CHAPTER 8

Hochschild homology

8.1. Formal Hochschild homology

The Hochschild homology is defined, formally, for any associative algebra. Thus if $\mathcal{A}$ is the algebra then the space of *formal* k-chains, for $k \in \mathbb{N}_0$ is the $(k+1)$-fold tensor product

$$\mathcal{A}^{\otimes(k+1)} = \mathcal{A} \otimes \mathcal{A} \otimes \cdots \otimes \mathcal{A}.$$

The 'formal' here refers to the fact that for the 'large' topological algebras we shall consider it is wise to replace this tensor product by an appropriate completion, usually the 'projective' tensor product. At the formal level the differential defining the cohomolgy is given in terms of the product, $\star$, by

$$\begin{gathered}b(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = b'(a_0 \otimes a_1 \otimes \cdots \otimes a_k) + (-1)^k (a_0 \star a_k) \otimes a_1 \otimes \cdots \otimes a_{k-1}, \\ b'(a_0 \otimes a_1 \otimes \cdots \otimes a_k) = \sum_{j=0}^{k-1} (-1)^j a_0 \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \star a_j \otimes a_{j+2} \otimes \cdots \otimes a_k.\end{gathered} \tag{8.1}$$

Lemma 8.1. *Both the partial map, b', and the full map, b, are differentials, that is*

$$(b')^2 = 0 \text{ and } b^2 = 0. \tag{8.2}$$

Proof. This is just a direct computation. From (8.1) it follows that

$$\begin{aligned}(8.3)\quad & (b')^2(a_0 \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_m) \\ &= \sum_{j=2}^{m-1} \sum_{p=0}^{j-2} (-1)^j (-1)^p (\cdots \otimes a_{p+1} \star a_p \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \star a_j \otimes a_{j+2} \otimes \cdots \otimes a_m) \\ &\quad - \sum_{j=1}^{m-1} (\cdots \otimes a_{j+1} \star a_j \star a_{j-1} \otimes \cdots) - \sum_{j=0}^{m-2} (\cdots \otimes a_{j+21} \star a_{j+1} \star a_j \star \otimes \cdots) \\ &+ \sum_{j=0}^{m-3} \sum_{p=j+2}^{m-1} (-1)^j (-1)^{p-1} (a_0 \otimes \cdots \otimes a_{j-1} \otimes a_{j+1} \star a_j \otimes a_{j+2} \otimes \cdots \otimes a_{p+1} \star a_p \otimes \cdots) = 0.\end{aligned}$$

Similarly, direct computation shows that

$$\begin{aligned}(b-b')b'(a_0\otimes\cdots\otimes a_m) &= (-1)^{m-1}(a_1\star a_0\star a_m\otimes\cdots a_{m-1})\\ &+\sum_{i=1}^{m-2}(-1)^{i+m-1}(a_0\star a_m\otimes\cdots\otimes a_{i+1}\star a_i\otimes\cdots)+(a_0\star a_m\star a_{m-1}\otimes\cdots),\\ b'(b-b')(a_0\otimes\cdots\otimes a_m) &= (-1)^m(a_1\star a_0\star a_m\otimes\cdots a_{m-1})\\ &+\sum_{i=1}^{m-2}(-1)^{i+m}(a_0\star a_m\otimes\cdots\otimes a_{i+1}\star a_i\otimes\cdots)\text{ and}\\ (b-b')^2(a_0\otimes\cdots\otimes a_m) &= -(a_0\star a_m\star a_{m-1}\otimes\cdots)\end{aligned}$$

so

$$(b-b')b'+b'(b-b')=-(b-b')^2. \tag{8.4}$$

□

The difference between these two differentials is fundamental, roughly speaking b' is 'trivial'.

LEMMA 8.2. *For any algebra with identity the differential b' is acyclic, since it satifies*

$$b's+sb'=\mathrm{Id}\ \textit{where} \tag{8.5}$$

$$s(a_0\otimes\cdots\otimes a_m)=\mathrm{Id}\otimes a_0\otimes\cdots\otimes a_m. \tag{8.6}$$

PROOF. This follows from the observation that

$$b'(\mathrm{Id}\otimes a_0\otimes\cdots\otimes a_m)=a_0\otimes\cdots\otimes a_m+\sum_{i=1}^{m}(-1)^i(\mathrm{Id}\otimes\cdots a_i\star a_{i-2}\otimes\cdots). \tag{8.7}$$

□

DEFINITION 8.1. *An associative algebra is said to be* H-unital *if its b' complex is acyclic.*

Thus the preceeding lemma just says that every unital algebra is H-unital.

8.2. Hochschild homology of polynomial algebras

Consider the algebra $\mathbb{C}[x]$ of polynomials in n variables[1], $x\in\mathbb{R}^n$ (or $x\in\mathbb{C}^n$ it makes little difference). This is not a finite dimensional algebra but it is filtered by the finite dimensional subspaces, $P_m[x]$, of polynomials of degree at most m;

$$\mathbb{C}[x]=\bigcup_{m=0}^{\infty}P_m[x],\ P_m[x]\subset P_{m+1}[x].$$

Furthermore, the Hochschild differential does not increase the total degree so it is enough to consider the formal Hochschild homology.

The chain spaces, given by the tensor product, just consist of polynomials in $n(k+1)$ variables

$$(\mathbb{C}[x])^{\hat{\otimes}(k+1)}=\mathbb{C}[x_0,x_1,\ldots,x_k],\ x_j\in\mathbb{R}^n.$$

[1] The method used here to compute the homology of a polynomial algebra is due to Sergiu Moroianu; thanks Sergiu.

Furthermore composition acts on the tensor product by

$$p(x_0)q(x_1) = p \otimes q \longmapsto p(x_0)q(x_0)$$

which is just restriction to $x_0 = x_1$. Thus the Hochschild differential can be written

$$b : \mathbb{C}[x_0, \dots, x_k] \longrightarrow \mathbb{C}[x_0, \dots, x_{k-1}],$$

$$(bq)(x_0, x_1, \dots, x_{k-1}) = \sum_{j=0}^{k-1} (-1)^j p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) \\ + (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0).$$

One of the fundamental results on Hochschild homology is

THEOREM 8.1. *The Hochschild homology of the polynomail algebra in n variables is*

$$\mathrm{HH}_k(\mathbb{C}[x]) = \mathbb{C}[x] \otimes \Lambda^k(\mathbb{C}^n), \tag{8.8}$$

with the identification given by the map from the chain spaces

$$\mathbb{C}[x_0, \dots, x_k] \ni q \longrightarrow \sum_{1 \le j_i \le n} \frac{\partial}{\partial x_1^{j_1}} \cdots \frac{\partial}{\partial x_k^{j_k}} p\big|_{x=x_0=x_1=\cdots=x_k} dx_1^{j_1} \wedge \cdots \wedge dx_k^{j_k}.$$

Note that the appearance of the original algebra $\mathbb{C}[x]$ on the left in (8.8) is not surprising, since the differential commutes with multilplication by polynomails in the first variable, x_0

$$b(r(x_0)q(x_0, \dots, x_k)) = r(x_0)(bq(x_0, \dots, x_k)).$$

Thus the Hochschild homology is certainly a module over $\mathbb{C}[x]$.

PROOF. Consider first the cases of small k. If $k = 0$ then b is identically 0. If $k = 1$ then again

$$(bq)(x_0) = q(x_0, x_0) - q(x_0, x_0) = 0$$

vanishes identically. Thus the homology in dimension 0 is indeed $\mathbb{C}[x]$.

