Introduction to Microlocal Analysis

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1991 *Mathematics Subject Classification.* All over the shop.

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Preface

This is a somewhat revised version of the lecture notes from a course in 1990; in particular I have included the corrections made by Mark Joshi (thanks Mark, hope you are having fun in Dear Old Blighty.) I am continuing to correct and update these notes, thanks to Edie Mooers, ...

Here is the earlier preface.

I'm sorry to have to admit that I did not get a chance to even properly proof-read large parts of these notes, especially Lecture XVI onward, and Lecture XV is essentially unwritten. They are offered for what they are worth as a 'conventional' course on pseudodifferential operators. Comments gratefully received, but there is absolutely no chance of my doing anything much with these until at least February 1991, if then.

Introduction

I shall assume some familiarity with distribution theory, with basic analysis and functional analysis and with (local) differential geometry. A passing knowledge of the theory of manifolds would also be useful. Any one or two of these prerequisites can be easily picked up along the way, but the prospective student with none of them should perhaps do some preliminary reading:

Distributions: I good introduction is Friedlander's book [**2**]. For a more exhaustive treatment see Volume I of Hörmander's treatise [**3**].

Analysis on manifolds: Most of what we need can be picked up from Munkres' book [**4**] or Spivak's little book [**5**].

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