CHAPTER 4

Wavefront set

We have now studied special distributions, the Schwartz kernels of pseudodifferential operators, to be able to apply them to the study of general distributions. Proceeding with this general idea, I shall next introduce wavefront set, a refinement of singular support, of general distributions. This notion is fundamental to the general idea of 'microlocalization.'

4.1. Characteristic variety

If $A \in \Psi^m_\infty(\mathbb{R}^n)$, the left-reduced symbol is elliptic at $(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ if there exists $\epsilon > 0$ such that

$$
\begin{gathered}
\mid \sigma_L(A)(x,\xi) \mid \geq \epsilon |\xi|^m \quad \text{in} \\
\big\{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})\,; |x - \overline{x}| \leq \epsilon,\ \Big| \frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\xi|} \Big| \leq \epsilon,\ |\xi| \geq \frac{1}{\epsilon}\big\}.
\end{gathered} \tag{4.1}
$$

Directly from the definition, ellipticity at $(\overline{x}, \overline{\xi})$ is actually a property of the principal symbol, $\sigma_m(A)$ and if A is elliptic at $(\overline{x}, \overline{\xi})$ then it is elliptic at $(\overline{x}, t\overline{\xi})$ for any $t > 0$. Clearly

$$
\big\{(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\});\ A \ \text{ is elliptic (of order } m) \text{ at } \ (\overline{x}, \overline{\xi})\big\}
$$

is an *open* cone in $\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$. The complement

$$
\Sigma_m(A) = \big\{(\overline{x}, \overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})\,;\ A \text{ is } \textit{not} \text{ elliptic of order } m \text{ at } (\overline{x}, \overline{\xi})\big\} \tag{4.2}
$$

is therefore a closed conic subset of $\mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$; it is the characteristic set (or variety) of A. From the composition properties of pseudodifferential operators

$$
\Sigma_{m+m'}(A \circ B) = \Sigma_m(A) \cup \Sigma_{m'}(B). \tag{4.3}
$$

4.2. Wavefront set

We adopt the following bald definition:

$$
\begin{gathered}
\text{If } u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n) = \big\{u \in \mathcal{S}'(\mathbb{R}^n);\ \operatorname{supp}(u) \Subset \mathbb{R}^n\big\} \text{ then} \\
\operatorname{WF}(u) = \bigcap \big\{\Sigma_0(A);\ A \in \Psi^0_\infty(\mathbb{R}^n) \text{ and } Au \in \mathcal{C}^\infty(\mathbb{R}^n)\big\}.
\end{gathered} \tag{4.4}
$$

Thus $\operatorname{WF}(u) \subset \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\})$ is always a closed conic set, being the intersection of such sets. The first thing we wish to show is that $\operatorname{WF}(u)$ is a refinement of $\operatorname{sing\,supp}(u)$. Let

$$
\pi :\ \mathbb{R}^n \times (\mathbb{R}^n\backslash\{0\}) \ni (x,\xi) \longmapsto x \in \mathbb{R}^n \tag{4.5}
$$

be projection onto the first factor.

Proposition 4.1. *If $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ then*

$$
\pi(\operatorname{WF}(u)) = \operatorname{sing\,supp}(u). \tag{4.6}
$$

PROOF. The inclusion $\pi(\mathrm{WF}(u)) \subset \operatorname{sing\,supp}(w)$ is straightforward. Indeed, if $\overline{x} \notin \operatorname{sing\,supp}(u)$ then there exists $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi(\overline{x}) \neq 0$ such that $\phi u \in \mathcal{C}^\infty(\mathbb{R}^n)$. Of course as a multiplication operator, $\phi \in \Psi_\infty^0(\mathbb{R}^n)$ and $\Sigma_0(\phi) \not\ni (\overline{x}, \overline{\xi})$ for any $\overline{\xi} \neq 0$. Thus the definition (4.4) shows that $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$ for all $\overline{\xi} \in \mathbb{R}^n \setminus 0$ proving the inclusion.

Using the calculus of pseudodifferential operators, the opposite inclusion,

$$\pi(\mathrm{WF}(u)) \supset \operatorname{sing\,supp}(u)$$

is only a little more complicated. Thus we have to show that if $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$ for all $\overline{\xi} \in \mathbb{R}^n \setminus 0$ then $\overline{x} \notin \operatorname{sing\,supp}(u)$. The hypothesis is that for each $(\overline{x}, \overline{\xi}), \overline{\xi} \in \mathbb{R}^n \setminus 0$, there exists $A \in \Psi_\infty^0(\mathbb{R}^n)$ such that A is elliptic at $(\overline{x}, \overline{\xi})$ and $Au \in \mathcal{C}^\infty(\mathbb{R}^n)$. The set of elliptic points is open so there exists $\epsilon = \epsilon(\overline{\xi}) > 0$ such that A is elliptic on

$$\big\{(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0); |x - \overline{x}| < \epsilon, \Big| \frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|} \Big| < \epsilon \big\}. \tag{4.7}$$

Let B_j, $j = 1, \dots, N$ be a finite set of such operators associated to $\overline{\xi}_j$ and such that the corresponding sets in (4.7) cover $\{\overline{x}\} \times (\mathbb{R}^n \setminus 0)$; the finiteness follows from the compactness of the sphere. Then consider

$$B = \sum_{j=1}^N B_j^* B_j \Longrightarrow Bu \in \mathcal{C}^\infty(\mathbb{R}^n).$$

This operator B is elliptic at $(\overline{x}, \xi)$, for all $\xi \neq 0$. Thus if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $0 \leq \phi(x) \leq 1$, has support sufficiently close to $\overline{x}$, $\phi(x) = 1$ in $|x - \overline{x}| < \epsilon/2$ then, since B has non-negative principal symbol

$$B + (1 - \phi) \in \Psi_\infty^0(\mathbb{R}^n)$$

is globally elliptic. Thus, by Lemma 3.4, there exists $G \in \Psi_\infty^0(\mathbb{R}^n)$ which is a parametrix for $B + (1 - \phi)$:

$$\mathrm{Id} = G \circ B + G(1 - \phi). \tag{4.8}$$

Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ be such that $\operatorname{supp}(\psi) \subset \{\phi = 1\}$ and $\psi(\overline{x}) \neq 0$. Then, from the reduction formula

$$\psi \circ G \circ (1 - \phi) \in \Psi_\infty^{-\infty}(\mathbb{R}^n).$$

Thus from (4.8) we find

$$\psi u = \psi G \circ Bu + \psi G(1 - \phi) u \in \mathcal{C}^\infty(\mathbb{R}^n).$$

Thus $\overline{x} \notin \operatorname{sing\,supp}(u)$ and the proposition is proved. □

We extend the definition to general tempered distributions by setting

$$\mathrm{WF}(u) = \bigcup_{\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)} \mathrm{WF}(\phi u), \ u \in \mathcal{S}'(\mathbb{R}^n). \tag{4.9}$$

Then (4.6) holds for every $u \in \mathcal{S}'(\mathbb{R}^n)$.

4.3. Essential support

Next we shall consider the notion of the essential support of a pseudodifferential operator. If $a \in S^m_\infty(\mathbb{R}^N;\mathbb{R}^n)$ we define the cone support of a by

$$
\begin{aligned}
\operatorname{cone\,supp}(a) = &\{(\overline{x},\overline{\xi}) \in \mathbb{R}^N \times (\mathbb{R}^n \setminus 0); \exists\ \epsilon > 0 \text{ and } \forall\ M \in \mathbb{R}, \exists\ C_M \text{ s.t.} \\
&|a(x,\xi)| \le C_M \langle\xi\rangle^{-M} \text{ if } |x-\overline{x}| \le \epsilon, \{|\frac{\xi}{|\xi|} - \frac{\overline{\xi}}{|\overline{\xi}|} \le \epsilon\}^{\complement}.
\end{aligned} \tag{4.10}
$$

This is clearly a closed conic set in $\mathbb{R}^N \times (\mathbb{R}^n \setminus 0)$. By definition the symbol decays rapidly outside this cone, in fact even more is true.

LEMMA 4.1. *If $a \in S^\infty_\infty(\mathbb{R}^N;\mathbb{R}^n)$ then*

$$
\begin{aligned}
&(\overline{x},\overline{\eta}) \notin \operatorname{cone\,supp}(a) \Longrightarrow \\
&\exists\ \epsilon > 0 \text{ s.t. } \forall\ M,\alpha,\beta\ \exists\ C_M \text{ with} \\
&|\ D^\alpha_x D^\beta_\xi a(x,\eta)\ | \le C_M\langle\eta\rangle^{-M} \text{ if } |x-\overline{x}| < \epsilon, |\ \frac{\eta}{|\eta|} - \frac{\overline{\eta}}{|\overline{\eta}|}\ | < \epsilon.
\end{aligned} \tag{4.11}
$$

PROOF. To prove (4.11) it suffices to show it to be valid for $D_{x_j}a$, $D_{\xi_k}a$ and then use an inductive argument, i.e. to show that

$$
\operatorname{cone\,supp}(D_{x_j}a), \operatorname{cone\,supp}(D_{\xi_k}a) \subset \operatorname{cone\,supp}(a). \tag{4.12}
$$

Arguing by contradiction suppose that $D_{x_\ell}a$ does not decay to order M in any cone around $(\overline{x},\overline{\eta}) \notin \operatorname{cone\,supp}$. Then there exists a sequence (x_j,η_j) with

$$
\begin{cases} x_j \longrightarrow \overline{x},\ |\ \frac{\eta_j}{|\eta_j|} - \frac{\overline{\eta}}{|\overline{\eta}|}\ | \longrightarrow 0,\ |\eta_j| \longrightarrow \infty \\ \text{and}\ \ |\ D_{x_\ell}a(x_j,\eta_j)\ | > j\langle\eta_j\rangle^M. \end{cases} \tag{4.13}
$$

We can assume that $M < m$, since $a \in S^m_\infty\left(\mathbb{R}^n;\mathbb{R}^N\right)$. Applying Taylor's formula with remainder, and using the symbol bounds on $D^2_{x_j}a$, gives

$$
a(x_j + te_\ell,\xi_j) = a(x_j,\xi_j) + it(D_{x_j}a)(x_j,\xi_j) + O\left(t^2\langle\xi_j\rangle^m\right),\ (e_\ell)_j = \delta_{\ell j} \tag{4.14}
$$

providing $|t| < 1$. Taking $t = \langle\xi_j\rangle^{M-m} \longrightarrow 0$ as $j \longrightarrow \infty$, the first and third terms on the right in (4.14) are small compared to the second, so

$$
|\ a\left(x_j + \langle\xi_j\rangle^{\frac{M-m}{2}},\xi_j\right)\ | > \langle\xi_j\rangle^{2M-m},
$$

contradicting the assumption that $(\overline{x},\overline{\xi}) \notin \operatorname{cone\,supp}(a)$. A similar argument applies to $D_{\xi_\ell}a$ so (4.11), and hence the lemma, is proved. □

For a pseudodifferential operator we define the *essential support* by

$$
\mathrm{WF}'(A) = \operatorname{cone\,supp}(\sigma_L(A)) \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0). \tag{4.15}
$$

LEMMA 4.2. *For every $A \in \Psi^m_\infty(\mathbb{R}^n)$*

$$
\mathrm{WF}'(A) = \operatorname{cone\,supp}(\sigma_R(A)). \tag{4.16}
$$

PROOF. Using (4.11) and the formula relating $\sigma_R(A)$ to $\sigma_L(A)$ we conclude that

$$
\operatorname{cone\,supp}(\sigma_L(A)) = \operatorname{cone\,supp}(\sigma_R(A)), \tag{4.17}
$$

from which (4.16) follows. □

A similar argument shows that

$$\mathrm{WF}'(A\circ B)\subset \mathrm{WF}'(A)\cap \mathrm{WF}'(B). \tag{4.18}$$

Indeed the asymptotic formula for $\sigma_L(A\circ B)$ in terms of $\sigma_L(A)$ and $\sigma_L(B)$ shows that

$$\operatorname{cone\,supp}(\sigma_L(A\circ B))\subset \operatorname{cone\,supp}(\sigma_L(A))\cap \operatorname{cone\,supp}(\sigma_L(B)) \tag{4.19}$$

which is the same thing.

