infty}_{\mathrm{scl}}(M/B;\mathcal{A}\otimes\mathbb{C}^N);(\mathrm{Id}+A)^{-1}=\mathrm{Id}+B\right\} & \\ \swarrow^{[\mathrm{Id}+\sigma_{\mathrm{scl}}(A)]} & & \searrow^{[(\mathrm{Id}+A)\big|_{\epsilon=\frac12}]} \\ \mathrm{K}^1_{\mathrm{c}}(T^*(Y/X)) & \xrightarrow{\quad\mathrm{ind}^1_{\mathrm{scl}}\quad} & \mathrm{K}^1_{\mathrm{c}}(X;\mathcal{A}). \end{array}$$

Proof. The space on the top line in (8.4) consists of the invertible perturbations of the identity by semiclassical smoothing operators, with the inverse of the same form. Thus it follows that $\mathrm{Id}+\sigma_{\mathrm{scl}}(A)$ is invertible as a smooth family of $N\times N$ matrices over $T^*(Y/X)$, reducing to the identity at infinity. It therefore defines an element of odd K-theory giving the map on the left. Conversely, the invertibility of $\mathrm{Id}+a$ for a symbol a implies, using the exactness of the symbol sequence, that $\mathrm{Id}+A$, where $\sigma_{\mathrm{scl}}(A) = a$, is invertible at least for small ϵ. Modifying the semiclassical family to remain invertible for $\epsilon \in (0,1)$ shows that this map is surjective. The map on the right, defined by restriction to $\epsilon = \frac{1}{2}$ (or any other positive value) immediately gives an element of the odd twisted K-theory of the base.

To see that the 'odd semiclassical index', or push-forward map, is defined from this diagram it suffices to note that the 'quantized' class on the right only depends on the class on the left up to homotopy and stability, which as usual follows directly from the properties of the algebra. ☐

For this odd index there is a companion even index map. Recall that a compactly supported K-class can be defined by a smooth map into $N \times N$ matrices which takes values in the idempotents and is constant outside a compact set, where the class can be identified with the difference of the projection and the limiting constant projection.

Proposition 9. *If* $a \in \mathcal{C}^\infty_c(T^*(Y/X);\mathbb{C}^N)$ *is such that* $\Pi_\infty + a$ *takes values in the idempotents, where* $\Pi_\infty \in M(N,\mathbb{C})$*, then* a *has a semiclassical quantization*

$$A \in \Psi^{-\infty}_{\mathrm{scl}}(Y/X;\mathcal{A}\otimes\mathbb{C}^N),\ \sigma_{\mathrm{scl}}(A) = a, \tag{8.5}$$

such that $(\Pi_\infty + A)^2 = \Pi_\infty + A$ *and this leads to a well-defined even semiclassical index map*

$$\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) \underset{\mathrm{ind}^0_a}{\overset{\mathrm{ind}^0_{\mathrm{scl}}}{\underset{=}{\rightrightarrows}}} \mathrm{K}^0_{\mathrm{c}}(X;\mathcal{A}) \tag{8.6}$$

analogous to (8.4) *and as indicated, equal to the analytic index as defined in* §*5.*

Proof. Certainly a quantization of a exists by the surjectivity of the symbol map. Moreover the idempotent $\Pi_\infty + a$ can be extended to a 'formal' idempotent, meaning that, using the symbol calculus, the quantization can be arranged to be idempotent up to infinite order error at $\epsilon = 0$. The error terms of order $-\infty$ in the semiclassical smoothing algebra are simply smoothing operators vanishing to infinite order with ϵ. Use of the functional calculus then allows one to perturb the quantization by such a term to give a true idempotent for small $\epsilon > 0$. Then stretching the parameter arranges this for $\epsilon \in (0,1)$. The pair of this projection, for any $\epsilon > 0$, and the limiting constant projection, Π_∞ defines a K-class. The existence of the map (8.6) then follows in view of the homotopy invariance and stability of this construction.

To see the equality with the analytic index as previously defined is the major step in the proof of the index theorem. This amounts to a construction giving both this semiclassical index map and the usual analytic index map at the same time. The two index maps, semiclassical and analytic are based on two different models for the compactly supported K-theory of $T^*(Y/X)$ – or more generally of a vector bundle W. The first reduces to the set of projection-valued smooth maps $W \longrightarrow M(N,\mathbb{C})$ into matrices which are constant outside a compact set. The second is defined in terms of triples, consisting of a pair of vector bundles over the base together with an isomorphism between their lifts to S^*W.

These two models can be combined into a larger one, in which the set of objects are triples (E,F,a) where E and F are vector bundles over $\overline{W}$, the radial compactification of W, given directly as smooth projection-valued matrices into some $\mathbb{C}^N$ and where a intertwines these two smooth families of projections over S^*W, the boundary of the radial compactification. The equivalence relation $(E_1,F_1,a_1) \simeq (E_2,F_2,a_2)$ is generated by isomorphisms, meaning smooth intertwinings of E_1, and E_2 and of F_1 and F_2 over $\overline{W}$ which also intertwine the isomorphisms over S^*W, the boundary, plus stability. This again gives $\mathrm{K}_{\mathrm{c}}(W)$.