Suppose that $k > 1$ and consider the subspace of $\mathbb{C}[x_0, x_1, \dots, x_k]$ consisting of the elements which are independent of x_1. Then the first two terms in the definition of b cancel and

$$(bq)(x_0, x_1, \dots, x_{k-1}) = \sum_{j=2}^{k-1} (-1)^j p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) \\ + (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0),\ \ \partial_{x_1} q \equiv 0.$$

It follows that bq is also independent of x_1. Thus there is a well-defined subcomplex on polynomails independend of x_1 given by

$$\mathbb{C}[x_0, x_2, \dots, x_k] \ni q \longmapsto (\tilde{b}q)(x_0, x_2, \dots, x_{k-1}) \\ = \sum_{j=2}^{k-1} (-1)^j p(x_0, x_2, x_2, x_3 \dots, x_{k-1}) + \sum_{j=3}^{k-1} (-1)^j \\ p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) + (-1)^k q(x_0, x_2, \dots, x_{k-1}, x_0)$$

The reordering of variables $(x_0, x_2, x_3, \dots, x_k) \longrightarrow (x_2, x_3, \dots, x_k, x_0)$ for each k, transforms $\tilde{b}$ to the reduced Hochschild differential b' acting in k variables. Thus $\tilde{b}$ is acyclic.

Similarly consider the subspace of $\mathbb{C}[x_0, x_1, \dots, x_k]$ consisting of the polynomials which vanish at $x_1 = x_0$. Then the first term in the definition of b vanishes and the action of the differential becomes

$$
\begin{aligned}
(8.9)\quad (bq)(x_0, x_1, \dots, x_{k-1}) = p(x_0, x_1, x_1, x_2, \dots, x_{k-1}) + \\
\sum_{j=2}^{k-1} (-1)^j p(x_0, \dots, x_{j-1}, x_j, x_j, x_{j+1}, \dots, x_{k-1}) \\
+ (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0), \text{ if } b(x_0, x_0, x_2, \dots) \equiv 0.
\end{aligned}
$$

It follows that bq also vanishes at $x_1 = x_0$.

By Taylor's theorem any polynomial can be written uniquely as a sum

$$q(x_0, x_1, x_2, \dots, x_k) = q_1'(x_0, x_1, x_2, \dots, x_k) + q''(x_0, x_2, \dots, x_k)$$

of a polynomial which vanishes at $x_1 = x_0$ and a polynomial which is independent of x_1. From the discussion above, this splits the complex into a sum of two subcomplexes, the second one of which is acyclic. Thus the Hochschild homology is the same as the homology of b, which is then given by (8.9), acting on the spaces

$$\{q \in \mathbb{C}[x_0, x_1, \dots, x_k]; q(x_0, x_1, \dots) = 0\}. \tag{8.10}$$

This argument can be extended iteratively. Thus, if $k > 2$ then b maps the subspace of (8.10) of functions independent of x_2 to functions independent of x_2 and on these subspaces acts as b' in $k-2$ variables; it is therefore acyclic. Similar it acts on the complementary spaces given by the functions which vanish on $x_2 = x_1$. Repeating this argument shows that the Hochschild homology is the same as the homology of b acting on the smaller subspaces

$$
\begin{gathered}
\{q \in \mathbb{C}[x_0, x_1, \dots, x_k]; q(\dots, x_{j-1}, x_j, \dots) = 0,\ j = 1, \dots, k\}, \\
(bq)(x_0, x_1, \dots, x_{k-1}) = (-1)^k q(x_0, x_1, \dots, x_{k-1}, x_0).
\end{gathered} \tag{8.11}
$$

Note that one cannot proceed further directly, in the sense that one cannot reduce to the subspace of functions vanishing on $x_k = x_0$ as well, since this subspace is not linearly independent of the previous ones[2]

$$x_k - x_0 = \sum_{j=0}^{k-1} (x_{j_1} - x_j).$$

It is precisely this 'non-transversality' of the remaining restriction map in (8.11) which remains to be analysed.

Now, let us we make the following change of variable in each of these reduced chain spaces setting

$$y_0 = x_0,\ y_1 = x_j - x_{j-1}, \text{ for } j = 1, \dots, k.$$

[2]Hence Taylor's theorem cannot be applied.

Then the differential can be written in terms of the pull-back operation

$$E_P : \mathbb{R}^{nk} \hookrightarrow \mathbb{R}^{n(k+1)},\ E_P(y_0, y_1, \ldots, y_{k-1}) = (y_0, y_1, \ldots, y_{k-1}, -\sum_{j=1}^{k-1} y_j),$$
$$bq = (-1)^k E_P^* q,$$

The variable $x_0 = y_0$ is a pure parameter, so can be dropped from the notation (and restored at the end as the factor $\mathbb{C}[x]$ in (8.8)). Also, as already noted, the degree of a polynomial (in all variables) does not increase under any of these pull-back operations, in fact they all preserve the total degree of homogeneity so it suffices to consider the differential b acting on the spaces of homogeneous polynomials which vanish at the origin in each factor

$$Q_k^m = \{q \in \mathbb{C}^m[y_1, \ldots, y_k]; q(sy) = s^m q(y),\ q(y_1, \ldots, y_{j-1}, 0, y_{j+1}, \ldots, y_k) = 0\}$$
$$b : Q_k^m \longrightarrow Q_{k-1}^m,\ bq = (-1)^* E_P^* q.$$

To analyse this non-transversality further, let $J_i \subset \mathbb{C}[y_1, \ldots, y_k]$ be the ideal generated by the n monomials y_i^l, $l = 1, \ldots, n$. Thus, by Taylor's theorem,

$$J_i = \{q \in \mathbb{C}[y_1, \ldots, y_k]; q(y_1, y_2, \ldots, y_{j-1}, 0, y_j, y_k) = 0.$$

Similary set

$$J_P = \{q \in \mathbb{C}[y_1, \ldots, y_k]; q(y_1, \ldots, -\sum_{j=1}^{k-1} y_j) = 0)$$

For any two ideals I and J, let $I \cdot J$ be the span of the products. Thus for these particular ideals an element of the product is a sum of terms each of which has a factor vanishing on the corresponding linear subspace. For each k there are $k+1$ ideals and, by Taylor's theorem, the intersection of any k of them is equal to the span of the product of those k ideals. For the k coordinate ideals this is Taylor's theorem as used in the reduction above. The general case of any k of the ideals can be reduced to this case by linear change of coordinates. The question then, is structure of the intersection of all $k+1$ ideals. The proof of the theorem is therefore completed by the following result. □

LEMMA 8.3. *The intersection* $Q_k^m \cap J_P = Q^m \cdot J_P$ *for every* $m \neq k$ *and*

$$Q_k^k \cap J_P = \Lambda^k(\mathbb{C}^n). \tag{8.12}$$

PROOF. When $m < k$ the ideal Q_k^m vanishes, so the result is trivial.

Consider the case in (8.12), when $m = k$. A homogeneous polynomial of degree k in k variables (each in $\mathbb{R}^n$) which vanishes at the origin in each variable is necessarily linear in each variable, i.e. is just a k-multilinear function. Given such a multilinear function $q(y_1, \ldots, y_k)$ the condition that $bq = 0$ is just that

$$q(y_1, \ldots, y_{k-1}, -y_1 - y_2 - \cdots - y_{k-1}) \equiv 0. \tag{8.13}$$

Using the linearity in the last variable the left side can be expanded as a sum of $k-1$ functions each quadratic in one variables y_j and linear in the rest. Thus the vanishing of the sum implies the vanishing of each, so

$$q(y_1, \ldots, y_{k-1}, y_j) \equiv 0\ \forall\ j = 1, \ldots, k-1.$$

This is the statement that the multlinear function q is antisymmetric between the jth and kth variables for each $j < k$. Since these exchange maps generate the permutation group, q is necessarily totally antisymmetric. This proves the isomorphism (8.12) since $\Lambda^k(\mathbb{C}^n)$ is the space of complex-valued totally antisymmetric k-linear forms.[3]