4.4. Microlocal parametrices

The concept of essential support allows us to refine the notion of a parametrix for an elliptic operator to that of a *microlocal parametrix.*

LEMMA 4.3. *If $A\in\Psi^m_\infty(\mathbb{R}^n)$ and $z\notin\Sigma_m(A)$ then there exists a microlocal parametrix at z, $B\in\Psi^{-m}_\infty(\mathbb{R}^n)$ such that*

$$z\notin \mathrm{WF}'(\mathrm{Id}-AB)\ \text{and}\ z\notin \mathrm{WF}'(\mathrm{Id}-BA). \tag{4.20}$$

PROOF. If $z=(\overline{x},\overline{\xi})$, $\overline{\xi}\neq 0$, consider the symbol

$$\gamma_\epsilon(x,\xi)=\phi\left(\frac{x-\overline{x}}{\epsilon}\right)(1-\phi)(\epsilon\xi)\phi\left((\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|})/\epsilon\right)$$

where as usual $\phi\in\mathcal{C}^\infty_c(\mathbb{R}^n)$, $\phi(\zeta)=1$ in $|\zeta|\le\frac12$, $\phi(\zeta)=0$ in $|\zeta|\ge 1$. Thus $\gamma_\epsilon\in S^0_\infty(\mathbb{R}^n;\mathbb{R}^n)$ has support in

$$|x-\overline{x}|\le\epsilon,\ |\xi|\ge\frac{1}{2\epsilon},\ \left|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right|\le\epsilon \tag{4.21}$$

and is identically equal to one, and hence elliptic, on a similar smaller set

$$|x-\overline{x}|<\frac{\epsilon}{2},|\xi|\ge\frac{1}{\epsilon},\ \left|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right|\le\frac{\epsilon}{2}. \tag{4.22}$$

Define $L_\epsilon\in\Psi^0_\infty(\mathbb{R}^n)$ by $\sigma_L(L_\epsilon)=\gamma_\epsilon$. Thus, for any $\epsilon>0$,

$$z\notin \mathrm{WF}'(\mathrm{Id}-L_\epsilon),\ \mathrm{WF}'(L_\epsilon)\subset\left\{(x,\xi);|x-\overline{x}|\le\epsilon\text{ and }\left|\frac{\xi}{|\xi|}-\frac{\overline{\xi}}{|\overline{\xi}|}\right|\le\epsilon\right\}. \tag{4.23}$$

Let $G_{2m}\in\Psi^{2m}_\infty(\mathbb{R}^n)$ be a globally elliptic operator with positive principal symbol. For example take $\sigma_L(G_{2m})=(1+|\xi|^2)^m$, so $G_s\circ G_t=G_{s+t}$ for any s, $t\in\mathbb{R}$. Now consider the operator

$$J=(\mathrm{Id}-L_\epsilon)\circ G_{2m}+A^*A\in\Psi^{2m}_\infty(\mathbb{R}^n). \tag{4.24}$$

The principal symbol of J is $(1-\gamma_\epsilon)(1+|\xi|^2)^m+|\sigma_m(A)|^2$ which is globally elliptic if $\epsilon>0$ is small enough (so that $\sigma_m(A)$ is elliptic on the set (4.21). According to Lemma 3.1, J has a global parametrix $H\in\Psi^{-2m}_\infty(\mathbb{R}^n)$. Then

$$B=H\circ A^*\in\Psi^{-m}_\infty(\mathbb{R}^n) \tag{4.25}$$

is a microlocal right parametrix for A in the sense that $B\circ A-\mathrm{Id}=R_R$ with $z\notin\mathrm{WF}'(R_R)$ since

$$\begin{aligned} R_R=B\circ A-\mathrm{Id}&=H\circ A^*\circ A-\mathrm{Id}\\ &=(H\circ J-\mathrm{Id})+H\circ(\mathrm{Id}-L_\epsilon)G_{2m}\circ A \end{aligned} \tag{4.26}$$

and the first term on the right is in $\Psi_\infty^{-\infty}(\mathbb{R}^n)$ whilst z is not in the operator wavefront set of $(\mathrm{Id} - L_\epsilon)$ and hence not in the operator wavefront set of the second term.

By a completely analogous construction we can find a left microlocal parametrix. Namely $(\mathrm{Id} - L_\epsilon) \circ G_{2m} + A \circ A^*$ is also globally elliptic with parametrix H' and then $B' = A^* \circ H'$ satisfies

$$B' \circ A - \mathrm{Id} = R_L,\ z \notin \mathrm{WF}'(R_L).$$

Then, as usual,

$$B = (B' \circ A - R_L)\, B = B'\,(A \circ B) - R_L B = B' + B' R_R - R_L B \tag{4.27}$$

so $z \notin \mathrm{WF}'(B - B')$, which implies that B is both a left and right microlocal parametrix. □

In fact this argument shows that such a left parametrix is essentially unique. See Problem 4.20.

4.5. Microlocality

Now we can consider the relationship between these two notions of wavefront set.

Proposition 4.2. *Pseudodifferential operators are* microlocal *in the sense that*

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \cap \mathrm{WF}(u) \quad \forall\ A \in \Psi_\infty^\infty(\mathbb{R}^n),\ u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n). \tag{4.28}$$

Proof. We need to show that

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \text{ and } \mathrm{WF}(Au) \subset \mathrm{WF}(u). \tag{4.29}$$

the second being the usual definition of microlocality. The first inclusion is easy. Suppose $(\overline{x}, \overline{\xi}) \notin \operatorname{cone\,supp} \sigma_L(A)$. If we choose $B \in \Psi_\infty^0(\mathbb{R}^n)$ with $\sigma_L(B)$ supported in a small cone around $(\overline{x}, \overline{\xi})$ then we can arrange

$$(\overline{x}, \overline{\xi}) \notin \Sigma_0(B),\ \mathrm{WF}'(B) \cap \mathrm{WF}'(A) = \emptyset.$$

Then from (4.18), $\mathrm{WF}'(BA) = \emptyset$ so $BA \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$ and $BAu \in \mathcal{C}^\infty(\mathbb{R}^n)$. Thus $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(Au)$.

Similarly suppose $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$. Then there exists $G \in \Psi_\infty^0(\mathbb{R}^n)$ which is elliptic at $(\overline{x}, \overline{\xi})$ with $Gu \in \mathcal{C}^\infty(\mathbb{R}^n)$. Let B be a microlocal parametrix for G at $(\overline{x}, \overline{\xi})$ as in Lemma 4.3. Thus

$$u = BGu + Su,\ (\overline{x}, \overline{\xi}) \notin \mathrm{WF}'(S). \tag{4.30}$$

Now apply A to this identity. Since, by assumption, $Gu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ the first term on the right in

$$Au = ABGu + ASu \tag{4.31}$$

is smooth. Since, by (4.18), $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}'(AS)$ it follows from the first part of the argument above that $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(ASu)$ and hence $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(Au)$. □

We can deduce from the existence of microlocal parametrices at elliptic points a partial converse of (8.24).

Proposition 4.3. *For any* $u \in \mathcal{C}^{-\infty}(\mathbb{R}^n)$ *and any* $A \in \Psi_\infty^m(\mathbb{R}^n)$

$$\mathrm{WF}(u) \subset \mathrm{WF}(Au) \cup \Sigma_m(A). \tag{4.32}$$

PROOF. If $(\overline{x},\overline{\xi}) \notin \Sigma_m(A)$ then, by definition, A is elliptic at $(\overline{x},\overline{\xi})$. Thus, by Lemma 4.3, A has a microlocal parametrix B, so

$$u = BAu + Su,\ (\overline{x},\overline{\xi}) \notin \mathrm{WF}'(S). \tag{4.33}$$

It follows that $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(Au)$ implies that $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(u)$ proving the Proposition. □

4.6. Explicit formulations

From this discussion of $\mathrm{WF}'(A)$ we can easily find a 'local coordinate' formulations of $\mathrm{WF}(u)$ in general.

LEMMA 4.4. *If $(\overline{x},\overline{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$ then $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(u)$ if and only if there exists $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with $\phi(\overline{x}) \neq 0$ such that for some $\epsilon > 0$, and for all M there exists C_M with*

$$\mid \widehat{\phi u}(\xi) \mid \leq C_M \langle\xi\rangle^M \ \text{in} \ \mid \frac{\xi}{|\xi|} - \frac{\bar{\xi}}{|\xi|} \mid < \epsilon. \tag{4.34}$$

PROOF. If $\zeta \in \mathcal{C}^\infty(\mathbb{R})$, $\zeta(\xi) \equiv 1$ in $|\xi| < \frac{\epsilon}{2}$ and $\mathrm{supp}(\zeta) \subset \left[\frac{-3\epsilon}{4}, \frac{3\epsilon}{4}\right]$ then

$$\gamma(\xi) = (1-\zeta)(\xi) \cdot \zeta(\frac{\xi}{|\xi|} - \frac{\overline{x}}{|\overline{x}|}) \in S^0_\infty(\mathbb{R}^n)$$

is elliptic at $\bar{\xi}$ and from (4.34)

$$\gamma(\xi) \cdot \widehat{\phi u}(\xi) \in \mathcal{S}(\mathbb{R}^n).$$

Thus if $\sigma_R(A) = \phi_1(x)\gamma(\xi)$ then $A(\phi_2 u) \in \mathcal{C}^\infty$ where $\phi_1\phi_2 = \phi$, $\phi_1(\overline{x}), \phi_2(\overline{x}) \neq 0$, $\phi_1, \phi_2 \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Thus $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(u)$. Conversely if $(\overline{x},\overline{\xi}) \notin \mathrm{WF}(u)$ and A is chosen as above then $A(\phi_1 u) \in \mathcal{S}(\mathbb{R}^n)$ and Lemma 4.4 holds. □

4.7. Wavefront set of K_A

At this stage, a natural thing to look at is the wavefront set of the kernel of a pseudodifferential operator, since these kernels are certainly an interesting class of distributions.

