CHAPTER 6

The wave kernel

Let us return to the subject of "good distributions" as exemplified by Dirac delta 'functions' and the Schwartz kernels of pseudodifferential operators. In fact we shall associate a space of "conormal distributions" with any submanifold of a manifold.

Thus let X be a $\mathcal{C}^\infty$ manifold and $Y \subset X$ a closed embedded submanifold – we can easily drop the assumption that Y is closed and even replace embedded by immersed, but let's treat the simplest case first! To say that Y is embedded means that each $\bar{y} \in Y$ has a coordinate neighbourhood U, in X, with coordinate $x_1, \dots, x_n$ in terms of which $\bar{y} = 0$ and

$$Y \cap U = \{x, = \dots = x_k = 0\}. \tag{6.1}$$

We want to define

$$I^*(X, Y; \Omega^{\frac{1}{2}}) \subset \mathcal{C}^{-\infty}(X; \Omega^{\frac{1}{2}}) \tag{6.2}$$

to consist of distributions which are singular only at Y and small "along Y."

So if $u \in \mathcal{C}_c^{-\infty}(U)$ then in local coordinates (6.1) we can identify u with $u' \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ so $u' \in H_c^s(\mathbb{R}^n)$ for some $s \in \mathbb{R}$. To say that u is 'smooth along Y' means we want to have

$$D_{x_{k+1}}^{\ell_1} \dots D_{x_n}^{\ell_{n-k}} u' \in H_c^{s'}(\mathbb{R}^n) \quad \forall\ \ell_1, \dots, \ell_{n-k} \tag{6.3}$$

and a fixed s', independent of ℓ (but just possibly different from the initial s); of course we can take $s = s'$. Now conditions like (6.3) do *not* limit the singular support of u' at all! However we can add a requirement that multiplication by a function which *vanishes* on Y makes u' smooth, by one degree, i.e.

$$x_1^{p_1} \dots x_k^{p_k} u' \in H^{s+|p|}(\mathbb{R}^n), |p| = p_1 + \dots + p_k. \tag{6.4}$$

This last condition implies

$$D_1^{q_1} \dots D_k^{q_k} x_1^{p_1} \dots x_k^{p_k} u' \in H^s(\mathbb{R}^n) \text{ if } |q| \le |p|. \tag{6.5}$$

Consider what happens if we rearrange the order of differentiation and multiplication in (6.5). Since we demand (6.5) for *all* p, q with $|q| \le |p|$ we can show in tial that

$$\forall\ |q| \le |p| \le L \tag{6.6}$$

$$\Longrightarrow$$

$$\prod_{i=1}^{L} (x_{j_i} D_{\ell_i}) u \in H^s(\mathbb{R}^n) \quad \forall \text{ pairs, } (j_{i,\ell_i}) \in (1, \dots, k)^2. \tag{6.7}$$

Of course we can combine (6.3) and (6.7) and demand

$$\prod_{i=1}^{L_2} D_{p_i} \prod_{i=1}^{L_1} (x_{j_i} D_{\ell_i}) u' \in H^s_c(\mathbb{R}^n)(j_j, \ell_i) \in (1, \ldots, k)^2 \tag{6.8}$$
$$\forall\ L_1, L_2\ p_i \in (k+1, \ldots u).$$

Problem 6.1. Show that (6.8) implies (6.3) and (6.4)

The point about (6.8) is that it is easy to interpret in a coordinate independent way. Notice that putting $\mathcal{C}^\infty$ coefficients in front of all the terms makes no difference.

Lemma 6.1. *The space of all $\mathcal{C}^\infty$ vector fields on $\mathbb{R}^n$ tangent to the submanifold $\{x_1 = \cdots = x_k = 0\}$ is spanning over $\mathcal{C}^\infty(\mathbb{R}^n)$ by*

$$x_i D_j, D_p\ \ i, j \le k, p > k. \tag{6.9}$$

Proof. A $\mathcal{C}^\infty$ vector field is just a sum

$$V = \sum_{j \le k} a_j D_j + \sum_{p > k} b_p D_p.$$

Notice that the D_p, for $p > k$, *are* tangent to $\{x_1 = \cdots = x_k = 0\}$, so we can assume $b_p = 0$. Tangency is then given by the condition

$$Vx)i = 0 \text{ and } \{x_1 = \cdots = x_k = 0\}, i = 1, \ldots, h$$

i.e. $a_j = \sum\limits_{\ell=1} a_{j\ell} x_\ell, 1 \le j \le h$. Thus

$$V = \sum_{\ell=1} a_{j\ell} x_\ell D_j$$

which proves (6.9). $\square$

This allows us to write (6.8) in the compact form

$$\mathcal{V}(\mathbb{R}^n, Y_k)^p u' \subset H^s_c(\mathbb{R}^n)\ \forall\ p \tag{6.10}$$

where $\mathcal{V}(\mathbb{R}^n, Y_k)$ is just the space of all $\mathcal{C}^\infty$ vector fields tangent to $Y_k = \{x_1 = \cdots = x_k = 0\}$. Of course the local coordinate just reduce vector fields tangent to Y to vector fields tangent to Y_k so the *invariant* version of (6.10) is

$$\mathcal{V}(X, Y)^p u \subset H^s(X; \Omega^{\frac{1}{2}})\ \forall\ p. \tag{6.11}$$

To interpret (6.11) we only need recall the (Lie) action of vector fields on half-densities. First for densities: The *formal* transpose of V is $-V$, so set

$$^L V\phi(\psi) = \phi(-V\psi)$$

if $\phi \in \mathcal{C}^\infty(X; \Omega), \psi \in \mathcal{C}^\infty(X)$. On $\mathbb{R}^n$ then becomes

$$\begin{aligned} \int {}^L V\phi \cdot \psi &= -\int \phi \cdot V\psi \\ &= -\int \phi(x) V\psi \cdot dx \\ &= \int (V\phi(x) + \delta_V \phi)\, \psi\ dx \\ \delta_V &= \sum_{i=1}^n D_i a_i \quad \text{ if } V = \Sigma a_i D_i. \end{aligned} \tag{6.12}$$

i.e.

$$L_V\left(\phi|dx|\right) = (V\phi)|dx| + \delta_V\phi. \tag{6.13}$$

Given the tensorial properties of density, set

$$L_V(\phi|dx|^t) = V\phi|dx|^t + t\delta_V\phi. \tag{6.14}$$

This corresponds to the *natural* trivialization in local coordinates.

DEFINITION 6.1. *If $Y \subset X$ is a closed embedded submanifold then*

$$\begin{aligned} IH^s\left(X, Y; \Omega^{\frac{1}{2}}\right) &= \left\{u \in H^s\left(X; \Omega^{\frac{1}{2}}\right) \text{ satisfying (11)}\right\} \\ I^*\left(X, Y; \Omega^{\frac{1}{2}}\right) &= \bigcup_s IH^s\left(X, Y; \Omega^{\frac{1}{2}}\right). \end{aligned} \tag{6.15}$$

Clearly

$$u \in I^*(X, Y; \Omega^{\frac{1}{2}}) \Longrightarrow u \restriction X \backslash Y \in \mathcal{C}^\infty\left(X\backslash Y; \Omega^{\frac{1}{2}}\right) \tag{6.16}$$

and

$$\bigcap_s IH^s\left(X, Y; \Omega^{\frac{1}{2}}\right) = \mathcal{C}^\infty\left(X; \Omega^{\frac{1}{2}}\right). \tag{6.17}$$

Let us try to understand these distributions *in some detail!* To do so we start with a very simple case, namely $Y = \{p\}$ is a point; so we only have one coordinate system. So construct $p = 0 \in \mathbb{R}^n$.

$$\begin{aligned} u \in I_c^*\left(\mathbb{R}^n, \{0\}; \Omega^{\frac{1}{2}}\right) &\Longrightarrow u = u'|dx|^{\frac{1}{2}} \text{ when} \\ x^\alpha D_x^\beta u' \in H_c^s(\mathbb{R}^n), &\quad s \text{ fixed } \ \forall\ |\alpha| \geq |\beta|. \end{aligned} \tag{6.18}$$

Again by a simple commutative argument this is equivalent to

$$D_x^\beta x^\alpha u' \in H_c^s(\mathbb{R}^n) \ \ \forall\ |\alpha| \geq |\beta|. \tag{6.19}$$

We can take the Fourier transform of (6.19) and get

$$\xi^\beta D_\xi^\alpha \hat{u}' \in \langle\xi\rangle^{-s} L^2(\mathbb{R}^n)\ \forall\ |\alpha| \geq |\beta|. \tag{6.20}$$

In this form we can just replace ξ^β by $\langle\xi\rangle^{|\beta|}$, i.e. (6.20) just says

$$D_\xi^\alpha \hat{u}'(\xi) \in \langle\xi\rangle^{-s-|\beta|} L^2(\mathbb{R}^n)\ \forall\ \alpha. \tag{6.21}$$

Notice that this is *very* similar to a symbol estimate, which would say

$$D_\xi^\alpha \hat{u}'(\xi) \in \langle\xi\rangle^{m-|\alpha|} L^\infty(\mathbb{R}^n)\ \forall\ \alpha. \tag{6.22}$$

LEMMA 6.2. *The estimate* (6.21) *implies* (6.22) *for any $m > -s-\frac{n}{2}$; conversely* (6.22) *implies* (6.21) *for any $s < -m - \frac{n}{2}$.*

PROOF. Let's start with the simple derivative, (6.22) implies (6.21). This really reduces to the case $\alpha = 0$. Thus

$$\langle\xi\rangle^M L^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \Longrightarrow M < -\frac{n}{2}$$

is the inequality

$$\left(\int |u|^2 d\xi\right)^{\frac{1}{2}} \leq \sup\langle\xi\rangle^{-M}|u| \left(\int \langle\xi\rangle^{2M} d\xi\right)^{\frac{1}{2}}$$

and

$$\int \langle\xi\rangle^{2M} d\xi = \int \left(1+|\xi|^2\right)^M d\xi < \infty \text{ iff } M < -\frac{n}{2}.$$

To get (6.22) we just show that (6.22) implies

$$\langle\xi\rangle^{s+|\alpha|} D_\xi^\alpha \hat{u}' \in \langle\xi\rangle^{m+s} L^\infty \subset L^2 \text{ if } m+s < -\frac{n}{2}. \tag{6.23}$$

The converse is a little trickier. To really see what is going on we can reduce (6.21) to a one dimensional version. Of course, near $\xi = 0$, (6.21) just says $\hat{u}'$ is $\mathcal{C}^\infty$, so we can assume that $|\xi| > 1$ on supp $\hat{u}'$ and introduce polar coordinates:

$$\xi = tw,\ w \in S^{n-1} t > 1. \tag{6.24}$$

Then

Exercise 2. Show that (6.21) (or maybe better, (6.20)) implies that

$$D_t^k P \hat{u}'(tw) \in t^{-s-k} L^2(\mathbb{R}^+ \times S^{n-1}; t^{n-1} dt dw)\ \forall\ k \tag{6.25}$$

for *any* $\mathcal{C}^\infty$ differential operator on S^{n-1}. □

In particular we can take P to be elliptic of any order, so (6.25) actually implies

$$\sup_w D_t^k P \hat{u}(t,w) \in t^{-s-k} L^2(\mathbb{R}^+; t^{n-1} dt) \tag{6.26}$$

or, changing the meaning to dt,

$$\sup_{w \in S^{n-1}} \mid D_t^k P \hat{u}(t,w) \mid \in t^{-s-k-\frac{n-1}{2}} L^2\left(\mathbb{R}^+, dt\right). \tag{6.27}$$

So we are in the one dimensional case, with s replaced by $s + \frac{n-1}{2}$. Now we can rewrite (6.27) as

$$D_t t^q D_t^k P \hat{u} \in t^r L^2,\ \forall\ k, r - q = -s - k - \frac{n-1}{2} - 1. \tag{6.28}$$

Now, observe the simple case:

$$f = 0 t < 1, D_t f \in t^r L^2 \Longrightarrow f \in L^\infty \text{ if } r < -\frac{1}{2} \tag{6.29}$$

since

$$\sup |f| = \int_{-\infty}^{t} t^r g \le \left(\int |g|^2\right)^{\frac{1}{2}} \cdot \left(\int_{-\infty}^{t} t^{2r}\right)^{\frac{1}{2}}.$$

Thus from (6.28) we deduce $\le (\int |g|^2)^{\frac{1}{2}}$

$$D_t^k P \hat{u} \in t^{-q} L^\infty \text{ if } r < -\frac{1}{2}, \text{ i.e. } -q > -s - k - \frac{n}{2}.$$

Finally this gives (6.22) when we go back from polar coordinates, to prove the lemma.

Definition 6.2. *Set, for* $m \in \mathbb{R}$,

$$I_c^m(\mathbb{R}^n, |[0\}) = \{u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n); \hat{u} \in S^{m-\frac{n}{4}}(\mathbb{R}^n)\}$$

with this definition,

$$IH^s(\mathbb{R}^n, \{0\}) \subset I_c^m(\mathbb{R}^n, \{0\}) \subset I_c^{s'}(\mathbb{R}^n, \{0\}) \tag{6.30}$$

provided

$$s > -m - \frac{n}{4} > s'.$$

Exercise 3. Using Lemma 24, prove (6.30) carefully.

So now what we want to do is to *define* $I_c^m(X, \{p\}; \Omega^{\frac{1}{2}})$ for any $p \in X$ by

$$u \in I_c^m(X, \{p\}; \Omega^{\frac{1}{2}}) \Longleftrightarrow F^*(\phi u) \in I_c^m(\mathbb{R}^n, \{0\}),\\ u \restriction X\backslash\{p\} \in \mathcal{C}^\infty(X\backslash\{p\}). \tag{6.31}$$

Here we have a little problem, namely we have to check that $I^m(\mathbb{R}^n, \{0\})$ is invariant under coordinate changes. Fortunately we can do this using (6.30).

