CHAPTER 5

Manifolds

The aim of this chapter is simply to remind you of enough material concerning differentiable manifolds to be able to prove the Hodge theorem (given the deRham theorem) and also to be able to talk freely, later, about symplectic manifolds and such things.

5.1. $\mathcal{C}^\infty$ structures

Let X be a paracompact Hausdorff topological space. A $\mathcal{C}^\infty$ structure on X is a subspace

$$\mathcal{F} \subset C^0(X) = \{u : X \longrightarrow \mathbb{R} \ \text{ continuous } \} \tag{5.1}$$

with the following property:

For each $\overline{x} \in X$ there exists elements $f_1, \dots, f_n \in \mathcal{F}$ such that for some open neighbourhood $\Omega \ni \overline{x}$

$$F : \Omega \ni x \longmapsto (f_1(x), \dots, f_n(x)) \in \mathbb{R}^n \tag{5.2}$$

is a homeomorphism onto an open subset of $\mathbb{R}^n$ and *every* $f \in \mathcal{F}$ satisfies

$$f \restriction \Omega = g \circ F \quad \text{ for some } g \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{5.3}$$

The map (5.2) is a coordinate system near $\overline{x}$. Two $\mathcal{C}^\infty$ structures $\mathcal{F}_1$ and $\mathcal{F}_2$ are 'compatible' if $\mathcal{F}_1 \cup \mathcal{F}_2$ is also a $\mathcal{C}^\infty$ structure. It therefore makes sense to say that:

Definition 5.1. *A $\mathcal{C}^\infty$ manifold is a (connected) paracompact Hausdorff topological space with a* maximal *$\mathcal{C}^\infty$ structure.*

The maximal $\mathcal{C}^\infty$ structure is conventionally denoted

$$\mathcal{C}^\infty(X) \subset C^0(X).$$

It is necessarily a ring. If we let $\mathcal{C}^\infty_c(W) \subset \mathcal{C}^\infty(X)$ denote the subspace of functions which vanish outside a compact subset of W then any local coordinates (5.2) have the property

$$F^* : \mathcal{C}^\infty_c(F(\Omega)) \longleftrightarrow \big\{u \in \mathcal{C}^\infty(X);\ u = 0 \ \text{ on } X \backslash K, K \subset\subset \Omega\big\}. \tag{5.4}$$

Futhermore $\mathcal{C}^\infty(X)$ is *local*:

$$\begin{gathered} u : X \longrightarrow \mathbb{R} \text{ and } \ \forall\, \overline{x} \in X \ \exists\ \Omega_{\overline{x}} \text{ open}, \Omega_{\overline{x}} \ni \overline{x}, \\ \text{s.t. } u = f_{\overline{x}} = 0 \text{ on } \Omega_{\overline{x}} \text{ for some } f_{\overline{x}} \in \mathcal{C}^\infty(X) \Longrightarrow u \in \mathcal{C}^\infty(X). \end{gathered} \tag{5.5}$$

A map $G : X \longrightarrow Y$ between $\mathcal{C}^\infty$ manifolds X and Y is $\mathcal{C}^\infty$ if

$$G^* :\ \mathcal{C}^\infty(Y) \longrightarrow \mathcal{C}^\infty(X)$$

i.e. $G \circ u \in \mathcal{C}^\infty(X)$ for all $u \in \mathcal{C}^\infty(Y)$.

5.2. Form bundles

A vector bundle is a triple $\pi : V \longrightarrow X$ consisting of two manifolds, X and V, and a $\mathcal{C}^\infty$ map π with each

$$V_x = \pi^{-1}(x)$$

having a *linear structure* such that

$$\mathcal{F} = \{u : V \longrightarrow \mathbb{R}, u \text{ is linear on each } V_x\}$$

is a $\mathcal{C}^\infty$ structure on V compatible with $\mathcal{C}^\infty(V)$ (i.e. contained in it, since it is maximal).