Standard arguments show that any such class in this general sense is equivalent to an 'analytic class' in which the bundles are lifted from the base, or a 'semiclassical class' in which the projections are constant outside a compact set and the isomorphism between them is the identity – in fact the second projection can be taken to be globally constant. Moreover equivalence is preserved under these reductions.

Using these more general triples a combined analytic-semiclassical quantization procedure may be defined by first taking semiclassical quantizations of the projections E, F to actual semiclassical families P, Q which are projections; the classical symbols of these projections can be chosen to be independent of ϵ. This is again the standard argument for quantizations of idempotents which is outlined above. Then the isomorphism a can be quantized to a pseudodifferential operator A in the ordinary sense but this can be chosen to satisfy $AP(\frac{1}{2}) = A = Q(\frac{1}{2})A$ so it 'acts between' the images of $P(\frac{1}{2})$ and $Q(\frac{1}{2})$. This is accomplished by choosing some A' with symbol a and replacing it by the 'Toeplitz operator' $A = Q(\frac{1}{2})A'P(\frac{1}{2})$ which necessarily has the same symbol.

Then A is relatively elliptic, in the sense that it has a parametrix B satisfying $BQ(\frac{1}{2}) = B = P(\frac{1}{2})B$ and with $AB - Q(\frac{1}{2})$ and $BA - P(\frac{1}{2})$ smoothing operators. The analytic-semiclassical index can now be defined using the using the same formula as the analytic index above. That it is well-defined involves the standard homotopy and stability arguments.

Finally then this map clearly reduces to the analytic and semiclassical index maps on the corresponding subsets of data and hence these two maps must be equal. The introduction of the Dixmier-Douady twisting makes essentially no difference to these constructions so the equality in (8.6) follows. □

Proposition 10. *The appropriate form of Bott periodicity can be proved directly giving commutative diagrams*

$$\begin{array}{ccc} \mathrm{K}^1_{\mathrm{c}}(T^*(Y/X)) & \longrightarrow & \mathrm{K}^0_{\mathrm{c}}(T^*(Y\times\mathbb{R})/(X\times\mathbb{R})) \\ \downarrow{\scriptstyle \mathrm{ind}^1_{\mathrm{scl}}} & & \downarrow{\scriptstyle \mathrm{ind}^0_{\mathrm{scl}}} \\ \mathrm{K}^1(X;\mathcal{A}) & \longrightarrow & \mathrm{K}^0(X\times\mathbb{R};\mathcal{A}) \end{array} \tag{8.7}$$

where the horizontal maps are the clutching construction and

$$\begin{array}{ccc} \mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) & \longrightarrow & \mathrm{K}^1_{\mathrm{c}}(T^*(Y\times\mathbb{R})/(X\times\mathbb{R})) \\ \downarrow{\scriptstyle \mathrm{ind}_{\mathrm{scl}}\,1^0} & & \downarrow{\scriptstyle \mathrm{ind}^1_{\mathrm{scl}}} \\ \mathrm{K}^0(X;\mathcal{A}) & \longrightarrow & \mathrm{K}^1(X\times\mathbb{R};\mathcal{A}) \end{array} \tag{8.8}$$

where the inverses of the horizontal isomorphism are the Toeplitz index maps.

Corollary 1. *To prove the equality of the analytic and topological index maps it suffices to prove the equality of the odd semiclassical and odd topological index maps.*

Proof. Suppose we have proved the equality of odd semiclassical and odd topological index maps

$$\mathrm{K}^1_{\mathrm{c}}(T^*(Y/X)) \underset{\mathrm{ind}^1_{\mathrm{t}}}{\overset{\mathrm{ind}^1_{\mathrm{scl}}}{\rightrightarrows}} \mathrm{K}^1_{\mathrm{c}}(X;\mathcal{A}). \qquad (=) \tag{8.9}$$

Both the topological and the semiclassical index maps give commutative diagrams as in Proposition 10, so it follows that the more standard, even, versions of these maps are also equal. ☐

Lemma 8. *For an iterated fibration of manifolds*

(8.10)
$$\begin{array}{ccc} Z' & \text{---} & M' \\ & & \downarrow \psi \\ Z & \text{---} & M \\ & & \downarrow \phi \\ & & B \end{array}$$

the semiclassical index gives a commutative diagram

(8.11)
$$\begin{array}{ccc} \mathrm{K}^1_c(T^*(M'/Y)) & \xrightarrow{\mathrm{ind}^1_{\mathrm{scl}}} & \mathrm{K}^1_c(T^*(M/Y)) \\ \downarrow \mathrm{ind}^1_{\mathrm{scl}} & \swarrow \mathrm{ind}^1_{\mathrm{scl}} & \\ \mathrm{K}^1_c(Y) & & \end{array}$$

where the map on the top right is the semiclassical index map for the fibration of M' over M pulled back to $T^*(M/Y)$.