Thus it remains to consider the case $m \geq k+1$. Consider a general element $q \in Q_k^m \cap J_P$. To show that it is in $Q_k^m \cdot J_P$ we manipulate it, working modulo $Q_k^m \cap J_P$, and use induction over k. Decompose q as a sum of terms q_l, each homogeneous in the first variable, y_1, of degree l. Since q vanishes at $y_1 = 0$ the first term is q_1, linear in y_1. The condition $bq = 0$, i.e. $q \in J_P$, is again just (8.13). Expanding in the last variable shows that the only term in bq which is linear in y_1 is

$$q_1(y_1, \dots, y_{k-1}, -y_2 - \cdots - y_{k-1}).$$

Thus the coefficient of $y_{1,i}$, the ith component of y_1 in q_1, is an element of Q_{k-1}^{m-1} which is in the ideal $J_P(\mathbb{R}^{k-1})$, i.e. for $k-1$ variables. This ideal is generated by the components of $y_2 + \cdots + y_k$. So we can proceed by induction and suppose that the result is true for less than k variables for all degrees of homogeneity. Writing $y_2 + \cdots + y_k = (y_1 + y_2 + \cdots + y_k) - y_1$ It follows that, modulo $Q_k^m \cdot J_P$, q_1 can be replaced by a term of one higher homogeneity in y_1. Thus we can assume that $q_i = 0$ for $i < 2$. The same argument now applies to q_2; expanded as a polynomial in y_1 the coefficients must be elements of $Q_{k-1}^{m-2} \cap J_P$. Thus, unless $m - 2 = k - 1$, i.e. $m = k+1$, they are, by the inductive hypothesis, in $Q_{k-1}^{m-2} \cdot J_P(\mathbb{R}^{k-1})$ and hence, modulo $Q_k^m \cdot J_P$, q_2 can be absorbed in q_3. This argument can be continued to arrange that $q_i \equiv 0$ for $i < m - k + 1$. In fact $q_i \equiv 0$ for $i > m - k + 1$ by the assumption that $q \in Q_k^m$.

Thus we are reduced to the assumption that $q = q_{m-k+1} \in Q_k^m \cap J_P$ is homogeneous of degree $m-k+1$ in the first variable. It follows that it is multilinear in the last $k-1$ variables. The vanishing of bq shows that it is indeed totally antisymmetric in these last $k-1$ variables. Now for each non-zero monomial consider the map $J : \{1, 2, \dots, n\} \longrightarrow \mathbb{N}_0$ such that $J(i)$ is the number of times a variable $y_{l,i}$ occurs for some $1 \leq l \leq k$. The decomposition into the sum of terms for each fixed J is preserved by b. It follows that we can assume that q has only terms corresponding to a fixed map J. If $J(i) > 1$ for any i then a factor $y_{1,i}$ must be present in q, since it is antisymmetric in the other $k-1$ variables. In this case it can be written $y_{1,i}q'$ where $bq' = 0$. Since q' is necessarily in the product of the indeals $J_2 \cdot \ldots J_k \cdot J_P$ it follows that $q' \in Q^m \cdot J_P$. Thus we may assume that $J(i) = 0$ or 1 for all i. Since the extra variables now play no rôle we may assume that $n = m$ is the degree of homogeneity and each index i occurs exactly once.

For convenience let us rotate the last $k-1$ variables so the last is moved to the first position. Polarizing q in the first variable, it can be represented uniquely as an n-multilinear function on $\mathbb{R}^n$ which is symmetric in the first $n-k+1$ variables, totally antisymmetric in the last $k-1$ and has no monomial with repeated index. Let $M_{k-1}(n)$ be the set of such multilinear funtions. The vanishing of bq now corresponds to the vanishing of the symmetrization of q in the first $n-k+2$ variables. By the antisymmetry in the second group of variables this gives a complex

$$M_n(n) \xrightarrow{b_n} M_{n-1}(n) \xrightarrow{b_{n-1}} \cdots \qquad \xrightarrow{b_2} M_1(n) \xrightarrow{b_1} M_0 \xrightarrow{b_0} 0.$$

[3]Really on the dual but that does not matter at this stage.

The remaining step is to show that this is exact.

Observe that $\dim(M_k(n)) = \binom{n}{k}$ since there is a basis of $M_k(n)$ with elements labelled by the subsets $I \subset \{1, \dots, n\}$ with k elements. Indeed let ω be a non-trivial k-multilinear function of k variables and let ω_I be this function on $\mathbb{R}^k \subset \mathbb{R}^n$ identified as the set of variables indexed by I. Then if $a \in M_0(n-k)$ is a basis of this 1-dimensional space and a_I is this function on the complementary $\mathbb{R}^{n-k}$ the tensor products $a_I \omega_I$ give a basis. Thus there is an isomorphism

$$M_k \ni q = \sum_{I \subset \{1,\dots,n\}, |I|=k} c_I a_I \otimes \omega_I \longmapsto \sum_{I \subset \{1,\dots,n\}, |I|=k} c_I \otimes \omega_I \in \Lambda^k(\mathbb{R}^n).$$

Transfered to the exterior algebra by this isomorphism the differential b is just contraction with the vector $e_1 + e_2 + \cdots + e_n$ (in the first slot). A linear transformation reducing this vector to e_1 shows immediately that this (Koszul) complex is exact, with the null space of b_k on $\Lambda^k(\mathbb{R}^n)$ being spanned by those ω_I with $1 \in I$ and the range of b_{k+1} spanned by those with $1 \notin I$. The exactness of this complex completes the proof of the lemma. $\square$

8.3. Hochschild homology of $\mathcal{C}^\infty(X)$

The first example of Hochschild homology that we shall examine is for the commutative algebra $\mathcal{C}^\infty(X)$ where X is any $\mathcal{C}^\infty$ manifold (compact or not). As noted above we need to replace the tensor product by some completion. In the present case observe that for any two manifolds X and Y

$$\mathcal{C}^\infty(X) \otimes \mathcal{C}^\infty(Y) \subset \mathcal{C}^\infty(X \times Y) \tag{8.14}$$

is dense in the $\mathcal{C}^\infty$ topology. Thus we simply declare the space of k-chains for Hochschild homology to be $\mathcal{C}^\infty(X^{k+1})$, which can be viewed as a natural completion[4] of $\mathcal{C}^\infty(X)^{\otimes(k+1)}$. Notice that the product of two functions can be written in terms of the tensor product as

$$a \cdot b = D^*(a \otimes b),\ a, b \in \mathcal{C}^\infty(X),\ D : X \ni z \longmapsto (z, z) \in X^2.$$

The variables in X^{k+1} will generally be denoted $z_0, z_1, \dots, z_k$. Consider the 'diagonal' submanifolds

$$D_{i,j} = \{(z_0, z_1, \dots, z_k); z_i = z_j\},\ i, j = 0, \dots, m,\ i \neq j. \tag{8.15}$$

We shall use the same notation for the natural embedding of X^k as each of these submanifolds, at least for $j = i+1$ and $i = 0$, $j = m$,

$$D_{i,i+1}(x_0, \dots, z_{m-1}) = (z_0, \dots, z_i, z_i, z_{i+1}, \dots, z_{m-1}) \in D_{i,i+1},\ i = 0, \dots, m-1$$
$$D_{m,0}(z_0, \dots, z_{m-1}) = (z_0, \dots, z_{m-1}, z_0).$$

[4] One way to justify this is to use results on smoothing operators. For finite dimensional linear spaces V and W the tensor product can be realized as

$$V \otimes W = \hom(W', V)$$

the space of linear maps from the dual of W to V. Identifying the topological dual of $\mathcal{C}^\infty(X)$ with $\mathcal{C}_c^{-\infty}(X; \Omega)$, the space of distributions of compact support, with the weak topology, we can identify the *projective* tensor product $\mathcal{C}^\infty(X) \hat{\otimes} \mathcal{C}^\infty(X)$ as the space of continuous linear maps from $\mathcal{C}_c^{-\infty}(X; \Omega)$ to $\mathcal{C}^\infty(X)$. These are precisely the smoothing operators, corresponding to kernels in $\mathcal{C}^\infty(X \times X)$.

Then the action of b' and b on the tensor products, and hence on all chains, can be written

$$b'\alpha = \sum_{i=0}^{m-1}(-1)^i D^*_{i,i+1}\alpha,\ b\alpha = b'\alpha + (-1)^m D^*_{m,0}\alpha. \tag{8.16}$$

THEOREM 8.2. *The differential b' is acyclic and the homology[5] of the complex*

$$\dots \xrightarrow{b} \mathcal{C}^\infty(X^{k+1}) \xrightarrow{b} \mathcal{C}^\infty(X^k) \xrightarrow{b} \dots \tag{8.17}$$

is naturally isomorphic to $\mathcal{C}^\infty(X;\Lambda^*)$.