PROPOSITION 4.4. *If $A \in \Psi^m_\infty(\mathbb{R}^n)$ then*

$$\begin{aligned} \mathrm{WF}(K_A) = \big\{ &(x,y,\xi,\eta) \in \mathbb{R}^{2n} \times \left(\mathbb{R}^{2n} \setminus 0\right); \\ &x = y,\ \xi + \eta = 0 \ \text{and} \ (x,\xi) \in \mathrm{WF}'(A)\big\}. \end{aligned} \tag{4.35}$$

In particular this shows that $\mathrm{WF}'(A)$ determines $\mathrm{WF}(K_A)$ and conversely.

PROOF. Using Proposition 4.1 we know that $\pi\left(\mathrm{WF}(K_A)\right) \subset \big\{(x,x)\big\}$ so

$$\mathrm{WF}(K_A) \subset \big\{(x,x;\xi,\eta)\big\}.$$

To find the wave front set more precisely consider the kernel

$$K_A(x,y) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} b(x,\xi) d\xi$$

where we can assume $|x-y| < 1$ on $\mathrm{supp}(K_A)$. Thus is $\phi \in \mathcal{C}_c^\infty(X)$ then

$$g(x,y) = K_A(x,y) \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$$

and

$$\hat{g}(\zeta,\eta) = (2\pi)^{-n}\int e^{-i\zeta x - i\eta y} e^{i(x-y)\cdot\zeta}(\phi b)(x,\xi)d\zeta dx dy \tag{4.36}$$

$$= \int e^{-i(\zeta+\eta)\cdot x}(\phi b)(x,-\eta)dx \tag{4.37}$$

$$= \widehat{\phi b}(\zeta+\eta,-\eta). \tag{4.38}$$

The fact that ϕb is a symbol of compact support in x means that for every M

$$\left| \widehat{\phi b}(\zeta+\eta,-\eta) \right| \leq C_M \left(\langle \zeta+\eta \rangle\right)^{-M} \langle \eta \rangle^m.$$

This is rapidly decreasing if $\zeta \neq -\eta$, so

$$\mathrm{WF}(K_A) \subset \{(x,x,\eta,-\eta)\} \text{ as claimed.}$$

Moreover if $(\overline{x},\overline{\eta}) \notin \mathrm{WF}'(A)$ then choosing ϕ to have small support near $\overline{x}$ makes $\widehat{\phi b}$ rapidly decreasing near $-\overline{\eta}$ for all ζ. This proves Proposition 4.4. □

4.8. Elementary calculus of wavefront sets

We want to achieve a reasonable understanding, in terms of wavefront sets, of three fundamental operations. These are

Pull-back: F^*u

Push-forward: F_*u and

Multiplication: $u_1 \cdot u_2$.

In order to begin to analyze these three operations we shall first introduce and discuss some other more "elementary" operations:

$$\text{Pairing: } (u,v) \longrightarrow \langle u,v\rangle = \int u(x)\overline{v(x)}dx \tag{4.39}$$

$$\text{Projection: } u(x,y) \longmapsto \int u(x,y)dy \tag{4.40}$$

$$\text{Restriction: } u(x,y) \longmapsto u(x,0) \tag{4.41}$$

$$\text{Exterior product: } (u,v) \longmapsto (u \boxtimes v)(x,y) = u(x)v(y) \tag{4.42}$$

$$\text{Invariance: } F^*u, \text{ for } F \text{ a diffeomorphism.} \tag{4.43}$$

Here (4.41) and (4.43) are special cases of (4.8), (4.40) of (4.8) and (4.42) is a combination of (4.8) and (4.8). Conversely the three fundamental operations can be expressed in terms of these elementary ones. We can give direct definitions of the latter which we then use to analyze the former. We shall start with the pairing in (4.39).

4.9. Pairing

We know how to 'pair' a distribution and a $\mathcal{C}^\infty$ function. If both are $\mathcal{C}^\infty$ and have compact supports then

$$\langle u_1,u_2\rangle = \int u_1(x)\overline{u_2(x)}dx \tag{4.44}$$

and in general this pairing extends by continuity to either $\mathcal{C}_c^{-\infty}(\mathbb{R}^n) \times \mathcal{C}^\infty(\mathbb{R}^n)$ or $\mathcal{C}^\infty(\mathbb{R}^n) \times \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ Suppose *both* u_1 and u_2 are distributions, when can we pair them?

PROPOSITION 4.5. *Suppose* $u_1,\ u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *satisfy*

$$\mathrm{WF}(u_1) \cap \mathrm{WF}(u_2) = \emptyset \tag{4.45}$$

then if $A \in \Psi_\infty^0(\mathbb{R}^n)$ *has*

$$\mathrm{WF}(u_1) \cap \mathrm{WF}'(A) = \emptyset,\ \mathrm{WF}(u_2) \cap \mathrm{WF}'(\mathrm{Id} - A^*) = \emptyset \tag{4.46}$$

the bilinear form

$$\langle u_1, u_2 \rangle = \langle Au_1, u_2 \rangle + \langle u_1, (\mathrm{Id} - A^*)u_2 \rangle \tag{4.47}$$

is independent of the choice of A.

Notice that A satisfying (4.46) does indeed exist, just choose $a \in S_\infty^0(\mathbb{R}^n;\mathbb{R}^n)$ to be identically 1 on $\mathrm{WF}(u_2)$, but to have $\operatorname{cone\,supp}(a) \cap \mathrm{WF}(u_1) = \emptyset$, possible because of (4.45), and set $A = q_L(a)$.

PROOF. Of course (4.47) makes sense because Au_1, $(\mathrm{Id} - A^*)u_2 \in \mathcal{C}^\infty(\mathbb{R}^n)$. To prove that this definition is independent of the choice of A, suppose A' also satisfies (4.46). Set

$$\langle u_1, u_2 \rangle' = \langle A'u_1, u_2 \rangle + \langle u_1, (\mathrm{Id} - A')^* u_2 \rangle.$$

Then

$$\mathrm{WF}'(A - A') \cap \mathrm{WF}(u_1) = \mathrm{WF}'((A - A')^*) \cap \mathrm{WF}(u_2) = \emptyset.$$

The difference can be written

$$\langle u_1, u_2 \rangle' - \langle u_1, u_2 \rangle = \langle (A - A')u_1, u_2 \rangle - \langle u_1, (A - A')^* u_2 \rangle. \tag{4.48}$$

Naturally we expect this to be zero, but this is not quite obvious since u_1 and u_2 are both distributions. We need an approximation argument to finish the proof.

Choose $B \in \Psi_\infty^0(\mathbb{R}^n)$ with

$$\begin{gathered} \mathrm{WF}'(B) \cap \mathrm{WF}(u_1) = \mathrm{WF}'(B) \cap \mathrm{WF}(u_2) = \emptyset \\ \mathrm{WF}'(\mathrm{Id} - B) \cap \mathrm{WF}(A - A') = \emptyset \end{gathered} \tag{4.49}$$

If $v_n \longrightarrow u_2$, in $\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$, $v_n \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then

$$w_n = \phi\big[(\mathrm{Id} - B)\, v_n + Bu_2\big] \longrightarrow u_2$$

if $\phi \equiv 1$ in a neighbourhood of $\mathrm{supp}(u_2)$, $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. Here $Bu_2 \in \mathcal{C}^\infty(\mathbb{R}^n)$, so

$$(A - A')w_n = (A - A')\phi(\mathrm{Id} - B) \cdot v_n + (A - A')\phi Bu_2 \longrightarrow (A - A')u_2 \text{ in } \mathcal{C}^\infty(\mathbb{R}^n),$$

since $(A - A')\phi(\mathrm{Id} - B) \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$. Thus

$$\begin{gathered} \langle (A - A')\, u_1, u_2 \rangle \longrightarrow \langle (A - A')\, u_1, u_2 \rangle \\ \langle u_1, (A - A')^*\, w_n \rangle \longrightarrow \langle u_1, (A - A')^*\, u_2 \rangle, \end{gathered}$$

since $w_n \longrightarrow u_2$ in $\mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ and $(A - A')^* w_n \longrightarrow (A - A')^*\, u_2$ in $\mathcal{C}^\infty(\mathbb{R}^n)$. Thus

$$\langle u_1, u_2 \rangle' - \langle u_1, u_2 \rangle = \lim_{n\to\infty} \big[\langle (A - A')\, u_1, w_n \rangle - \langle u_1, (A - A')^*\, w_n\big] = 0.$$

□

Here we are using the *complex* pairing. If we define the real pairing by

$$(u_1, u_2) = \langle u_1, \overline{u}_2 \rangle \tag{4.50}$$

then we find

Proposition 4.6. *If* $u_1,\ u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *satisfy*

$$(x,\xi) \in \mathrm{WF}(u_1) \Longrightarrow (x,-\xi) \notin \mathrm{WF}(u_2) \tag{4.51}$$

then the real pairing, defined by

$$(u_1,u_2) = (Au_1,u_2) + (u_1,A^tu_2), \tag{4.52}$$

where A *satisfies* (4.46), *is independent of* A.

Proof. Notice that

$$\mathrm{WF}(\overline{u}) = \big\{(x,-\xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0); (x,\xi) \in \mathrm{WF}(u)\big\}. \tag{4.53}$$

We can write (4.51), using (4.50), as

$$(u_1,u_2) = \langle Au_1,\overline{u}_2\rangle + \langle u_1,\overline{A^tu_2}\rangle.$$

Since, by definition, $\overline{A^tu_2} = A^*\overline{u}_2$,

$$(u_1,u_2) = \langle Au_1,\overline{u}_2\rangle + \langle u_1,A^*\overline{u}_2\rangle = \langle u_1,\overline{u}_2\rangle$$

is defined by (4.47), since (4.51) translates to (4.45). □

4.10. Multiplication of distributions

The pairing result (4.52) can be used to define the *product* of two distributions under the same hypotheses, (4.51).

Proposition 4.7. *If* $u_1,\ u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *satisfy*

$$(x,\xi) \in \mathrm{WF}(u_1) \Longrightarrow (x,-\xi) \notin \mathrm{WF}(u_2) \tag{4.54}$$

then the product of u_1 *and* $u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *is well-defined by*

$$u_1u_2(\phi) = (u_1,\phi u_2) = (\phi u_1,u_2)\ \forall\ \phi \in \mathcal{C}_c^{\infty}(\mathbb{R}^n) \tag{4.55}$$

using (4.52).