Proof. The commutativity of (8.11) follows from the use of a double semiclassical quantization, with different parameters in the two fibres (see the extensive discussion in [17]). ☐

Lemma 9. *For any complex, or real-symplectic, vector bundle W over a manifold X the semiclassical index implements the Thom isomorphism*

(8.12)
$$\mathrm{K}^1_c(W) \underset{\mathrm{Thom}}{\overset{\mathrm{ind}^1_{\mathrm{scl}}}{\underset{\longrightarrow}{\overset{\longrightarrow}{=}}}} \mathrm{K}^1_c(X).$$

Proof. This again follows from the use of semiclassical quantization in the 'isotropic' of (pseudodifferential) Weyl algebra of operators on a symplectic vector space. The resulting symbol map is shown, in [17], to be an isomorphism using the argument of Atiyah. Since the Thom map constructed this way is homotopy invariant it applies to to case of a complex vector bundle where the 'positive' sympectic structure on the underlying real bundle is fixed up to homotopy. ☐

9. The index theorem

The odd topological index is defined as the composite map arising from an embedding so we wish to prove the commutativity of the diagram

$$
\begin{array}{ccccc}
\mathrm{K}^1_{\mathrm{c}}(T^*(Y/X)) & \longrightarrow & \mathrm{K}^1_{\mathrm{c}}(T^*(\Omega/X);\mathcal{A}) & \xrightarrow{\iota^*} & \mathrm{K}^1_{\mathrm{c}}(T^*(\mathbb{R}^M)\times X);\mathcal{A}) \\
 & \searrow^{\mathrm{ind}_{\mathrm{scl}}} & & \searrow^{\mathrm{ind}_{\mathrm{scl}}} & \mathrm{ind}_{\mathrm{scl}} \downarrow = \downarrow \mathrm{Thom} \\
 & & & & \mathrm{K}^1_{\mathrm{c}}(X;\mathcal{A}).
\end{array}
\tag{9.1}
$$

Here Ω is a collar neighbourhood of Y embedded in $\mathbb{R}^M$, so is isomorphic to the normal bundle to Y. Thus, it suffices to prove commutativity in three places. The last of these is equality of the two maps on the right, that the semiclassical index map implements the Thom isomorphism (or in this trivial case, Bott periodicity). The second is 'excision' which is immediate from the definition of the semiclassical index. The first commutativity, for the triangle on the left corresponds to multiplicativity of the semiclassical index which in this case reduces to (a special case of) Lemma 8.

This leads to the main theorem. Here we tacitly identify the tangent and cotangent bundles via a Riemannian metric.

Theorem 1 (The index theorem in K-theory)**.** *Let $\phi : Y \longrightarrow X$ be a fiber bundle of compact manifolds, together with the other data in* (3) – (6)*. Let $\mathcal{A}$ be the smooth Azumaya bundle over X as defined in §3 and $P \in \Psi^{\bullet}(Y/X, \mathcal{A}\otimes\mathbb{E})$ be a projective family of elliptic pseudodifferential operators acting on the projective Hilbert bundle $\mathbb{P}(\phi_*(\mathbb{E}\otimes K_\tau))$ over X, with symbol $p \in \mathrm{K}_c(T(Y/X))$, then*

$$\mathrm{ind}_a(P) = \mathrm{ind}_t(p) \in K^0(X, \mathcal{A}). \tag{9.2}$$

10. The Chern character of the index

As discussed above, the index map in K-theory can be considered as acting on the untwisted K-theory, with compact supports, of $T^*(Y/X)$, via the identification with the (trivially) twisted K-theory coming from the original choice of data (3) – (6). The Chern character for the symbol class in the standard setting,

$$\mathrm{K}^0_{\mathrm{c}}(T^*(Y/X)) \longrightarrow \mathrm{H}^{\mathrm{even}}_{\mathrm{c}}(T^*(Y/X)) \tag{10.1}$$

can be represented explicitly in terms of symbol data and connections in a relative version of the formulæ in §7 following Fedosov [11]. A K-class is represented by bundles (E_+, E_-) over Y and an elliptic symbol a identifying them over $S^*(Y/X)$. It is convenient to use the relative interpretation of the cohomology from [1]. Thus one can take the explicit representative

$$
\begin{gathered}
\mathrm{Ch}([(E_+,E_-,a)] = (\widetilde{\mathrm{Ch}}(a), \mathrm{Ch}(E_+) - \mathrm{Ch}(E_-)),\\
\widetilde{\mathrm{Ch}}(a) = -\frac{1}{2\pi i}\int_0^1 \mathrm{tr}\left(a^{-1}(\nabla a)e^{w(t)}\right) dt \text{ where}\\
w(t) = (1-t)F_+ + ta^{-1}F_-a + \frac{1}{2\pi i}t(1-t)(a^{-1}\nabla a)^2 \text{ and}\\
\mathrm{Ch}(E_\pm) = \mathrm{tr}\exp(F_\pm/2\pi i), \quad F_\pm = \nabla_\pm^2.
\end{gathered}
\tag{10.2}
$$

Here $\nabla_\pm$ are connections on $E_\pm$ over Y and ∇ is the induced connection on $\hom(E_+, E_-)$ lifted to $S^*(Y/X)$. Note that the underlying relative complex is the direct sum of the deRham complexes with differential

$$\mathcal{C}^\infty(S^*(Y/X);\Lambda^*)\oplus\mathcal{C}^\infty(Y;\Lambda^*),\quad D=\begin{pmatrix} d & \pi^* \\ 0 & -d\end{pmatrix}. \tag{10.3}$$

In our twisted case, as shown in §7 the line bundle J over $Y^{[2]}$ decomposes as $\tilde J\boxtimes\tilde J'$ when lifted to $\tilde Y^{[2]}$ which has an additional fiber factor of $\tilde L$. The discussion of the Chern character therefore carries over directly to this relative setting.