LEMMA 6.3. *If $F : \Omega \longrightarrow \mathbb{R}^n$ is a diffeomorphism of a neighbourhood of* 0 *onto its range, with* $F(0) = 0$, *then*

$$F^*\{u \in I_c^m(\mathbb{R}^n, \{0\}; \mathrm{supp}(u) \subset F(\Omega)\} \subset I_c^m(\mathbb{R}^n, \{0\}). \tag{6.32}$$

PROOF. Start with a simple case, that F is *linear*. Then

$$u = (2\pi)^{-n} \int e^{ix\xi} a(\xi) d\xi, a \in S^{m-\frac{n}{4}}(\mathbb{R}^n).$$

so

$$\begin{aligned} F^*u &= (2\pi)^{-n} \int e^{iAx\cdot\xi} a(\xi) d\xi \;\; Fx = Ax \\ &= (2\pi)^{-n} \int i^{ix\cdot A^t\xi} a(\xi) d\xi \\ &= (2\pi)^{-n} \int e^{ix\cdot\eta} a((A^t)^{-1}\eta)|\det A|^{-1} d\eta. \end{aligned} \tag{6.33}$$

Since $a((A^t)^{-1}\eta)|\det A|^{-1} \in S^{m-\frac{n}{4}}\mathbb{R}^n)$ we have proved the result for linear transformations. We can always factorize F is

$$F = G \cdot A, \qquad A = (F_*) \tag{6.34}$$

so that the differential of G at 0 is the identity, i.e.

$$G(x) = x + O(|x|^2). \tag{6.35}$$

Now (6.35) allows us to use an homotopy method, i.e. set

$$G_s(x) = x + s(G(x) - x) \qquad s \in [0, 1) \tag{6.36}$$

so that $G_0 = \mathrm{Id}$, $G_s = G$. Such a 1-parameter family is given by integration of a vector field:

$$G_s^*\phi = \int_0^s \frac{d}{ds} G_s^* \phi dx \tag{6.37}$$

$$= \int_0 s\frac{d}{ds}\phi(G_x(x))ds \tag{6.38}$$

$$= \sum_1 \int_0^s \frac{d}{G_{s,i}}^{\xi} ds\, (\partial x_j \phi)\, (G_\delta(x))\, ds \tag{6.39}$$

$$= \int_0^s G_s^*\, (V_s\phi)\, ds \tag{6.40}$$

when the coefficients of V_s are

$$G_s^* V_{s,j} = \frac{d}{ds} G_{s,i}. \tag{6.41}$$

Now by (6.36) $\frac{d}{ds} G_{s,i} = \Sigma x_i x_j a_{ij}^s(x)$, so the same is true of the $V_{s,i}$, again using (6.36).

We can apply (6) to compute

$$G^* u = \int_0^1 G_s^* \left(V_s u\right) ds \tag{6.42}$$

when $u \in I_c^m\left(\mathbb{R}^n, \{0\}\right)$ has support near 0. Namely, by (6.30), $u \in IH_c^s\left(\mathbb{R}^n, \{0\}\right)$, with $s < -m - \frac{n}{4}$, but then

$$V_s u \in IH_c^{s+1}(\mathbb{R}^n, \{0\})$$

since $V = \sum_{i,j=1}^{n} b_{ij}^s(x) x_i x_j D_j$. Applying (6.30) again gives

$$G_s^*(V_s u) \in I^{m'}\left(\mathbb{R}^n, \{0\}\right), \ \forall \ m' > m - 1. \tag{6.43}$$

This proves the coordinates invariance. □

Last time we defined the space of conormal distributions associated to a closed embedded submanifold $Y \subset X$:

$$IH^s(X, Y) = \{u \in H^s(X); \mathcal{V}(X, Y)^k u \subset H^s(X) \ \forall \ k\} \tag{6.44}$$

$$IH^*(X, Y) = I^*(X, Y) = \bigcup sIH^s(X, Y). \tag{6.45}$$

Here $\mathcal{V}(X, Y)$ is the space of $\mathcal{C}^\infty$ vector fields on X tangent to Y. In the special case of a point in $\mathbb{R}^n$, say 0, we showed that

$$u \in I_c^*(\mathbb{R}^n), \{0\}) \Longleftrightarrow u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n) \text{ and } \hat{u} \in S^M(\mathbb{R}^n), M = M(u). \tag{6.46}$$

In fact we then defined the "standard order filtration" by

$$u \in I_c^m(\mathbb{R}^n, \{0\}) = \left\{u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n); \hat{u} \in S^{m-\frac{n}{4}}(\mathbb{R}^n)\right\}, \tag{6.47}$$

and found that

$$IH_c^s(\mathbb{R}^n, \{0\}) \subset I_c^{-s-\frac{n}{4}}(\mathbb{R}^n, \{0\}) \subset IH_c^{s'}(\mathbb{R}^n, \{0\}) \ \forall \ s' < s. \tag{6.48}$$

Our next important task is to show that $I_c^m(\mathbb{R}^n, \{0\})$ is invariant under coordinate changes. That is, if $F : U_1 \longrightarrow \mathbb{R}^n$ is a diffeomorphism of a neighbourhood of 0 to its range, with $F(0) = 0$, then we want to show that

$$F^* u \in I_c^m(\mathbb{R}^n, \{0\}) \ \forall \ u \in I_c^m(\mathbb{R}^n, \{0\}), \operatorname{supp}(u) \subset F(U_1). \tag{6.49}$$

Notice that we already know the coordinate independence of the Sobolev-based space, so using (6.48), we deduce that

$$F^* u \in I_c^{m'}(\mathbb{R}^n, \{0\}) \ \forall \ u \in I_c^m(\mathbb{R}^n, \{0\}), n' > m, \operatorname{supp}(u) \subset F(U_1). \tag{6.50}$$

In fact we get quite a lot more for our efforts:

LEMMA 6.4. *There is a coordinate-independent symbol map:*

$$I^m(X,\{p\};\Omega^{\frac12})@>\sigma^m_Y>>S^{m+\frac{n}{4}-[J]}\left(T^*_p\mathbb{R}^n;\Omega^{\frac12}\right) \tag{6.51}$$

given by the local prescription

$$\sigma^m_Y(u)=\hat{u}(\xi)|d\xi|^{\frac12} \tag{6.52}$$

where $u=v|dx|^{\frac12}$ *is local coordinate based at* 0, *with* ξ *the dual coordinate in* T^*_pX.

PROOF. Our definition of $I^m(X,\{p\};\Omega^{\frac12})$ is just that in any local coordinate based at p

$$u\in I^m(X,\{p\};\Omega^{\frac12})\Longrightarrow \phi u=v|dx|^{\frac12},v\in I^m_c(\mathbb{R}^n,\{0\}) \tag{6.53}$$

and $u\in\mathcal{C}^\infty(X\backslash\{p\};\Omega^{\frac12})$. So the symbol map is clearly supposed to be

$$\sigma^m(u)^{(\zeta)}\equiv_\downarrow \hat{v}(\xi)|d\xi|^{\frac12}\in S^{m+\frac{n}{4}-[1]}(\mathbb{R}^n;\Omega^{\frac12}) \tag{6.54}$$

where $\zeta\in T^*_pX$ is the 1-form $\zeta=\xi\cdot dx$ in the local coordinates. Of course we have to show that (6.54) is independent of the choice of coordinates. We already know that a change of coordinates changes $\hat{v}$ by a term of order $m-\frac{n}{4}-1$, which disappears in (6.54) so the residue class is determined by the Jacobian of the change of variables. From (6.33) we see exactly how $\hat{v}$ transforms under the Jacobian, namely as a density on

$$T^*_0\mathbb{R}^n: A\in GL(n,\mathbb{R})\Longrightarrow \widehat{A^*v}(\eta)|d\eta|^{\frac12}$$
$$=\hat{v}((A^t)^{-1}\eta)|\det A|^{-1}|dy|$$

so $\eta=A^t\xi\Longrightarrow$

$$\widehat{A^*v}(\eta)|dy|=\hat{v}(\xi)|d\xi|. \tag{6.55}$$

However recall from (6.53) that u is a half-density, so actually in the new coordinates $v'=A^*v\cdot|\det A|^{\frac12}$. This shows that (6.54) is well-defined.

Before going on to consider the general case let us note a few properties of $I^m(X,\{p\},\Omega^{\frac12})$: □

Exercise: Prove that

$$\begin{gathered}\text{If } P\in \mathrm{Diff}^m(X;\Omega^{\frac12}) \text{ then}\\ P: I^m(X,\{p\};\Omega^{\frac12})\longrightarrow I^{m+M}(X,\{p\};\Omega^{\frac12})\ \forall\ m\\ \sigma^{m+M}(Pu)=\sigma^M(P)\cdot\sigma^m(u).\end{gathered} \tag{6.56}$$

To pass to the general case of $Y\subset X$ we shall proceed in two steps. First let's consider a rather 'linear' case of $X=V$ a vector bundle over Y. Then Y can be identified with the zero section of V. In fact V is locally trivial, i.e. each $p\in y$ has a neighbourhood U s.t.

$$\pi^{-1}(U)\simeq\mathbb{R}^n_x\times U'_{y'},U'\subset\mathbb{R}^p \tag{6.57}$$

by a fibre-linear diffeomorphism projecting to a coordinate system on this base. So we want to define

$$I^m(V,Y;\Omega^{\frac12})=\{u\in I^*(V,Y;\Omega^{\frac12});$$

of $\phi \in \mathcal{C}_c^\infty(U)$ then under *any* trivialization (6.57)

$$\begin{gathered}\phi u(x,y) \equiv (2\pi)^{-n} \int e^{ix\cdot\xi} a(y,\xi) d\xi |dx|^{\frac{1}{2}}, \quad \mod \mathcal{C}^\infty, \\ a \in S^{m-\frac{n}{2}-\frac{p}{4}}(\mathbb{R}^p_y, \mathbb{R}^n_\xi).\end{gathered} \tag{6.58}$$

Here $p = \dim Y, p+n = \dim V$. Of course we have to check that (6.58) is coordinate-independent. We can write the order of the symbol, corresponding to u having order m as

$$m - \frac{\dim V}{4} + \frac{\dim Y}{2} = m + \frac{\dim V}{4} - \frac{\operatorname{codim} Y}{2}. \tag{6.59}$$

These additional shifts in the order are only put there to confuse you! Well, actually they make life easier later.

Notice that we know that the space is invariant under *any* diffeomorphism of the fibres of V, varying smoothly with the base point, it is also obvious that (6.58) in independent the choice of coordinates is U', since that just transforms these variables. So a general change of variables preserving Y is

$$(y,x) \longmapsto (f(y,x), X(y,x)) \quad X(y,0) = 0.$$

In particular f is a local diffeomorphism, which just changes the base variables in (6.58), so we can assume $f(y) \equiv y$. Then $X(y,x) = A(y)\cdot x + O(x^2)$. Since $x \longmapsto A(y)\cdot x$ is a fibre-by-fibre transformation it leaves the space invariant too, So we are reduced to considering

$$G : (y,x) \longmapsto (y, x + \Sigma a_{ij}(x,y)x_i x_j)y + \Sigma b_i(x,y)x_i. \tag{6.60}$$

To handle these transformations we can use the same homotopy method as before i.e.

$$G_s(x, y = (y+s)\sum_i b_i(x,y)x_i, x + s\sum_{i,j} a_{ij}(x,y)x_i x_j) \tag{6.61}$$

is a 1-parameter family of diffeomorphisms. Moreover

$$\frac{d}{ds} G_s^* u = G_s^* V_s k$$

where

$$V_s = \sum_{i,\ell} \beta_{i,\ell}(s,x,y)x_i \partial_{y_\ell} + \sum_{i,j,k} \alpha_{i,j,k} + \sum_{i,j,k} \alpha_{ijk}(\alpha,y,s)\ell_i, \ell_j \frac{\partial}{\partial x_k}.$$

So all we really have to show is that

$$V_s : I^M(U' \times \mathbb{R}^n, U' \times \{0\}) \longrightarrow I^{M-1}(U' \times \mathbb{R}^n, U' \times \{0\}) \;\forall\; M. \tag{6.62}$$

Again the spaces are $\mathcal{C}^\infty$-modules so we only have to check the action of $x_i \partial_{y_\ell}$ and $x_i x + j\partial_{x_k}$. These change the symbol to

$$D_{\xi_i} \partial_{y_\ell} a \text{ and } iD_{\xi_i} D_{\xi_j} \cdot \xi_k a$$

respectively, all one order lower.

This shows that the definition (6.58) is actually a reasonable one, i.e. as usual it suffices to check it for any covering by coordinate partition.

Let us go back and see what the symbol showed before.

LEMMA 6.5. *If*

$$u \in I^m(V, Y; \Omega^{\frac{1}{2}})u = v|dx|^{\frac{1}{2}}|d\xi|^{\frac{1}{2}}$$

defines an element

$$\sigma^m(u) \in S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(V^*; \Omega^{\frac{1}{2}})$$

independent of choices.

Last time we discussed the invariant symbol for a conormal distribution associated to the zero section of a vector bundle. It turns out that the general case is not any more complicated thanks to the "tubular neighbourhood" or "normal fibration" theorem. This compares $Y \hookrightarrow X$, a closed embedded submanifold, to the zero section of a vector bundle.

Thus at each point $y \in Y$ consider the normal space:

$$N_y Y = N_y\{X, Y\} = T_y x / T_y Y. \tag{6.63}$$

That is, a normal vector is just *any* tangent vector to X modulo tangent vectors to Y. These spaces define a vector bundle over Y :

$$NY = N\{X; Y\} = \bigsqcup_{y \in Y} N_y Y \tag{6.64}$$

where smoothness of a section is inherited from smoothness of a section of $T_y X$, i.e.

$$NY = T_y X / T_y Y. \tag{6.65}$$

Suppose $Y_i \subset X_i$ are $\mathcal{C}^\infty$ submanifolds for $i = 1, 2$ and that $F : X_1 \longrightarrow X_2$ is a $\mathcal{C}^\infty$ map such that

$$F(Y_1) \subset Y_2. \tag{6.66}$$

Then $F_* : T_y X_1 \longrightarrow T_{F(y)} X_2$, must have the property

$$F_* : T_y Y_1 \longrightarrow T_{F(y)} Y_2 \ \forall\ y \in Y_1. \tag{6.67}$$

This means that F_* defines a map of the normal bundles

$$\begin{tikzcd} F_* : NY_1 \arrow[r] \arrow[d] & NY_2 \arrow[d] \\ Y_1 \arrow[r, "F"'] & Y_2. \end{tikzcd} \tag{6.68}$$

Notice the very special case that $W \longrightarrow Y$ is a vector bundle, and we consider $Y \hookrightarrow W$ as the zero section. Then

$$N_y\{W; Y\} \longleftrightarrow W_y \quad \forall\ y \in Y \tag{6.69}$$

since

$$T_y W = T_y Y \oplus T_y(W_y) \quad \forall\ y \in W. \tag{6.70}$$

That is, the normal bundle to the zero section is *naturally* identified with the vector bundle itself.