The basic example is the *cotangent bundle* which we defined before. Namely for each $\overline{x} \in X$ set

$$\mathcal{I}_{\overline{x}} = \{u \in \mathcal{C}^\infty(X); u(\overline{x}) = 0\}$$
$$\mathcal{I}_{\overline{x}}^2 = \{u = \sum_{\text{finite}} u_i u_i';\ u_i, u_i' \in \mathcal{I}_{\overline{x}}\}$$
$$T_{\overline{x}}^* X = \mathcal{I}_{\overline{x}} / \mathcal{I}_{\overline{x}}^2,\ T^*X = \bigcup_{\overline{x} \in X} T_{\overline{x}}^* X.$$

So $\pi : T^*X \longrightarrow X$ just maps each $T_{\overline{x}}^*X$ to $\overline{x}$. We need to give T^*X a $\mathcal{C}^\infty$ structure so that "it" (meaning $\pi : T^*X \longrightarrow X$) becomes a vector bundle. To do so note that the *differential* of any $f \in \mathcal{C}^\infty(X)$

$$df : X \longrightarrow T^*X \quad df(\overline{x}) = [f - f(\overline{x})] \in T_{\overline{x}}^*X$$

is a section ($\pi \circ df = \mathrm{Id}$). Put

$$\mathcal{F} = \{u : T^*X \longrightarrow \mathbb{R}; u \circ df : X \longrightarrow \mathbb{R} \text{ is } \mathcal{C}^\infty\ \forall\ f \in \mathcal{C}^\infty(X)\}.$$

Then $\mathcal{F} = \mathcal{C}^\infty(T^*X)$ is a maximal $\mathcal{C}^\infty$ structure on T^*X and

$$\mathcal{F}_{\text{lin}} = \{u : T^*X \longrightarrow \mathbb{R}, \text{ linear on each } T_{\overline{x}}^*X; u \in \mathcal{F}\}$$

is therefore compatible with it. Clearly df is $\mathcal{C}^\infty$.

Any (functorial) operation on finite dimensional vector spaces can be easily seen to generate new vectors bundles from old. Thus *duality, tensor product, exterior powers* all lead to new vector bundles:

$$T_xX = (T_x^*X)^*,\ TX = \bigcup_{x \in X} T_xX$$

is the tangent bundle

$$\Lambda_x^k X = \{u : \overbrace{T_xX \times \cdots \times T_xX}^{k \text{ factors}} \longrightarrow \mathbb{R}; u \text{ is multilinear and antisymmetric }\}$$

leads to the k-form bundle

$$\Lambda^k X = \bigcup_{x \in X} \Lambda_x^k X,\ \Lambda^1 X \simeq T^*X$$

where equivalence means there exists (in this case a *natural*) $\mathcal{C}^\infty$ diffeomorphism mapping fibres to fibres linearly (and in this case projecting to the identity on X).

A similar construction leads to the *density bundles*

$$\Omega_x^\alpha X = \{u : \overbrace{T_xX \wedge \cdots \wedge T_xX}^{n = \dim X,\ textfactors} \longrightarrow \mathbb{R};\ \text{absolutely homogeneous of degree } \alpha\}$$

that is

$$u(tv_1 \wedge \ldots v_n) = |t|^\alpha u(v, \wedge \cdots \wedge v_n).$$

These are important because of integration. In general if $\pi : V \longrightarrow X$ is a vector bundle then

$$\mathcal{C}^\infty(X;V) = \{u : \; X \longrightarrow V; \; \pi \circ u = \mathrm{Id}\}$$

is the space of sections. It has a natural linear structure. Suppose $W \subset X$ is a coordinate neighbourhood and $u \in \mathcal{C}^\infty(X;\Omega)$, $\Omega = \Omega^1 X$, has compact support in W. Then the coordinate map gives an identification

$$\Omega^*_x X \longleftrightarrow \Omega^*_{F(x)}\mathbb{R}^n \quad \forall \; \alpha$$

and

$$\int u = \int\limits_{\mathbb{R}^n} g_u(x), \quad u = g_u(x)|dx|$$

is defined independent of coordiantes. That is the integral

$$\int : \mathcal{C}^\infty_c(X;\Omega) \longrightarrow \mathbb{R}$$

is well-defined.