Proposition 11. *For any element of* $\mathrm{K}^0_c(T^*(Y/X))$ *represented by (untwisted) data* (E_+,E_-,a) *the twisted Chern character of the image in* $\mathrm{K}_c(T^*(Y/X);\rho^*\phi^*\mathcal{A})$ *is represented by the pair of forms after lifting to* $\tilde L$ *and trivializing* J *as in* (7.17)

$$\mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}([(E_+,E_-,a)])=(\widetilde{\mathrm{Ch}}_{\mathcal{A}}(a),\mathrm{Ch}_{\mathcal{A}}(E_+)-\mathrm{Ch}_{\mathcal{A}}(E_-)) \tag{10.4}$$

in the subcomplex of the relative deRham complex fixed by (1.10)*, and* $\rho\colon T^*(Y/X)\longrightarrow Y$ *is the projection.*

Of course the point of this discussion is that these forms do give the analogue of the index formula in (twisted) cohomology.

Theorem 2. *For the twisted index map* (5.5) *the twisted Chern character is given by the push-forward of the differential form in* (10.4)

$$\begin{aligned}&\mathrm{Ch}_{\mathcal{A}}\circ\mathrm{ind}:\mathrm{K}^0_c(T^*(Y/X);\rho^*\phi^*\mathcal{A})\simeq\mathrm{K}^0_c(T^*(Y/X))\longrightarrow\mathrm{H}^{even}(X,\delta),\\ &\mathrm{Ch}_{\mathcal{A}}\circ\mathrm{ind}(p)=(-1)^n\phi_*\rho_*\left\{\rho^*\mathrm{Todd}(T^*(Y/X)\otimes\mathbb{C})\wedge\mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\},\end{aligned} \tag{10.5}$$

where $\mathrm{Todd}(T^*(Y/X)\otimes\mathbb{C})$ *denotes the Todd class of the complexified vertical cotangent bundle and* $p=[(E_+,E_-,a)]$ *as identified in Proposition 11.*

Proof. By the Index Theorem in K-theory, Theorem 1 of the previous section, it suffices to compute the twisted Chern character of the topological index of the projective family of elliptic pseudodifferential operators. We begin with by recalling the basic properties of the twisted Chern character. As before, we assume that the primitive line bundle J defining the smooth Azumaya bundle $\mathcal{A}$ is endowed with a fixed connection respecting the primitive property. It gives a homomorphism,

$$\mathrm{Ch}_{\mathcal{A}}:\mathrm{K}^0(X,\mathcal{A})\longrightarrow\mathrm{H}^{even}(X,\delta), \tag{10.6}$$

satisfying the following properties.

(1) The Chern character is *functorial* under smooth maps in the sense that if $f\colon W\longrightarrow X$ is a smooth map between compact manifolds, then the following diagram commutes:

$$\begin{array}{ccc}\mathrm{K}^0(X,\mathcal{A}) & \xrightarrow{f^!} & \mathrm{K}^0(W,f^*\mathcal{A})\\ \downarrow{\scriptstyle \mathrm{Ch}_{\mathcal{A}}} & & \downarrow{\scriptstyle \mathrm{Ch}_{f^*\mathcal{A}}}\\ \mathrm{H}^{even}(X,\delta) & \xrightarrow{f^*} & \mathrm{H}^{even}(W,f^*\delta).\end{array} \tag{10.7}$$

Here the pullback primitive line bundle ${f^{[2]}}^*J$ defining the pullback smooth Azumaya bundle $f^*\mathcal{A}$ is endowed with the pullback of the fixed connection respecting the primitive property.

(2) The Chern character respects the structure of $\mathrm{K}^0(X,\mathcal{A})$ as a module over $\mathrm{K}^0(X)$, in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{K}^0(X)\times \mathrm{K}^0(X,\mathcal{A}) & \longrightarrow & \mathrm{K}^0(X,\mathcal{A}) \\ \Big\downarrow{\scriptstyle \mathrm{Ch}\times\mathrm{Ch}_{\mathcal{A}}} & & \Big\downarrow{\scriptstyle \mathrm{Ch}_{\mathcal{A}}} \\ \mathrm{H}^{even}(X,\mathbb{Q})\times \mathrm{H}^{even}(X,\delta) & \longrightarrow & \mathrm{H}^{even}(X,\delta) \end{array} \tag{10.8}$$

where the top horizontal arrow is the action of $\mathrm{K}^0(X)$ on $\mathrm{K}^0(X,\mathcal{A})$ given by tensor product and the bottom horizontal arrow is given by the cup product.

The theorem now follows rather routinely from the index theorem in K-theory, Theorem 1. The key step to getting the formula is the analog of the Riemann-Roch formula in the context of twisted K-theory, which we now give details.