So, suppose we consider $\mathcal{C}^\infty$ maps

$$f : B \longrightarrow N\{X; Y\} = NY \tag{6.71}$$

where $B \subset X$ is an open neighbourhood of the submanifold Y. We can demand that

$$f(y) = (y, 0) \in N_y Y \quad \forall\ y \in Y \tag{6.72}$$

which is to say that f induces the natural identification of Y with the zero section of NY and moreover we can demand

$$f_* : NY \longrightarrow NY \text{ is the identity.} \tag{6.73}$$

Here f_* is the map (6.68), so maps NY to the normal bundle to the zero section of NY, which we have just observed is naturally just NY again.

THEOREM 6.1. *For any closed embedded submanifold $Y \subset X$ there exists a normal fibration, i.e. a diffeomorphism (onto its range)* (6.71) *satisfing* (6.72) *and* (6.73)*; two such maps f_1, f_2 are such that $g = f_2 \circ f_1^{-1}$ is a diffeomorphism near the zero section of NY, inducing the identity on Y and inducing the identity* (6.73)*.*

PROOF. Not bad, but since it uses a little Riemannian geometry I will *not* prove it, see [ ], [ ]. (For those who know a little Riemannian geometry, f^{-1} can be taken as the exponential map near the zero section of NY, identified as a subbundle of $T_Y X$ using the metric.) Of course the uniqueness part is obvious. □

Actually we do *not* really need the global aspects of this theorem. Locally it is immediate by using local coordinates in which $Y = \{x_1 = \cdots = x_k = 0\}$.

Anyway using such a normal fibration of X near Y (or working locally) we can simply *define*

$$\begin{aligned} I^m(X, Y; \Omega^{\frac{1}{2}}) = \{&u \in \mathcal{C}^{-\infty}(X; \Omega^{\frac{1}{2}}); u \text{ is } \mathcal{C}^\infty \text{ in } X \backslash Y \text{ and} \\ &(f^{-1})^*(\phi u) \in I^m(NY, Y; \Omega^{\frac{1}{2}}) \text{ if } \phi \in \mathcal{C}^\infty(X), \operatorname{supp}(\phi) \subset B\}. \end{aligned} \tag{6.74}$$

Naturally we should check that the definition doesn't depend on the choice of f. This means knowing that $I^m(NY, Y; \Omega^{\frac{1}{2}})$ is invariant under g, as in the theorem, but we have already checked this. In fact notice that g is exactly of the type of (6.60). Thus we actually know that

$$\sigma^m(g^*u) = \sigma^m(u) \text{ in } S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(N^*Y; \Omega^{\frac{1}{2}}).$$

So we have shown that there is a coordinate invariance symbol map

$$\sigma^m : I^m(X, Y; \Omega^{\frac{1}{2}}) \longrightarrow S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(N^*Y; \Omega^{\frac{1}{2}}) \tag{6.75}$$

giving a short exact sequence

(6.76)

$$0 \hookrightarrow I^{m-1}(X, Y; \Omega^{\frac{1}{2}}) \longrightarrow I^m(X, Y; \Omega^{\frac{1}{2}}) @ > \sigma^m >> S^{m+\frac{n}{4}+\frac{p}{4}-[1]}(N^*Y; \Omega^{\frac{1}{2}}) \longrightarrow 0$$

$$\text{where } n = \dim X - \dim Y, p = \dim Y.$$

Asymptotic completeness carries over immediately. We also need to go back and check the extension of (6.56):

PROPOSITION 6.1. *If $Y \hookrightarrow X$ is a closed embedded submanifold and $A \in \Psi_c^m(X; \Omega^{\frac{1}{2}})$ then*

$$A : I^M(X, Y; \Omega^{\frac{1}{2}}) \longrightarrow I^{M+m}(X, Y; \Omega^{\frac{1}{2}}) \ \forall\ M \tag{6.77}$$

and

$$\sigma^{m+M}(Au) = \sigma^m(A)\sigma^m(A) \restriction N^*Y \sigma^M(u). \tag{6.78}$$

Notice that $\sigma^m(A) \in S^{m-[1]}(T^*X)$ so the product here makes perfectly good sense.

PROOF. Since everything in sight is coordinate-independent we can simply work in local coordinates where

$$X \sim \mathbb{R}^p_y \times \mathbb{R}^n_x, Y = \{x = 0\}. \tag{6.79}$$

Then $u \in I^m_c(X, Y; \Omega^{\frac{1}{2}})$ means just

$$u = (2\pi)^{-n} \int e^{ix\cdot\xi} a(y, \xi) d\xi \cdot |dx|^{\frac{1}{2}}, a \in S^{m-\frac{n}{4}+\frac{p}{4}}(\mathbb{R}^p, \mathbb{R}^n). \tag{6.80}$$

Similarly A can be written in the form

$$A = (2\pi)^{-n-p} \int e^{i(x-x')\cdot\xi + i(y-y')\cdot\eta} b(x, y, \xi, \eta) d\xi d\eta. \tag{6.81}$$

Using the invariance properties of the Sobolev based space if we write

$$A = A_0 + \Sigma x_j B_j, A_0 = q_L(b(0, y, \xi, \eta))$$

we see that $Au \in I^{m+M}(X, Y; \Omega^{\frac{1}{2}})$ is equivalent to $A_0 u \in I^{m+M}(X, Y; \Omega^{\frac{1}{2}})$. Then

$$A_0 u = (2\pi)^{-n-p} \int e^{ix\cdot\xi + i(y-y')\cdot\eta} b(0, y', \xi, \eta) b(y', \xi) dy' d\eta d\xi,$$

where we have put A_0 in right-reduced form. This means

$$A_0 u = (2\pi)^{-n} \int e^{ix\cdot\xi} c(y, \xi) d\xi$$

where

$$c(y, \xi) = (2\pi)^{-p} \int e^{i(y-y')\cdot\eta} b(0, y', \xi, \eta) a(y', \xi) dy' d\eta.$$

Regarding ξ as a parameter, this is, before y' integration, the kernel of a pseudo-differential operator is y. It can therefore be written in left-reduced form, i.e.

$$c(y, \xi) = (2\pi)^{-p} \int e^{i(y-y')\eta} e(y, \xi, \eta) d\eta dy' = e(y, \xi, 0) \tag{6.82}$$

where $e(y, \xi, \eta) = b(0, y, \xi, \eta) a(y, \xi)$ plus terms of order at most $m + M - \frac{n}{4} + \frac{p}{4} - 1$. This proves the formula (6.78). $\square$

Notice that if A is elliptic then $Au \in \mathcal{C}^\infty$ implies $u \in \mathcal{C}^\infty$, i.e. there are *no* singular solutions. Suppose that P is say a *differential* operator which is not elliptic and we look for solutions of

$$Pu \in \mathcal{C}^\infty(X\Omega^{\frac{1}{2}}). \tag{6.83}$$

How can we find them? Well suppose we try

$$u \in I^M(X, Y; \Omega^{\frac{1}{2}}) \tag{6.84}$$

for some submanifold Y. To know that u is singular we will want to have

$$\sigma(u) \text{ is elliptic on } N^*Y \tag{6.85}$$

(which certainly implies that $u \notin \mathcal{C}^\infty$).

The simplest case would be Y a hypersurface. In any case from (6.78) and (6.83) we deduce

$$\sigma^m(P) \cdot \sigma^M(u) \equiv 0. \tag{6.86}$$

So if we assume (6.85) then we *must* have

$$\sigma^m(P) \upharpoonright N^*Y = 0. \tag{6.87}$$

Definition 6.3. *A submanifold is said to be* characteristic *for a given operator* $P \in \mathrm{Diff}^m(X; \Omega^{\frac{1}{2}})$ *if* (6.87) *holds.*

Of course even if P *is* characteristic for y, and so (6.86) holds we do *not* recover (6.83), just

$$Pu \in I^{m+M-1}(X, Y; \Omega^{\frac{1}{2}}) \tag{6.88}$$

i.e., one order smoother than it "should be". The task might seem hopeless, but let us note that these are examples, and important ones at that!!

Consider the (flat) wave operator

$$P = P_t^2 - \sum_{i=1}^{n} D_i^2 = D_t^2 - \Delta \text{ on } \mathbb{R}^{n+1}. \tag{6.89}$$

A hypersurface in $\mathbb{R}^{n+1}$ looks like

$$H = \{h(t, x) = 0\}, (dh \neq 0 \text{ on } H). \tag{6.90}$$

The symbol of P is

$$\sigma^2(P) = \tau^2 - |\xi|^2 = \tau^2 - \xi_1^2 - \cdots - \xi_n^2, \tag{6.91}$$

where τ, ξ are the dual variables to t, x. So consider (6.87),

$$N^*Y = \{(t, x; \lambda dh(t, y)); h(t, x) = 0\}. \tag{6.92}$$

Inserting this into (6.91) we find:

$$\left(\lambda \frac{\partial h}{\partial t}\right)^2 - \left(\lambda \frac{\partial h}{\partial x_1}\right)^2 - \cdots - \left(\lambda \frac{\partial h}{\partial x_n}\right)^2 = 0 \text{ on } h = 0 \tag{6.93}$$

i.e. simply:

$$\left(\frac{\partial h}{\partial t}\right)^2 = |d_x h|^2 \text{ on } h(t, x) = 0. \tag{6.94}$$

This is the "eikonal equation" for h (and hence H).

Solutions to (6.94) are easy to find – we shall actually find all of them (locally) next time. Examples are given by taking h to be linear:

$$H = \{h = at + b \cdot x = 0\} \text{ is characteristic for } P \Longleftrightarrow a^2 = |b|^2. \tag{6.95}$$

Since h/a defines the same surface, all the linear solutions correspond to planes

$$t = \omega \cdot x, \omega \in \mathbb{S}^{n-1}. \tag{6.96}$$

So, do solutions of $Pu \in \mathcal{C}^\infty$ which are conormal with respect to such hypersurfaces exist? Simply take

$$u = v(t - \omega \cdot x) \quad v \in I^*(\mathbb{R}, \{0\}; \Omega^{\frac{1}{2}}). \tag{6.97}$$

Then

$$Pu = 0, u \in I^*(\mathbb{R}^{n+1}, H; \Omega^{\frac{1}{2}}). \tag{6.98}$$

For example $v(s) = \delta(s), u = \delta(t - \omega \cdot x)$ is a "travelling wave".

6.1. Hamilton-Jacobi theory

Let X be a $\mathcal{C}^\infty$ manifold and suppose $p \in \mathcal{C}^\infty(T^*X\backslash 0)$ is homogeneous of degree m. We want to find characteristic hypersurfaces for p, namely hypersurfaces (locally) through $\bar{x} \in X$

$$H = \{f(x) = 0\} \quad h \in \mathcal{C}^\infty(x) h(\bar{x}) = 0, dh(\bar{x}) \neq 0 \tag{6.99}$$

such that

$$p(x, dh(x)) = 0. \tag{6.100}$$

Here we demand that (6.100) hold near $\bar{x}$, not just on H itself. To solve (6.100) we need to impose some additional conditions, we shall demand

$$p \text{ is real-valued} \tag{6.101}$$

and

$$d_{\text{fibre}} p \neq 0 \text{ or } \Sigma(p) = \{p = 0\} \subset T^*X\backslash 0. \tag{6.102}$$

This second condition is actually stronger than really needed (as we shall see) but in any case it implies that

$$\Sigma(P) \subset T^*X\backslash 0 \text{ is a } \mathcal{C}^\infty \text{ conic hypersurface} \tag{6.103}$$

by the implicit function theorem.

The strategy for solving (6.100) is a geometric one. Notice that

$$\Lambda_h = \{(x, dh(x)) \in T^*X\backslash 0\} \tag{6.104}$$

actually determines h up to an additive constant. The first question we ask is – precisely which submanifold $\Lambda \subset T^*X\backslash 0$ corresponds to graphs of differentials of $\mathcal{C}^\infty$ functions? The answer to this involves the tautologous *contact* form.

$$\begin{aligned} \alpha :\ & T^*X \longrightarrow T^*(T^*X) \not\subset \tilde{\pi} \circ \alpha = \mathrm{Id} \\ & \alpha(x, \xi) = \tilde{\pi}^* \xi \in T^*_{(x,\xi)}(T^*X). \end{aligned} \tag{6.105}$$

Here $\tilde{\pi} : T^*(T^*X) \longrightarrow T^*X$ is the projection. Notice that if $x_1, \dots, x_n$ are local coordinates in X then $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ are local coordinates T^*X, where $\xi \in T^*_x X$ is written

$$\xi = \sum_{i=1}^n \xi_i dx_i. \tag{6.106}$$

Since $x_1, \dots, x_n, \xi_1, \dots, \xi_n$ are local coordinates in T^*X they together with the dual coordinates $\Xi_1, \dots, \Xi_n$, $X_1, \dots, X_n$ are local coordinates in $T^*(T^*X)$ where

$$\zeta \in T^*_{(x,\xi)}(T^*X) \Longrightarrow \zeta = \sum_{j=1}^n \Xi_j dx_j + \sum_{j=1}^n X_j d\xi_j. \tag{6.107}$$

In these local coordinates

$$\alpha = \sum_{j=1}^n \xi_j dx_j! \tag{6.108}$$

The first point is that α is independent of the original choice of coordinates, as is evident from (6.105).