Before proceeding to the proof proper we note two simple lemmas.

LEMMA 8.4. [6]*For any $j = 0,\dots,m-1$, each function $\alpha \in \mathcal{C}^\infty(X^{k+1})$ which vanishes on $D_{i,i+1}$ for each $i \le j$ can be written uniquely in the form*

$$\alpha = \alpha' + \alpha'',\ \alpha',\ \alpha'' \in \mathcal{C}^\infty(X^{k+1})$$

where α'' vanishes on $D_{i,i+1}$ for all $i \le j+1$ and α' is independent of z_{j+1}.

PROOF. Set $\alpha' = \pi^*_{j+1}(D^*_{j,j+1}\alpha)$ where $\pi_j : X^{k+1} \longrightarrow X^k$ is projection off the jth factor. Thus, essentially by definition, α' is independent of z_{j+1}. Moreover, $\pi_{j+1}D_{j,j+1} = \mathrm{Id}$ so $D^*_{j,j+1}\alpha' = D^*_{j,j+1}\alpha$ and hence $D^*_{j,j+1}\alpha'' = 0$. The decomposition is clearly unique, and for $i < j$,

$$D_{j,j+1} \circ \pi_{j+1} \circ D_{i,i+1} = D_{i,i+1} \circ F_{i,j} \tag{8.18}$$

for a smooth map $F_{i,j}$, so α' vanishes on $D_{i,i+1}$ if α vanishes there. $\square$

LEMMA 8.5. *For any finite dimensional vector space, V, the k-fold exterior power of the dual, $\Lambda^k V^*$, can be naturally identified with the space of functions*

$$\begin{aligned}\big\{u \in \mathcal{C}^\infty(V^k);\, u(sv) = s^k v,\ s \ge 0,\ u \restriction (V^i \times \{0\} \times V^{k-i-1}) = 0 \text{ for } i = 0,\dots,k-1 \\ \text{and } u \restriction G = 0,\ G = \{(v_1,\dots,v_k) \in V^k;\, v_1 + \dots + v_k = 0\}\big\}.\end{aligned} \tag{8.19}$$

PROOF. The homogeneity of the smooth function, u, on V^k implies that it is a homogeneous polynomial of degree k. The fact that it vanishes at 0 in each variable then implies that it is multlinear, i.e. is linear in each variable. The vanishing on G implies that for any j and any $v_i \in V$, $i \ne j$,

$$\sum_{i\ne j} u(v_1,\dots,v_{j-1},v_i,v_{j+1},\dots,v_k) = 0. \tag{8.20}$$

Since each of these terms is quadratic (and homogeneous) in the corresponding variable v_i, they must each vanish identically. Thus, u vanishes on $v_i = v_j$ for each $i \ne j$; it is therefore totally antisymmetric as a multlinear form, i.e. is an element of $\Lambda^k V^*$. The converse is immediate, so the lemma is proved. $\square$

[5] This homology is properly referred to as the *continuous* Hochschild homology of the topological algebra $\mathcal{C}^\infty(X)$.

[6] As pointed out to me by Maciej Zworski, this is a form of Hadamard's lemma.

Proof of Theorem 8.2. The H-unitality[7] of $\mathcal{C}^\infty(X)$ follows from the proof of Lemma 8.59 which carries over *verbatim* to the larger chain spaces.

By definition the Hochschild homology in degree k is the quotient

$$\mathrm{HH}_k(\mathcal{C}^\infty(X)) = \{u \in \mathcal{C}^\infty(X^{k+1}); bu = 0\}/b\mathcal{C}^\infty(X^{k+2}). \tag{8.21}$$

The first stage in identifying this quotient is to apply Lemma 8.4 repeatedly. Let us carry through the first step separately, and then do the general case.

For $j = 0$, consider the decomposition of $u \in \mathcal{C}^\infty(X^{k+1})$ given by Lemma 8.4, thus

$$u = u_0 + u_{(1)},\ u_0 \in \pi_1^*\mathcal{C}^\infty(X^k),\ u_{(1)} \in J_1^{(k)} = \{u \in \mathcal{C}^\infty(X^{k+1}); u \restriction D_{0,1} = 0\}. \tag{8.22}$$

Now each of these subspaces of $\mathcal{C}^\infty(X^{k+1})$ is mapped into the corresponding subspace of $\mathcal{C}^\infty(X^k)$ by b; i.e. they define subcomplexes. Indeed,

$$u \in \pi_1^*\mathcal{C}^\infty(X^k) \Longrightarrow D_{0,1}^* u = D_{1,2}^* u \text{ so}$$
$$u = \pi_1^* v \Longrightarrow bu = \pi_1^* Bv,\ B^*v = -\sum_{i=1}^{k-1}(-1)^i D_{i,i+1}^* u + (-1)^k D_{k-1,0}^* v.$$

For the other term

$$bu_{(1)} = \sum_{i=1}^{k-1}(-1)^i D_{i,i+1}^* u_{(1)} + (-1)^k D_{k,0}^* u_{(1)} \Longrightarrow bu_{(1)} \in J_1^{(k-1)}. \tag{8.23}$$

Thus, $bu = 0$ is equivalent to $bu_0 = 0$ and $bu_{(1)} = 0$. From (8.3), defining an isomorphism by

$$E_{(k-1)} : \mathcal{C}^\infty(X^k) \longrightarrow \mathcal{C}^\infty(X^k),\ E_{(k-1)}v(z_1, \dots, z_k) = v(z_2, \dots, z_k, z_1), \tag{8.24}$$

it follows that

$$B = -E_{(k-1)}^{-1} b' E_{(k-1)} \tag{8.25}$$

is conjugate to b'. Thus B is acyclic so in terms of (8.22)

$$bu = 0 \Longrightarrow u - u_{(1)} = bw,\ w = \pi_1^* v'. \tag{8.26}$$

As already noted this is the first step in an inductive procedure, the induction being over $1 \le j \le k$ in Lemma 8.4. Thus we show inductively that

$$\begin{aligned} bu = 0 \Longrightarrow u - u_{(j)} = bw,& \\ u_{(j)} \in J_j^{(k)} = \{u \in \mathcal{C}^\infty(X^{k+1}); u \restriction D_{i,i+1} = 0,\ 0 \le i \le j-1\}.& \end{aligned} \tag{8.27}$$

For $j = 1$ this is (8.26). Proceeding inductively we may suppose that $u = u_{(j)}$ and take the decomposition of Lemma 8.4, so

$$u_{(j)} = u' + u_{(j+1)},\ u_{(j+1)} \in J_{j+1}^{(k)},\ u' = \pi_{j+1}^* v \in J_j^{(k)}. \tag{8.28}$$

Then, as before, $bu_{(j)} = 0$ implies that $bu' = 0$. Furthermore, acting on the space $\pi_{j+1}^*\mathcal{C}^\infty(X^k) \cap J_{(j)}^k$, b is conjugate to b' acting in $k+1-j$ variables. Thus, it is again acyclic, so $u_{(j)}$ and $u_{(j+1)}$ are homologous as Hochschild k-cycles.

[7] Meaning here the *continuous* H-unitality, that is the acyclicity of b' on the chain spaces $\mathcal{C}^\infty(X^{k+1})$.