Let $\pi : E \longrightarrow X$ be a spin$\mathbb{C}$ vector bundle over X, $i : X \longrightarrow E$ the zero section embedding, and $F \in \mathrm{K}^0(X,\mathcal{A})$. Then using the properties of the twisted Chern character as above, we compute,

$$\begin{aligned} \mathrm{Ch}_{\pi^*\mathcal{A}}(i_!F) &= \mathrm{Ch}_{\pi^*\mathcal{A}}(i_!1\otimes\pi^*F) \\ &= \mathrm{Ch}(i_!1)\wedge \mathrm{Ch}_{\pi^*\mathcal{A}}(\pi^*F). \end{aligned}$$

The standard Riemann-Roch formula asserts that

$$\mathrm{Ch}(i_!1) = i_*\mathrm{Todd}(E)^{-1}.$$

Therefore we deduce the following Riemann-Roch formula for twisted K-theory,

$$\mathrm{Ch}_{\pi^*\mathcal{A}}(i_!F) = i_*\left\{\mathrm{Todd}(E)^{-1}\wedge \mathrm{Ch}_{\mathcal{A}}(F)\right\}. \tag{10.9}$$

We need to compute $\mathrm{Ch}_{\mathcal{A}}(\mathrm{ind}_t\, p)$ where $p = [E_+,E_-,a] \in \mathrm{K}^0_{\mathrm{c}}(T(Y/X)) \cong \mathrm{K}^0_{\mathrm{c}}(T(Y/X),\rho^*\phi^*\mathcal{A})$. We will henceforth identify $T(Y/X)\cong T^*(Y/X)$ via a Riemannian metric. Recall from §6 that the topological index, $\mathrm{ind}_t = j_!^{-1}\circ(Di)_!$ where $i : Y \hookrightarrow X\times\mathbb{R}^{2N}$ is an embedding that commutes with the projections $\phi : Y \longrightarrow X$ and $\pi_1 : X\times\mathbb{R}^{2N}\longrightarrow X$, and $j : X \hookrightarrow X\times\mathbb{R}^{2N}$ is the zero section embedding. Therefore

$$\mathrm{Ch}_{\mathcal{A}}(\mathrm{ind}_t\, p) = \mathrm{Ch}_{\mathcal{A}}(j_!^{-1}\circ(Di)_!p)$$

By the Riemann-Roch formula for twisted K-theory (10.9),

$$\mathrm{Ch}_{\pi_1^*\mathcal{A}}(j_!F) = j_*\,\mathrm{Ch}_{\mathcal{A}}(F)$$

since $\pi_1 : X\times\mathbb{R}^{2N}\longrightarrow X$ is a trivial bundle. Since $\pi_{1*}j_*1 = (-1)^n$, it follows that for $\xi\in \mathrm{K}^0_{\mathrm{c}}(X\times\mathbb{R}^{2N},\pi_1^*\mathcal{A})$, one has

$$\mathrm{Ch}_{\mathcal{A}}(j_!^{-1}\xi) = (-1)^n\pi_{1*}\,\mathrm{Ch}_{\pi_1^*\mathcal{A}}(\xi)$$

Therefore

$$\mathrm{Ch}_{\mathcal{A}}(j_!^{-1}\circ(Di)_!p) = (-1)^n\pi_{1*}\,\mathrm{Ch}_{\pi_1^*\mathcal{A}}((Di)_!p) \tag{10.10}$$

By the Riemann-Roch formula for twisted K-theory (10.9),

$$\mathrm{Ch}_{\pi_1^*\mathcal{A}}((Di)_!p) = (Di)_*\left\{\rho^*\mathrm{Todd}(\mathcal{N})^{-1}\wedge \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\} \tag{10.11}$$

where $\mathcal{N}$ is the complexified normal bundle to the embedding $Di : T(Y/X)\longrightarrow X\times T\mathbb{R}^{2N}$, that is, $\mathcal{N} = X\times T\mathbb{R}^{2N}/Di(T(Y/X))\otimes\mathbb{C}$. Therefore $\mathrm{Todd}(\mathcal{N})^{-1} = \mathrm{Todd}(T(Y/X)\otimes\mathbb{C})$ and (10.11) becomes

$$\mathrm{Ch}_{\pi_1^*\mathcal{A}}((Di)_!p) = (Di)_*\left\{\rho^*\mathrm{Todd}(T(Y/X)\otimes\mathbb{C})\wedge \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\}.$$

Therefore (10.10) becomes

(10.12)
$$\begin{aligned}\mathrm{Ch}_{\mathcal{A}}(j_!^{-1}\circ(Di)_!p) &= (-1)^n \pi_{1*}(Di)_* \left\{\rho^*\mathrm{Todd}(T(Y/X)\otimes\mathbb{C})\wedge \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\} \\ &= (-1)^n \phi_*\rho_* \left\{\rho^*\mathrm{Todd}(T(Y/X)\otimes\mathbb{C})\wedge \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\}\end{aligned}$$

since $\phi_*\rho_* = \pi_{1*}(Di)_*$. Therefore

(10.13) $$\mathrm{Ch}_{\mathcal{A}}(\mathrm{ind}_t\, p) = (-1)^n \phi_*\rho_* \left\{\rho^*\mathrm{Todd}(T(Y/X)\otimes\mathbb{C})\wedge \mathrm{Ch}_{\rho^*\phi^*\mathcal{A}}(p)\right\},$$

proving Theorem 2.