Proof. We only need to observe that if $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ and $A \in \Psi_\infty^m(\mathbb{R}^n)$ has $\mathrm{WF}'(A) \cap \mathrm{WF}(u) = \emptyset$ then for any fixed $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$\|\psi A\phi u\|_{C^k} \le C\|\phi\|_{C^p} \quad p = k+N \tag{4.56}$$

for some N, depending on m. This implies the continuity of $\phi \longmapsto u_1u_2(\phi)$ defined by (4.55). □

4.11. Projection

Here we write $\mathbb{R}^n_z = \mathbb{R}^p_x \times \mathbb{R}^k_y$ and define a continuous linear map, which we write rather formally as an integral

$$\mathcal{C}_c^{-\infty}(\mathbb{R}^n) \ni u \longmapsto \int u(x,y)dy \in \mathcal{C}_c^{-\infty}(\mathbb{R}^p) \tag{4.57}$$

by pairing. If $\phi \in \mathcal{C}^\infty(\mathbb{R}^p)$ then

$$\pi_1^*\phi \in \mathcal{C}^\infty(\mathbb{R}^n), \quad \pi_1 :\ \mathbb{R}^n \ni (x,y) \longmapsto x \in \mathbb{R}^p \tag{4.58}$$

and for $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ we define the formal 'integral' in (4.57) by

$$(\int u(x,y)dy,\phi) = ((\pi_1)_*u,\phi) := u(\pi_1^*\phi). \tag{4.59}$$

In this sense we see that the projection is dual to pull-back (on functions) under π_1, so is "push-forward under π_1," a special case of (4.8). The support of the projection satisfies

$$\operatorname{supp}\left((\pi_1)_* u\right) \subset \pi_1\left(\operatorname{supp}(u)\right)\ \forall\ u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n), \tag{4.60}$$

as follows by duality from

$$\operatorname{supp}(\pi_1^* \phi) \subset \pi_1^{-1}\left(\operatorname{supp}\phi\right). \tag{4.61}$$

PROPOSITION 4.8. *Let* $\pi_1 : \mathbb{R}^{p+k} \longrightarrow \mathbb{R}^p$ *be projection, then for every* $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^{p+k})$

$$\begin{aligned}\operatorname{WF}\left((\pi_1)_* u\right) \subset \big\{&(x,\xi) \in \mathbb{R}^p \times (\mathbb{R}^p \backslash 0)\,;\\ &\exists\ y \in \mathbb{R}^k \text{ with } (x,y,\xi,0) \in \operatorname{WF}(u)\big\}.\end{aligned}$$

PROOF. First notice that

$$(\pi_1)_* :\ \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R}^p). \tag{4.62}$$

Combining this with (4.60) we see that

$$\operatorname{sing\,supp}\left((\pi_1)_* u\right) \subset \pi_1\left(\operatorname{sing\,supp} u\right) \tag{4.63}$$

which is at least consistent with Proposition 4.8. To prove the proposition in full let me restate the local characterization of the wavefront set, in terms of the Fourier transform:

LEMMA 4.5. *Suppose* $K \subset\subset \mathbb{R}^n$ *and* $\Gamma \subset \mathbb{R}^n \setminus 0$ *is a* closed *cone, then*

$$\begin{aligned}u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n),\ \operatorname{WF}(u) \cap (K \times \Gamma) = \emptyset,\ A \in \Psi_\infty^m(\mathbb{R}^n),\ \operatorname{WF}'(A) \subset K \times \Gamma\\ \Longrightarrow Au \in \mathcal{S}(\mathbb{R}^n).\end{aligned} \tag{4.64}$$

In particular

$$\begin{aligned}u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n),\ \operatorname{WF}(u) \cap (K \times \Gamma) = \emptyset,\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^n), \operatorname{supp}(\phi) \subset K\\ \Longrightarrow \widehat{\phi u}(\xi) \text{ is rapidly decreasing in } \Gamma.\end{aligned} \tag{4.65}$$

Conversely suppose $\Gamma \subset \mathbb{R}^n \setminus 0$ *is a closed cone and* $u \in \mathcal{S}'(\mathbb{R}^n)$ *is such that for some* $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$

$$\widehat{\phi u}(\xi) \text{ is rapidly decreasing in } \Gamma \tag{4.66}$$

then

$$\operatorname{WF}(u) \cap \left\{x \in \mathbb{R}^n; \phi(x) \neq 0\right\} \times \operatorname{int}(\Gamma) = \emptyset. \tag{4.67}$$

With these local tools at our disposal, let us attack (4.8). We need to show that

$$\begin{aligned}(\overline{x},\overline{\xi}) \in \mathbb{R}^p \times (\mathbb{R}^p \backslash 0) \text{ s.t. } (\overline{x}, y, \overline{\xi}, 0) \notin \operatorname{WF}(u)\ \forall\ y \in \mathbb{R}^n\\ \Longrightarrow (\overline{x},\overline{\xi}) \notin \operatorname{WF}\left((\pi_1)_* u\right).\end{aligned} \tag{4.68}$$

Notice that, $\operatorname{WF}(u)$ being conic and $\pi(\operatorname{WF}(u))$ being compact, $\operatorname{WF}(u) \cap (\mathbb{R}^n \times S^{n-1})$ is compact. The hypothesis (4.68) is the statement that

$$\{\overline{x}\} \times \mathbb{R}^k \times \mathbb{S}^{n-1} \times \{0\} \cap \operatorname{WF}(u) = \emptyset.$$

Thus $\overline{x}$ has an open neighbourhood, W, in $\mathbb{R}^p$, and $(\overline{\xi}, 0)$ a conic neighbourhood γ_1 in $(\mathbb{R}^n \backslash 0)$ such that

$$(W \times \mathbb{R}^k \times \gamma_1) \cap \operatorname{WF}(u) = \emptyset.$$

Now if $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^p)$ is chosen to have support in W

$$\widehat{(\pi_1^*\phi)u}(\xi,\eta) \text{ is rapidly decreasing in } \gamma_1. \tag{4.69}$$

Set $v = \phi(\pi_1)_*u$. From the definition of projection and the identity

$$v = \phi(\pi_1)_*u = (\pi_1)_*[(\pi_1^*\phi)u],$$

we have

$$\hat{v}(\xi) = v(e^{-ix\cdot\xi}) = (\widehat{(\pi_1^*\phi)u})(\xi,0). \tag{4.70}$$

Now (4.69) shows that $\hat{v}(\xi)$ is rapidly decreasing in $\gamma_1 \cap (\mathbb{R}^p \times \{0\})$, which is a cone around $\overline{\xi}$ in $\mathbb{R}^p$. Since $v = \phi(\pi_1)_*u$ this shows that $(\overline{x},\overline{\xi}) \notin \mathrm{WF}\left((\pi_1)_*u\right)$, as claimed. □

Before going on to talk about the other operations, let me note a corollary of this which is useful and, even more, helps to explain what is going on:

Corollary 4.1. *If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ and*

$$\mathrm{WF}(u) \cap \left\{(x,y,\xi,0); x \in \mathbb{R}^p, y \in \mathbb{R}^k, \xi \in \mathbb{R}^p\backslash 0\right\} = \emptyset$$

then $(\pi_1)_*(u) \in \mathcal{C}_c^\infty(\mathbb{R}^n)$.

Proof. Indeed, (4.8) says $\mathrm{WF}\left((\pi_1)_*u\right) = \emptyset$. □

Here, the vectors $(x,y,\xi,0)$ are the ones "normal" (as we shall see, really *conormal*) to the surfaces over which we are integrating. Thus Lemma 4.5 and Corollary 4.1 both state that the only singularities that survive integration are the ones which are conormal to the surface along which we integrating; the ones even partially in the direction of integration are wiped out. This in particular fits with the fact that if we integrate in *all variables* then there are no singularities left.

4.12. Restriction

Next we wish to consider the *restriction* of a distribution to a subspace

$$\mathcal{C}_c^{-\infty}(\mathbb{R}^n) \ni u \longmapsto u \restriction \{y=0\} \in \mathcal{C}_c^{-\infty}(\mathbb{R}^p). \tag{4.71}$$

This is *not* always defined, i.e. no reasonable map (4.71) exists for all distributions. However under an appropriate condition on the wavefront set we can interpret (4.71) in terms of pairing, using our definition of products. Thus let

$$\iota : \mathbb{R}^p \ni x \longmapsto (x,0) \in \mathbb{R}^n$$

be the *inclusion* map. We want to think of $u \restriction \{y=0\}$ as ι^*u. If $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ then for any $\phi' \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ the identity

$$\iota^*u(\iota^*\phi') = u\left(\phi'\delta(y)\right) \tag{4.72}$$

holds.

The restriction map $\iota^* : \mathcal{C}_c^\infty(\mathbb{R}^n) \longrightarrow \mathcal{C}_c^\infty(\mathbb{R}^p)$ is surjective. If $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ satisfies the condition

$$\mathrm{WF}(u) \cap \left\{(x,0,0,\eta); x \in \mathbb{R}^p, \eta \in \mathbb{R}^{n-p}\right\} = \emptyset \tag{4.73}$$

then we can interpret the pairing

$$\begin{gathered}\iota^*u(\phi) = u\left(\phi'\delta(y)\right)\ \forall\ \phi \in \mathcal{C}_c^\infty(\mathbb{R}^p)\\ \text{where } \phi' \in \mathcal{C}_c^\infty(\mathbb{R}^n) \text{ and } \iota^*\phi' = \phi\end{gathered} \tag{4.74}$$

to define $\iota^* u$. Indeed, the right side makes sense by Proposition 4.7.

Thus we have directly proved the first part of

PROPOSITION 4.9. *Set* $\mathcal{R} = \{u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n); (4.73)$ *holds*$\}$ *then* (4.74) *defines a linear* restriction *map* $\iota^* : \mathcal{R} \longrightarrow \mathcal{C}_c^{-\infty}(\mathbb{R}^p)$ *and*

$$\mathrm{WF}(\iota^* u) \subset \{(x, \xi) \in \mathbb{R}^p \times (\mathbb{R}^p \setminus 0);\ \exists\ \eta \in \mathbb{R}^n \text{ with } (x, 0, \xi, \eta) \in \mathrm{WF}(u)\}. \tag{4.75}$$

PROOF. First note that (4.73) means *precisely* that

$$\hat{u}(\xi, \eta) \text{ is rapidly decreasing in a cone around } \{0\} \times \mathbb{R}^k \backslash 0. \tag{4.76}$$

When $u \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ taking Fourier transforms in (4.72) gives

$$\widehat{\iota^* u}(\xi) = \frac{1}{(2\pi)^k} \int \hat{u}(\xi, \eta) d\eta. \tag{4.77}$$

In general (4.76) ensures that the integral in (4.77) converges, it will then hold by continuity.