5.3. Pseudodifferential operators

Let X be a $\mathcal{C}^\infty$ manifold, and let $\mathcal{C}^\infty_c(X) \subset \mathcal{C}^\infty(X)$ be the space of $\mathcal{C}^\infty$ functions of compact support. Then, for any $m \in \mathbb{R}$, $\Psi^m(X)$ is the space of linear operators

$$A : \mathcal{C}^\infty_c(X) \longrightarrow \mathcal{C}^\infty(X) \tag{5.6}$$

with the following properties. First,

$$\begin{gathered}\text{if } \phi, \; \psi \in \mathcal{C}^\infty(X) \text{ have disjoint supports then } \exists \; K \in \mathcal{C}^\infty(X \times X; \Omega_R) \\ \text{such that } \forall \; u \in \mathcal{C}^\infty_c(X) \; \phi A \psi u = \int\limits_X K(x,y)u(y),\end{gathered} \tag{5.7}$$

and secondly if $F : W \longrightarrow \mathbb{R}^n$ is a coordinate system in X and $\psi \in \mathcal{C}^\infty_c(X)$ has support in W then

$$\begin{gathered}\exists \; B \in \Psi^m_\infty(\mathbb{R}^n), \; \mathrm{supp}(B) \subset W \times W \text{ s.t.} \\ \psi A \psi u \restriction W = F^*(B((F^{-1})^*(\psi u))) \; \forall \; u \in \mathcal{C}^\infty_c(X).\end{gathered}$$

This is a pretty horrible definition, since it requires us to check *every* coordinate system, at least in principle. In practice the coordinate-invariance we proved earlier means that this is *not* necessary and also that there are plenty of examples!

Any open cover of a $\mathcal{C}^\infty$ manifold has a partition of unity subordinate to it, i.e. if $A_r \subset X$ are open sets for $r \in R$ and

$$X = \bigcup_{r \in R} A_r$$

there exists $\phi_i \in \mathcal{C}^\infty_c(X)$, all non-negative with locally finite support:

$$\forall \; i \; \mathrm{supp}(\phi_i) \cap \mathrm{supp}(\phi_j) \neq \emptyset \text{ for a finite set of indices } j,$$

where each $\mathrm{supp}(\phi_i) \subset A_r$ for some $r = r(i)$ and

$$\sum_i \phi_i(x) = 1 \quad \forall \; x.$$

In fact one can do slightly better than this, for a covering by coordinate neighbourhoods.

LEMMA 5.1. *There exists a partition of unity on X (a $\mathcal{C}^\infty$ manifold) ϕ_i s.t. for every i*

$$\bigcup_j \{\mathrm{supp}(\phi_j); \mathrm{supp}(\phi_j) \cap \mathrm{supp}(\phi_i) \neq 0\}$$

is contained in a coordinate neighbourhood!

Using such a partition of unity we see that every element of $\Psi^m(X)$ can be written in the form

$$A = \sum_i \sum_j \phi_j F^* K_A \phi_i$$

where the terms have smooth kernels if the supports of ψ_i and ψ_j do not meet, or else are pseudodifferential operators in any local coordinates in a A patch containing both supports. Below we shall use this to prove:

THEOREM 5.1. *Let X be a compact $\mathcal{C}^\infty$ manifold then the pseudodifferential operators $\Psi^*(X)$ form a symbol-filtered ring.*

If X is a $\mathcal{C}^\infty$ manifold we have defined the space $\Psi^m(X)$ as consisting of those linear operators

$$A : \mathcal{C}_c^\infty(X) \longrightarrow \mathcal{C}^\infty(X) \tag{5.8}$$

which are given locally by pseudodifferential operators of order m on X, in a precise sense. Let us recast this definition in terms of the Schwartz kernel theorem. Over the product, X^2, consider the right density bundle, $\Omega_R = \pi_R^* \Omega$. Here we use the pull-back operation on vector bundles:

THEOREM 5.2. *If $W \longrightarrow Y$ is a $\mathcal{C}^\infty$ vector bundle and $F : X \longrightarrow Y$ is a $\mathcal{C}^\infty$ map then $F^*W \longrightarrow X$ is a well-defined $\mathcal{C}^\infty$ vector bundle over X with total space*

$$F^*W = \bigcup_{x \in X} W_{F(x)}; \tag{5.9}$$

if $\phi \in \mathcal{C}^\infty(Y; W)$ then $F^*\phi$ is a section of F^*W and $\mathcal{C}^\infty(X; F^*W)$ is spanned by $\mathcal{C}^\infty(X) \cdot F^*\mathcal{C}^\infty(Y; W)$.