LEMMA 6.6. *A submanifold $\Lambda \subset T^*X\backslash 0$ is, near $(\bar{x}, \bar{\xi}) \in \Lambda$, of the form* (6.104) *for some $h \in \mathcal{C}^\infty(X)$, if*

$$\pi : \Lambda \longrightarrow X \text{ is a local diffeomorphism} \tag{6.109}$$

and

$$\alpha \text{ restricted to } \Lambda \text{ is exact.} \tag{6.110}$$

PROOF. The first condition, (6.109), means that Λ is locally the image of a section of T^*X :

$$\Lambda = \{(x, \zeta(x)), \zeta \in \mathcal{C}^\infty(X; T^*X)\}.$$

Thus the section ζ gives an inverse Z to π in (6.109). It follows from (6.105) that

$$Z^*\alpha = \zeta.$$

Thus if α is exact on Λ then ζ is exact on X, $\zeta = dh$ as required. □

Of course if we are only working locally near some point $(\bar{x}, \bar{\xi}) \in \Lambda$ then (6.110) can be replaced by the condition

$$\omega = d\alpha = 0 \text{ on } X. \tag{6.111}$$

Here $\omega = d\alpha$ is the symplectic form on T^*X :

$$\omega = \sum_{j=1}^n d\xi_j \wedge dx_j. \tag{6.112}$$

DEFINITION 6.4. *A submanifold $\Lambda \subset T^*X$ of dimension equal to that of X is said to be* Lagrangian *if the fundamental 2-form, ω, vanishes when pulled back to Λ.*

By definition a symplectic manifold is a $\mathcal{C}^\infty$ manifold S with a $\mathcal{C}^\infty$ 2-form $\omega \in \mathcal{C}^\infty(S; \Lambda^2)$ fixed satisfying two constraints

$$d\omega = 0 \tag{6.113}$$

$$\underbrace{\omega \wedge \cdots \wedge \omega}_{n \text{ factors}} \neq 0 \qquad \dim S = 2n. \tag{6.114}$$

A particularly simple example of a symplectic manifold is a real vector space, necessarily of even dimension, with a non-degenerate antisymmetric 2-form:

$$\begin{cases} \omega : E \times E \longrightarrow \mathbb{R} \\ \tilde{\omega} : E \longleftrightarrow E^*. \end{cases} \tag{6.115}$$

Here $\tilde{\omega}(v)(w) = \omega(v, w)\ \forall\ w \in E$. Now (6.113) is trivially true if we think of ω as a translation-invariant 2-form on E, thought of as a manifold.

Then a subspace $V \subset E$ is Lagrangian if

$$\omega(v, w) = 0\ \forall\ v, w \in V \tag{6.116}$$

$$2\dim V = \dim E. \tag{6.117}$$

Of course the point of looking at symplectic vector spaces and Lagrangian subspaces is:

LEMMA 6.7. *If S is a symplectic manifold then T_zS is a symplectic vector space for each $z \in S$. A submanifold $\Lambda \subset S$ is Lagrangian iff $T_z\Lambda \subset T_zS$ is a Lagrangian subspace $\forall\ z \in \Lambda$.*

We can treat ω, the antisymmetric 2-form on E, as though it were a Euclidean inner product, at least in some regards! Thus if $W \subset E$ is any subspace set

$$W^\omega = \{v \in E; \omega(v, w) = 0 \ \forall \ w \in W\}. \tag{6.118}$$

LEMMA 6.8. *If $W \subset E$ is a linear subspace of a symplectic vector space then* $\dim W^\omega + \dim W = \dim E$; *$W$ is Lagrangian if and only if*

$$W^\omega = W. \tag{6.119}$$

PROOF. Let $W^0 \subset E^*$ be the usual annihilator:

$$W^0 = \{\alpha \in E^*; \alpha(v) = 0 \ \forall \ v \in W\}.$$

Then $\dim W^0 = \dim E - \dim W$. Observe that

$$\tilde{\omega} : W^\omega \longleftrightarrow W^0. \tag{6.120}$$

Indeed if $\alpha \in W^0$ and $\tilde{\omega}(v) = \alpha$ then

$$\alpha(w) = \tilde{\omega}(v)(w) = \omega(v, w) = 0 \ \forall \ w \in W$$

implies that $v \in W^\omega$. Conversely if $v \in W^\omega$ then $\alpha = \tilde{\omega}(v) \in W^0$. Thus $\dim W^\omega + \dim W = \dim E$.

Now if W is Lagrangian then $\alpha = \tilde{\omega}(w), w \in W$ implies

$$\alpha(v) = \tilde{\omega}(w)(v) = \omega(w, v) = 0 \ \forall \ v \in w.$$

Thus $\tilde{\omega}(W) \subset W^0 \Longrightarrow W \subset W^\omega$, by (6.120), and since $\dim W = \dim W^\omega$, (6.119) holds. The converse follows similarly. ☐

The "lifting" isomorphism $\tilde{\omega} : E \longleftrightarrow E^*$ for a symplectic vector space is like the Euclidean identification of vectors and covectors, but "twisted". It is of fundamental importance, so we give it several names! Suppose that S is a symplectic manifold. Then

$$\tilde{\omega}_z : T_z S \longleftrightarrow T_z^* S \ \forall \ z \in S. \tag{6.121}$$

This means that we can associate (by the inverse of (6.121)) a vector field with each 1-form. We write this relation as

$$\begin{gathered} H_\gamma \in \mathcal{C}^\infty(S; TS) \text{ if } \gamma \in \mathcal{C}^\infty(S; T^*S) \text{ and} \\ \tilde{\omega}_z(H_\gamma) = \gamma \ \forall \ z \in S. \end{gathered} \tag{6.122}$$

Of particular importance is the case $\gamma = df, f \in \mathcal{C}^\infty(S)$. Then H_{df} is written H_f and called the Hamilton vector field of f. From (6.122)

$$\omega(H_f, v) = df(v) = vf \ \forall \ v \in T_z S, \ \forall \ z \in S. \tag{6.123}$$

The identity (6.123) implies one important thing immediately:

$$H_f f \equiv 0 \ \forall \ f \in \mathcal{C}^\infty(S) \tag{6.124}$$

since

$$H_f f = df(H_f) = \omega(H_f, H_f) = 0$$

by the antisymmetry of ω. We need a generalization of this:

LEMMA 6.9. *Suppose $L \subset S$ is a Lagrangian submanifold of a symplectic manifold then for each $f \in \mathcal{I}(S) = \{f \in \mathcal{C}^\infty(X); f \restriction \{s = 0\}$, H_f is tangent to Λ.*

PROOF. H_f tangent to Λ means $H_f(z) \in T_z\Lambda \ \forall \ z \in \Lambda$. If $f = 0$ on Λ then $df = 0$ on $T_z\Lambda$, i.e. $df(z) \in (T_z\Lambda)^0 \subset (T_z S) \ \forall \ z \in \Lambda$. By (6.119) the assumption that Λ is Lagrangian means $\tilde{\omega}_z(df(z)) \in T_z\Lambda$, i.e. $H_f(z) \in T_\zeta\Lambda$ as desired. ☐

This lemma gives us a necessary condition for our construction of a Lagrangian submanifold

$$\Lambda \subset \Sigma(P). \tag{6.125}$$

Namely H_p must be tangent to Λ! We use this to construct Λ as a union of integral curves of H_p. Before thinking about this seriously, let's look for a moment at the conditions we imposed on p, (6.101) and (6.102). If p is real then H_p is real (since ω is real). Notice that

$$\text{If } S = T^*X \text{ then each fibre } T_x^*X \subset T^*X \text{ is Lagrangian .} \tag{6.126}$$

Remember that on $T^*X, \omega = d\alpha, \alpha = \xi \cdot dx$ the canonical 1-form. Thus T_x^*X is just $x = \text{const}$, so $dx = 0$, so $\alpha = 0$ on T_x^*X and hence in particular $\omega = 0$, proving (6.126). This allows us to interpret (6.102) in terms of H_p as

$$(6.102) \longleftrightarrow H_p \text{ is everywhere transversal to the fibres } T_x^*X. \tag{6.127}$$

Now we want to construct a little piece of Lagrangian manifold satisfying (6.125). Suppose $z \in \Sigma(P) \subset T^*X\backslash 0$ and we want to construct a piece of Λ through z. Since $\pi_*(H_p(z)) \neq 0$ we can choose a local coordinate, $t \in \mathcal{C}^\infty(X)$, such that

$$\pi_*(H_p(z))t \neq 0, \text{ i.e. } H_p(\pi^*t)(z) \neq 0. \tag{6.128}$$

Consider the hypersurface through $\pi(z) \in X$,

$$H = \{t = t(z)\} \Longrightarrow \pi(z) \in H. \tag{6.129}$$

Suppose $f \in \mathcal{C}^\infty(H), df(\pi(z)) = 0$. In fact we can choose f so that

$$f = f' \restriction H, f' \in \mathcal{C}^\infty(X), \; df'(\pi(z)) = z \tag{6.130}$$

where $z \in \Xi(P)$ was our chosen base point.

Theorem 6.2. *(Hamilton-Jacobi) Suppose $p \in \mathcal{C}^\infty(T^*X\backslash 0)$ satisfies (6.101) and (6.102) near $z \in T^*X\backslash 0$, H is a hypersurface through $\pi(z)$ as in (6.128), (6.125) and $f \in \mathcal{C}^\infty(H)$ satisfies (6.130), then there exists $\tilde{f} \in \mathcal{C}^\infty(X)$ such that*

$$\begin{aligned} \Lambda = \; & \textit{graph } (d\tilde{f}) \subset \Sigma(P) \textit{ near } z \\ & \tilde{f} \restriction H = f \textit{ near } \pi(z) \\ & d\tilde{f}(\pi(z)) = z \end{aligned} \tag{6.131}$$

and any other such solution, $\tilde{f}'$, is equal to $\tilde{f}$ in a neighbourhood of $\pi(z)$.

Proof. We need to do a bit more work to prove this important theorem, but let us start with the strategy. First notice that $\Lambda \cap \pi^{-1}(H)$ is already determined, near $\pi(z)$.

To see this we have to understand the relationship between $df(h) \in T^*H$ and $d\tilde{f}(h) \in T^*X$, $h \in H, \tilde{f} \restriction H = f$. Observe that $H = \{t = 0\}$ lifts to $T_H^*X \subset T^*X$ a *hypersurface*. By (6.124), H_t is tangent to T_H^*X and non-zero. In local coordinates $t, x, \ldots, x_{n-1}$, the x's in H,

$$H_t = -\frac{\partial}{\partial \tau}$$

where $\tau, \xi_1, \ldots, \xi_n$ are the dual coordinates. Thus we see that

$$\pi_H : T_H^*X \longrightarrow T^*H \qquad \pi_H(\beta)(v) = \beta(v), v \in T_h H \subset T_h X, \tag{6.132}$$

is projection along ∂_τ. Now starting from $f \in \mathcal{C}^\infty(H)$ we have

$$\Lambda_f \subset T^*H.$$

Notice that if $\tilde{f} \in \mathcal{C}^\infty(X), \tilde{f} \mid H = f$ then

$$\Lambda_{\tilde{f}} \cap T^*_H X \text{ has dimension } n-1 \tag{6.133}$$

and

$$\pi_H(\Lambda_{\tilde{f}} \cap T^*_H X) = \Lambda_f. \tag{6.134}$$

The first follows from the fact that $\Lambda_{\tilde{f}}$ is a graph over X and the second from the definition, (6.132). So we find □

Lemma 6.10. *If $z \in \Sigma(P)$ and H is a hypersurface through $\pi(z)$ satisfying* (6.128) *and* (6.129) *then $\pi^P_H : (\Sigma(P) \cap T^*_H X) \longrightarrow T^*H$ is a local diffeomorphism in a neighbourhood z; if* (6.130) *is to hold then*

$$\Lambda_{\tilde{f}} \cap T^*_H X = (\pi^P_H)^{-1}(\Lambda_f) \textit{ near } z. \tag{6.135}$$

Proof. We know that H_p is tangent to $\Sigma(P)$ but, by assumption (6.130) is *not* tangent to $T^*_H X$ at z. Then $\Sigma(P) \cap T^*_H X$ does have dimension $2n-1-1 = 2(n-1)$. Moreover $\pi_H$is projection along ∂_τ which cannot be tangent to $\Sigma(P) \cap T^*_H X$ (since it would be tangent to $\Sigma(P)$). Thus π^P_H has injective differential, hence is a local isomorphism.

So this is our strategy:

Start with $f \in \mathcal{C}^\infty(H)$, look at $\Lambda_f \subset T^*H$, lift to $\Lambda \cap T^*_H X \subset \Sigma(P)$ by π^P_H. Now let

$$\Lambda = \bigcup \{H_p - \text{curves through } (\pi^P_H)^{-1}(\Lambda_f)\}.$$

This we will show to be Lagrangian and of the form $\Lambda_{\tilde{f}}$, it follows that

$$p(x, d\tilde{f}) = 0, \tilde{f} \restriction H = f. \tag{6.136}$$

□

6.2. Problems

Problem 6.2. Prove that if E is a symplectic vector space, with non-degenerate bilinear form ω then there is a basis $v_1, \dots, v_n, w_1, \dots, w_n$ of E such that in terms of the dual basis of E^*

$$\omega = \sum_j v^*_j \wedge w^*_j. \tag{6.137}$$

Hint: Construct the w_j, v_j successive. Choose $v_1 \neq 0$. Then choose w_1 so that $\omega(v_1, w_1) = 1$. Then choose v_2 so $\omega(v_1, v_2) = \omega(w_1, v_2) = 0$ (why is this possible?) and w_2 so $\omega(v_2, w_2) = 1$, $\omega(v_1, w_2) = \psi(w_1, w_2) = 0$. Then proceed and conclude that (6.137) must hold.

Deduce that there is a linear transformation $T : E \longrightarrow \mathbb{R}^{2n}$ so that $\omega = T^*\omega_D$, with ω_D given by (6.112).

Problem 6.3. Extend problem 6.2 to show that T can be chosen to map a given Lagrangian plane $V \subset E$ to

$$\{x = 0\} \subset \mathbb{R}^{2n}$$

Hint: Construct the basis choosing $v_j \in V \ \forall\ j$!

PROBLEM 6.4. Suppose S is a symplectic manifold. Show that the *Poisson bracket*

$$\{f,g\} = H_f g$$

makes $\mathcal{C}^\infty(S)$ into a Lie algebra.

6.3. Riemann metrics and quantization

Metrics, geodesic flow, Riemannian normal form, Riemann-Weyl quantization.