The end point of this inductive procedure is that each k-cycle is homologous to an element of

$$J^{(k)} = J_k^{(k)} = \{u \in \mathcal{C}^\infty(X^{k+1}); D^*_{i,i+1}u = 0,\ i \le i \le k-1\}. \tag{8.29}$$

Acting on this space $bu = (-1)^k D^*_{k,0}u$, so we have shown that

(8.30)
$$\mathrm{HH}_k(\mathcal{C}^\infty(X)) = M^{(k)}/(M^{(k)} \cap b\mathcal{C}^\infty(X^{k+1})),\ M^{(k)} = \{u \in J^{(k)}; D^*_{k,o}u = 0\}.$$

Now consider the subspace

$$\begin{aligned}(8.31)\quad \tilde{M}^{(k)} = \{u \in \mathcal{C}^\infty(X^{k+1});\\ u = \sum_{\text{finite},\ 0\le j\le k-1} (f(z_j) - f(z_{j+1}))\, u_{f,j},\ u_{f,j} \in M^{(k)},\ f \in \mathcal{C}^\infty(X).\}.\end{aligned}$$

If $u = (f(z_j) - f(z_{j+1}))v$, with $v \in M^{(k)}$ set

$$\begin{aligned}(8.32)\quad w(z_0, z_1, \dots, z_j, z_{j+1}, z_{j+2}, \dots, z_{k+1})\\ = (-1)^j (f(z_j) - f(z_{j+1}))v(z_0, \dots, z_j, z_{j+2}, z_{j+3}, \dots, z_k).\end{aligned}$$

Then, using the assumed vanishing of v, $bw = u$.[8] Thus all the elements of $\tilde{M}^{(k)}$ are exact.

Let us next compute the quotient $M^{(k)}/\tilde{M}^{(k)}$. Linearizing in each factor of X around the submanifold $z_0 = z_1 = \cdots = z_k$ in V^k defines a map

$$\mu : M^{(k)} \ni u \longrightarrow u' \in \mathcal{C}^\infty(X; TX \otimes \cdots \otimes T^*X). \tag{8.33}$$

The map is defined by taking the term of homogeneity k in a normal expansion around the submanifold. The range space is therefore precisely the space of sections of the k-fold tensor product bundle which vanish on the subbundle defined in each fibre by $v_1 + \cdots + v_k = 0$. Thus, by Lemma 8.5, μ actually defines a sequence

$$0 \longrightarrow \tilde{M}^{(k)} \hookrightarrow M^{(k)} \xrightarrow{\mu} \mathcal{C}^\infty(X; \Lambda^k X) \longrightarrow 0. \tag{8.34}$$

Lemma 8.6. *For any k, (8.34) is a short exact sequence.*

Proof. So far I have a rather nasty proof by induction of this result, there should be a reasonably elementary argument. Any offers? □

From this the desired identification, induced by μ,

$$\mathrm{HH}_k(\mathcal{C}^\infty(X)) = \mathcal{C}^\infty(X; \Lambda^k X) \tag{8.35}$$

follows, once is is shown that no element $u \in M^{(k)}$ with $\mu(u) \neq 0$ can be exact. This follows by a similar argument. Namely if $u \in M^{(k)}$ is exact then write $u = bv$, $v \in \mathcal{C}^\infty(X^k)$ and apply the decomposition of Lemma 8.4 to get $v = v_0 + v_{(1)}$. Since $u = 0$ on $D_{1,0}$ it follows that $bv_0 = 0$ and hence $u = bv_{1)}$. Proceeding inductively we conculde that $u = bv$ with $v \in M^{(k+1)}$. Now, $\mu(bv) = 0$ by inspection. □

[8] Notice that $v(z_0, \dots, z_j, z_{j+2}, \dots, z_{k+1})$ vanishes on $z_{i+1} = z_i$ for $i < j$ and $i > j+1$ and also on $z_0 + z_1 + \cdots + z_{k+1} = 0$ (since it is independent of z_{j+1} and $bv = 0$.

8.4. Commutative formal symbol algebra

As a first step towards the computation of the Hochschild homology of the algebra $\mathcal{A} = \Psi^{\mathbb{Z}}(X)/\Psi^{-\infty}(X)$ we consider the formal algebra of symbols with commutative product. Thus,

$$\mathcal{A} = \{(a_j)_{j=-\infty}^{\infty}; a_j \in \mathcal{C}^\infty(S^*X; P^{(j)}),\ a_j = 0 \text{ for } j >> 0\}. \tag{8.36}$$

Here $P^{(k)}$ is the line bundle over S^*X with sections consisting of the homogeneous functions of degree k on $T^*X \setminus 0$. The multiplication is as functions on $T^*X \setminus 0$, so

$$(a_j) \cdot (b_j) = (c_j),\ c_j = \sum_{k=-\infty}^{\infty} a_{j-k} b_k$$

using the fact that $P^{(l)} \otimes P^{(k)} \equiv P^{(l+k)}$. We take the completion of the tensor product to be

$$\begin{aligned}\mathcal{B}^{(k)} = \{u \in \mathcal{C}^\infty((T^*X \setminus 0)^{k+1}); u = \sum_{\text{finite}} u_I,\\ u_I \in \mathcal{C}^\infty(S^*X; P^{(I_0)} \otimes P^{(I_1)} \otimes \cdots \otimes P^{(I_k)}),\ |I| = k\}.\end{aligned} \tag{8.37}$$

That is, an element of $\mathcal{B}^{(k)}$ is a finite sum of functions on the $(k+1)$-fold product of $T^*X \setminus 0$ which are homogeneous of degree I_j on the jth factor, with the sum of the homogeneities being k. Then the Hochschild homology is the cohomology of the subcomplex of the complex for $\mathcal{C}^\infty(T^*X)$

$$\cdots \xrightarrow{b} \mathcal{B}^{(k)} \xrightarrow{b} \mathcal{B}^{(k-1)} \xrightarrow{b} \cdots \tag{8.38}$$

THEOREM 8.3. *The cohomology of the complex* (8.38) *for the commutative product on $\mathcal{A}$ is*

$$\mathrm{HH}_k(\mathcal{A}) \equiv \{\alpha \in \mathcal{C}^\infty(T^*X \setminus 0; \Lambda^k(T^*X); \alpha \text{ is homogeneous of degree } k\}. \tag{8.39}$$

8.5. Hochschild chains for $\Psi^{\mathbb{Z}}_{\mathrm{phg}}(X)/\Psi^{-\infty}_{\mathrm{phg}}(X)$

The completion of the tensor product that we take to define the Hochschild homology of the 'full symbol algebra' is the same space as in (8.37) but with the non-commutative product derived from the quantization map for some Riemann metric on X. Since the product is given as a formal sum of bilinear differential operators it can be take to act on an pair of factors.

$$\ldots \xrightarrow{b^{(*)}} \mathcal{B}^{(k)} \xrightarrow{b^{(*)}} \mathcal{B}^{(k-1)} \xrightarrow{b^{(*)}} \ldots \tag{8.40}$$

The next, and major, task of this chapter is to describe the cohomology of this complex.

THEOREM 8.4. *The Hochschild homolgy of the algebra, $\Psi^{\mathbb{Z}}_{\mathrm{phg}}(X)/\Psi^{-\infty}_{\mathrm{phg}}(X)$, of formal symbols of pseudodifferential operators of integral order, identified as the cohomology of the complex* (8.40), *is naturally identified with two copies of the cohomology of S^*X*[9]

$$\mathrm{HH}_k(\mathcal{A}; \circ) \equiv H^{2n-k}(S^*X) \oplus H^{2n-1-k}(S^*X). \tag{8.41}$$

[9] In particular the Hochschild homology vanishes for $k > 2\dim X$. For a precise form of the identification in (8.41) see (9.1).

8.6. Semi-classical limit and spectral sequence

The 'classical limit' in physics, especially quantuum mechanics, is the limit in which physical variables become commutative, i.e. the non-commutative coupling between position and momentum variables vanishes in the limit. Formally this typically involves the replacement of Planck's constant by a parameter $h \to 0$. A phenomenon is 'semi-classical' if it can be understood at least in Taylor series in this parameter. In this sense the Hochschild homology of the full symbol algebra is semi-classical and (following [**1**]) this is how we shall compute it.