We actually apply (4.77) to a localized version of u; if $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^p)$ then

$$\widehat{\psi \iota^*(u)}(\xi) = (2\pi)^{-k} \int \widehat{\psi}(\xi) \widehat{u}(\xi, \eta) d\eta. \tag{4.78}$$

Thus suppose $(\overline{x}, \overline{\xi}) \in \mathbb{R}^p \times (\mathbb{R}^p \backslash 0)$ is such that $(\overline{x}, 0, \overline{\xi}, \eta) \notin \mathrm{WF}(u)$ for *any* η. If ψ has support close to $\overline{x}$ and $\zeta \in \mathcal{C}_c^\infty(\mathbb{R}^{n-p})$ has support close to 0 this means

$$\widehat{\psi\zeta u}(\xi, \eta) \text{ is rapidly decreasing in a cone around each } (\overline{\xi}, \eta). \tag{4.79}$$

We also have rapid decrease around $(0, \eta)$ from (4.76) (make sure you understand this point) as

$$\widehat{\psi\zeta u}(\xi, \eta) \text{ is rapidly decreasing in } \gamma \times \mathbb{R}^p$$

for a cone, γ, around $\overline{\xi}$. From (4.77)

$$\widehat{\psi \iota^*(\zeta u)}(\xi) \text{ is rapidly decreasing in } \gamma.$$

Thus $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(\iota^*(\zeta u))$. Of course if we choose $\zeta(y) = 1$ near 0, $\iota^*(\zeta u) = \iota^*(u)$ so $(\overline{x}, \overline{\xi}) \notin \mathrm{WF}(u)$, provided $(\overline{x}, 0, \overline{\xi}, \eta) \notin \mathrm{WF}(u)$, for all η. This is what (4.75) says. □

Try to picture what is going on here. We can restate the main conclusion of Proposition 4.9 as follows.

Take $\mathrm{WF}(u) \cap \{(x, 0, \xi, \eta) \in \mathbb{R}^p \times \{0\} \times (\mathbb{R}^n \backslash 0)\}$ and let Z denote projection *off* the η variable:

$$\mathbb{R}^p \times \{0\} \times \mathbb{R}^p \times \mathbb{R}^k \xrightarrow{Z} \mathbb{R}^p \times \mathbb{R}^p \tag{4.80}$$

then

$$\mathrm{WF}(\iota^* u) \subset Z(\mathrm{WF}(u) \cap \{y = 0\}).$$

We will want to think more about these operations later.

4.13. Exterior product

This is maybe the easiest of the elementary operators. It is always defined

$$(u_1 \boxtimes u_2)(\phi) = u_1\,(u_2(\phi(x,\cdot)) = u_2(u_1(\phi(\cdot,y)). \tag{4.81}$$

Moreover we can easily compute the Fourier transform:

$$\widehat{u_1 \boxtimes u_2}(\xi,\eta) = \hat{u}_1(\xi)\hat{u}_2(\eta). \tag{4.82}$$

PROPOSITION 4.10. *The (exterior) product*

$$\mathcal{C}_c^{-\infty}(\mathbb{R}^p) \times \mathcal{C}_c^{-\infty}(\mathbb{R}^k) \longleftarrow \mathcal{C}_c^{-\infty}(\mathbb{R}^{p+k}) \tag{4.83}$$

is a bilinear map such that

$$\begin{aligned}\mathrm{WF}(u_1 \boxtimes u_2) \subset {} & [(\mathrm{supp}(u_1) \times \{0\}) \times \mathrm{WF}(u_2)] \\ & \cup [\mathrm{WF}(u_1) \times (\mathrm{supp}(u_2) \times \{0\})] \cup [\mathrm{WF}(u_1) \times \mathrm{WF}(u_2)].\end{aligned}$$

PROOF. We can localize near any point $(\overline{x},\overline{y})$ with $\phi_1(x)\phi_2(y)$, where ϕ_1 is supported near $\overline{x}$ and ϕ_2 is supported near $\overline{y}$. Thus we only need examine the decay of

$$\phi_1\widehat{u_1 \boxtimes \phi_2}u_2 = \widehat{\phi_1 u_1}(\xi) \cdot \widehat{\phi_2 u_2}(\eta). \tag{4.84}$$

Notice that if $\widehat{\phi_1 u_1}(\xi)$ is rapidly decreasing around $\overline{\xi} \neq 0$ then the product is rapidly decreasing around *any* $(\overline{\xi},\eta)$. This gives (4.10). □

4.14. Diffeomorphisms

We next turn to the question of the extension of F^*, where $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map, from $\mathcal{C}^\infty(\Omega_2)$ to some elements of $\mathcal{C}^{-\infty}(\Omega_2)$. The simplest example of pull-back is that of transformation by a diffeomorphism.

We have already noted how pseudodifferential operators behave under a diffeomorphism: $F :\ \Omega_1 \longrightarrow \Omega_2$ between open sets of $\mathbb{R}^n$. Suppose $A \in \Psi^m_\infty(\mathbb{R}^n)$ has Schwartz kernel of *compact* support in $\Omega_1 \times \Omega_1$ then we define

$$A_F : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}_c^\infty(\Omega_2) \tag{4.85}$$

by $A_F = G^* \cdot A \cdot F^*$, $G = F^{-1}$. In § 3.13 we showed that $A_F \in \Psi^m_\infty(\mathbb{R}^n)$. In fact we showed much more, namely we computed a (very complicated) formula for the full symbols. Recall the definition of the *cotangent bundle* of $\mathbb{R}^n$

$$T^*\mathbb{R}^n \simeq \mathbb{R}^n \times \mathbb{R}^n \tag{4.86}$$

identified as pairs of points $(\overline{x},\overline{\xi})$, where $\overline{x} \in \mathbb{R}^n$ and

$$\overline{\xi} = df(\overline{x}) \text{ for some } f \in \mathcal{C}^\infty(\mathbb{R}^n). \tag{4.87}$$

The *differential* $df(\overline{x})$ of f at $\overline{x} \in \mathbb{R}^n$ is just the equivalence class of $f(x) - f(\overline{x}) \in \mathcal{I}_{\overline{x}}$ modulo $\mathcal{I}^2_{\overline{x}}$. Here

$$\begin{cases}\mathcal{I}_{\overline{x}} = \{g \in \mathcal{C}^\infty(\mathbb{R}^n);\ g(\overline{x}) = 0\} \\ \mathcal{I}^2_{\overline{x}} = \{\sum\limits_{\text{finite}} g_i h_i,\ g_i, h_i \in \mathcal{I}_{\overline{x}}\}.\end{cases} \tag{4.88}$$

The identification of $\overline{\xi}$, given by (4.86) and (4.87), with a point in $\mathbb{R}^n$ is obtained using Taylor's formula. Thus if $f \in \mathcal{C}^\infty(\mathbb{R}^n)$

$$f(x) = f(\overline{x}) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_j}(\overline{x})(x - \overline{x})_j + \sum_{i,j=1} g_{ij}(x) x_i x_j. \tag{4.89}$$

The double sum here is in $\mathcal{I}_{\overline{x}}^2$, so the residue class of $f(x) - f(\overline{x})$ in $\mathcal{I}_{\overline{x}}/\mathcal{I}_{\overline{x}}^2$ is the same as that of

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_j}(\overline{x})(x - \overline{x})_j.$$

That is, $d(x - \overline{x})_j = dx_j$, $j = 1, \dots, n$ form a basis for $T^*_{\overline{x}}\mathbb{R}^n$ and in terms of this basis

$$df(\overline{x}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_j}(\overline{x}) dx_j.$$

Thus the entries of $\overline{\xi}$ are just $\left(\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_n}\right)$ for some f. Another way of saying this is that the *linear functions* $\xi \cdot x = \xi_1 x_1 + \xi_2 x_2 \cdots \xi_n x_n$ have differentials spanning $T^*_{\overline{x}}\mathbb{R}^n$.

So suppose $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map. Then

$$F^* : T^*_{\overline{y}}\Omega_2 \longrightarrow T^*_{\overline{x}}\Omega_1, \ \overline{y} = F(\overline{x})$$

is defined by $F^* df(\overline{y}) = d(F^* f)(\overline{x})$ since $F^* : \mathcal{I}_{\overline{y}} \longrightarrow \mathcal{I}_{\overline{x}}$, $F^* : \mathcal{I}_{\overline{y}}^2 \longrightarrow \mathcal{I}_{\overline{x}}^2$. In coordinates $F(x) = y \Longrightarrow$

$$\frac{\partial}{\partial x_j}(F^* f(x)) = \frac{\partial}{\partial y} f(F(x)) = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(y) \frac{\partial F_k}{\partial x_j}$$

i.e. $F^*(\eta \cdot dy) = \xi \cdot dx$ if

$$\xi_j = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_j}(x) \cdot \eta_k. \tag{4.90}$$

Of course if F is a diffeomorphism then the Jacobian matrix $\frac{\partial F}{\partial x}$ is invertible and (4.90) is a linear isomorphism. In this case

$$\begin{gathered} F^* : T^*_{\Omega_2}\mathbb{R}^n \longleftrightarrow T^*_{\Omega_1}\mathbb{R}^n \\ (x, \xi) \longleftrightarrow (F(x), \eta) \end{gathered} \tag{4.91}$$

with ξ and η connected by (4.90). Thus $(F^*)^* : \mathcal{C}^\infty(T^*\Omega_1) \longrightarrow \mathcal{C}^\infty(T^*\Omega_2)$.

Proposition 4.11. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a diffeomorphism of open sets of $\mathbb{R}^n$ and $A \in \Psi^m_\infty(\mathbb{R}^n)$ has Schwartz kernel with compact support in $\Omega_1 \times \Omega_2$ then*

$$\sigma_m(A_F) = (F^*)^* \sigma_m(A) \tag{4.92}$$

and

$$F^*\left(\mathrm{WF}'(A_F)\right) = \mathrm{WF}'(A). \tag{4.93}$$

It follows that symbol $\sigma_m(A)$ of A is well-defined as an element of $S^{m-[1]}_\infty(T^*\mathbb{R}^n)$ independent of coordinates and $\mathrm{WF}'(A) \subset T^*\mathbb{R}^n\backslash 0$ is a well-defined closed conic set, independent of coordinates. The elliptic set and the characteristic set Σ_m are therefore also well-defined complementary conic subsets of $T^*\Omega\backslash 0$.

Proof. Look at the formulae. □

The main use we make of this invariance result is the freedom it gives us to *choose* local coordinates adapted to a particular problem. It also suggests that there should be neater ways to write various formulae, e.g. the wavefront sets of push-forward and pull-backs.