6.4. Transport equation

The first thing we need to do is to finish the construction of characteristic hypersurfaces using Hamilton-Jacobi theory, i.e. prove Theorem XIX.37. We have already defined the submanifold Λ as follows:

1) We choose $z \in \Sigma(P)$ and $t \in \mathcal{C}^\infty(X)$ s.t. $H_p\pi^*(t) \neq 0$ at dz, then selected $f \in \mathcal{C}^\infty(H)$, $H = \{t=0\} \cap \Omega, \Omega \ni \pi z$ s.t.

$$z(v) = df(v) \ \forall \ v \in T_{\pi z}H. \tag{6.138}$$

Then we consider

$$\Lambda_f = \ \mathrm{graph}\{df\} = \{(x, df(x)), x \in H\} \subset T^*H \tag{6.139}$$

as our "initial data" for Λ. To move it into $\Sigma(P)$ we noted that the map

$$\Sigma(P) \cap \underset{\{t=0 \text{ in } T^*X\}}{\underset{\|}{T^*_H X}} \longrightarrow T^*H \tag{6.140}$$

is a local diffeomorphism near z, $df(\pi(z))$ by (6.138). The inverse image of Λ_f in (6.140) is therefore a submanifold $\tilde{\Lambda}_f \subset \Sigma(p) \cap T^*_H X$ of dimension $\dim X - 1 = \dim H$. We define

$$\Lambda = \bigcup \{H_p - \text{ curves of length } \epsilon \text{ starting on } \tilde{\Lambda}_f\}. \tag{6.141}$$

So we already know:

$$\Lambda \subset \Sigma(P) \text{ is a manifold of dimension } n, \tag{6.142}$$

and

$$\pi : \Lambda \longrightarrow X \text{ is a local diffeomorphism near } n, \tag{6.143}$$

What we need to know most of all is that

$$\Lambda \text{ is Lagrangian.} \tag{6.144}$$

That is, we need to show that the symplectic two form vanishes identically on $T_{z'}\Lambda$, $\forall \ z' \in \Lambda$ (at least near z). First we check this at z itself! Now

$$T_z\Lambda = T_z\tilde{\Lambda}_f + \ \mathrm{sp}(H_p). \tag{6.145}$$

Suppose $v \in T_z\tilde{\Lambda}_f$, then

$$\omega(v, H_p) = -dp(v) = 0 \text{ since } \tilde{\Lambda}_f \subset \Sigma(P). \tag{6.146}$$

Of course $\omega(H_p, H_p) = 0$ so it is enough to consider

$$\omega|(T_z\tilde{\Lambda}_f \times T_z\tilde{\Lambda}_f). \tag{6.147}$$

Recall from our discussion of the projection (6.140) that we can write it as projection along ∂_τ. Thus

$$\omega_X(v,w) = \omega_H(v',w') \text{ if } v,w \in T_z(T_H X), \qquad (c_H^*)_* v = v' (c_H^*)_* w = w' \in T_z(T^*H) \tag{6.148}$$

where $z = df(\pi(z))$. Thus the form (6.147) vanishes identically because Λ_f is Lagrangian.

In fact the same argument applies at every point of the initial surface $\tilde{\Lambda}_f \subset \Lambda$:

$$T_{z'}\Lambda \text{ is Lagrangian } \;\forall\; z' \in \tilde{\Lambda}_f. \tag{6.149}$$

To extend this result out into Λ we need to use a little more differential geometry. Consider the local diffeomorphisms obtained by exponentiating H_p :

$$\exp(\epsilon H_p)(\Lambda \cap \Omega) \subset \Lambda \;\forall\; \epsilon \text{ small}, \Omega \ni z \text{ small.} \tag{6.150}$$

This indeed is really the definition of Λ_j more precisely,

$$\Lambda = \bigcup_{\epsilon \text{ small}} \exp(\epsilon H_p)(\tilde{\Lambda}_f). \tag{6.151}$$

The main thing to observe is that, on T^*H, the local diffeomorphisms $\exp(\epsilon H_p)$ are *symplectic*:

$$\exp(\epsilon H_p)^* \omega_X = \omega_X. \tag{6.152}$$

Clearly (6.152), (6.150) and (6.149) prove (6.144). The most elegant wary to prove (6.152) is to use Cartan's identity (valid for H_p any vector field, ω any form)

$$\frac{d}{d\epsilon} \exp(\epsilon H_p)^* \omega = \exp(\epsilon H_p)^*(\mathcal{L}_{H_p}\omega) \tag{6.153}$$

where the Lie derivative is given explicitly by

$$\mathcal{L}_V = d \circ \iota_V + \iota_V \circ d, \tag{6.154}$$

c_V being contradiction with V (i.e. $\alpha(\cdot,\cdot,\dots) \longrightarrow \alpha(V,\cdot,\cdot,\dots)$). Thus

$$\mathcal{L}_{H_p}\omega = d(\omega(H_p,\cdot)) + \underset{\substack{\| \\ 0}}{\iota_V(d\omega)} = d(dp) = 0. \tag{6.155}$$

Thus from (6.142), (6.143) and (6.144) we know that

$$\Lambda = \text{ graph}(d\tilde{f}), \tilde{f} \in \mathcal{C}^\infty(X), \text{ near } \pi(z), \tag{6.156}$$

must satisfy the eikonal equation

$$p(x, d\tilde{f}(x)) = 0 \text{ near } \pi(z), H\tilde{f} \restriction H = f \tag{6.157}$$

where we may actually have to add a constant to $\tilde{f}$ to get the initial condition – since we only have $d\tilde{f} = df$ on TH.

So now we can return to the construction of travelling waves: We want to find

$$u \in I^*(X, G; \Omega^{\frac{1}{2}}) \qquad G = \{f = 0\} \tag{6.158}$$

such that u is elliptic at $z \in \Sigma(p)$ and

$$Pu \in \mathcal{C}^\infty(X). \tag{6.159}$$

So far we have noticed that

$$\sigma_{m+M}(Pu) = \sigma_m(P) \restriction N^*G \cdot \sigma(u) \tag{6.160}$$

so that

$$N^*G \subset \Sigma(p) \Longleftrightarrow p(x, df) = 0 \text{ on } f = 0 \tag{6.161}$$

implies

$$Pu \in I^{m+M-1}(X, G; \Omega^{\frac{1}{2}}) \text{ near } \pi(z) \tag{6.162}$$

which is one order smoother than without (6.161).

It is now clear, I hope, that we need to make the "next symbol" vanish as well, i.e. we want

$$\sigma_{m+M-1}(Pu) = 0. \tag{6.163}$$

Of course to arrange this it helps to know what the symbol is!

PROPOSITION 6.2. *Suppose $P \in \Psi^m(X; \Omega^{\frac{1}{2}})$ and $G \subset X$ is a $\mathcal{C}^\infty$ hypersurface characteristic for P (i.e. $N^*G \subset \Sigma(P)$) then $\forall\ u \in I^M(X, G; \Omega^{\frac{1}{2}})$*

$$\sigma_{m+M-1}(Pu) = (-iH_p + a)\sigma_m(u) \tag{6.164}$$

where $a \in S^{m-1}(N^*G)$ and H_p is the Hamilton vector field of $p = \sigma_m(P)$.

PROOF. Observe first that the formula makes sense since $\Lambda = N^*G$ is Lagrangian, $\Lambda \subset \Sigma(p)$ implies H_p is tangent to Λ and if p is homogeneous of degree m (which we are implicitly assuming) then

$$\mathcal{L}_{H_p} : S^r(\Lambda; \Omega^{\frac{1}{2}}) \longrightarrow S^{r+m-1}(\Lambda; \Omega^{\frac{1}{2}})\ \forall\ m \tag{6.165}$$

where one can ignore the half-density terms. So suppose $G = \{x_1 = 0\}$ locally, which we can always arrange by choice of coordinates. Then

$$X = N^*G = \{(0, x', \xi_1, 0) \in T^*X\}.$$

To say $N^*G \subset \Sigma(p)$ means $p = 0$ on Λ, i.e.

$$p = x_1 q(x, \xi) + \sum_{j>1} \xi_j p_j(x, \xi) \text{ near } z \tag{6.166}$$

with q homogeneous of degree m and the p_j homogeneous of degree $m-1$. Working microlocally we can choose $Q \in \Psi^m(X, \Omega^{\frac{1}{2}})$, $P_j \in \Psi^{m-1}(X, \Omega^{\frac{1}{2}})$ with

$$\sigma_m(Q) = q, \sigma_{m-1}(P_j) = p_j \text{ near } z. \tag{6.167}$$

Then, from (6.166)

(6.168)

$$P = x_1 Q + D_{x_j} P_j + R + P', \quad R \in \Psi^{m-1}(X; \Omega^{\frac{1}{2}}) z \notin WF'(P'), P' \in \Psi^m(X, \Omega^{\frac{1}{2}}).$$

Of course P' does not affect the symbol near z so we only need observe that

$$\sigma_{r-1}(x, u) = -d_{\xi_1} \sigma_r(u) \tag{6.169}$$

$$\forall\ u \in I^r(X, G; \Omega^{\frac{1}{2}}) \tag{6.170}$$

$$\sigma_r(D_{x_j} u) = D_{x_j} \sigma_r(u). \tag{6.171}$$

This follows from the local expression

$$u(x) = (2\pi)^{-1} \int e^{ix_1\xi_1} a(x', \xi_1) d\xi_1. \tag{6.172}$$

Then from (6.168) we get

$$\begin{aligned}\sigma_{m+M-1}(Pu) &= -D_{\xi_1}(q\sigma_M(u)) + \sum_j D_{x_j}(p_j\sigma_M(u)) + r\cdot\sigma_m(u)\\ &= -i\left(\sum_{j>1} p_j \restriction \Lambda\frac{\partial}{\partial x_j} - q\restriction\Lambda\frac{\partial}{\partial \xi_i}\right)\sigma_M(u) + a'\sigma_M(u).\end{aligned} \tag{6.173}$$

Observe from (6.166) that the Hamilton vector field of p, at $x_1 = \xi' = 0$ is just the expression in parenthesis. This proves (6.164). □

So, now we can solve (6.163). We just set

$$\sigma_M(u)(\exp(\epsilon H_p)z') = e^{i\epsilon A}\exp(\epsilon H_p)^*[b]\ \forall\ z' \in \tilde{\Lambda}_f = \Lambda\cap\{t=0\}. \tag{6.174}$$

where A is the solution of

$$H_pA = a,\ A\restriction t=0=0 \quad \text{on } \Lambda_0 \tag{6.175}$$

and $b \in S^r(\Lambda_0)$ is a symbol defined on $\Lambda_0 = \Lambda\cap\{t=0\}$ near z.

Proposition 6.3. *Suppose $P \in \Psi^m(X;\Omega^{\frac12})$ has homogeneous principal symbol of degree m satisfying*

$$p = \sigma_m(P) \text{ is real} \tag{6.176}$$

$$d_{\text{fibre}}\, p \neq 0 \text{ on } p = 0 \tag{6.177}$$

and $z \in \Sigma(p)$ is fixed. Then if $H \ni \pi(z)$ is a hypersurface such that $\pi_*(H_p)\cap H$ and $G \subset H$ is an hypersurface in H s.t.

$$\bar{z} = c_H^*(z) \in H^*_{\pi z}G$$

there exist a characteristic hypersurface $\tilde{G} \subset X$ for P such that $\tilde{G}\cap H = G$ near $\pi(z)$, $z \in N^*_{\pi z}\tilde{G}$. For each

$$u_0 \in I^{m+\frac14}(H,G;\Omega^{\frac12}) \text{ with } WF(u_0) \subset \gamma, \tag{6.178}$$

γ a fixed small conic neighbourhood of $\bar{z}$ n T^*H there exists

$$u \in I(X,\tilde{G};\Omega^{\frac12}) \text{ satisfying} \tag{6.179}$$

$$u\restriction G = u_0 \text{ near } \pi z \in H \tag{6.180}$$

$$Pu \in \mathcal{C}^\infty \text{ near } \pi z \in X. \tag{6.181}$$

Proof. All the stuff about G and $\tilde{G}$ is just Hamilton-Jacobi theory. We can take the symbol of u_0 to be the b in (6.174), once we think a little about half-densities, and thereby expect (6.180) and (6.181) to hold, modulo certain singularities. Indeed, we would get

$$u_1\restriction G - u_0 \in I^{r+\frac14-1}(H,G;\Omega^{\frac12}) \text{ near } \pi z \in H \tag{6.182}$$

$$Pu \in I^{r+m-2}(X,\tilde{G};\Omega^{\frac12}) \text{ near } \pi z \in X. \tag{6.183}$$

So we have to work a little to remove lower order terms. Let me do this informally, without worrying too much about (6.180) for a moment. In fact I will put (6.182) into the exercises!

All we really have to observe to improve (6.183) to (6.181) is that

$$g \in I^r(X, \tilde{G}; \Omega^{\frac{1}{2}}) \Longrightarrow \exists \quad u \in I^{r+m-1}(X; \tilde{G}; \Omega^{\frac{1}{2}}) \\ \text{s.t.} \quad Pu - g \in I^{r-1}(X, \tilde{G}; \Omega^{\frac{1}{2}}) \tag{6.184}$$

which we can then iterate and asymptotically sum. In fact we can choose the solution so $u \restriction H \in \mathcal{C}^\infty$, near $\pi\bar{z}$. To solve (6.184) we just have to be able to solve

$$-i(H_p + a)\sigma(u) = \sigma(g) \tag{6.185}$$

which we can do by integration (duHamel's principle). □

The equation (6.185) for the symbol of the solution is the transport equation. We shall use this construction next time to produce a microlocal parametrix for P!

6.5. Problems

PROBLEM 6.5. Let X be a $\mathcal{C}^\infty$ manifold, $G \subset X$ on $\mathcal{C}^\infty$ hypersurface and $t \in \mathcal{C}^\infty(X)$ a real-valued function such that

$$dt \neq 0 \text{ on } T_pG \;\forall\; p \in L = G \cap \{t = 0\}. \tag{6.186}$$

Show that the transversality condition (6.186) ensures that $H = \{t = 0\}$ and $L = H \cap G$ are both $\mathcal{C}^\infty$ submanifolds.

PROBLEM 6.6. Assuming (6.186) show that dt gives an isomorphism of line bundles

$$\Omega^{\frac{1}{2}}(H) \equiv \Omega_H^{\frac{1}{2}}(X) \sim \Omega_H^{\frac{1}{2}}(X)/|dt|^{\frac{1}{2}}$$

and hence one can define a restriction map,

$$\mathcal{C}^\infty(X; \Omega^{\frac{1}{2}}) \longrightarrow \mathcal{C}^\infty(H; \Omega^{\frac{1}{2}}).$$

PROBLEM 6.7. Assuming 1 and 2, make sense of the restriction formula

$$\restriction H : I^m\left(X, G; \Omega^{\frac{1}{2}}\right) \longrightarrow I^{m+\frac{1}{4}}\left(H, L; \Omega^{\frac{1}{2}}\right)$$

and prove it, and the corresponding symbolic formula

$$\sigma_{m+\frac{1}{4}}(u \restriction H) = (\iota_H^*)^*\left(\sigma_m(u) \restriction N_L^*G\right)/|d\tau|^{\frac{1}{2}}.$$

NB. Start from local coordinates and try to understand restriction at that level before going after the symbol formula!