Distributional sections of any $\mathcal{C}^\infty$ vector bundle can be defined in two equivalent ways:

$$\text{“Algebraically”}\ \mathcal{C}^{-\infty}(X; W) = \mathcal{C}^{-\infty}(X) \bigotimes_{\mathcal{C}^\infty(X)} \mathcal{C}^\infty(X; W) \tag{5.10}$$

or

$$\text{“Analytically”}\ \mathcal{C}^{-\infty}(X; W) = \left[\mathcal{C}_c^\infty(X; \Omega \otimes W')\right] \tag{5.11}$$

where W' is the dual bundle and Ω the density bundle over X. In order to use (5.11) we need to define a topology on $\mathcal{C}_c^\infty(X; U)$ for any vector bundle U over X. One can do this by reference to local coordinates.

If $W \longrightarrow X$ is a vector bundle the spaces $S^m(W)$ of symbols on W is well-defined for each $m \in \mathbb{R}$.

PROPOSITION 5.1. *A pseudodifferential operator $A \in \Psi^m(X)$ can be written in terms of its kernel*

$$K_A \in \mathcal{C}^\infty(X^2; \Omega_R)$$

where

$$K_A \text{ is } \mathcal{C}^\infty \text{ in } X^2\backslash\Delta \tag{5.12}$$

and if $X_i \subset X$ is a coordinate partition, $\rho_i \in \mathcal{C}_c^\infty(X)$ has support in X_i then in terms of the same coordinates $x_i = F_i^*(x_i)$ and $y_i = F^*(x_i)$ in the two factors of X

$$\rho_i(x)\rho(y)K_A = K_i(x,y)|dy|,\ K_i \in \Psi^m(\mathbb{R}^n). \tag{5.13}$$

Suppose ρ_i^2 is a partition of unity of X, subordinate to a coordinate covering. For each i the symbol of K_i in (5.13) is an equivalence class on $\mathbb{R}^n \times (\mathbb{R}^n\backslash 0)$, with support in $\operatorname{supp}(\rho_i) * (\mathbb{R}^2\backslash 0)$. Set

$$\sigma_m(\tau) = \sum_{\{i,\pi(\tau)\in\operatorname{supp}(\rho_i)\}} a_i(x^{(i)}, \xi^{(i)}) \tag{5.14}$$

where

$$\tau = F_i^*(\sum_j \xi_j^{(i)} \cdot dx_j) \quad \xi^{(i)} \cdot dx \in T^*_{x^{(i)}}\mathbb{R}^n,\ x^{(i)} = F_i(\pi(\tau))$$

and the a_i are representations of the symbols of the K_i. This defines a function on $T^*X\backslash 0$, in fact the equivalence class

$$\sigma_m(A) \in S^{m-[1]}(T^*X) \tag{5.15}$$

is well-defined.

Proposition 5.2. *The principal symbol map in* (5.15)*, defined as in* (5.14)*, gives a short exact sequence:*

$$0 \hookrightarrow \Psi^{m-1}(X) \hookrightarrow \Psi^m(X) \xrightarrow{\sigma_m} S^{m-[1]}(T^*X) \longrightarrow 0. \tag{5.16}$$

Proof. First we need to check that $\sigma_m(A)$ is indeed well-defined. This involves checking what happens under a change of coordinate covering and a change of partition of unity subordinate to it. First, under a change of partition of unity, subordinate to a fixed covering note that

$$\begin{aligned} \rho_j'(x)\rho_j'(y)K_A &= \sum_i \rho_j'(x)\rho_i^2(x)\rho_j'(y)K_A \\ &\equiv \sum_i \rho_j'(x)\rho_i(x)\rho_i(y)\rho_j'(y)K_A \\ &\equiv (\rho_j')^2 \sum_i \rho_i(x)K_A\rho_j(y). \end{aligned}$$

where equality is modulo Ψ^{m-1}, since $[\phi, \Psi^m] \subset \Psi^{m-1}$ for any $\mathcal{C}^\infty$ function ϕ.