□

Appendix A. Differential characters

We will refine Lemma 1 to an equality of differential characters. For an account of differential characters, see [10, 14].

We first relate a connection $\tilde{\gamma}$ on $\tilde{L}$ to the 1-form γ on Y. Consider the commutative diagram

(A.1) $$\begin{array}{ccc} \phi^*(\tilde{L}) & \xrightarrow{\tilde{\phi}} & \tilde{L} \\ {\scriptstyle pr_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{\phi} & X \end{array}$$

where $\phi^*(\tilde{L}) = Y\times\mathbb{S}$ as observed earlier, and $pr_1 : Y\times\mathbb{S}\longrightarrow Y$ denotes projection to the first factor. Then the connection 1-form $\tilde{\gamma}$ on $\tilde{L}$ with curvature equal to β is related to the 1-form γ on Y by $\tilde{\phi}^*(\tilde{\gamma}) = \gamma+\theta$ where θ is the Cartan-Maurer 1-form on $\mathbb{S}$. If $\iota : Y\to Y\times\mathbb{S}$ denotes the inclusion map into the first factor, then $\iota^*\tilde{\phi}^*(\tilde{\gamma}) = \gamma$.

Now the circle bundle $\tilde{L}$ has a section $\tilde{\tau} : X\setminus M_1 \to \tilde{L}$, where M_1 is a codimension 2 submanifold of X. We define a section $\tau : X\setminus M_1\to Y$ such that $\tilde{\phi}\circ\iota\circ\tau = \tilde{\tau}$. Then we have a well defined singular 1-form $\varphi_1 := \tilde{\tau}^*(\tilde{\gamma}) = \tau^*(\gamma)$ on X with the property that $d\varphi_1 = \beta$. The differential character associated to φ_1 is (cf. [9])

$$S(\varphi_1)(z) = \varphi_1(z') + \beta(c)$$

where $z, z' \in Z_1(X,\mathbb{Z})$ and $c\in C_2(X,\mathbb{Z})$ is such that $\partial c = z - z'$, where $z'\cap M_1 = \emptyset$.

The smooth map $u : X\to\mathbb{R}/\mathbb{Z}$ gives rise to a singular function φ_0 on X as follows. If $t\in\mathbb{R}/\mathbb{Z}$ is a regular value for u, then $M_0 := u^{-1}(t)$ is a codimension 1 submanifold of X, and the Cartan-Maurer 1-form θ on $\mathbb{R}/\mathbb{Z}$ is exact on $\mathbb{R}/\mathbb{Z}\setminus\{t\}$, say dg, where g is a smooth function on $\mathbb{R}/\mathbb{Z}\setminus\{t\}$. Then the pullback function $\varphi_0 = u^*(g)$ is a smooth function on $X\setminus M_0$, ie it is a singular function on X such that $d\varphi_0 = u^*(\theta) = \alpha$ is the associated global smooth 1-form on X with integer periods.

With μ as in Lemma 1, $\varphi_2 := \tau^*(\mu) = d\varphi_0\wedge\varphi_1$ is a singular 2-form on X, whose associated differential character is

$$S(\varphi_2)(z) = d\varphi_0\wedge\varphi_1(z') + \alpha\wedge\beta(c)$$

where $z, z'\in Z_2(X,\mathbb{Z})$ and $c\in C_3(X,\mathbb{Z})$ is such that $\partial c = z-z'$, where $z'\cap M_1 = \emptyset$. By the argument given above, it is also the differential character associated to the Azumaya bundle $\mathcal{A}$ with connection.

Lemma 10. *In the notation above, $S(\varphi_2) = S(\varphi_0) \star S(\varphi_1)$, where $\star$ denotes the Cheeger-Simons product of differential characters.*

Proof. First note that by [10], the field strength of $S(\varphi_0) \star S(\varphi_1)$ is $\bar{\alpha} \wedge \bar{\beta}$, which by Lemma 1 is equal to $\bar{\delta}$ which is the field strength of $S(\varphi_2)$. That is,

$$S(\varphi_0) \star S(\varphi_1)(\partial c) = S(\varphi_2)(\partial c)$$

for every degree 3 integral cochain c.

By [10], we see that the characteristic class of $S(\varphi_0) \star S(\varphi_1)$ is equal to the cup product $\alpha \cup \beta$, and by Appendix B, also equal to δ, which is the characteristic class of $S(\varphi_2)$. Note that the image in real cohomology of α and β is equal to $[\bar{\alpha}]$ and $[\bar{\beta}]$ respectively.