6.6. The wave equation

We shall use the construction of travelling wave solutions to produce a parametrix, and then a fundamental solution, for the wave equation. Suppose X is a Riemannian manifold, e.g. $\mathbb{R}^n$ with a 'scattering' metrice:

$$g = \sum_{i,j=1}^{n} g_{ij}(x)dx^i dx^j, g_{ij} = \delta_{ij}|x|R. \tag{6.187}$$

Then the associates Laplacian, on functions, i.e.

$$\Delta u = -\sum_{i,j=1}^{n} \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_j}(\delta g g^{ij}(x)) \frac{\partial}{\partial x_i} u \tag{6.188}$$

where $g^{ij}(x) = (g_{ij}(x))^{-1}$ and $g = \det g_{ij}$. We are interested in the wave equation

$$Pu = (D_t^2 - \Delta)u = f \qquad \text{on } \mathbb{R} \times X \tag{6.189}$$

For simplicity we assume X is either compact, or $X = \mathbb{R}^n$ with a metric of the form (6.187).

The cotangent bundle of $\mathbb{R} \times X$ is

$$T^*(\mathbb{R} \times X) \simeq T^*\mathbb{R} \times T^*X$$

with canonical coordinates (t, x, τ, ξ). In terms of this

$$\sigma(P) = \tau^2 - |\xi|^2|\xi| = \sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j. \tag{6.190}$$

Thus we certainly have an operator satisfying the conditions of (6.189) and (6.190), since

$$d_{\text{ fibre}}\, p = \left(\frac{\partial p}{\partial \tau}, \frac{\partial p}{\partial \xi}\right) = 0 \Longrightarrow \tau = 0 \text{ and } g^{ij}(x)\xi_i = 0 \Longrightarrow \xi = 0.$$

As initial surface we consider the obvious hypersurface $\{t = 0\}$ (although it will be convenient to consider others). We are after the two theorems, one local and global, the other microlocal, although made to look global.

THEOREM 6.3. *If X is a Riemannian manifold, as above, then for every $f \in \mathcal{C}_c^{-\infty}(\mathbb{R} \times X)$ $\exists!$ $u \in \mathcal{C}^{-\infty}(\mathbb{R} \times X)$ satisfying*

$$Pu = f, u = 0 \text{ in } t < \inf\{\bar{t}; \quad \exists(\bar{t}, x) \in \operatorname{supp}(f)\}. \tag{6.191}$$

THEOREM 6.4. *If X is a Riemannian manifold, as above, then for every $u \in \mathcal{C}^{-\infty}(\mathbb{R} \times X)$,*

$$WF(u)\backslash WF(Pu) \subset \Sigma(P)\backslash WF(Pu) \tag{6.192}$$

is a union of maximally extended H_o-curves in the open subset $\Sigma(P)\backslash WF(Pu)$ of $\Sigma(P)$.

Let us think about Theorem 6.3 first. Suppose $\bar{x}X$ is fixed on $\delta_{\bar{x}} \in \mathcal{C}^{-\infty}(X; \Omega)$ is the Dirac delta (g measure) at $\bar{x}$. Ignoring, for a moment, the fact that this is not quite a generalized function we can look for the "forward fundamental solution" of P with pole at $(0, \bar{x})$:

$$\begin{gathered} PE_{\bar{x}}(t, x) = \delta(t)\delta_{\bar{x}}(x) \\ E_{\bar{x}} = 0 \text{ in } t < 0. \end{gathered} \tag{6.193}$$

Theorem 6.3 asserts its existence and uniqueness. Conversely if we can construct $E_{\bar{x}}$ for each $\bar{x}$, and get reasonable dependence on $\bar{x}$ (continuity is almost certain once we prove uniqueness) then

$$K(t, x; \bar{t}, \bar{x}) = E_{\bar{x}}(t - \bar{t}, x) \tag{6.194}$$

is the kernel of the operator $f \mapsto u$ solving (6.191).

So, we want to solve (6.193). First we convert it (without worrying about rigour) to an initial value problem. Namely, suppose we can solve instead

$$\begin{gathered} PG_{\bar{x}}(t, x) = 0 \text{ in } \mathbb{R} \times X \\ G_{\bar{x}}(0, x) = 0,\ D_tG_{\bar{x}}(0, x) = \delta_{\bar{x}}(x) \text{ in } X. \end{gathered} \tag{6.195}$$

Note that

$$(g(t,x,\tau,0)\notin\Sigma(P)\Longrightarrow(t,x;\tau,0)\notin WF(G). \tag{6.196}$$

This means the restriction maps, to $t=0$, in (6.195) are well-defined. In fact so is the product map:

$$E_{\bar{x}}(t,x)=H(t)G_{\bar{x}}(t,x). \tag{6.197}$$

Then if G satisfied (6.195) a simple computation shows that $E_{\bar{x}}$ satisfies (6.193). Thus we want to solve (6.195).

Now (6.195) seems very promising. The initial data, $\delta_{\bar{x}}$, is certainly conormal to the point $\{\bar{x}\}$, so we might try to use our construction of travelling wave solutions. However there is a serious problem. We already noted that, for the wave equation, there cannot be any smooth characteristic surface other than a hypersurface. The point is that if H has codimension k then

$$N^*_{\bar{x}}H\subset T^*_{\bar{x}}(\mathbb{R}\times X)\text{ has dimension }k.$$

To be characteristic we must have

$$N^*_{\bar{x}}H\subset\Sigma(P)\Longrightarrow k=1 \tag{6.198}$$

Since the *only* linear space contained in a (proper) cone is a line.

However we can easily 'guess' what the characteristic surface corresponding to the point $(x,\bar{x})$ is – it is the *cone* through that point:

This certainly takes us *beyond* our conormal theory. Fortunately there is a way around the problem, namely the possibility of superposition of conormal solutions.

To see where this comes from consider the representation in terms of the Fourier transform:

$$\delta(x)=(2\pi)^{-n}\int e^{ix\xi}d\xi. \tag{6.199}$$

The integral of course is not quite a proper one! However introduce polar coordinates $\xi=r\omega$ to get, at least formally

$$\delta(x)=(2\pi)^{-n}\int_0^\infty\int_{\mathbb{S}^{n-1}}e^{irx\cdot\omega}r^{n-1}dr\ d\omega. \tag{6.200}$$

In odd dimensions r^{n-1} is even so we can write

$$\delta(x)=\frac{1}{2(2\pi)^n}\int_{\mathbb{S}^{n-1}}\int_{-\infty}^{\infty}e^{irx\cdot\omega}r^{n-1}dr\ d\omega, n\text{ odd }. \tag{6.201}$$

Now we can interpret the r integral as a 1-dimensional inverse Fourier transform so that, always formally,

$$\delta(x)=\frac{1}{2(2\pi)^{n-1}}\int_{\mathbb{S}^{n-1}}f_n(x\cdot\omega)d\omega \tag{6.202}$$

$$n\text{ odd} \tag{6.203}$$

$$f_n(s)=\frac{1}{(2\pi)}\int e^{irs}\gamma^{n-1}dr. \tag{6.204}$$

In even dimensions we get the same formula with

$$f_n(s) = \frac{1}{2\pi} \int e^{irs} |r|^{n-1} dr. \tag{6.205}$$

These formulas show that

$$f_n(s) = |D_s|^{n-1} \delta(s). \tag{6.206}$$

Here $|S_s|^{n-1}$ is a pseudodifferential operator for n even or differential operator $(= D_s^{n-1})$ if n is odd. In any case

$$f_n \in I^{n-1+\frac{1}{4}} (\mathbb{R}, \{0\})! \tag{6.207}$$

Now consider the map

$$\mathbb{R}^n \times \mathbb{S}^{n-1} \ni (x, \omega) \mapsto x \cdot \omega \in \mathbb{R}. \tag{6.208}$$

Thus $\mathcal{C}^\infty$ has different

$$\omega \cdot dx + x \cdot d\omega \neq 0 \text{ or } x \cdot \omega = 0$$

So the inverse image of $\{0\}$ is a smooth hypersurface R.

Lemma 6.11. *For each $n \geq 2$*

$$f_n(x, \omega) = \frac{1}{2\pi} \int e^{i(x \cdot \omega) r} |r|^{n-1} dr \in I^{\frac{n}{4} - \frac{1}{4}} \left(\mathbb{R} \times \mathbb{S}^{n-1}, R \right). \tag{6.209}$$

Proof. Replacing $|r|^{n-1}$ by $\rho(r)|r|^{n-1} + (1 - \rho(r))|r|^{n-1}$, where $\rho(r) = 0$ n $r < \frac{1}{2}, \rho(r) = 1$ in $r > 1$, expresses f_n as a sum of a $\mathcal{C}^\infty$ term and a conormal distribution. Check the order yourself! □

Proposition 6.4. *(Radon inversion formula). Under pushforward corresponding to $\mathbb{R}^n \times \mathbb{S}^{n-1} @ > \pi_1 >> \mathbb{R}^n$*

$$(\pi_1)_* f'_n = 2(2\pi)^{n-1} \delta(x),$$
$$f'_n = f_n |d\omega||dx|.$$

Proof. Pair with a test function $\phi \in \mathcal{S}(\mathbb{R}^n)$:

$$(\pi_1)_* f'_n = \iint f_n(x \cdot \omega) \phi(x) dx \; d\omega$$

by the Fourier inversion formula. □

So now we have a superposition formula expressing $\delta(x)$ as an integral:

$$\delta(x) = \frac{1}{2(2\pi)^{n-1}} \int_{\mathbb{S}^{n-1}} f_n(x \cdot \omega) d\omega \tag{6.210}$$

where for each fixed ω $f_n(x \cdot \omega)$ is conormal with respect to $x \cdot \omega = 0$. This gives us a strategy to solve (6.195).

Proposition 6.5. *Each $\bar{x} \in X$ has a neighbourhood, $U_{\bar{x}}$, such that for $\bar{t} > 0$ (independent of $\bar{x}$) there are two characteristic hypersurfaces for each $\omega \in \mathbb{S}^{n-1}$*

$$H^{\pm}_{\bar{x}, \omega)} \subset (-\bar{t}, \bar{t}) \times U_{\bar{x}}$$

depending on $\bar{x}, \omega$, and there exists

$$u^{\pm}(t, x; \bar{x}, \omega) \in I^*((-\bar{t}|\bar{t}| \times U_{\bar{x}}, H^{\pm}_{(\bar{x}, \omega)})$$

such that

$$Pu^{\pm} \in \mathcal{C}^{\infty} \tag{6.211}$$

$$\begin{cases} u^{+} + \bar{u} \restriction t = 0 = \delta_{\bar{x}}(x \cdot \omega) & \textit{in } U_{\bar{x}} \\ D_t(u^{+} + u^{-}) \restriction \{t = 0\} = 0 & \textit{in } U_{\bar{x}}. \end{cases} \tag{6.212}$$

PROOF. The characteristic surfaces are constructed through Hamilton-Jacobi theory:

$$N^{*}H^{\pm} \subset \Sigma(P),$$
$$H_0 = H^{\pm} \cap \{t = 0\} = \{x \cdot \omega = 0\}.$$

There are two or three because the conormal direction to H_0 at 0; ωdx, has two $\Sigma(P)$:

$$\tau = \pm 1,\ (\tau, \omega) \in T_0^{*}(\mathbb{R} \times X). \tag{6.213}$$

With *each* of these two surfaces we can associate a microlocally unique conormal solution

$$\begin{gathered} Pu^{\pm} = 0, \quad u^{\pm} \restriction \{t = 0\} = u_0^{\pm} \\ u_0^{\pm} \in I^{*}(\mathbb{R}^n, \{x \cdot \omega = 0\}) \end{gathered} \tag{6.214}$$

Now, it is easy to see that there are unique choices

$$\begin{gathered} u_{\delta}^{+} + u_0^{-} = \delta(x \cdot \omega) \\ D_t u^{+} + D_t u^{-} \restriction \{t = 0\} = 0. \end{gathered} \tag{6.215}$$

(See exercise 2.) This solves (6.212) and proves the proposition (modulo a fair bit of hard work!).

□

So now we can use the superposition principle. Actually it is better to add the variables ω to the problem and see that

$$\begin{gathered} u^{\pm}(t, x; \omega, \bar{x}) \in I^{*}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n; H^{\pm}) \\ H^{\pm} \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^{n-1} \times \mathbb{R}^n \end{gathered} \tag{6.216}$$

being fixed by the condition that

$$H^{\pm} \cap \mathbb{R} \times \mathbb{R}^n \times \{\omega\} \times \{\bar{x}\} = H^{\pm}_{\bar{x},\omega}.$$

Then we set

$$G'_{\bar{x}}(t, x) = \int_{\mathbb{S}^{n-1}} (u^{+} + u^{-})(x, x; \omega, \bar{x}). \tag{6.217}$$

This satisfies (6.195) locally near $\bar{x}$ and modulo $\mathcal{C}^{\infty}$. i.e.

$$\begin{cases} PG'_{\bar{x}} \in \mathcal{C}^{\infty}((-\bar{t}(\bar{t})) \times U_{\bar{x}}) \\ G'_{\bar{x}} \restriction \{t = 0\} = xv, \\ & v_i \in \mathcal{C}^{\infty} \\ D_t G'_{\bar{x}} = \delta_{\bar{x}}(x) + v_2 \end{cases} \tag{6.218}$$

Let us finish off by doing a calculation. We have (more or less) shown that $u^{\pm}$ are conormal with respect to the hypersurfaces $H^{\pm}$. A serious question then is, what is (a bound one) the wavefront set of $G'_{\bar{x}}$? This is fairly easy provided we understand the geometry. First, since $u^\pm$ are conormal,

$$WF(u^\pm) \subset N^*H^\pm.$$

Then the push-forward theorem says

$$WF(G^\pm) \subset \{(t,x,\tau,\xi);\ \exists \quad (t,x,\tau,\xi,\omega,w) \in WF(u^\pm)\}$$

$$G^\pm = (\pi_1)_* u^\pm = \int_{\mathbb{S}^{n-1}} u^\pm(t,s;\omega,\bar{x})d\omega \tag{6.219}$$

so here

$$(t,x,\tau,\xi,\omega,w) \in T^*(\mathbb{R}\times\mathbb{R}^n\times\mathbb{S}^{n-1}) = T^*(\mathbb{R}\times\mathbb{R}^n)\times T^*\mathbb{S}^{n-1}.$$

We claim that the singularities of $G'_{\bar{x}}$ lie on a cone:

$$WF(G'_{\bar{x}}) \subset \Lambda_{\bar{x}} \subset T^*(\mathbb{R}\times\mathbb{R}^n)$$

where $\Lambda_{\bar{x}}$ is the conormal bundle to a cone:

$$\begin{aligned}\Lambda_{\bar{x}} = \mathrm{cl}\{(t,x;\tau,\xi);\ t\neq 0, D(x,\bar{x}) = \pm t,\\ (\tau,\xi) = \tau(1,\mp d_x D(x,\bar{x}))\end{aligned} \tag{6.220}$$

where $D(x,\bar{x})$ is the Riemannian distance from x to $\bar{x}$.