It follows from (5.16) that the principal symbols, defined by (5.14), for the two partitions of unity agree.

For a change of coordinate covering it suffices to use the transformation law for the principal symbol under a diffeomorphism of $\mathbb{R}^n$ and the freedom, just established, to choose the partition of unity to be subordinates to *both* coordinate coverings. Thus σ_m is well-defined.

Certainly if $A \in \Psi^{m-1}(X)$ then $\sigma_m(N) \equiv 0$. Moreover if $A \in \Psi^m(X)$ and $\sigma_m(A) \equiv 0$ then all the operators $\rho_i(x)K\rho_i(y)$ are actually of order $m-1$. Since

$$A \equiv \sum_i \sum_i \rho_i'\rho_j(X)A\rho_i'\rho_j \qquad \operatorname{mod} \Psi^{m-1}(X) \tag{5.17}$$

for any two partitions of units $\rho_i'^2, (\rho_i')^2$ we can choose the ρ_i' to each have support in a region where $\rho_j \neq 0$ for some j. Then (5.17) shows that $A \in \Psi^{m-1}(X)$.

Thus it only remains to show that the map σ_m is surjective. If $a \in S^m(T^*X)$ choose $A_i \in \Psi^m_\infty(\mathbb{R}^n)$ by

$$\sigma_L(A_i) = \rho_i(x)(F^*)^{-1}a_i\rho_i(y) \in S^m_\infty(\mathbb{R}^n \times \mathbb{R}^n)$$

and set

$$A = \sum_i F_i^* A_i G_i^* \qquad G_i = F_i^{-1}. \tag{5.18}$$

Then, from (5.14) $\sigma_m(A) \equiv a$ by invariance of the principal symbol. □

The other basic properties of the calculus are easily established. For example

$$\sigma_{m+m'}(A \cdot B) = \sigma_m(A) \cdot \sigma_m(B) \tag{5.19}$$

if $A \in \Psi^m(X), B \in \Psi^{m'}(X), X$ compact. Similarly note that

$$AB = \sum_{i,j} \rho_i^2 A \rho_j^2 B = \sum_{i,j} \rho_i A \rho_i \cdot \rho_j B \rho_i \qquad \text{mod } \Psi^{m'_m - 1}$$

which gives (5.19).

In § 4.4 we used the symbol calculus to construct a left and right parametrix for an elliptic element of $\Psi^m(X)$, where X is compact, i.e. an element $B \in \Psi^{-m}(X)$, such that

$$AB - \mathrm{Id}, BA - \mathrm{Id} \in \Psi^{-\infty}(X). \tag{5.20}$$

As a consequence of this construction note that:

Proposition 5.3. *If $A \in \Psi^m(X)$ is elliptic, and X is compact, then*

$$A : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X) \tag{5.21}$$

is Fredholm, *i.e. has finite dimensional null space and closed range with finite dimensional complement. If ν is a non-vanishing $\mathcal{C}^\infty$ measure on X and a generalized inverse of A is defined by*

$$\begin{aligned} &Gu = f \text{ if } u \in \mathrm{Ran}(A), Af = u, f \perp_\nu \mathrm{Nul}(A) \\ &Gu = 0 \text{ if } u \perp_\nu \mathrm{Ran}(A) \end{aligned} \tag{5.22}$$

then $G \in \Psi^{-m}(X)$ satisfies

$$\begin{aligned} GA &= \mathrm{Id} - \pi_N \\ AG &= \mathrm{Id} - \pi_R \end{aligned} \tag{5.23}$$

where π_N and π_R are ν-orthogonal projections onto the null space of A and the ν-orthocomplement of the range of A respectively.