The parameter h is introduced directly as an isomorphism of the space $\mathcal{A}$

$$L_h : \mathcal{A} \longrightarrow \mathcal{A},\ L_h(a_j)_{j=-\infty}^{*} = (h^j a_j)_{j=-\infty}^{*},\ h > 0.$$

Clearly $L_h \circ L_{h'} = L_{hh'}$. For $h \neq 1$, L_h is *not* an algebra morphism, so induces a 1-parameter family of products

$$\alpha \star_h \beta = (L_h^{-1})(L_h\alpha \star L_h\beta). \tag{8.42}$$

In terms of the differential operators, associated to quantization by a particular choice of Riemann metric on X this product can be written

$$\alpha \star_h \beta = (c_j)_{j=-\infty}^{*},\ c_j = \sum_{k=0}^{*} \sum_{l=-*}^{*} h^k P_k(a_{j-l-k}, b_l). \tag{8.43}$$

It is important to note here that the P_k, as differential operators on functions on T^*X, do only depend on k, which is the difference of homogeneity between the product $a_{j-l+k}b_l$, which has degree $j+k$ and c_j, which has degree j.

Since $\mathcal{A}$ with product $\star_h$ is a 1-parameter family of algebras, i.e. a deformation of the algebra $\mathcal{A}$ with product $\star = \star_1$, the Hochschild homology is 'constant' in h. More precisely the map L_h induces a canonical isomorphism

$$L_h^* : \mathrm{HH}_k(\mathcal{A}; \star_h) \equiv \mathrm{HH}_k(\mathcal{A}; \star).$$

The dependence of the product on h is smooth, so it is reasonable to expect the cycles to have smooth representatives as $h \to 0$. To investigate the consider Taylor series in h and define

$$\begin{aligned} F_{p,k} = \big\{\alpha \in \mathcal{B}^{(k)};\ \exists\ \alpha(h) \in \mathcal{C}^\infty([0,1)_h; \mathcal{B}^{(k)}) \text{ with } \alpha(0) = \alpha \text{ and}\\ b_h\alpha \in h^p\mathcal{C}^\infty([0,1)_h; \mathcal{B}^{k-1)}\big\}, \end{aligned} \tag{8.44}$$

$$\begin{aligned} G_{p,k} = \big\{\alpha \in \mathcal{B}^{(k)};\ \exists\ \beta(h) \in \mathcal{C}^\infty([0,1)_h; \mathcal{B}^{(k+1)}) \text{ with}\\ b_h\beta(h) \in h^{p-1}\mathcal{C}^\infty([0,1)_h; \mathcal{B}^{(k)} \text{ and } (t^{-p+1}b_h\beta)(0) = \alpha\big\}. \end{aligned} \tag{8.45}$$

Here b_h is the differential defined by the product $\star_h$.

Notice that the $F_{p,k}$ decrease with increasing p, since the condition becomes stronger, while $G_{p,k}$ increases with p, the condition becoming weaker.[10] We define the 'spectral sequence' corresponding to this filtration by

$$E_{p,k} = F_{p,k}/G_{p,k}.$$

[10] If $\alpha \in G_{p,k}$ and $\beta(h)$ is the 1-parameter family of chains whose existence is required for the definition then $\beta'(h) = h\beta(h)$ satisfies the same condition with p increased to show that $\alpha \in G_{p+1,k}$.

These can also be defined successively, in the sense that if

$$F'_{p,k} = \{u \in E_{p-1,k}; u = [u'],\ u' \in F_{p,k}\}$$
$$G'_{p,k} = \{e \in E_{p-1,k}; u = [u'],]u' \in G_{p,k}\}$$
$$\text{then } E_{p,k} \equiv F'_{p,k}/G'_{p,k}.$$

The basic idea[11] of a spectral sequence is that each $E_p = \bigoplus_k E_{p,k}$, has defined on it a differential such that the next spaces, forming E_{p+1}, are the cohomology space for the complex. This is easily seen from the definitions of $F_{p,k}$ as follows. If $\alpha \in F_{p,k}$ let $\beta(t)$ be a 1-parameter family of chains as in the defintion. Then consider

$$\gamma(t^{-p}b_h\beta)(0) \in \mathcal{B}^{(k-1)}. \tag{8.46}$$

This depends on the choice of β, but only up to a term in $G_{p,k-1}$. Indeed, let $\beta'(t)$ is another choice of extension of α satisfying the condition that $b_h\beta' \in h^p\mathcal{C}^\infty([0,1);\mathcal{B}^{(k-1)}$ and let γ' be defined by (8.46) with β replaced by β'. Then $\delta(t) = t^{-1}(\beta(h) - \beta'(h))$ satisfies the requirements in the definition of $G_{p,k-1}$, i.e. the difference $\gamma' - \gamma \in G_{p,k-1}$. Similarly, if $\alpha \in G_{p,k}$ then $\gamma \in G_{p,k}$.[12] The map so defined is a differential

$$b_{(p)} : E_{p,k} \longrightarrow E_{p,k-1},\ b^2_{(p)} = 0.$$

This follows from the fact that if $\mu = b_{(p)}\alpha$ then, by definition, $\mu = (t^{-p}b_h\beta)(0)$, where $\alpha = \beta(0)$. Taking $\lambda(t) = t^{-p}b_h\beta(t)$ as the extension of μ it follows that $b_h\lambda = 0$, so $b_{(p)}\mu = 0$.

Now, it follows directly from the definition that $F_{0,k} = E_{0,k} = \mathcal{B}^{(k)}$ since $G_{0,k} = \{0\}$. Furthermore, the differential $b_{(0)}$ induced on E_0 is just the Hochschild differential for the limiting product, $\star_0$, which is the commutative product on the algebra. Thus, Theorem 8.3 just states that

$$E_{1,k} = \bigoplus_{k=-\infty}^{*} \{u \in \mathcal{C}^\infty(T^*X \setminus 0; \Lambda^k); u \text{ is homogeneous of degree } k\}.$$

To complete the proof of Theorem 8.4 it therefore suffices to show that

$$E_{2,k} \equiv H^{2n-k}(S^*X) \oplus H^{2n-1-k}(S^*X), \tag{8.47}$$

$$E_{p,k} = E_{2,k},\ \forall\ p \geq 2, \text{ and} \tag{8.48}$$

$$\mathrm{HH}_k(\Psi^{\mathbb{Z}}_{\mathrm{phg}}(X)/\Psi^{-\infty}_{\mathrm{phg}}(X)) = \lim_{p\to\infty} E_{p,k}. \tag{8.49}$$

The second and third of these results are usually described, respectively, as the 'degeneration' of the spectral sequence (in this case at the 'E_2 term') and the 'convergence' of the spectral sequence to the desired cohomology space.

8.7. The E_2 term

As already noted, the $E_{1,k}$ term in the spectral sequence consists of the formal sums of k-forms, on $T^*X \setminus 0$, which are homogeneous under the $\mathbb{R}^+$ action. The E_2 term is the cohomology of the complex formed by these spaces with the differential $b_{(1)}$, which we proceed to compute. For simplicity of notation, consider the formal

[11] Of Leray I suppose, but I am not really sure.

[12] Indeed, α is then the value at $h = 0$ of $\beta(t) = t^{-p+1}b_h\phi(t)$ which is by hypothesis smooth; clearly $b_h\beta \equiv 0$.

tensor prodoct rather than its completion. As already noted, for any $\alpha \in \mathcal{B}^{(k)}$ the function $b_h\alpha$ is smooth in h and from the definition of b,

$$(8.50)\quad \frac{d}{dh}b_h\alpha(0) = \sum_{i=0}^{k-1}(-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes P_1(a_{i+1}, a_i) \otimes a_{i+2} \otimes \cdots \otimes a_k \\ + (-1)^k P_1(a_0, a_k) \otimes a_1 \otimes \cdots \otimes a_{k-1},\ \alpha = a_0 \otimes \cdots \otimes a_k.$$

The general case is only more difficult to write, not different.[13] This certainly determines $b_1\alpha$ if α is a superposition of such terms with $b_0\alpha = 0$. Although (8.50) is explicit, it is not given directly in terms of the representation of α, assumed to satisfy $b_0\alpha = 0$ as a form on $T^*X \setminus 0$.