6.7. Forward fundamental solution

Last time we constructed a local parametrix for the Cauchy problem:

$$\begin{cases} PG'_{\bar{x}} = f \in \mathcal{C}^\infty(\Omega) & (0,\bar{x}) \in \Omega \subset \mathbb{R}\times X\\ G'_{\bar{x}} \restriction t = 0 = u' & \\ D_t G'_{\bar{x}} \restriction \{t=0\} = \delta_{\bar{x}}(x) + u'' & u',u'' \in \mathcal{C}^\infty(\Omega_0)\end{cases} \tag{6.221}$$

where $P = D_t^2 - \Delta$ is the wave operator for a Riemann metric on X. We also computed the wavefront set, and hence singular support of $G_{\bar{x}}$ and deduced that

$$\mathrm{sing}\cdot\mathrm{supp}\,.(G_{\bar{x}}) \subset \{(t,x); d(x,\bar{x}) = |t|\} \tag{6.222}$$

in terms of the Riemannian distance.

(6.223)

This allows us to improve (6.221) in a very significant way. First we can chop $G_{\bar{x}}$ off by replacing it by

$$\phi\left(\frac{t^2 - d^2(x,\bar{x})}{\epsilon^2}\right). \tag{6.224}$$

where $\phi \in \mathcal{C}^\infty(\mathbb{R})$ has support near 0 and is identically equal to 1 in some neighbourhood of 0. This gives (6.221) again, with $G'_{\bar{x}}$ now supported in say $d^2 < t^2 + \epsilon^2$.

(6.225)

Next we can improve (6.221) a little bit by arranging that

$$u' = u'' = 0,\ D_t^k f\,\Big|_{t=0} = 0\ \forall\ k. \tag{6.226}$$

This just requires adding to G' a $\mathcal{C}^\infty$, v, function, so that

$$v\,\Big|_{t=0} = u',\ D_t v\,\Big|_{t=0} = -u'',\quad D_t^k(Pu)\,\Big|_{t=0} = -D_t^k f\,\Big|_{t=0}\quad k>0. \tag{6.227}$$

Once we have done this we consider

$$E'_{\bar{x}} = iH(t)G'_{\bar{x}}$$

which now satisfies

$$\begin{aligned} PE'_{\bar{x}} &= \delta(t)\delta_{\bar{t}}(x) + F_{\bar{x}},\ F_{\bar{x}} \in \mathcal{C}^\infty(\Omega_{\bar{x}}) \\ \operatorname{supp}(E'_x) &\subset \{d^2(x,\bar{x}) \le t^2 + \epsilon^2\} \cap \{t \ge 0\}. \end{aligned} \tag{6.228}$$

Here F vanishes in $t < 0$, so vanishes to infinite order at $t = 0$.

Next we remark that we can actually do all this with smooth dependence of $\bar{x}$. This should really be examined properly, but I will not do so to save time. Thus we actually have

$$\begin{cases} E'(t,x,\bar{x}) \in \mathcal{C}^{-\infty}(P(-\infty,\epsilon) \times X \times X) \\ PE' = \delta(t)\sigma_{\bar{x}}(x) + F \\ \operatorname{supp} E' \subset \{d^2(x,\bar{x}) \ge t^2 + \epsilon^2\} \cap \{t \ge 0\}. \end{cases} \tag{6.229}$$

We can, and later shall, estimate the wavefront set of E. In case $X = \mathbb{R}^n$ we can take E to be the *exact* forward fundamental solution where $|x|$ or $\bar{x} \ge R$, so

$$\begin{gathered} \operatorname{supp}(F) \subset \{t \ge 0\} \cap \{|x|, |\bar{x}| \le R\} \cap \{d^2 \le t^2 + \epsilon^2\} \\ F \in \mathcal{C}^\infty((-\infty,\epsilon) \times X \times X). \end{gathered} \tag{6.230}$$

Of course we want to remove F, the error term. We can do this because it is a *Valterra operator*, very similar to an upper triangular metric. Observe first that the operators of the form (6.230) form an algebra under t-convolution:

$$F = F_1 \circ F_1,\ F(t,x,\bar{x}) = \int_0^t \int F_1(t,-t',x,x')F_2(t^1,x^1,\bar{x})dx'dt'. \tag{6.231}$$

In fact if one takes the iterates of a fixed operator

$$F^{(k)} = F^{(k-1)} \circ F$$

One finds exponential convergence:

$$\mid D_x^\alpha D_t^p F^{(k)}(t,x,\bar{x}) \mid \le \frac{C^{k+1}N,\delta}{k!}|t|^N \quad \text{in } t < \epsilon - \delta\ \forall\ N. \tag{6.232}$$

Thus if F is as in (6.230) then $Id + F$ has inverse $Id + \tilde{F}$,

$$\tilde{F} = \sum_{j \ge 1} (-1)^j F^{(j)} \tag{6.233}$$

again of this form.

Next note that the composition of E' with $\tilde{F}$ is again of the form (6.230), with R increased. Thus

$$E = E' + E' \circ F \tag{6.234}$$

is a forward fundamental solution, satisfying (6.229) with $F \equiv 0$.

In fact E is also a left parametrix, in an appropriate sense:

Proposition 6.6. *Suppose $u \in \mathcal{C}^{-\infty}((-\infty,\epsilon) \times X)$ is such that*

$$\operatorname{supp}(u) \cap [-T,\tau] \times X \text{ is compact } \forall\ T \text{ and for } \tau < \epsilon \tag{6.235}$$

then $Pu = 0 \Longrightarrow u = 0$.

PROOF. The trick is to make sense of the formula

$$0 = E \cdot Pu = u. \tag{6.236}$$

In fact the operators G with kernel $G(t, x, \bar{x})$, defined in $t < \epsilon$ and such that $G * \phi \subset \mathcal{C}^\infty \ \forall\ \phi \in \mathcal{C}^\infty$ and

$$\{t \geq 0\} \cap \{d(x, \bar{x}) \leq R\} \supset \operatorname{supp}(G) \tag{6.237}$$

act on the space (6.235) as t-convolution operators. For this algebra $E * P = \mathrm{Id}$ so (6.236) holds! □

We can use this proposition to prove that E itself is unique. Actually we want to do more.

THEOREM 6.5. *If X is either a compact Riemann manifold or $\mathbb{R}^n$ with a scattering metric then P has a unique forward fundamental solution, ω.*

$$\operatorname{supp}(E) \subset \{t \geq 0\}, P^E = \mathrm{Id} \tag{6.238}$$

and

$$\operatorname{supp}(E) \subset \{(t, x, \bar{x}) \in \mathbb{R} \times X \times X; d(x, \bar{x}) \leq t\} \tag{6.239}$$

and further

$$WF'(E) \subset \mathrm{Id} \cup \mathcal{F}_+ \tag{6.240}$$

where $\mathcal{F}_+$ is the forward bicharacteristic relation on $T^*(\mathbb{R} \times X)$

$$\begin{gathered} \zeta = (t, x, \tau, \xi) \notin \Sigma(P) \Longrightarrow \mathcal{F}_+(\zeta) = \emptyset \\ \zeta = (t, x, \tau, \xi) \in \Sigma(P) \Longrightarrow \mathcal{F}_+(\zeta) = \{\zeta' = (t', x', \tau', \xi') \\ t' \geq t \times \zeta' = \exp(TH_p)\zeta \textit{ for some } T\}. \end{gathered} \tag{6.241}$$

PROOF. 1. Use E_1 defined in $(-\infty, \epsilon \times X$ to continue E globally.
2. Use the freedom of choice of $\{t = 0\}$ and uniqueness of E to show that (6.239)can be arranged for small, and hence all,
3. Then get (6.241) by checking the wavefront set of G.

□

As corollary we get proofs of (6.225) and (6.226).

PROOF OF THEOREM XXI.5.

$$u(t, x) = \int E(t - t', x, x') f(t', x') dx' dt'.$$

□

PROOF OF THEOREM XXI.6. We have to show that if both $\mathrm{WF}(Pu) \not\ni z$ and $\mathrm{WF}(u) \not\ni z$ then $\exp(\delta H_p)z \notin WF(u)$ for small δ. The general case that follows from the (assumed) connectedness of H_p curves. This involves microlocal uniqueness of solutions of $Pu = f$. Thus if $\phi \in \mathcal{C}^\infty(\mathbb{R})$ has support in $t > -\delta$, for $\delta > 0$ small enough, $\pi^* t(z) = \bar{t}$

$$P(\phi(t - \bar{t})u) = g \text{ has } z \notin WF(g),$$

and vanishes in $t < \delta$. Then

$$\begin{gathered} \phi(t - \bar{t})u = E \times g \\ \Longrightarrow \exp(\tau H_p)(z) \notin WF(u) \text{ for small } \tau. \end{gathered}$$

□

6.8. Operations on conormal distributions

I want to review and refine the push-forward theorem, in the general case, to give rather precise results in the conormal setting. Thus, suppose we have a projection

$$X \times Y @>x>> X \tag{6.242}$$

where we can view $X \times Y$ as compact manifolds or Euclidean spaces as desired, since we actually work locally. Suppose

$$Q \subset X \times Y \text{ is an embeded submanifold.} \tag{6.243}$$

Then we know how to define and examine the *conormal* distribution associated to Q. If

$$u \in I^m(X \times Y, Q; \Omega) \tag{6.244}$$

when is $\pi_*(u) \in \mathcal{C}^{-\infty}(X; \Omega)$ conormal? The obvious thing we ned is a submanifold with respect to what it should be conormal! From our earlier theorem we know that

$$WF(\pi_*(u)) \subset \{(x,\xi); \quad \exists \quad (x,\xi,y,0) \in WF(u) \subset N^*Q\}. \tag{6.245}$$

So suppose $Q = \{q_j(x,y) = 0, j = 1,\dots,k\}$, $k = \operatorname{codim} Q$. Then we see that

$$(\bar{x},\bar{\xi},\bar{y},0) \in N^*Q \Longleftrightarrow (\bar{x},\bar{y}) \in Q, \bar{\xi} = \sum_{j=1}^{k} \tau_j d_x q_j, \sum_{j=1}^{k} \tau_j dyq_j = 0. \tag{6.246}$$

Suppose for a moment that Q has a hypersurface, i.e. $k = 1$, and that

$$Q \longrightarrow \pi(Q) \text{ is a fibration} \tag{6.247}$$

then we expect

Theorem 6.6. *$\pi_* : I^m(X \times Y, Q, \Omega) \longrightarrow I^{m'}(X, \pi(Q))$.*

Proof. Choose local coordinates so that

$$Q = \{x_1 = 0\} \tag{6.248}$$

$$u = \frac{1}{2\pi}\int e^{ix_1\xi_1} a(x', y, \xi_1) d\xi_1 \tag{6.249}$$

$$\pi^* u = \frac{1}{2\pi}\int e^{ix_1\xi_1} b(x', \xi_1) d\xi_1 \tag{6.250}$$

$$b = \int a(x', y, \xi) dy. \tag{6.251}$$

□

Next consider the case of restriction to a submanifold. Again let us suppose $Q \subset X$ is a hypersurface and $Y \subset X$ is an embedded submanifold *transversal* to Q :

$$Q \pitchfork Y = QY \tag{6.252}$$

$$\text{i.e. } T_qQ + T_qY = T_qX \quad \forall\, q \in Qy \tag{6.253}$$

$$\Longrightarrow Q_y \qquad \text{is a hypersurface in } X. \tag{6.254}$$

Indeed locally we can take coordinates in which

$$Q = \{x_1 = 0\}, Y = \{x'' = 0\}, \quad x = (x_1, x', x''). \tag{6.255}$$

THEOREM 6.7.
$$C_Y^* : I^m(X,Q) \longrightarrow I^{m+\frac{k}{4}}(Y,Q_Y)k = \operatorname{codim} Y \ \textit{in}\ X.$$

PROOF. In local coordinates as in (6.255)
$$u = \frac{1}{2\pi}\int e^{ix_1\xi_1} a(x(x',x'',\xi_1))d\xi,$$
$$c^*u = \frac{1}{2\pi}\int e^{ix_1\xi_1} a(x',0,\xi_1)d\xi_1.$$
Now let's apply this to the fundamental solution of the wave equation. Well rather consider the solution of the initial value problem

$$\begin{cases} PG(t,x,\bar{x}) = 0 \\ G(0,x,\bar{x}) = 0 \\ D_tG(0,x,\bar{x}) = \delta_{\bar{x}}(x). \end{cases} \tag{6.256}$$

We know that G exists for *all* time and that for short time it is

$$G - \int\limits_{\mathbb{S}^{n-1}} (u_+(t,x,\bar{x};\omega) + u_-(t,x,\bar{x};\omega))d\omega + \mathcal{C}^\infty \tag{6.257}$$

where $u_\pm$ are conormal for the term characteristic hypersurfaces H_p satisfying

$$\begin{gathered} N^*H_\pm \subset \Sigma(P) \\ H_\pm \cap \{t=0\} = \{(x-\bar{x})\cdot\omega = 0\} \end{gathered} \tag{6.258}$$

Consider the 2×2 matrix of distribution

$$U(t) = \begin{pmatrix} D_tG & G \\ D_t^2G & D_tG \end{pmatrix}. \tag{6.259}$$

Since $WFU \subset \Sigma(P)$, in polar $\tau \neq 0$ we can consider this as a smooth function of t, with values in distribution on $X\times X$. □

THEOREM 6.8. *For each $t\in\mathbb{R}$ $U(t)$ is a boundary operator on $L^2(X)\oplus H'(X)$ such that*

$$U(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u(t) \\ D_tu(t) \end{pmatrix} \tag{6.260}$$

where $u(t,x)$ is the unique solution of

$$\begin{aligned} (D_t^2 - \Delta)u(t) &= 0 \\ u(0) &= u_0 \\ D_t + u(0) &= u_1. \end{aligned} \tag{6.261}$$

PROOF. Just check it! □

Consider again the formula (6.257). First notice that at $x=\bar{x}$, $t=0$, $dH^\pm = dt \pm d(x-\bar{x})\omega)$ (by construction). so

$$H_\pm \pitchfork \{x=\bar{x}\} = \{t=0\} \subset \mathbb{R}\times X \hookrightarrow \mathbb{R}\times X\times Y\times\mathbb{S}^{n-1}. \tag{6.262}$$

Moreover the projection

$$\mathbb{R}\times X\times\mathbb{S}^{n-1} \longrightarrow \mathbb{R} \tag{6.263}$$

clearly fibres $\{t = 0\}$ over $\{t = 0\} \in= \{0\} \subset \mathbb{R}$. Then we can apply the two theorems, on push-forward and pull-back, above to conclude that

$$T(t) = \int_X G(t, x, \bar{x}) \restriction x = \bar{x} dx \in \mathcal{C}^{-\infty}(\mathbb{R}) \tag{6.264}$$

is conormal near $t = 0$ i.e. $\mathcal{C}^\infty$ in $(-\epsilon, \epsilon)\backslash\{0\}$ for some $\epsilon > 0$ and conormal at 0. Moreover, we can, at least in principle, work at the *symbol* of $T(t)$ at $t = 0$. We return to this point next time.