Proof. The main point to note is that $E \in \Psi^{-\infty}(X)$ is smoothing,

$$E : \mathcal{C}^{-\infty}(X) \longrightarrow \mathcal{C}^\infty(X) \quad \forall\, E \in \Psi^{-\infty}(X). \tag{5.24}$$

Such a map is *compact* on $L^2(X)$, i.e. maps bounded sets into precompact sets by the theorem of Ascoli and Arzela. The second thing to recall is that a Hilbert space with a compact unit ball is finite dimensional. Then

$$\mathrm{Nul}(A) = \{u \in \mathcal{C}^\infty(X); Au = 0\} = \{u \in L^2(X); Au = 0\} \tag{5.25}$$

since, from (5.25) $Au = 0 \Longrightarrow (BA - \mathrm{Id})u = -Eu$, $E \in \Psi^{\infty}(X)$, so $Au = 0$, $u \in \mathcal{C}^{-\infty}(X) \Longrightarrow u \in \mathcal{C}^{\infty}(X)$. Then

$$\mathrm{Nul}(A) = \{u \in L^2(X); Au = 0 \int |u|^2\nu = 1\} \subset L^2(X)$$

is compact since it is closed (A is continuous) and so $\mathrm{Nul}(A) = E(\mathrm{Nul}(A))$ is *precompact*. Thus $\mathrm{Nul}(A)$ is finite dimensional.

Next let us show that $\mathrm{Ran}(A)$ is closed. Suppose $f_j = Au_j \longrightarrow f$ in $\mathcal{C}^{\infty}(X)$, $u_j \in \mathcal{C}^{\infty}(X)$. By what we have just shown we can assume that $u_j \perp_{\nu} \mathrm{Nul}(A)$. Now if B is the parametrix

$$u_j = Bf_j + Eu_j,\ E \in \Psi^{-\infty}(X).$$

Suppose, along some subsequence, $\|u_j\|_{\nu} \longrightarrow \infty$. Then

$$\frac{u_j}{\|u_j\|_{\nu}} = B\left(\frac{f_j}{\|u_j\|_{\nu}}\right) + E\left(\frac{u_j}{\|u_j\|_{\nu}}\right)$$

shows that $\frac{u_j}{\|u_j\|_{\nu}}$ lies in a precompact subset of L^2, $\frac{u_j}{\|u_j\|_{\nu}} \longrightarrow u$. This is a contradiction, since $Au = 0$ but $\|u\| = 1$ and $u \perp_{\nu} \mathrm{Nul}(A)$. Thus the norm sequence $\|u_j\|$ is bounded and therefore the sequence has a weakly convergent subsequence, which we can relabel as u_j. The parametrix shows that $u = Bf_j + Eu_j$ is strongly convergent with limit u, which satisfies $Au = f$.

Finally we have to show that $\mathrm{Ran}(A)$ has a finite dimensional complement. If π_R is orthogonal projection off $\mathrm{Ran}(A)$ then from the second part of (5.20) $f = \pi_R E' f$ for some smoothing operator E. This shows that the orthocomplement has compact unit ball, hence is finite dimensional. □

Notice that it follows that the two projections in (5.23) are both smoothing operators of finite rank.

5.4. Pseudodifferential operators on vector bundles

We have just shown that any elliptic pseudodifferential operator, $A \in \Psi^m(X)$ on a compact manifold X has a generalized inverse $B \in \Psi^{-m}(X)$ where

$$\begin{aligned} BA &= \mathrm{Id} - \pi_N \\ AB &= \mathrm{Id} - \pi_R \end{aligned} \tag{5.26}$$

where π_N and π_R are the orthogonal projections onto the null space of A and the orthocomplement of the range of A with respect to a prescribed $\mathcal{C}^{\infty}$ positive density ν, both are elements of $\Psi^{-\infty}(X)$ and have finite rank. To use this theorem in geometric situations we need first to make the "trivial" extension to operators on sections of vector bundles.

As usual there are two ways (at least) to approach this extension; the high road and the low road. The "low" road is to go back to the definition of $\Psi^m(X)$ and to generalize to $\Psi^m(X; V, W)$. This just requires to take the definition, following (5.6), but using a covering with respect to which the bundles V, W are *both* locally trivial. The local coordinate representatives of the pseudodifferential operator are then matrices of pseudodifferntial opertors. The symbol mapping becomes

$$\Psi^m(X; V, W) \longrightarrow S^{m-[1]}\left(T^*X; \mathrm{Hom}(V, W)\right) \tag{5.27}$$

where $\mathrm{Hom}(V,W) \simeq V \otimes W'$ is the bundle of homomorphisms from V to W and the symbol space consists of symbolic sections of the lift of this bundle to T^*X. We leave the detailed description and proof of these results to the enthusiasts.