According to [9], if $z \in Z_2(X, \mathbb{Z})$ is transverse to M_1, then

$$S(\varphi_0) \star S(\varphi_1)(z) = -d\varphi_0 \wedge \varphi_1(z) + \sum_{p \in z \cap M_1} \varphi_0(p)$$

In particular, if $z \cap M_1 = \emptyset$, then

$$S(\varphi_0) \star S(\varphi_1)(z) = S(\varphi_2)(z),$$

proving the lemma. □

Appendix B. Čech class of the Azumaya bundle

Suppose that there is a line bundle K over Y such that $J \cong K \boxtimes K'$. That is, in this case, $\mathcal{C}^\infty(X, \mathcal{A}) \cong \Psi^{-\infty}(Y/X, K)$ is the algebra of smoothing operators along the fibres of $\phi : Y \longrightarrow X$ acting on sections of K, and therefore has trivial Dixmier-Douady invariant. Here $\mathcal{A}_x = \Psi^{-\infty}(\phi^{-1}(x), K\big|_{\phi^{-1}(x)})$ for all $x \in X$.

Conversely, suppose that the Dixmier-Douady invariant of $\mathcal{A}$ is trivial, where $\mathcal{C}^\infty(X, \mathcal{A}) = \mathcal{C}^\infty(Y^{[2]}, J)$. Then there is a line bundle K over Y such that $J \cong K \boxtimes K'$. To see this, we use the connection ∇^J preserving the primitive property of J, and Lemma 1 to see that $d\mu = \phi^* dB$, for some global 2-form $B \in \Omega^2(X)$. Then $d(\mu - \phi^*(B)) = 0$, and $\pi_1^*(\mu - \phi^*(B)) - \pi_2^*(\mu - \phi^*(B)) = F_{\nabla^J}$. So $\mu - \phi^*(B)$ is a closed 2-form on Y and can be chosen to have integral periods, since F_{∇^J} has integral periods (this is clear from the Čech description below). Therefore there is a line bundle K on Y with connection, whose curvature is equal to $\mu - \phi^*(B)$ such that $J \cong K \boxtimes K'$.

Suppose that J_1 and J_2 are two primitive line bundles over $Y^{[2]}$ and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the corresponding Azumaya bundles, that is, $\mathcal{C}^\infty(X, \mathcal{A}_j) = \mathcal{C}^\infty(Y^{[2]}, J_j)$, $j = 1, 2$. Then we conclude by the argument above that $\mathcal{A}_1 \cong \mathcal{A}_2$ if and only if there is a line bundle K over Y such that $J_1 \cong J_2 \otimes (K \boxtimes K')$.

The main result that we want to show here is the following.

Lemma 11. *Suppose $\phi : Y \longrightarrow X$, $L \longrightarrow X$ and $u : X \longrightarrow \mathrm{U}(1)$ are as in the introduction. Then the Dixmier-Douady class of the Azumaya bundle $\mathcal{A}$ constructed from this data as in §3, is equal to $\alpha \cup \beta$, where $\alpha \in \mathrm{H}^1(X, \mathbb{Z})$ is the cohomology defined by u and $\beta \in \mathrm{H}^2(X, \mathbb{Z})$ is the Chern class of L.*

Proof. As noted above, the Dixmier-Douady invariant of $\mathcal{A}$ is the degree 3 cohomology class on X associated to the primitive line bundle J over $Y^{[2]}$.

As argued in §2, the line bundle L gives rise to a character $s : Y^{[2]} \to \mathrm{U}(1)$. Suppose that $\tau_i : U_i \to Y$ are local sections of Y. Then it is clear from §2 that

$c_{ij} := s(\tau_i, \tau_j)$ defines a U(1)-valued Cech 1-cocycle representing the first Chern class of L.

Using the same local sections of Y, we see that $J_{ij} := (\tau_i \times \tau_j)^* J = L_j^{-n_{ij}}$, where $n_{jk} : U_j \cap U_k \to \mathbb{Z}$ denotes the transition functions of $\hat{X}$. If s_j is a local nowhere zero section of L_j, then $\sigma_{ij} := s_j^{-n_{ij}}$ is a local nowhere zero section of J_{ij}. We compute,

$$\sigma_{ij}\sigma_{jk} = s_j^{-n_{ij}} s_k^{-n_{jk}} \tag{B.1}$$

$$= c_{jk}^{-n_{ij}} s_k^{-n_{ij}} s_k^{-n_{jk}} \tag{B.2}$$

$$= c_{kj}^{-n_{ji}} s_k^{-n_{ik}} = c_{kj}^{-n_{ji}} \sigma_{ik}. \tag{B.3}$$

Therefore the U(1)-valued Cech 2-cocycle associated to J is $d_{ijk} := c_{kj}^{-n_{ji}}$ But it is well known (cf. equation (1-18), page 29, [7]) that the right hand side represents the cup product of the Cech cocycles $[c]$ and $[-n]$, that is, $[d] = [c] \cup [-n] = -\beta \cup \alpha = \alpha \cup \beta \in \mathrm{H}^3(X, \mathbb{Z})$, proving the lemma. □

Appendix C. The universal case

Let $\phi : Y \to X$ be a fibre bundle of compact manifolds, $L \to X$ a line bundle over X with the property that the pullback $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$, where $\beta \in \mathrm{H}^2(X, \mathbb{Z})$ is the first Chern class of L.