PROPOSITION 4.12. *If* $u \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *has* $\mathrm{supp}(u) \subset \Omega_2$ *and* $F : \Omega_1 \longrightarrow \Omega_2$ *is a diffeomorphism then*

(4.94)
$$\mathrm{WF}(F^*u) \subset \big\{(x,\xi) \in \mathbb{R}^n \times (\mathbb{R}^n\backslash 0); (F(x),\eta) \in \mathrm{WF}(u), \eta_j = \sum_i \frac{\partial F_i}{\partial x_j}(x)\xi_i\big\}.$$

PROOF. Just use the standard definition

$$\mathrm{WF}(F^*u) = \bigcap\big\{\Sigma(A);\ A(F^*u) \in \mathcal{C}^\infty\big\}.$$

To test the wavefront set of F^*u it suffices to consider A's with kernels supported in $\Omega_1 \times \Omega_1$ since $\mathrm{supp}(F^*u \Subset \Omega_1$ and for a general pseudodifferential operator A' there exists A with kernel supported in Ω_1 such that $A'u - Au \in \mathcal{C}^\infty(\mathbb{R}^n)$. Then $AF^*u \in \mathcal{C}_c^\infty(\Omega_1) \Longleftrightarrow A_F u \in \mathcal{C}_c^\infty(\Omega_2)$. Thus

$$\mathrm{WF}(F^*u) = \bigcap\big\{\Sigma(A);\ A_F u \in \mathcal{C}^\infty\big\} \tag{4.95}$$

$$= \bigcap\big\{F^*(\Sigma(A_F)); A_F u \in \mathcal{C}^\infty\big\} \tag{4.96}$$

$$= F^*\,\mathrm{WF}(u) \tag{4.97}$$

since, for u, it is enough to consider operators with kernels supported in $\Omega_2 \times \Omega_2$. □

4.15. Products

Although we have discussed the definition of the product of two distributions we have not yet analyzed the wavefront set of the result.

PROPOSITION 4.13. *If* $u_1,\ u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$ *are such that*

$$(x,\xi) \in \mathrm{WF}(u_1) \Longrightarrow (x,-\xi) \notin \mathrm{WF}(u_2)$$

then the product $u_1u_2 \in \mathcal{C}_c^{-\infty}(\mathbb{R}^n)$*, defined by Proposition 4.7 satisfies*

$$\begin{aligned}\mathrm{WF}(u_1u_2) \subset &\big\{(x,\xi); x \in \mathrm{supp}(u_1) \text{ and } (x,\xi) \in \mathrm{WF}(u_2)\big\}\\ &\cup\big\{(x,\xi); x \in \mathrm{supp}(u_2) \text{ and } (x,\xi) \in \mathrm{WF}(u_1)\big\}\\ &\cup\big\{(x,\xi); \xi = \eta_1 + \eta_2, (x,\eta_i) \in \mathrm{WF}(u_i), i = 1,2\big\}.\end{aligned}$$

PROOF. We can represent the product in terms of three 'elementary' operations.

$$u_1u_2(x) = \iota^*\big[F^*(u_1 \boxtimes u_2)\big] \tag{4.98}$$

where $F : \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2n}$ is the linear transformation

$$F(x,y) = (x+y, x-y)$$

and $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}$ is inclusion as the first factor. Thus (4.98) corresponds to the 'informal' notation

$$u_1u_2(x) = u_1(x+y)u_2(x-y) \restriction \{y = 0\} \tag{4.99}$$

and will follow by continuity once we analyse the wavefront set properties.

We know from Proposition 4.10 that

$$\begin{aligned}\mathrm{WF}\,(u_1 \boxtimes u_2) \subset \big\{&(X,Y,\Xi,H)\,; X \in \mathrm{supp}(u_1), \Xi = 0, (Y,H) \in \mathrm{WF}(u_2)\big\}\\ \cup\big\{&(X,Y,\Xi,H)\,;\ (X,\Xi) \in \mathrm{WF}(u_1),\ Y \in \mathrm{supp}(u_2), H = 0\big\}\\ \cup\big\{&(X,Y,\Xi,H)\,;\ (X,\Xi) \in \mathrm{WF}(u_1),\ (Y,H) \in \mathrm{WF}(u_2)\big\}.\end{aligned}$$

Since F is a diffeomorphism, by Proposition 4.12,

$$\begin{aligned}\mathrm{WF}(F^*(u_1 \boxtimes u_2)) = \big\{(x,y,\xi,\eta); (F^t(x,y),\Xi,H) \in \mathrm{WF}(u_1 \boxtimes u_2),&\\ (\xi,\eta) = A^t(\Xi,H)\big\}.&\end{aligned}$$

where F^t is the transpose of F as a linear map. In fact $F^t = F$, so

$$\begin{aligned}&\mathrm{WF}(F^*(u_1 \boxtimes u_2)) \subset\\ &\quad\big\{(x,y,\xi,\eta); x+y \in \mathrm{supp}(u_1),\ \xi+\eta = 0, (x-y, \frac{1}{2}(\xi-\eta)) \in \mathrm{WF}(u_2)\big\}\\ &\quad\cup\big\{(x,y,\xi,\eta); (x+y, \frac{1}{2}(\xi+\eta)) \in \mathrm{WF}(u_1),\ (x-y, \frac{1}{2}(\xi-\eta)) \in \mathrm{WF}(u_2)\big\}\end{aligned}$$

and so using Proposition 4.9

$$\begin{aligned}\mathrm{WF}(F^*(u_1 \boxtimes u_2)) &\restriction \{y=0\}\\ &\subset \big\{(x,0,\xi,-\xi); x \in \mathrm{supp}(u_1), (x,\xi) \in \mathrm{WF}(u_2)\big\}\\ &\cup \big\{(x,0,\xi,\eta); (x \in \mathrm{supp}(u_2), (x,\xi) \in \mathrm{WF}(u_2)\big\}\\ &\cup \big\{(x,0,\xi,\eta); (x, \frac{1}{2}(\xi+\eta)) \in \mathrm{WF}(u_2), (x, \frac{1}{2}(\xi-\eta)) \in \mathrm{WF}(u_1)\big\}\end{aligned}$$

Notice that

$$(x,0,0,\eta) \in \mathrm{WF}\,(F^*(u_1 \boxtimes u_2)) \Longrightarrow (x, \frac{1}{2}\eta) \in \mathrm{WF}(u_1) \text{ and } (x, \frac{1}{2}\eta)\,\mathrm{WF}(u_2)$$

which introduces the assumption under which u_1u_2 is defined. Finally then we see that

(4.100)

$$\begin{aligned}\mathrm{WF}(u_1u_2) \subset &\big\{(x,\xi); x \in \mathrm{supp}(u_1), (x,\xi) \in \mathrm{WF}(u_2)\big\}\\ &\cup\big\{(x,\xi); x \in \mathrm{supp}(u_2), (x,\xi) \in \mathrm{WF}(u_1)\big\}\\ &\cup\big\{(x,\xi); (x,\eta_1) \in \mathrm{WF}(u_1), (x,\eta_2) \in \mathrm{WF}(u_2) \text{ and } \xi = \eta_1+\eta_2\big\}.\end{aligned}$$

which is another way of writing the conclusion of Proposition 4.13. □

4.16. Pull-back

Now let us consider a general $\mathcal{C}^\infty$ map

$$F : \Omega_1 \longrightarrow \Omega_2,\ \Omega_1 \subset \mathbb{R}^n, \Omega_2 \subset \mathbb{R}^m. \tag{4.101}$$

Thus even the dimension of domain and range spaces can be different. When can we *define* F^*u, for $u \in \mathcal{C}_c^{-\infty}(\Omega_2)$ and what can we say about $\mathrm{WF}(F^*u)$? For a general map F it is not possible to give a sensible, i.e. consistent, definition of F^*u for all distributions $u \in \mathcal{C}^{-\infty}(\Omega_2)$.

For smooth functions we have defined

$$F^* : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}^\infty(\Omega_1)$$

but in general $F^*\phi$ does not have compact support, even if ϕ does. We therefore impose the condition that F be *proper*

$$F^{-1}(K) \Subset \Omega_2 \ \forall \ K \Subset \Omega_2, \tag{4.102}$$

(mostly just for convenience). In fact if we want to understand F^*u near $\overline{x}_1 \in \Omega_1$ we only need to consider u near $F(\overline{x}_1) \in \Omega_2$.

The problem is that the map (4.101) may be rather complicated. However *any* smooth map can be decomposed into a product of simpler maps, which we can analyze locally. Set

$$\operatorname{graph}(F) = \big\{(x,y) \in \Omega_1 \times \Omega_2;\ y = F(x)\big\} \xrightarrow{\iota_F} \Omega_1 \times \Omega_2.$$

This is *always* an embedded submanifold of $\Omega_1 \times \Omega_2$ the functions $y_i - F_i(x)$, $i = 1, \dots, N$ are independent defining functions for $\operatorname{graph}(F)$ and $x_1, \dots, x_n$ are coordinates on it. Now we can write

$$F = \pi_2 \circ \iota_F \circ g \tag{4.103}$$

where $g : \Omega_1 \longleftrightarrow \operatorname{graph}(F)$ is the diffeomorphism onto its range $x \longmapsto (x, F(x))$. This decomposes F as a projection, an inclusion and a diffeomorphism. Now consider

$$F^*\phi = g^* \cdot \iota_F^* \cdot \pi_2^*\phi \tag{4.104}$$

i.e. $F^*\phi$ is obtained by pulling ϕ back from Ω_2 to $\Omega_1 \times \Omega_2$, restricting to $\operatorname{graph}(F)$ and then introducing the x_i as coordinates. We have directly discussed $(\pi_2^*\phi)$ but we can actually write it as

$$\pi_2^*\phi = 1 \boxtimes \phi(y), \tag{4.105}$$

so the result we have proved can be applied to it. So let us see what writing (4.104) as

$$F^*\phi = g^* \circ \iota_F^*(1 \boxtimes \phi) \tag{4.106}$$

tells us. If $u \in \mathcal{C}_c^{-\infty}(\Omega_2)$ then

$$\operatorname{WF}(1 \boxtimes u) \subset \big\{(x, y, 0, \eta); (y, \eta) \in \operatorname{WF}(u)\big\} \tag{4.107}$$

by Proposition 4.10. So we have to discuss $\iota_F^*(1 \boxtimes u)$, i.e. restriction to $y = F(x)$. We can do this by making a diffeomorphism:

$$T_F(x, y) = (x, y + F(x)) \tag{4.108}$$

so that $T_F^{-1}(\operatorname{graph}(F)) = \{(x, 0)\}$. Notice that $g \circ T_F = \pi_1$, so

$$F^*\phi = \iota_{\{y=0\}}^* \left(T_F^*(1 \boxtimes u)\right). \tag{4.109}$$

Now from Proposition 4.12 we know that

$$\begin{aligned}\operatorname{WF}(T_F^*(1 \boxtimes u)) &= T_F^*(\operatorname{WF}(1 \boxtimes u)) \\ &= \big\{(X, Y, \Xi, H); (X, Y + F(X), \xi, \eta) \in \operatorname{WF}(1 \boxtimes u), \\ &\qquad\qquad \eta = H, \xi_i = \Xi_i + \Sigma \frac{\partial F_j}{\partial x_i} H_j\big\}\end{aligned} \tag{4.110}$$

i.e.