So what is the "high" road. This involves only a little sheaf-theoretic thought. Namely we want to define the space $\Psi^m(X;V,W)$ using $\Psi^m(X)$ by:

$$\Psi^m(X;V,W) = \Psi^m(X) \bigotimes_{\mathcal{C}^\infty(X^2)} \mathcal{C}^\infty(X^2; V \boxtimes W'). \tag{5.28}$$

To make sense of this we first note that $\Psi^m(X)$ is a $\mathcal{C}^\infty(X^2)$-module as is the space $\mathcal{C}^\infty(X^2; V \boxtimes W')$ where $V \boxtimes W'$ is the "exterior" product:

$$(V \boxtimes W')_{(x,y)} = V_x \otimes W'_y. \tag{5.29}$$

The tensor product in (5.28) means that

$$A \in \Psi^m(X;V,W) \text{ is of the form } A = \sum_i A_i \cdot G_i \tag{5.30}$$

where $A_i \in \Psi^m(X)$, $G_i \in \mathcal{C}^\infty(X^2; V \boxtimes W')$ and equality is fixed by the relation

$$\phi A \cdot G - A \cdot \phi G \equiv 0. \tag{5.31}$$

Now what we really need to note is:

Proposition 5.4. *For any compact $\mathcal{C}^\infty$ manifold Y and any vector bundle U over Y*

$$\mathcal{C}^{-\infty}(Y;U) \equiv \mathcal{C}^{-\infty}(Y) \bigotimes_{\mathcal{C}^\infty(Y)} \mathcal{C}^\infty(Y;U). \tag{5.32}$$

Proof. $\mathcal{C}^{-\infty}(Y;U) = (\mathcal{C}^\infty(Y;\Omega \otimes U'))'$ is the definition. Clearly we have a mapping

$$\mathcal{C}^{-\infty}(Y) \bigotimes_{\mathcal{C}^\infty(Y)} \mathcal{C}^\infty(Y;U) \ni \sum_i A_i \cdot g_i \longrightarrow \mathcal{C}^{-\infty}(Y;U)$$

given by

$$\sum_i u_i \cdot g_i(\psi) = \sum_i u_i(g_i \cdot \psi) \tag{5.33}$$

since $g_i\psi \in \mathcal{C}^\infty(Y;\Omega)$ and linearity shows that the map descends to the tensor product. To prove that the map is an isomorphism we construct an inverse. Since Y is compact we can find a finite number of sections $g_i \in \mathcal{C}^\infty(Y;U)$ such that *any* $u \in \mathcal{C}^\infty(Y;U)$ can be written

$$u = \sum_i h_i g_i \quad h_i \in \mathcal{C}^\infty(Y).$$

By reference to local coordinates the same is true of distributional sections with

$$h_i = u \cdot q_i \quad q_i \in \mathcal{C}^\infty(Y;U').$$

This gives a left and right inverse. □

Theorem 5.3. *The calculus extends to operators sections of vector bundles over any compact $\mathcal{C}^\infty$ manifolds.*

5.5. Laplacian on forms

Since this is more differential geometry than differential analysis I will be brief. We already defined the exterior derivative

$$d : \mathcal{C}^\infty(X) \longrightarrow \mathcal{C}^\infty(X; T^*X) \tag{5.34}$$

as part of the definition of T^*X, i.e.

$$df(\overline{x}) = [f(x - f(\overline{x})] \in T^*_{\overline{x}}X \ \forall \ \overline{x} \in X.$$

The importance of the form bundles is tha they give a *resolution* of (5.34).

[Actually I didn't mean to be this brief, but was interupted while writing the lecture!]

5.6. Hodge theorem

5.7. Heat kernel

5.8. Resolvent

5.9. Complex powers