To get such an explicit formula we shall use the symplectic analogue of the Hodge isomorphism. Recall that in any local coordinates on X, x_i, $i = 1, \dots, n$, induce local coordinates x_i, ξ_i in the part of T^*X lying above the coordinate patch. In these canonical coordinates the symplectic form (which determines the Poisson bracket) is given by

$$(8.51)\qquad \omega = \sum_{k=1}^{n} d\xi_k \wedge dx_k.$$

This 2-form is non-degenerate, i.e. the n-fold wedge product $\omega^n \neq 0$. In fact this volume form fixes an orientation on T^*X. The symplectic form can be viewed as a non-degenerate antisymmetric bilinear form on $T_q(T^*X)$ at each point $q \in T^*X$, and hence by duality as a bilineear form on $T_q^*(T^*X)$. We denote this form in the same way as the Poisson bracket, since with the convention

$$\{a, b\}(q) = \{da, db\}_q$$

they are indeed the same. As a non-degenerate bilinear form on T^*Y, $Y = T^*X$ this also induces a bilinear form on the tensor algebra, by setting

$$\{e_1 \otimes \cdots \otimes e_k, f_1 \otimes \cdots \otimes f_k,\} = \prod_j \{e_j, f_j\}.$$

These bilinear forms are all antisymmetric and non-degenerate and restrict to be non-degnerate on the antisymmetric part, $\Lambda^k Y$, of the tensor algebra. Thus each of the form bundles has a bilinear form defined on it, so there is a natural isomorhism

$$(8.52)\qquad W_\omega : \Lambda_q^k Y \longrightarrow \Lambda_q^{2n-k} Y,\ \alpha \wedge W_\omega\beta = \{\alpha, \beta\}\omega^n,\ \alpha, \beta \in \mathcal{C}^\infty(Y, \Lambda^k Y),$$

for each k.

Lemma 8.7. *In canonical coordinates, as in* (8.51)*, consider the basis of k-forms given by all increasing subsequences of length k, $I : \{1, 2, \dots, k\} \longrightarrow \{1, 2, \dots, 2n\}$, and setting*

$$(8.53)\qquad \alpha_I = dz_{I(1)} \wedge dz_{I(2)} \wedge \cdots \wedge dz_{I(k)}, \\ (z_1, z_2, \dots, z_{2n}) = (x_1, \xi_1, x_2, \xi_2, \dots, x_n, \xi_n).$$

In terms of this ordering of the coordinates

$$(8.54)\qquad W_\omega(\alpha_I) = (-1)^{N(I)}\alpha_{W(I)}$$

[13] If you feel it necessary to do so, resort to an argument by continuity towards the end of this discussion.

where $W(I)$ is obtained from I by considering each pair $(2p-1, 2p)$ for $p = 1, \dots, n$, erasing it if it occurs in the image of I, inserting it into I if neither $2p-1$ nor $2p$ occurs in the range of I and if exactly one of $2p-1$ and $2p$ occurs then leaving it unchanged; $N(I)$ is the number of times $2p$ appears in the range of I without $2p-1$.

PROOF. The Poisson bracket pairing gives, on 1-forms,

$$-\{dx_j, d\xi_j\} = 1 = \{d\xi_j, dx_j\}$$

with all other pairings zero. Extending this to k-forms gives

$$\begin{aligned}\{\alpha_I, \alpha_J\} &= 0 \text{ unless } (I(j), J(j)) = (2p-1, 2p) \text{ or } (2p, 2p-1)\ \forall\ j \text{ and}\\ \{\alpha_I, \alpha_J\} &= (-1)^N, \text{ if } (I(j), J(j)) = (2p-1, 2p) \text{ for } N \text{ values of } j\\ &\qquad \text{and } (I(j), J(j)) = (2p-1, 2p) \text{ for } N-k \text{ values of } j.\end{aligned}$$

From this, and (8.52), (8.54) follows. □

From this proof if also follows that $N(W(I)) = N(I)$, so $W_\omega^2 = \mathrm{Id}$. We shall let

$$\delta_\omega = W_\omega \circ d \circ W_\omega \tag{8.55}$$

denote the differential operator obtained from d by conjugation,

$$\delta_\omega : \mathcal{C}^\infty(T^*X \setminus 0; \Lambda^k) \longrightarrow \mathcal{C}^\infty(T^*X \setminus 0, \Lambda^{k-1}).$$

By construction $\delta_\omega^2 = 0$. The exterior algebra of a symplectic manifold with this differential is called the Koszul complex.[14] All the α_I are closed so

$$\begin{aligned}\delta_\omega(a\alpha_I) &= W_\omega\big(\sum_j \frac{\partial a}{\partial z_j} dz_j\big) \wedge (-1)^{N(I)} \alpha_{W(I)}\\ &= \sum_j \frac{\partial a}{\partial z_j} (-1)^{N(I)} W_\omega(dz_j \wedge \alpha_{W(I)}),\end{aligned} \tag{8.56}$$

Observe that[15]

$$\begin{aligned}W_\omega\big(dz_{2p-1} \wedge \alpha_{W(I)}\big) &= \iota_{\partial/\partial z_{2p}} \alpha_I\\ W_\omega\big(dz_{2p} \wedge \alpha_{W(I)}\big) &= \iota_{\partial/\partial z_{2p-1}} \alpha_I,\end{aligned}$$

where, ι_v denotes contraction with the vector field v. We therefore deduce the following formula for the action of the Koszul differential

$$\delta_\omega(a\alpha_I) = \sum_{i=1}^{2n} (H_{z_i} a) \iota_{\partial/\partial z_i} \alpha_I. \tag{8.57}$$

LEMMA 8.8. *With E_1 identified with the formal sums of homogeneous forms on $T^*X \setminus 0$, the induced differential is*

$$b_{(1)} = \frac{1}{i} \delta_\omega. \tag{8.58}$$

---

[14] Up to various sign conventions of course!

[15] Check this case by case, as the range of I meets the pair $\{2p-1, 2p\}$ in $\{2p-1, 2p\}$, $\{2p-1\}$, $\{2p\}$ or $\emptyset$. Both sides of the first equation are zero in the second and fourth case as are both sides of the second equation in the third and fourth cases. In the remaining four individual cases it is a matter of checking signs.

PROOF. We know that the bilinear differential operator $2iP_1$ is the Poisson bracket of functions on T^*X. Thus (8.50) can be written

$$(8.59)\quad 2ib_1\alpha = \sum_{i=0}^{k-1}(-1)^i a_0\otimes\cdots\otimes a_{i-1}\otimes\{a_{i+1},a_i\}\otimes a_{i+2}\otimes\cdots\otimes a_k \\ +(-1)^k\{a_0,a_k\}\otimes a_1\otimes\cdots\otimes a_{k-1},\ \alpha=a_0\otimes\cdots\otimes a_k.$$

The form to which this maps under the identification of E_2 is just

$$(8.60)\quad 21b_1\alpha = \sum_{i=0}^{k-1}(-1)^i a_0\wedge da_{i-1}\wedge\cdots\wedge d\{a_{i+1},a_i\}\wedge da_{i+2}\wedge a_k \\ +(-1)^k\{a_0,a_k\}\wedge da_1\wedge\cdots\wedge da_{k-1}$$

Consider the basis elements α_I for k-forms. These arise as the images of the corresponding functions in local coordinates on X^{k+1}

$$\tilde\alpha_I(z_0,z_1,\dots,z_k)=\sum_\sigma(-1)^{\operatorname{sgn}\sigma}(z_{1,\sigma I(1)}-z_{0,\sigma I(1)}) \\ \times z_{2,\sigma I(1)}-z_{1,\sigma I(1)})\dots(z_{1,\sigma I(m)}-z_{0,\sigma I(m-1)}).$$

Since these functions are defined in local coordinates they are not globally defined on $(T^*X\backslash 0)^{k+1}$. Nevertheless they can be localized away from $z_0=\cdots=z_m$ and then, with a coefficient $(a_j(z_0))^*_{j=-\infty}$, $a_j\in\mathcal{C}^\infty(T^*X\setminus 0)$ homogeneous of degree j with support in the coordinate patch, unambiguously define elements of E_1 which we can simply denote as $a(z_0)\tilde\alpha_I\in E_1$. These elements, superimposed over a coordinate cover, span E_1. Consider $b_{(1)}\tilde\alpha$ given by (8.60). In the sum, the terms with P_1 contracting between indices other than $0,1$ or $m,0$ must give zero because the Poisson bracket is constant in the 'middle' variable. Futhermore, by the antisymmetry of $\tilde\alpha$, the two remaining terms are equal so

$$ib_{(1)}(a\tilde\alpha_I)=\sum_{\sigma\in\mathcal{P}_k}(H_{z_{\sigma I(1)}}a)(-1)^{\operatorname{sgn}(\sigma)}dz_{\sigma I(2)}\wedge\cdots\wedge dz_{\sigma I(k)} \\ =\sum_i(H_{z_i}a)\iota_{\partial/\partial_i}\alpha_I.$$

Since this is just (8.57) the lemma follows. ☐

With this lemma we have identified the differential on the E_1 term in the spectral sequence with the exterior differential operator. To complete the identification (8.47) we need to compute the corresponding deRham groups.