$$\operatorname{WF}(T_F^*(1 \boxtimes u)) = \big\{(x, y, \xi, \eta); \xi_i = \sum_j \frac{\partial F_j}{\partial x_j}(x)\eta_j, (F(x), \eta) \in \operatorname{WF}(u)\big\}. \tag{4.111}$$

So consider our existence condition for restriction to $y = 0$, that $\xi \neq 0$ on $\mathrm{WF}(T_F^*(1 \boxtimes u))$ i.e.

$$(F(x), \eta) \in \mathrm{WF}(u) \Longrightarrow \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j \neq 0. \tag{4.112}$$

If (4.112) holds then, from (4.110) and Proposition 4.9

$$\mathrm{WF}(F^*u) \subset \big\{(x, \xi);\ \exists\ (F(x), \eta) \in \mathrm{WF}(u) \text{ and } \xi_j = \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j\big\}. \tag{4.113}$$

We can reinterpret (4.112) and (4.113) more geometrically. The differential of F gives a map

$$F^* : T^*_{F(x)}\Omega_2 \longrightarrow T^*_x\Omega_1 \ \forall\ x \in \Omega_1$$
$$(F(x), \eta) \longmapsto (x, \xi) \text{ where } \xi_i = \Sigma \frac{\partial F_j}{\partial x_i}\eta_j.$$

Thus (4.112) can be restated as:

$$\begin{gathered} \forall\ x \in \Omega_1, \text{ the null space of } F_x^* : T^*_{F(x)}\Omega_2 \longrightarrow T^*_x\Omega_1 \\ \text{does not meet } \mathrm{WF}(u) \end{gathered} \tag{4.114}$$

and then (4.113) becomes

$$\mathrm{WF}(F^*u) \subset \bigcup_{x \in \Omega_1} F_x^*[\mathrm{WF}(u) \cap T^*_{F(x)}\Omega_2] = F^*(\mathrm{WF}(u)) \tag{4.115}$$

(proved we are a little careful in that F^* is *not* a map; it is a *relation* between $T^*\Omega_2$ and $T^*\Omega_1$) and in this sense (4.114) holds. Notice that (4.113) is a sensible "consequence" of (4.114), since otherwise $\mathrm{WF}(F^*u)$ would contain some zero directions.

Proposition 4.14. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a proper $\mathcal{C}^\infty$ map then F^* extends (by continuity) from $\mathcal{C}^\infty_c(\Omega_2)$ to*

$$\big\{u \in \mathcal{C}^{-\infty}_c(\Omega_2); F^*(\mathrm{WF}(u)) \cap (\Omega_1 \times 0) = \emptyset \textit{ in } T^*\Omega_1\big\} \tag{4.116}$$

and (4.115) *holds.*

4.17. The operation F_*

Next we will look at the dual operation, that of push-forward. Notice the basic properties of pull-back:

$$\text{Maps } \mathcal{C}^\infty_c \text{ to } \mathcal{C}^\infty_c (\text{if } F \text{ is proper}) \tag{4.117}$$

$$\textit{Not} \text{ always defined on distributions.} \tag{4.118}$$

Dually we find

Proposition 4.15. *If $F : \Omega_1 \longrightarrow \Omega_2$ is a $\mathcal{C}^\infty$ map of an open subset of $\mathbb{R}^n$ into an open subset of $\mathbb{R}^n$ then for any $u \in \mathcal{C}^{-\infty}_c(\Omega_1)$*

$$F_*(u)(\phi) = u(F^*\phi) \tag{4.119}$$

is a distribution of compact support and

$$F_* : \ \mathcal{C}^{-\infty}_c(\Omega_1) \longrightarrow \mathcal{C}^{-\infty}_c(\Omega_2) \tag{4.120}$$

has the property:

$$\begin{aligned}\mathrm{WF}(F_*u) \subset \big\{(y,\eta); y \in F(\mathrm{supp}(u)), y = F(x), F_x^*\eta = 0\big\} \cup \\ \big\{(y,\eta); y = F(x), (x, F_x^*\eta) \in \mathrm{WF}(u)\big\}.\end{aligned} \tag{4.121}$$

PROOF. Notice that the 'opposite ' of (4.117) and (4.118) hold, i.e. F_* is *always* defined but even if $u \in \mathcal{C}_c^\infty(\Omega_1)$ in general $F_*u \notin \mathcal{C}_c^\infty(\Omega_2)$. All we really have to prove is (4.121). As usual we look for a formula in terms of elementary operations. So suppose $u \in \mathcal{C}_c^\infty(\Omega_1)$

$$\begin{aligned}F_*u(\phi) &= u(F^*\phi) \quad \phi \in \mathcal{C}_c^\infty(\Omega_2) \\ &= \int u(x)\ \phi(F(x))\ dx \\ &= \int u(x)\delta(y - F(x))\ \phi(y)\ dydx.\end{aligned} \tag{4.122}$$

Thus, we see that

$$F_*u = \pi_*H^*(u \boxtimes \delta) \tag{4.123}$$

where $\delta = \delta(y) \in \mathcal{C}_c^{-\infty}(\mathbb{R}^m)$, H is the diffeomorphism

$$H(x,y) = (x, y - F(x))$$

and $\pi : \mathbb{R}^{n+m} \longrightarrow \mathbb{R}^m$ is projection *off* the first factor.

Thus (4.123) is the desired decomposition into elementary operations, since $u\boxtimes\delta \in \mathcal{C}_c^{-\infty}(\mathbb{R}^{n+m})$, $\pi_*H^*(u\boxtimes\delta)$ is always defined and indeed the map is continuous, which actually proves (4.123).

So all we need to do is estimate the wavefront set using our earlier results. From Proposition 4.10 it follows that

$$\begin{aligned}\mathrm{WF}(u \boxtimes \delta) \subset &\big\{(x,0,\xi,\eta); x \in \mathrm{supp}(u), \xi = 0\big\} \cup \big\{(x,0,\xi,0); (x,\xi) \in \mathrm{WF}(u)\big\} \\ &\cup \big\{(x,0,\xi,\eta); (x,\xi) \in \mathrm{WF}(u)\big\} \\ &= \big\{(x,0,\xi,\eta); x \in \mathrm{supp}(u), \xi = 0\big\} \cup \big\{(x,0,\xi,\eta); (x,\xi) \in \mathrm{WF}(u)\big\}.\end{aligned} \tag{4.124}$$

Then consider what happens under H^*. This is a diffeomorphism so the wavefront set transforms under the pull-back:

$$\begin{aligned}&\mathrm{WF}(H^*(u \boxtimes \delta)) = \mathrm{WF}(u(x)\delta(y - F(x)) \\ &= \big\{(x, F(x), \Xi, \eta); \Xi_i = \xi_i - \sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j, (x,0,\xi,\eta) \in \mathrm{WF}(u \boxtimes \delta)\big\} \\ &= \big\{(x, F(x), \Xi, \eta); x \in \mathrm{supp}(u), \Xi_i = -\sum_j \frac{\partial F_j}{\partial x_i}(x)\eta_j)\big\} \\ &\quad \cup \big\{(x, F(x), \Xi, \eta); \eta \in \mathbb{R}^m, (x,\xi) \in \mathrm{WF}(u)), \Xi_i = \xi_i - \sum_j \frac{\partial F_i}{\partial x_j}\eta_j\big\}.\end{aligned}$$

Finally recall the behaviour of wavefront sets under projection, to see that

$$\begin{aligned}
&\mathrm{WF}(F_*u) \subset \big\{(y,\eta);\ \exists\ (x,y,0,\eta) \in \mathrm{WF}(H^*(u \boxtimes \delta))\big\} \\
&= \big\{(y,\eta); y = F(x) \text{ for some } x \in \operatorname{supp}(u) \text{ and} \\
&\qquad\qquad \sum_j \frac{\partial F_j}{\partial x_i}\eta_j = 0,\ i = 1,\dots,n\big\} \\
&\cup\big\{(y,\eta); y = F(x) \text{ for some } (x,\xi) \in \mathrm{WF}(u) \text{ and} \\
&\qquad\qquad \xi_i = \sum_j \frac{\partial F_i}{\partial x_i}\eta_j, i = 1,\dots,n)\big\}.
\end{aligned}$$

This says

$$\mathrm{WF}(F_*u) \subset \big\{(y,\eta); y \in F(\operatorname{supp}(u)) \text{ and } F_x^*(\eta) = 0\big\} \tag{4.125}$$

$$\cup \big\{(y,\eta); y = F(x) \text{ with } (x, F_x^*\eta) \in \mathrm{WF}(u)\big\} \tag{4.126}$$

which is just (4.121). □

As usual one should note that the two terms here are "really the same".

Now let us look at F_* as a linear map,

$$F_* : \mathcal{C}_c^\infty(\Omega_1) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_2). \tag{4.127}$$

As such it has a Schwartz kernel, indeed (4.123) is just the usual formula for an operator in terms of its kernel:

$$F_*u(y) = \int K(y,x)u(x)dx,\ K(y,x) = \delta(y - F(x)). \tag{4.128}$$

So consider the wavefront set of the kernel:

$$\mathrm{WF}(\delta(y - F(x)) = \mathrm{WF}(H^*\delta(y)) = \big\{(y,x;\eta,\xi); y = F(x), \xi = F_x^*\eta\big\}. \tag{4.129}$$

Now changing the order of the factors we can regard this as a subset

(4.130)
$$\mathrm{WF}'(K) = \big\{((y,\eta),(x,\xi)); y = F(x), \xi = F^*\eta\big\} \subset (\Omega_2 \times \mathbb{R}^m) \times (\Omega_1 \times \mathbb{R}^n).$$

As a subset of the product we can regard $\mathrm{WF}'(K)$ as a *relation*: if $\Gamma \subset \Omega_2 \times (\mathbb{R}^n \backslash 0)$ set

$$\begin{gathered}
\mathrm{WF}'(K) \circ \Gamma = \\
\big\{(y,\eta) \in \Omega_2 \times (\mathbb{R}^m \backslash 0); \exists\ ((y,\eta)),(x,\xi)) \in \mathrm{WF}'(K) \text{ and } (x,\xi) \in \Gamma\big\}
\end{gathered}$$

Indeed with this definition

$$\mathrm{WF}(F_*u) \subset \mathrm{WF}'(K) \circ \mathrm{WF}(u), \quad K = \text{ kernel of } F_*. \tag{4.131}$$

4.18. Wavefront relation

One serious application of our results to date is:

THEOREM 4.1. *Suppose* $\Omega_1 \subset \mathbb{R}^n$, $\Omega_2 \subset \mathbb{R}^m$ *are open and* $A \in \mathcal{C}^{-\infty}(\Omega_1 \times \Omega_2)$ *has proper support, in the sense that the two projections*

$$\begin{tikzcd} & \mathrm{supp}(A) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \Omega_1 & & \Omega_2 \end{tikzcd} \tag{4.132}$$

are proper, then A *defines a linear map*

$$A : \mathcal{C}_c^\infty(\Omega_2) \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_1) \tag{4.133}$$

and extends by continuity to a linear map

$$A : \big\{u \in \mathcal{C}_c^{-\infty}(X); \mathrm{WF}(u) \cap \big\{(y,\eta) \in \Omega_2 \times (\mathbb{R}^n \backslash 0); \tag{4.134}$$

$$\exists\ (x,0,y,-\eta) \in \mathrm{WF}(K)\big\} = \emptyset\big\} \longrightarrow \mathcal{C}_c^{-\infty}(\Omega_1) \tag{4.135}$$

for which

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(A) \circ \mathrm{WF}(u), \tag{4.136}$$

where

$$\begin{aligned} \mathrm{WF}'(A) = \big\{((x,\xi),(y,\eta)) \in &(\Omega_1 \times \mathbb{R}^n) \times (\Omega_2 \times \mathbb{R}^m); (\xi,\eta) \neq 0 \\ &\textit{and } (x,y,\xi,-\eta) \in \mathrm{WF}(K)\big\}. \end{aligned} \tag{4.137}$$

PROOF. The action of the map A can be written in terms of its Schwartz kernel as

$$Au(x) = \int K(x,y)u(y)dy = (\pi_1)_*(K \cdot (1 \boxtimes u)). \tag{4.138}$$

Here $1 \boxtimes u$ is always defined and

$$\mathrm{WF}(1 \boxtimes u) \subset \big\{(x,y,0,\eta); (y,\eta) \in \mathrm{WF}(u)\big\}.$$

So the main question is, when is the product defined? Our sufficient condition for this is:

$$(x,y,\xi,\eta) \in \mathrm{WF}(K) \Longrightarrow (x,y,-\xi,-\eta) \notin \mathrm{WF}(1 \boxtimes u)$$

which is

$$(x,y,0,\eta) \in \mathrm{WF}(K) \Longrightarrow (x,y,0,-\eta) \notin \mathrm{WF}(1 \boxtimes u)$$
$$\text{i.e. } (y,-\eta) \notin \mathrm{WF}(u)$$

This of course is (4.135):

$$Au \text{ is defined (by continuity) if}$$
$$\big\{(y,\eta) \in \mathrm{WF}(u);\ \exists\ (x,0,y,-\eta) \in \mathrm{WF}(A)\big\} = \emptyset.$$

Then from our bound on the wavefront set of a product

$$\begin{aligned} &\mathrm{WF}\,(K \cdot (1 \boxtimes u)) \subset \\ &\big\{(x,y,\xi,\eta); (\xi,\eta) = (\xi',\eta') + (0,\eta'') \text{ with} \\ &\qquad (x,y,\xi',\eta') \in \mathrm{WF}(K) \text{ and } (x,\eta'') \in \mathrm{WF}(u)\big\} \\ &\cup\big\{(x,y,\xi,\eta); (x,y,\xi,\eta) \in \mathrm{WF}(K),\ y \in \mathrm{supp}(u)\big\} \\ &\cup\big\{(x,y,0,\eta); (x,y) \in \mathrm{supp}(A)(y,\eta) \in \mathrm{WF}(u)\big\}. \end{aligned}$$

This gives the bound

$$\begin{aligned}(4.139)\quad \mathrm{WF}\left(\pi_*(K\cdot(1\boxtimes u))\right) &\subset \left\{(x,\xi);(x,y,\xi,0)\in \mathrm{WF}(K\cdot(1\boxtimes u)) \text{ for some } y\right\}\\ (4.140)\quad &\subset \mathrm{WF}'(A)\circ \mathrm{WF}(u).\end{aligned}$$

□

4.19. Applications

Having proved this rather general theorem, let us note some examples and applications.

First, for pseudodifferential operators we know that

$$\mathrm{WF}'(A)\subset\{(x,x,\xi,\xi)\}$$

i.e. corresponds to the identity relation (which is a map). Then (4.136) is the microlocality of pseudodifferential operators. The next result also applies to all pseudodifferential operators.

Corollary 4.2. *If $K\in\mathcal{C}^{-\infty}(\Omega_1\times\Omega_2)$ has proper support and*

$$\mathrm{WF}'(K)\cap\{(x,y,\xi,0)\}=\emptyset \tag{4.141}$$

then the operator with Scwartz kernel K defines a continuous linear map

$$A:\mathcal{C}^\infty_c(\Omega_2)\longrightarrow\mathcal{C}^\infty_c(\Omega_1). \tag{4.142}$$

If

$$\mathrm{WF}'(K)\cap\{(x,y,0,\eta)\}=\emptyset \tag{4.143}$$

then A extends by continuity to

$$A:\mathcal{C}^{-\infty}_c(\Omega_2)\longrightarrow\mathcal{C}^{-\infty}_c(\Omega_1). \tag{4.144}$$

Proof. Immediate from (4.133)-(4.144). □

4.20. Problems

Problem 4.1. Show that the general definition (4.9) reduces to

$$\mathrm{WF}(u)=\bigcap\left\{\Sigma_0(A);\ A\in\Psi^0_\infty(\mathbb{R}^n) \text{ and } Au\in\mathcal{C}^\infty(\mathbb{R}^n)\right\},\ u\in\mathcal{S}'(\mathbb{R}^n)$$

and prove the basic result of 'microlocal elliptic regularity:'

$$\begin{gathered}\text{If } u\in\mathcal{S}'(\mathbb{R}^n) \text{ and } A\in\Psi^m_\infty(\mathbb{R}^n) \text{ then}\\ \mathrm{WF}(u)\subset\Sigma(A)\cup\mathrm{WF}(Au).\end{gathered} \tag{4.145}$$

Problem 4.2. Compute the wavefront set of the following distributions:

$$\delta(x)\in\mathcal{S}'(\mathbb{R}^n),\ |x|\in\mathcal{S}'(\mathbb{R}^n) \text{ and}$$

$$\chi_{\mathbb{B}^n}(x)=\begin{cases}1 & |x|\le 1\\ 0 & |x|>1.\end{cases}$$

Problem 4.3. Let $\Gamma\subset\mathbb{R}^n\times(\mathbb{R}^n\setminus 0)$ be an open cone and define

$$\begin{aligned}(4.146)\quad \mathcal{C}^{-\infty}_{c,\Gamma}(\mathbb{R}^n)=&\left\{u\in\mathcal{C}^{-\infty}_c(\mathbb{R}^n);Au\in\mathcal{C}^\infty(\mathbb{R}^n)\right.\\ (4.147)\quad &\left.\forall\ A\in\Psi^0_\infty(\mathbb{R}^n) \text{ with } \mathrm{WF}'(A)\cap\Gamma=\emptyset\right\}.\end{aligned}$$

Describe a complete topology on this space with respect to which $\mathcal{C}^\infty_c(\mathbb{R}^n)$ is a dense subspace.

PROBLEM 4.4. Show that, for any pseudodifferential operator $A \in \Psi^m_\infty(\mathbb{R}^n)$, $\mathrm{WF}'(A) = \mathrm{WF}'(A^*)$.

PROBLEM 4.5. Give an alternative proof to Lemma 4.3 along the following lines (rather than using Lemma 3.1). If $\sigma_L(A)$ is the left reduced symbol then for $\epsilon > 0$ small enough

$$b_0 = \gamma_\epsilon / \sigma_L(A) \in S^{-m}_\infty(\mathbb{R}^n; \mathbb{R}^n).$$

If we choose $B_0 \in \Psi^{-m}_\infty(\mathbb{R}^n)$ with $\sigma_L(B_0) = b_0$ then

$$\mathrm{Id} - A \circ B_0 = G \in \Psi^0_\infty(\mathbb{R}^n) \tag{4.148}$$

has principal symbol

$$\sigma_0(G) = 1 - \sigma_L(A) \cdot b_0.$$

From (4.22)

$$\gamma_{\epsilon/4}\sigma_0(G) = \gamma_{\epsilon/4}.$$

Thus we conclude that if $\sigma_L(C) = \gamma_{\epsilon/4}$ then

$$G = (\mathrm{Id} - C)G + CG \text{ with } CG \in \Psi^{-1}_\infty(\mathbb{R}^n).$$

Thus (4.148) becomes

$$\mathrm{Id} - AB_0 = CG + R_1 \quad \mathrm{WF}'(R_1) \not\ni z. \tag{4.149}$$

Let $B_1 \sim \sum\limits_{j \geq 1} (CG)^j$, $B_1 \in \Psi^{-1}$ and set

$$B = B_0 (\mathrm{Id} + B_1) \in \Psi^{-m}_\infty(\mathbb{R}^n).$$

From (4.149)

$$\begin{aligned} AB &= AB_0(I + B_1) && (4.150)\\ &= (\mathrm{Id} - CG)(I + B_1) - R_1 (\mathrm{Id} + B_1) && (4.151)\\ &= \mathrm{Id} + R_2, \quad \mathrm{WF}'(R_2) \not\ni z. && (4.152) \end{aligned}$$

Thus B is a right microlocal parametrix as desired. Write out the construction of a left parametrix using the same method, or by finding a right parametrix for the adjoint of A and then taking adjoints using Problem 4.4.

PROBLEM 4.6. Essential uniqueness of left and right parametrices.

PROBLEM 4.7. If $(\bar{x}, \bar{\xi}) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus 0)$ is a given point, construct a distribution $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ which has

$$\mathrm{WF}(u) = \left\{ (\bar{x}, t\bar{\xi}); t > 0 \right\} \subset \mathbb{R}^n \times (\mathbb{R}^n \setminus 0).$$

PROBLEM 4.8. Suppose that $A \in \Psi^m_\infty(\mathbb{R}^n)$ has Schwartz kernel of compact support. If $u \in \mathcal{C}^{-\infty}_c(\mathbb{R}^n)$ use the four 'elementary operations' (and an earlier result on the wavefront set of kernels) to investigate under what conditions

$$\kappa(x, y) = K_A(x, y)u(y) \text{ and then } \gamma(x) = (\pi_1)_* \kappa$$

make sense. What can you say about $\mathrm{WF}(\gamma)$?

PROBLEM 4.9. Consider the projection operation under $\pi_1 : \mathbb{R}^p \times \mathbb{R}^k \longrightarrow \mathbb{R}^p$. Show that $(\pi_1)_*$ can be extended to some distributions which do not have compact support, for example

$$\left\{ u \in \mathcal{S}'(\mathbb{R}^n); \mathrm{supp}(u) \cap K \times \mathbb{R}^k \text{ is compact for each } K \subset\subset \mathbb{R}^n \right\}.$$