For the moment let us think of a more 'fundamental analytic' interpretation of (6.264). By this I mean

$$T(t) = trU(t). \tag{6.265}$$

Remark 6.1. Trace class operators $\Delta\lambda$; Smoother operators are trace order, $tr = \int K(x, x)$

$$\int U(t)\phi(t) \text{ is smoothing}$$
$$\langle T(t), \phi(t)\rangle = tr\langle U(t), \phi(t)\rangle.$$

6.9. Weyl asymptotics

Let us summarize what we showed last time, and a little more, concerning the trace of the wave group

Proposition 6.7. *Let X be a compact Riemann manifold and $U(t)$ the wave group, so*

$$U(t) : \mathcal{C}^\infty(X) \times \mathcal{C}^\infty(X) \ni (u_0, u_1) \mapsto (u, (t), D + tu(t)) \in \mathcal{C}^\infty(X) \times \mathcal{C}^\infty(X)$$

where u is the solution to

$$(D_t^2 - \Delta)u(t) = 0 \tag{6.266}$$
$$u(0) = u_0 \tag{6.267}$$
$$D_t u(0) = u_1. \tag{6.268}$$

The trace of the wave group, $T \in \mathcal{S}'(\mathbb{R})$, is well-defined by

$$T(\phi) = \operatorname{Tr} U(\phi), U(\phi) = \int U(t)\phi(t)dt \ \forall \ \phi \in \mathcal{S}(\mathbb{R}) \tag{6.269}$$

and satisfies

$$T = Y(\left(1 + \sum_{j=1}^{\infty} 2\cos(t\lambda_j)\right) \tag{6.270}$$

$$\textit{where } 0 = \lambda_0 < \lambda_1^2 \le \lambda_2^2 \ldots \qquad \lambda_j \ge 0$$

is the spectrum of the Laplacian repeated with multiplicity

$$\operatorname{sing.supp}(T) \subset \mathcal{L} \cup \{0\} \cup -\mathcal{L} \tag{6.271}$$

where $\mathcal{L}$ is the set of lengthes of closed geodesics of X and

$$\textit{if } \psi \in \mathcal{C}_c^\infty(\mathbb{R}), \psi(t) = 0 \textit{ if } |t| \ge \ \textit{inf}\, \mathcal{L} - \epsilon, \epsilon > 0,$$
$$\psi T \in I(\mathbb{R}, \{0\}) \tag{6.272}$$
$$\sigma(\psi T) =$$

Proof. We have already discussed (6.270) and the first part of (6.272) (given (6.271)). Thus we need to show (6.271), the *Poisson relation*, and compute the symbol of T as a cononormal distribution at 0 .

Let us recall that if G is the solution to

$$\begin{aligned} (D_t^2 = \Delta)G(t,x,\bar{x}) &= 0 \\ G(0,x,\bar{x}) &= 0 \\ D_t G(0,x,\bar{x}) &= \delta_{\bar{x}}(x) \end{aligned} \tag{6.273}$$

then

$$T = \pi_*(\iota_\Delta^* 2D_t G), \tag{6.274}$$

where

$$\iota_\Delta : \mathbb{R} \times X \hookrightarrow \mathbb{R} \times X \times X \tag{6.275}$$

is the embedding of the diagonal and

$$\pi : \mathbb{R} \times X \longrightarrow \mathbb{R} \tag{6.276}$$

is projective. We also know about the wavefront set of G. That is,

$$\begin{aligned} WF(G) \subset \{(t,x,\bar{x},\tau,\xi,\bar{\xi}); \tau^2 = |\xi|^2 = |\bar{\xi}|^2, \\ \exp(sH_p)(0,\bar{x},\tau,\bar{\xi}) = (t,x,\tau,\xi), \text{ some } s\}. \end{aligned} \tag{6.277}$$

Let us see what (6.277) says about the wavefront set of T. First under the restriction map to $\mathbb{R} \times \Delta$

$$\begin{aligned} WF(\iota_\Delta^* D_t G) \subset \{(t,y,\tau,\eta); \quad \exists \\ (t,x,y,\tau,\xi,\bar{\xi});\ \eta = \xi - \bar{\xi}\}. \end{aligned} \tag{6.278}$$

Then integration gives

$$WF(T) \subset \{(t,\tau); \quad \exists \quad (t,y,\tau,0) \in WF(D_t G)\}. \tag{6.279}$$

Combining (6.278) and (6.279) we see

$$\begin{aligned} t \in \operatorname{sing.supp}(T) \Longrightarrow \quad \exists \quad (t,\tau) \in WF(T) \\ \Longrightarrow \quad \exists \quad (t,x,x,\tau,\xi,\xi) \in WF(D_t G) \\ \Longrightarrow \quad \exists \quad s \quad \text{s.t.} \quad \exp(sH_p)(0,x,\tau,\xi) = (t,x,\tau,\xi). \end{aligned} \tag{6.280}$$

Now

$$p = \tau^2 - |\xi|^2, \text{ so } H_p = 2\tau\partial_t - H_g,\ g = |\xi|^2,$$

H_g being a vector field on T^*X. Since WF is *conic* we can take $|\xi| = 1$ in the last condition in (6.280). Then it says

$$s = 2\tau t, \qquad \exp(tH_{\frac{1}{2}g})(x,\xi) = (x,\xi), \tag{6.281}$$

since $\tau^2 = 1$.

The curves in X with the property that their tangent vectors have unit length and the lift to T^*X is an integral curve of $H_{\frac{1}{2}g}$ are *by definition* geodesic, parameterized by arclength. Thus (6.281) is the statement that $|t|$ is the length of a closed geodesic. This proves (6.271).

So now we have to compute the symbol of T at 0. We use, of course, our local representation of G in terms of conormal distributions. Namely

$$G = \sum_j \phi_j G_j, \quad \phi_j \in \mathcal{C}^\infty(X), \tag{6.282}$$

where the ϕ_j has support in coordinate particles in which

$$
\begin{aligned}
G_j(t,x,\bar{x}) &= \int\limits_{\mathbb{S}^{n-1}} (u_+(t,x,\bar{x};\omega)+u_-(t,x,\bar{x};\omega))\,d\omega,\\
u_p m &= \frac{1}{2\pi}\int\limits_{\xi} e^{ih_\pm(t,x,\bar{x},\omega)\xi} a_\pm(x,\bar{x},\xi,\omega)d\xi.
\end{aligned}
\tag{6.283}
$$

Here $h_\pm$ are solutions of the eikonal equation (i.e. are characteristic for P)

$$
\begin{gathered}
|\partial_t h_\pm|^2 = |h_\pm|^2\\
h_\pm \Big|_{t=0} = (x-\bar{x})\cdot\omega\\
\pm\partial_t h_\pm > 0,
\end{gathered}
\tag{6.284}
$$

which fixes them locally uniquely. The $a_\pm$ are chosen so that

$$
(u_+ + u_\pm \Big|_{t=0} = 0, (D_t u_+ D_t u_-) \Big|_{t=0} \delta((x-\bar{x})\cdot\omega) P u_\pm \in \mathcal{C}^\infty. \tag{6.285}
$$

Now, from (6.283)

$$
u_+ + u_- \Big|_{t=0} = \frac{1}{2\pi}\int e^{((x-x\bar{x})\cdot\omega)\xi}(a_+ + a_-)(x,\bar{x},\xi,\omega)d\xi = 0
$$

so $a_+ - a_-$. Similarly

$$
D_t u_+ + D_t u_- \Big|_{t=0} = \frac{1}{2\pi}\int e^{i((x-\bar{x})\cdot\omega)\xi}\big[(D_t h_+)a_+ + (D_t h_-)a_-\big]d\xi \tag{6.286}
$$

$$
= \frac{1}{2(2\pi)^{n-1}} f_n((x-\bar{x})\cdot\omega) \tag{6.287}
$$

From (6.284) we know that $D_t h_\pm = \mp i|d_x(x-\bar{x})\cdot\omega| = \mp i|\omega|$ where the length is with respect to the Riemann measure. We can compute the symbols or both sides in (6.9) and consider that

$$
-2i|\omega|a_+ \equiv \frac{1}{2(2\pi)^{n-1}}|\xi|^{n-1} = D_t h_+ a_+ + D_t h_- a_- \Big|_{t=0} \tag{6.288}
$$

is necessary to get (6.285). Then

$$
T(t) = 2\pi_*(\iota_\Delta^* D_t G) \tag{6.289}
$$

$$
= \frac{1}{2\pi}\sum_{j,\pm} 2\int\limits_X \int\limits_{\mathbb{S}^{n-1}} e^{ih_\pm(t,x,x,\omega)\xi}(D_t h_\pm a_\pm)(x,\bar{x},\omega,\xi)d\xi d\omega dx. \tag{6.290}
$$

Here dx is really the Riemann measure on X. From (6.288) the leading part of this is

$$
\frac{2}{2\pi}\sum_{j\pm}\int\limits_X \int\limits_{\mathbb{S}^{n-1}} e^{ih_\pm(t,x,x,\omega)\xi}\frac{1}{4(2\pi)^{n-1}}|\xi|^{n-1}d\xi d\omega dx
$$

since any term vanishes at t contributes a weaker singularity. Now

$$
h_\pm = \pm|\omega|t + (x-\bar{x})\cdot\omega + 0(t^2).
$$

From which we deduce that

$$
\begin{aligned}
\psi(t)T(t) &= \frac{1}{2\pi}\int e^{it\tau}a(\tau)d\tau\\
a(\tau) &\sim C_n \operatorname{Vol}(X)|\tau|^{n-1} C_n =
\end{aligned}
\tag{6.291}
$$

where C_n is a universal constant depending only on dimension. Notice that if n is odd this is a "little" function.

The final thing I want to do is to show how this can be used to describe the asymptotic behaviour of the eigenvalue of Δ : □

Proposition 6.8. *("Weyl estimates with optimal remainder".) If $N(\lambda)$ is the number of eigenvalues at Δ satisfying $\lambda_1^2 \le \lambda$, counted with multiplicity, the*

$$N(\lambda) = C_n \operatorname{Vol}(X)\lambda^n + o(\lambda^{n-1}) \tag{6.292}$$

The estimate of the remainder terms is the here – weaker estimates are easier to get.

Proof. (Tauberian theorem). Note that

$$T = \mathcal{F}(\mu) \text{ where } N(\lambda) = \int_0^\lambda \mu(\lambda),$$

$\mu(\lambda)$ being the measure

$$\mu(\lambda) = \sum_{\lambda_j^2 \in \operatorname{spec}(\Delta)} \delta(\lambda - \lambda_j).$$

Now suppose $\rho \in \mathcal{S}(\mathbb{R})$ is even and $\int \rho = 1$, $\rho \ge 0$. Then $N_\rho(\lambda) = \int(\lambda')\rho(\lambda - \lambda')$ is a $\mathcal{C}^\infty$ function. Moreover

$$\widehat{\frac{d}{d\lambda} N_\rho(\lambda)} = \hat{\mu} \cdot \hat{\rho}. \tag{6.293}$$

Suppose we can choose ρ so that

$$\rho \ge 0, \int \rho = 1, \rho \in \mathcal{S},\ \hat{\rho}(t) = 0,\ |t| > \epsilon \tag{6.294}$$

for a given $\epsilon > 0$. Then we *know* $\hat{\mu}\hat{\rho}$ is conormal and indeed

$$\begin{aligned} &\frac{d}{d\lambda} N\rho(\lambda) \sim C \operatorname{Vol}(X)\lambda^{n-1} + \dots \\ &\Longrightarrow N_\rho(\lambda) \sim C' \operatorname{Vol}(X)\lambda^n + \textit{ lots.} \end{aligned} \tag{6.295}$$

So what we need to do is look at the *difference*

$$N_\rho(\lambda) - N(\lambda) = \int N(\lambda - \lambda')\rho(\lambda') - N(\lambda)\rho(\lambda'). \tag{6.296}$$

It follows that a bound for N

$$|N(\lambda + \mu) - N(\lambda)| \le ((1 + |\lambda| + |\mu|)^{n-1}(1 + |\lambda|) \tag{6.297}$$

gives

$$N(\lambda) - N_\rho(\lambda) \le C\lambda^{n-1}$$

which is what we want. Now (6.298) follows if we have

$$N(\lambda + 1) - N(\lambda) \le C(1 + |\lambda|) \qquad t/\lambda. \tag{6.298}$$

This in turn follows from the *positivity* of ρ, since

$$\int \rho(\lambda - \lambda')\mu(\lambda') \le C(1 + |\lambda|)^{n-1}. \tag{6.299}$$

Finally then we need to check the existence of ρ as in (6.294). If ϕ is real and even so is $\hat{\phi}$. Take ϕ with support in $(-\frac{\epsilon}{2}, \frac{\epsilon}{2})$ and construct $\phi * \phi$, real and even with ϕ. $\square$