PROPOSITION 8.1. *The cohomology of the complex*

$$\dots\xrightarrow{d}\sum_{j=-\infty}^{*}\mathcal{C}^\infty_{hom(j)}(T^*X\setminus 0;\Lambda^k)\xrightarrow{d}\sum_{j=-\infty}^{*}\mathcal{C}^\infty_{hom(j)}(T^*X\setminus 0;\Lambda^{k+1})\xrightarrow{d}\dots$$

in dimension k is naturally isomorphic to $H^k(S^*X)\oplus H^{k-1}(S^*X)$.

PROOF. Choose a metric on X and let $R=|\xi|$ denote the corresponding length function on $T^*X\setminus 0$. Thus, identifying the quotient $S^*X=(T^*X\setminus 0)/\mathbb{R}^+$ with

$\{R = 1\}$ gives an isomorphism $T^*X \setminus 0 \equiv S^*X \times (0, \infty)$. Under this map the smooth forms on $T^*X \setminus 0$ which are homogeneous of degree j are identified as sums

$$\begin{aligned} &\mathcal{C}^\infty_{\mathrm{hom}(j)}(T^*X \setminus 0, \Lambda^k) \ni \alpha_j \\ &= R^j\big(\alpha'_j + \alpha''_j \wedge \frac{dR}{R}\big),\ \alpha'_j \in \mathcal{C}^\infty(S^*X; \Lambda^k),\ \alpha''_j \in \mathcal{C}^\infty(S^*X; \Lambda^{k-1}). \end{aligned} \tag{8.61}$$

The action of the exterior derivative is then easily computed

$$\begin{gathered} d\alpha_j = \beta_j,\ \beta_j = R^j\big(\beta'_j + \beta_j -'' \wedge \frac{dR}{R}\big), \\ \beta'_j = d\alpha'_j,\ \beta''_j = d\alpha''_j + j(-1)^{k-1}\alpha'_j. \end{gathered}$$

Thus a k-form $(\alpha_j)^*_{j=-\infty}$ is closed precisely if it satisfies

$$j\alpha'_j = (-1)^k d\alpha''_j,\ d\alpha'_j = 0 \forall j. \tag{8.62}$$

It is exact if there exists a $(k-1)$-form $(\gamma_j)^*_{j=-\infty}$ such that

$$\alpha'_j = d\gamma'_j,\ \alpha''_j = d\gamma''_j + j(-1)^k\gamma'_j. \tag{8.63}$$

Since the differential preserves homogeneity it is only necessary to analyze these equations for each integral j. For $j \neq 0$, the second equation in (8.62) follows from the first and (8.63) then holds with $\gamma'_j = \frac{1}{j}(-1)^k\alpha''_j$ and $\gamma''_j = 0$. Thus the cohomology lies only in the subcomplex of homogeneous forms of degree 0. Then (8.62) and (8.63) become

$$d\alpha'_0 = 0,\ d\alpha''_0 = 0 \text{ and } \alpha'_0 = d\gamma'_0,\ \alpha''_0 = d\gamma''_0$$

respectively. This gives exactly the direct sum of $H^k(S^*X)$ and $H^{k-1}(S^*X)$ as the cohomology in degree k. The resulting isomorphism is independent of the choice of the radial function R, since another choice replaces R by Ra, where a is a smooth positive function on S^*X. In the decomposition (8.61), for $j = 0$, α''_0 is unchanged whereas α'_0 is replaced by $\alpha'_0 + \alpha''_0 \wedge d\log a$. Since the extra term is exact whenever α''_0 is closed it has no effect on the identification of the cohomology. □

Combining Proposition 8.1 and Lemma 8.8 completes the proof of (8.47). We make the identification a little more precise by locating the terms in E_2.

PROPOSITION 8.2. *Under the identification of E_1 with the sums of homogeneous forms on $T^*X \setminus 0$, E_2, identified as the cohomology of δ_ω, has a basis of homogeneous forms with the homogeneity degree j and the form degree k confined to*

$$k - j = \dim X,\ -\dim X \leq j \leq \dim X,\ \dim X \geq 2. \tag{8.64}$$

PROOF. Provided $\dim X \geq 2$, the cohomology of S^*X is isomorphic to two copies of the cohomology of X, one in the same degree and one shifted by $\dim X - 1$.[16] The classes in the first copy can be taken to be the lifts of deRham classes from X, while the second is spanned by the wedge of these same classes with the Todd class of S^*X. This latter, $n-1$, class restricts to each fibre to be non-vanishing. Thus in local representations the first forms involve only the base variable and in the second each terms has the maximum number, $n-1$, of fibre forms. The cohomology of the complex in Proposition 8.1 therefore consists of four copies of $H^*(X)$ consisting of these forms and the same forms wedged with dR/R.

[16] That is, just as though $S^*X = \mathbb{S}^{n-1} \times X$, where $n = \dim X$.

With this decomoposition of the cohomology consider the effect on it of the map W_ω. In each case the image forms are again homogeneous. A deRham class on X in degree l therefore has four images in E_2. One is a form of degree $k_1 = 2n - l$ which is homogeneous of degree $j_1 = n - l$. The second is a form of degree $k_2 = 2n - l - 1$ which is homogeneous of degree $j_2 = n - l - 1$. The third image is of form degree $k_3 = n - l + 1$ and homogeneous of degree $j_3 = -l + 1$ and the final image is of form degree $k_4 = n - l$ and is homogeneous of degree $j_4 = -l$. This gives the relations (8.64). $\square$

8.8. Degeneration and convergence

Now that the E_2 term in the spectral sequence has been explicitly computed, consider the induced differential, $b_{(2)}$ on it. Any homogeneous form representing a class in E_2 can be represented by a Hochshild chain α of the same homogeneity. Thus an element of E_2 in degree k corresponds to a function on $\mathcal{C}^\infty((T^*X)\backslash)^{k+1})$ which is separately homogeneous in each variable and of total homogeneity $k - n$. Furthermore it has an extension $\beta(t)$ as a function of the parameter h, of the same homogeneity, such that $b_t\beta(t) = t^2\gamma(t)$. Then $b_{(2)}\alpha = [\gamma(0)]$, the class of $\gamma(0)$ in E_2. Noting that the differential operator, P_j, which is the jth term in the Taylor series of the product $\star_h$ reduces homogeneity by j and that b_h depends multilinearly on $\star_h$ it follows tha $b_{(r)}$ must decrease homogeneity by r. Thus if the class $[\gamma(0)]$ must vanish in E_2 by (8.64). We have therefore shown that $b_{(2)} \equiv 0$, so $E_3 = E_2$. The same argument applies to the higher differentials, defining the $E_r \equiv E_2$ for $r \geq 2$, proving the 'degeneration' of the spectral sequence, (8.48).

The 'convergence' of the spectral sequence, (8.49), follows from the same analysis of homogneities. Thus, we shall define a map from E_2 to the Hochschild homology and show that it is an isomorphism.

8.9. Explicit cohomology maps

8.10. Hochschild holomology of $\Psi^{-\infty}(X)$

8.11. Hochschild holomology of $\Psi^{\mathbb{Z}}(X)$

8.12. Morita equivalence