Lemma 12. *In the notation above, $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$ if and only if there is a $\tilde{\beta} \in \mathrm{H}^2(B\mathrm{Diff}(Z), \mathbb{Z})$ such that $\beta = f^*(\tilde{\beta})$ in $\mathrm{H}^2(X, \mathbb{Z})$, where $f \colon X \to B\mathrm{Diff}(Z)$ is the classifying map for $\phi : Y \to X$, and Z is the typical fiber of $\phi : Y \to X$.*

This follows in a straightforward way from standard algebraic topology. The direction that we will mainly use is trivial to prove, viz. if there is a class $\tilde{\beta} \in \mathrm{H}^2(B\mathrm{Diff}(Z), \mathbb{Z})$ such that $\beta = f^*(\tilde{\beta})$ in $\mathrm{H}^2(X, \mathbb{Z})$, then $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$.

Therefore we see that given any fibre bundle of compact manifolds $\phi : Y \to X$ with typical fiber Z, and $\beta \in f^*(\mathrm{H}^2(B\mathrm{Diff}(Z), \mathbb{Z})) \subset \mathrm{H}^2(X, \mathbb{Z})$ (that is, if β is a characteristic class of the fiber bundle $\phi : Y \to X$), then $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y, \mathbb{Z})$, satisfying the hypotheses of our main index theorem.

But what are line bundles on $B\mathrm{Diff}(Z)$? Since roughly speaking, $B\mathrm{Diff}(Z) = \mathrm{Metrics}(Z)/\mathrm{Diff}(Z)$, where $\mathrm{Metrics}(Z)$ denotes the contractible space of all Riemanian metrics on Z, the theory of anomalies in gravity constructs line bundles on $B\mathrm{Diff}(Z)$ via determinant line bundles of index bundles of families of twisted Dirac operators obtained by varying the Riemannian metric on Z, cf. [2].

In particular, let $\phi : Y \to X$ be a fibre bundle of compact manifolds, with typical fiber a compact Riemann surface Σ_g of genus $g \geq 2$. Then $T(Y/X)$ is an oriented rank 2 bundle over Y. Define $\beta = \phi_*(e \cup e) \in \mathrm{H}^2(X, \mathbb{Z})$, where $e := e(T(Y/X)) \in \mathrm{H}^2(Y, \mathbb{Z})$ is the Euler class of $T(Y/X)$. By naturality of this construction, $\beta = f^*(e_1)$, where $e_1 \in \mathrm{H}^2(B\mathrm{Diff}(\Sigma_g), \mathbb{Z})$ and $f \colon X \to B\mathrm{Diff}(\Sigma_g)$ is the classifying map for $\phi : Y \to X$. e_1 is known as the universal first Mumford-Morita-Miller class, and β is the first Mumford-Morita-Miller class of $\phi : Y \to X$, cf. Chapter 4 in [18]. Therefore by Lemma 12, we have the following.

Lemma 13. *In the notation above, let $\phi : Y \to X$ be a fibre bundle of compact manifolds, with typical fiber a compact Riemann surface Σ_g of genus $g \geq 2$, and let*

$\beta \in \mathrm{H}^2(X,\mathbb{Z})$ be a multiple of the first Mumford-Morita-Miller class of $\phi : Y \to X$. Then $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y,\mathbb{Z})$.

Such choices satisfy the hypotheses of our main index theorem. In fact, if $\phi : Y \longrightarrow X$ be as above, and in addition let X be a closed Riemann surface. Then Proposition 4.11 in [18] asserts that $\langle e_1, [X]\rangle = \mathrm{Sign}(Y)$, where $\mathrm{Sign}(Y)$ is the signature of the 4-dimensional manifold Y, which is originally a result of Atiyah. As a consequence, Morita is able to produce infinitely many surface bundles Y over X that have non-trivial first Mumford-Morita-Miller class.

On the other hand, given any $\beta \in \mathrm{H}^2(X,\mathbb{Z})$, we know that there is a fibre bundle of compact manifolds $\phi : Y \to X$ such that $\phi^*(\beta) = 0$ in $\mathrm{H}^2(Y,\mathbb{Z})$. In fact we can choose Y to be the total space of a principal $\mathrm{U}(n)$ bundle over X with first Chern class β. Here we can also replace $\mathrm{U}(n)$ by any compact Lie group G such that $\mathrm{H}^1(G,\mathbb{Z})$ is nontrivial and torsion-free, such as the torus $\mathbb{T}^n$.

Lemma 14. *Let $\phi : Y \to X$ be a fibre bundle of compact manifolds with typical fiber Z and $\beta \in \mathrm{H}^2(X,\mathbb{Z})$. Let $\pi : P \to X$ be a principal $\mathrm{U}(n)$-bundle whose first Chern class is β. Then the fibred product $\phi\times\pi : Y \times_X P \to X$ is a fiber bundle with typical fiber $Z \times \mathrm{U}(n)$, and has the property that $(\phi\times\pi)^*(\beta) = 0$ in $\mathrm{H}^2(Y \times_X P,\mathbb{Z})$.*

This follows from the obvious commutativity of the following diagram,

$$\begin{array}{ccc} Y \times_X P & \xrightarrow{pr_1} & Y \\ {\scriptstyle pr_2}\downarrow & & \downarrow{\scriptstyle \phi} \\ P & \xrightarrow{\pi} & X. \end{array} \tag{C.1}$$

Hence this data also satisfy the hypotheses of our main index theorem.

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